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Weighted Clément operator and application to the finite element discretization of the axisymmetric Stokes problem

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Abstract We consider the Stokes problem in an axisymmetric three-dimensional domain with data which are axisymmetric and have angular component equal to zero. We observe that the solution is also axisymmetric and the velocity has also zero angular component, hence the solution satisfies a system of equations in the meridian domain. The weak three-dimensional problem reduces to a two-dimensional one with weighted integrals. The latter is discretized by Taylor–Hood type finite elements. A weighted Clément operator is defined and approximation results are proved. This operator is then used to derive the discrete inf–sup condition and optimal a priori error estimates.

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1 Introduction

Numerical simulation of three-dimensional flows with finite elements may feature a very high computational complexity. Reducing the dimension of the problem is sometimes of paramount interest. A simple approach consists of

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using Stokes equations in two dimensions and solve them with finite elements. This majorly reduces the size of the problem, but several three-dimensional features are not present in the model. If the problem is set in a domain which is symmetric by rotation around an axis, it is proved in [4] that, when using a Fourier expansion with respect to the angular variable, the three-dimensional problem is equivalent to a system of two-dimensional problems on the meridian domain, each problem being satisfied by a Fourier coefficient of the solution. So it is possible to reduce its size without losing three-dimensional features.

Here we are going to present an axisymmetric model which supposes data with angular component equal to zero. The advantage is that its discretization results in a linear system of the same size as a two-dimensional one. In this case, all the Fourier coefficients of the solution but the one of order zero vanish. So the number of unknowns in its discretization is the same as in the Cartesian two-dimensional one. The only further difficulty is that the variational formulation requires weighted Sobolev spaces.

For the discretization of the Stokes problem we have chosen to work with Taylor–Hood finite elements, also called P1isoP2/P1 elements: The approximation of the pressure makes use of continuous piecewise affine functions and the approximation of the velocity relies also on continuous piecewise affine functions but on a finer mesh. We refer to [14,16] for the numerical analysis of the discretization by other types of finite elements in a similar framework.

As usual, the numerical analysis of the discrete problem relies on an inf–sup condition of Babuška [1] and Brezzi [6] type. For the analogous discretization of the two-dimensional Stokes problem in the Cartesian case, Bercovier and Pironneau in [2] prove an inf–sup condition and Verfürth in [15] refines the analysis of these elements. Our aim is to extend these results to the axisymmetric case.

The proof of the inf–sup condition in [15] needs a very accurate approximation property of the discrete spaces, involving both the usual Lagrange interpolation operator and the Clément projection operator (see [8]). One of the main parts of this paper is devoted to the extension of the properties of these operators to the weighted Sobolev spaces. A first work in this subject is due to Mercier and Raugel (see [13]). However the results therein are not sufficient for our needs.

Once these results are established we prove an optimal inf–sup condition between discrete spaces and optimal a priori error estimates.

An extension of these results to Navier–Stokes equations may be found in [9] and an other extension to the case of non-axisymmetric solutions issued from non-axisymmetric data is under consideration.

The outline of the paper is as follows:

– In Sect. 2 we introduce the basic assumptions and the Stokes problem in the axisymmetric framework. Then we recall the definition of the weighted Sobolev spaces and some of their properties. This allows to write the variational formulation of the Stokes problem in the meridian domain. At the end of the section, the finite element formulation is presented.

- In Sect. 3 we prove weighted inverse inequalities for the finite element functions.
- Sect. 4 is devoted to the definition of weighted Clément operators and the proof of some of their fundamental properties.
- In Sect. 5 we prove the discrete inf–sup condition related to the Taylor–Hood elements and a priori error estimates.

2 Axisymmetric Stokes equations

In this section we introduce the model, the notation and we recall some results from [4].

2.1 Model description

We are interested in modeling a flow through a domain $\check{\Omega}$ symmetric with respect to the z axis (see Fig. 1). We use cylindrical coordinates (r, θ, z) and we note Ω the half section $(r, 0, z)$. On the boundary $\check{\Gamma}$ of the physical domain $\check{\Omega}$ we impose a Dirichlet boundary condition. Γ notes the half section of $\check{\Gamma}$ and Γ_0 the intersection of $\check{\Omega}$ with the axis, such that $\partial\Omega$ is the union of Γ and Γ_0. All vector fields on $\check{\Omega}$ are expressed in cylindrical coordinates.

The fluid is modeled by Stokes equations in the domain $\check{\Omega}$ and we suppose that the boundary condition and the external forces are axisymmetric and that their angular component is zero.

Scalar functions $\check{p}$ or vector fields $\check{\boldsymbol{u}}$ on $\check{\Omega}$ are *axisymmetric* (with respect to the z-axis) if for any rotation $\mathcal{R}_\eta$ around the z-axis and arbitrary angle η in $[-\pi, \pi)$, it holds
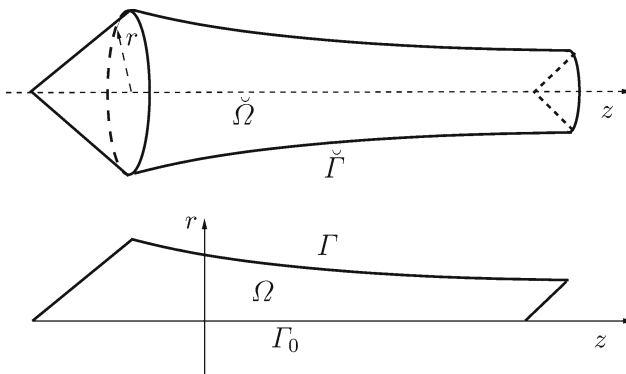

Fig. 1 The axisymmetric domain $\check{\Omega}$ and its half section Ω

$$\check{p} \circ \mathcal{R}_\eta = \check{p},$$
$$\mathcal{R}_{-\eta} \left(\check{\boldsymbol{u}} \circ \mathcal{R}_\eta \right) = \check{\boldsymbol{u}}.$$

In particular each cylindrical component of $\check{\boldsymbol{u}}$ is also axisymmetric.

An axisymmetric function $\check{p}$ on $\check{\Omega}$ depends only on the radial and axial coordinates, therefore we associate a function p on Ω such that $p(r,z) = \check{p}(r,0,z)$. An axisymmetric vector field $\check{\boldsymbol{u}}$ depends on (r,z). For any vector field $\check{\boldsymbol{u}}$, we denote by $\check{u}_r, \check{u}_\theta, \check{u}_z$ its radial, angular and axial components. If it has zero angular component ($\check{u}_\theta = 0$), we associate a vector field $\boldsymbol{u} = (u_r, u_z)$ on Ω such that $u_r = \check{u}_r$ and $u_z = \check{u}_z$.

2.2 The Stokes problem and its weak form

Suppose that the axisymmetric domain $\check{\Omega}$ is bounded, has a Lipschitz–continuous boundary, that Γ_0 is a finite union of segments of positive length and that the data are axisymmetric with zero angular component.

The stationary homogeneous three-dimensional Stokes problem reads

$$\begin{cases} -\nu \Delta \check{\boldsymbol{u}} + \boldsymbol{\nabla} \check{p} = \check{\boldsymbol{f}} & \text{in } \check{\Omega}, \\ \text{div } \check{\boldsymbol{u}} = 0 & \text{in } \check{\Omega}, \\ \check{\boldsymbol{u}} = \boldsymbol{0} & \text{on } \partial \check{\Omega}, \end{cases} \tag{1}$$

where $\check{\boldsymbol{f}}$ is in $H^{-1}(\check{\Omega})^3$. For simplicity we have chosen to work with zero boundary data, however the further analysis extends without difficulty to axisymmetric boundary data $\check{\boldsymbol{g}}$ with zero angular component and zero flux through $\partial \check{\Omega}$. Differential equation (1) writes in weak form as:

Find $(\check{\boldsymbol{u}}, \check{p})$ in $H_0^1(\check{\Omega})^3 \times L_0^2(\check{\Omega})$ such that for all $(\check{\boldsymbol{v}}, \check{q})$ in $H_0^1(\check{\Omega})^3 \times L_0^2(\check{\Omega})$

$$\begin{cases} \check{a}(\check{\boldsymbol{u}}, \check{\boldsymbol{v}}) + \check{b}(\check{\boldsymbol{v}}, \check{p}) = \int\limits_{\check{\Omega}} \check{\boldsymbol{f}} \cdot \check{\boldsymbol{v}} \, d\check{\boldsymbol{x}}, \\ \check{b}(\check{\boldsymbol{u}}, \check{q}) = 0, \end{cases} \tag{2}$$

where the bilinear forms $\check{a}$ and $\check{b}$ are defined as

$$\check{a}(\check{\boldsymbol{u}}, \check{\boldsymbol{v}}) = \nu \int\limits_{\check{\Omega}} \left(\boldsymbol{\nabla} \check{\boldsymbol{u}} : \boldsymbol{\nabla} \check{\boldsymbol{v}} \right) \, d\check{\boldsymbol{x}},$$
$$\check{b}(\check{\boldsymbol{u}}, \check{q}) = - \int\limits_{\check{\Omega}} (\text{div } \check{\boldsymbol{u}}) \, \check{q} \, d\check{\boldsymbol{x}},$$

$H_0^1(\check{\Omega})$ stands for the space of functions in $H^1(\check{\Omega})$ with zero trace and $L_0^2(\check{\Omega})$ for the space of functions in $L^2(\check{\Omega})$ with integral equal to zero.

Bernardi et al. have shown in [4, Sect. IX.1] that this problem has a unique axisymmetric solution and that it can be split in two separate problems on Ω, one for the angular component $\check{u}_\theta$ and the other for $(\check{u}_r, \check{u}_z, p)$. If the data have

no rotation as supposed, i.e., the angular component $\check{f}_\theta$ is equal to zero, then $\check{u}_\theta$ is also zero.

2.3 Weighted Sobolev spaces

In this section we introduce some weighted Sobolev spaces (see [12] and [4, Sect. II.1]) that we use in the weak formulation of the problem.

For any real number α and $1 \leqslant p < \infty$, the space $L_\alpha^p(\Omega)$ is defined as the set of measurable functions w such that

$$\|w\|_{L_\alpha^p(\Omega)} = \left(\int_\Omega |w|^p r^\alpha \mathrm{d}\boldsymbol{x} \right)^{\frac{1}{p}} < \infty,$$

where $r = r(\boldsymbol{x})$ is the radial coordinate of $\boldsymbol{x}$, i.e., the distance of a point $\boldsymbol{x}$ in Ω from the symmetry axis and $\mathrm{d}\boldsymbol{x} = \mathrm{d}r\mathrm{d}z$. For $p = \infty$, $L_\alpha^\infty(\Omega)$ is simply equal to $L^\infty(\Omega)$. The subspace $L_{1,0}^2(\Omega)$ of $L_1^2(\Omega)$ denotes the functions q with weighted integral equal to zero:

$$\int_\Omega q\,r\,\mathrm{d}\boldsymbol{x} = 0.$$

Let ℓ be a positive integer. We define the weighted Sobolev space $W_1^{\ell,p}(\Omega)$ as the space of functions in $L_1^p(\Omega)$ such that their partial derivatives of order less or equal to ℓ belong to $L_1^p(\Omega)$. The space $W_1^{\ell,p}(\Omega)$ is a Banach space endowed with the following semi-norm $|\cdot|_{W_1^{\ell,p}(\Omega)}$ and norm $\|\cdot\|_{W_1^{\ell,p}(\Omega)}$:

$$|w|_{W_1^{\ell,p}(\Omega)} = \left(\sum_{k=0}^{\ell} \|\partial_r^k \partial_z^{\ell-k} w\|_{L_1^p(\Omega)}^p \right)^{\frac{1}{p}},$$

$$\|w\|_{W_1^{\ell,p}(\Omega)} = \left(\sum_{k=0}^{\ell} |w|_{W_1^{k,p}(\Omega)}^p \right)^{\frac{1}{p}}.$$

For any positive real number s, $W_1^{s,p}(\Omega)$ is defined as the interpolation space of index $[s] + 1 - s$ between the spaces $W_1^{[s]+1,p}(\Omega)$ and $W_1^{[s],p}(\Omega)$, where $[s]$ is the integral part of s.

When $p = 2$, we note as in the standard case $W_1^{\ell,2}(\Omega)$ by $H_1^\ell(\Omega)$. We also need another weighted space $V_1^1(\Omega)$ defined as

$$V_1^1(\Omega) = \left\{ w \in H_1^1(\Omega); \; w \in L_{-1}^2(\Omega) \right\}$$

and endowed with the norm

$$\|w\|_{V_1^1(\Omega)} = \left(|w|_{H_1^1(\Omega)}^2 + \|w\|_{L_{-1}^2(\Omega)}^2 \right)^{\frac{1}{2}}.$$

It can be proved that all functions in $V_1^1(\Omega)$ have a null trace on Γ_0 (see [13]). The traces on Γ are defined in a nearly standard way, see [3, Sect. I, Theorem a.5]. Let $H_1^s(\Gamma)$, $s \geqslant 0$, be the scale of Sobolev spaces built from

$$L_1^2(\Gamma) = \left\{ g : \Gamma \to \mathbb{R} \text{ measurable} ; \int_\Gamma g^2 \, r(t) \, \mathrm{d}t < \infty \right\},$$

where $r(t)$ is the value of the radial coordinate at the point with tangential coordinate t. The trace operator $v \mapsto v|_\Gamma$ is continuous from $H_1^1(\Omega)$ onto $H_1^{\frac{1}{2}}(\Gamma)$ and also from $V_1^1(\Omega)$ into the same space.

2.4 Dimension reduction

In this section we are going to underline the correspondence of the standard three-dimensional and weighted two-dimensional Sobolev spaces. See [4, Sect. II.4] for the proofs of the following statements.

The subspace of axisymmetric functions in $H^1(\check{\Omega})$ is isomorphic to $H_1^1(\Omega)$. In the original three-dimensional problem, to take into account the boundary condition, the subspace $H_0^1(\check{\Omega})$ of zero trace functions is introduced. As explained in [4, Sect. II.4], the counterpart for the axial component of the velocity is the weighted subspace

$$H_{1\diamond}^1(\Omega) = \left\{ w \in H_1^1(\Omega); \ w = 0 \text{ on } \Gamma \right\},$$

and the one for the radial component is

$$V_{1\diamond}^1(\Omega) = \left\{ w \in V_1^1(\Omega); \ w = 0 \text{ on } \Gamma \right\}.$$

We describe the axisymmetric domain $\check{\Omega}$ with cylindrical coordinates (r, θ, z). It is possible to define two isomorphisms, which maps axisymmetric functions and vector-fields on $\check{\Omega}$ to functions and vector-fields on Ω. These isomorphisms are called *reduction operators* and are defined in the scalar case as

$$\left\{ \check{w} \in H_0^1(\check{\Omega}) \text{ axisymmetric} \right\} \longrightarrow H_{1\diamond}^1(\Omega),$$
$$\check{w} \longmapsto w : w(r, z) = \check{w}(r, \theta, z)$$

and in the vector case as

$$\left\{\begin{array}{l}\check{\boldsymbol{w}} \in H_0^1(\check{\Omega})^3 \text{ axisym. and } \check{w}_\theta = 0\end{array}\right\} \longrightarrow V_{1\diamond}^1(\Omega) \times H_{1\diamond}^1(\Omega),$$

$$\check{\boldsymbol{w}} \longmapsto \boldsymbol{w} : w_r = \check{w}_r, \, w_z = \check{w}_z.$$

Proposition 1 *The space of axisymmetric vector fields in $H^1(\check{\Omega})^3$ with zero angular component is isomorphic to $V_1^1(\Omega) \times H_1^1(\Omega)$. The space of axisymmetric vector fields in $H_0^1(\check{\Omega})^3$ with zero angular component is isomorphic to $V_{1\diamond}^1(\Omega) \times H_{1\diamond}^1(\Omega)$.*

We refer to [4, Sect. II.4] and to [9] for its proof.

2.5 The weak axisymmetric form

The Stokes problem (2) on Ω for $(\check{u}_r, \check{u}_z, p)$ is equivalent to the following weak formulation of the Stokes axisymmetric problem.

Find $(\boldsymbol{u}, p)$ in $V_{1\diamond}^1(\Omega) \times H_{1\diamond}^1(\Omega) \times L_{1,0}^2(\Omega)$ such that, for all $(\boldsymbol{v}, q)$ in $V_{1\diamond}^1(\Omega) \times H_{1\diamond}^1(\Omega) \times L_{1,0}^2(\Omega)$,

$$\begin{cases} a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = \int\limits_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, r \, \mathrm{d}\boldsymbol{x}, \\ b(\boldsymbol{u}, q) = 0, \end{cases} \tag{3}$$

where the forms a and b are defined by

$$a(\boldsymbol{u}, \boldsymbol{v}) = \nu \int\limits_\Omega (\nabla_a \boldsymbol{u} : \nabla_a \boldsymbol{v}) \, r \, \mathrm{d}\boldsymbol{x} + \nu \int\limits_\Omega u_r v_r \frac{1}{r} \mathrm{d}\boldsymbol{x}, \tag{4}$$

$$b(\boldsymbol{u}, q) = - \int\limits_\Omega (\mathrm{div}_a \boldsymbol{u}) \, q \, r \, \mathrm{d}\boldsymbol{x} - \int\limits_\Omega u_r q \, \mathrm{d}\boldsymbol{x},$$

where $\nabla_a \boldsymbol{u}$ is equal to $\begin{pmatrix} \partial_r u_r & \partial_r u_z \\ \partial_z u_r & \partial_z u_z \end{pmatrix}$ and $\mathrm{div}_a \boldsymbol{u} = \partial_r u_r + \partial_z u_z$ (the index a in this notation means "axisymmetric"). Indeed it can be checked that $a(\boldsymbol{u}, \boldsymbol{v}) = \frac{1}{2\pi} \check{a}(\check{\boldsymbol{u}}, \check{\boldsymbol{v}})$ and $b(\boldsymbol{u}, q) = \frac{1}{2\pi} \check{b}(\check{\boldsymbol{u}}, \check{q})$.

In [4, Sect. IX.1] it is proved that this problem has a unique solution. In particular it is easily derived from its analogue on $\check{\Omega}$ by using the reduction operator that the following inf–sup condition holds: There exists a constant β such that for all q in $L_{1,0}^2(\Omega)$,

$$\sup_{\boldsymbol{v} \in V_{1\diamond}^1(\Omega) \times H_{1\diamond}^1(\Omega)} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{V_1^1(\Omega) \times H_1^1(\Omega)}} \geqslant \beta \|q\|_{L_1^2(\Omega)}. \tag{5}$$
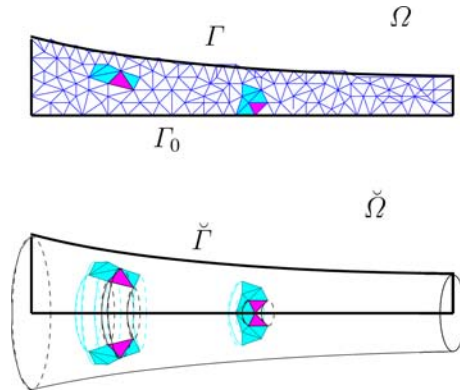
2.6 Finite element formulation

In this section we introduce the finite element that we employ to numerically solve the Stokes problem.

The half section Ω represents the computational domain (see Fig. 2). From now on we suppose that the domain Ω is polygonal and we introduce a *regular family of triangulations* $(\mathcal{T}_h)_h$ of Ω with the following properties:

(i)   The domain $\bar{\Omega}$ is the union of the elements of $\mathcal{T}_h$.
(ii) $T_k \cap T_j$ is a side, a node or empty for all triangles T_k, T_j, $k \neq j$, in $\mathcal{T}_h$.
(iii) There exists a constant σ independent of h, such that for all T in $\mathcal{T}_h$, its diameter h_T is smaller than h and T contains a circle of radius σh_T.

We also suppose that each triangle T in $\mathcal{T}_h$ has at least one vertex inside Ω (not on $\Gamma \cup \Gamma_0$). In all that follows, $c, c', \ldots$, denote generic constants that may depend on σ and vary from one line to the next one but are always independent of h.

Each triangulation $\mathcal{T}_h$ is used for P1 elements for the pressure. Moreover $\mathcal{T}_{h/2}$ denotes the triangulation obtained from $\mathcal{T}_h$ by dividing each triangle into four equal triangles by joining the midpoints of the edges. Indeed $\mathcal{T}_{h/2}$ is used for P1 elements for the velocity.

Let $P_k(T)$ denote the set of restrictions to T of polynomials of degree less than or equal to k; then the finite element spaces for the velocity and the pressure are

$$V_{h/2} = \left\{ \boldsymbol{v}_h \in \mathcal{C}^0(\bar{\Omega})^2 \; : \; \boldsymbol{v}_h|_\Gamma = 0, \, v_{h,r}|_{\Gamma_0} = 0; \, \forall T \in \mathcal{T}_{h/2} \; \boldsymbol{v}_h|_T \in P_1(T)^2 \right\},$$

$$Q_h = \left\{ q_h \in \mathcal{C}^0(\bar{\Omega}) \; : \; \int_\Omega q_h r \, d\boldsymbol{x} = 0; \, \forall T \in \mathcal{T}_h \; q_h|_T \in P_1(T) \right\}.$$

It is easily checked that $V_{h/2} \subset V^1_{1\diamond}(\Omega) \times H^1_{1\diamond}(\Omega)$ and $Q_h \subset L^2_{1,0}(\Omega)$. Then the discrete problem associated with formulation (3) reads:

Find $(\boldsymbol{u}_h, p_h)$ in $V_{h/2} \times Q_h$ such that for all $(\boldsymbol{v}_h, q_h)$ in $V_{h/2} \times Q_h$

$$\begin{cases} a(\boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, p_h) = \int\limits_{\Omega} \boldsymbol{f} \cdot \boldsymbol{v}_h \, r \, \mathrm{d}\boldsymbol{x}, \\ b(\boldsymbol{u}_h, q) = 0. \end{cases} \tag{6}$$

3 Weighted inverse inequalities

3.1 Preliminary results

In this section we are going to prove the inverse inequality for vector fields in $V_{h/2}$. We need the following classification of the triangles. For any T in $\mathcal{T}_h$, let F_T denote an affine mapping from a reference triangle $\hat{T}$ onto T. Then the vertices of $\hat{T}$, $\hat{\boldsymbol{a}}_i$, $i = 1, 2, 3$, are mapped by F_T onto the vertices of T, $\boldsymbol{a}_i$, $i = 1, 2, 3$. Let $\hat{\lambda}_i$ be the barycentric coordinate associated with $\hat{\boldsymbol{a}}_i$. We also define a scalar number r_T which is the minimum of the radial coordinate of the vertices of T not belonging to the axis.

Lemma 1 *For any triangle T of $\mathcal{T}_h$, there exists a constant c, such that $ch_T \leqslant r_T$.*

Proof Since the family of triangulations is regular, the distance of a vertex (r, z) to the opposite side of T is larger or equal to $2\sigma h_T$. Since r_T is the distance of a vertex from the axis and no triangle crosses the axis, $r_T \geqslant 2\sigma h_T$. The only exception is when the vertex is on a triangle which crosses the axis and has a side on Γ. In this case $r_T \geqslant Ch_T$, where C depends on the angles between Γ and Γ_0. Since there is a finite number of intersections between Γ and Γ_0, the constant C is bounded from below by a positive constant. $\qquad\square$

From now on and for brevity, we say that two nonnegative functions f_1 and f_2 on a domain T are equivalent if there exist two constants c and c', called equivalence constants, such that the inequalities $c f_1(\boldsymbol{x}) \leq f_2(\boldsymbol{x}) \leq c' f_1(\boldsymbol{x})$ hold for almost every $\boldsymbol{x}$ in T. It can be noted that, on all triangles T of $\mathcal{T}_h$, the constant functions $\max_{\boldsymbol{x} \in T} r(\boldsymbol{x})$ and r_T are equivalent. For more precise results, we observe that the triangles T of $\mathcal{T}_{h/2}$ can be parted into three types:

- Type 1. If $T \cap \Gamma_0$ is empty, the ratio $\frac{\max_{\boldsymbol{x} \in T} r(\boldsymbol{x})}{\min_{\boldsymbol{x} \in T} r(\boldsymbol{x})}$ is smaller than a constant only depending on the regularity parameter σ of the family of triangulations. Then r is equivalent to r_T.
- Type 2. If $T \cap \Gamma_0$ is an edge, for instance with endpoints $\boldsymbol{a}_2$ and $\boldsymbol{a}_3$, the function r is equal to $\alpha_1^T \lambda_1$, with the constant α_1^T equal to r_T, so that the ratio α_1^T / h_T is bounded from above and from below by positive constants only depending on σ.
- Type 3. If $T \cap \Gamma_0$ is a vertex, for instance $\boldsymbol{a}_1$, the function r is equal to $\alpha_2^T \lambda_2 + \alpha_3^T \lambda_3$, with the constants α_i^T equal to $r(\boldsymbol{a}_i)$, so that the ratio α_i^T / h_T is bounded from above and from below by positive constants only depending on σ. So, the function r is equivalent to $h_T(\lambda_2 + \lambda_3)$, with equivalence constants only depending on σ.

Here we introduce some technical lemmas which will be used to establish the approximation properties of the Lagrange interpolation operator and the Clément operator.

We fix an integer $k \geqslant 1$ and, with each T in $\mathcal{T}_h$, we associate its lattice of order k. Let $\Sigma_h = \{ \boldsymbol{a}_i, 1 \leqslant i \leqslant N_h \}$ be the union of these lattices on all T in $\mathcal{T}_h$.

Lemma 2 *Let φ_i denote the Lagrange function in $P_k(T)$ associated to the node $\boldsymbol{a}_i = (r_i, z_i)$ of Σ_h. Then there exists a constant c independent of h_T, such that for all T in $\mathcal{T}_h$ containing $\boldsymbol{a}_i$ the following inequalities hold*

$$\|\varphi_i\|_{L^p_1(T)} \leqslant c \left(\max_T r^{\frac{1}{p}} \right) h_T^{\frac{2}{p}}, \quad \|\varphi_i\|_{W^{1,p}_1(T)} \leqslant c \left(\max_T r^{\frac{1}{p}} \right) h_T^{\frac{2}{p}-1}. \tag{7}$$

Proof Since the proof is similar for both inequalities we only give it for the first one. Indeed, it is readily checked by going to the reference element that

$$\|\varphi_i\|_{L^p_1(T)} \leqslant \left(\max_T r^{\frac{1}{p}} \right) h_T^{\frac{2}{p}} \|\hat{\varphi}_i\|_{L^p(\hat{T})}.$$

$\square$

Note that if T intersect Γ_0 then r_T and $\max_{x \in T} r$ are of the same order as h_T.

3.2 Inverse inequalities

Firstly inequalities are proved for the norm of $L^p_{-1}(T)$, then the semi-norm of $W^{\ell,p}_1(T)$ and finally the proof is carried out in the norms of $V^1_1(\Omega) \times H^1_1(\Omega)$.

For a triangle T in $\mathcal{T}_h$, note its area by $|T|$. Let f be a polynomial defined in T, then $\hat{f}$ stands for $f \circ F_T$. In particular $\rho_T = r \circ F_T$ is the affine function representing the radial coordinate.

Lemma 3 *Let $1 \leqslant p < \infty$ and k be an integer. There exists a constant c, such that for every triangle T in $\mathcal{T}_h$ and any polynomial f in $P_k(T)$, vanishing on the axis if T is of type 2 or 3,*

$$\|f\|_{L^p_{-1}(T)} \leqslant c\, r_T^{-2/p} \|f\|_{L^p_1(T)}. \tag{8}$$

Proof If T is of type 1, then for any point in T, $r_T \leqslant r$ and

$$\int_T |f|^p \frac{1}{r} \mathrm{d}\boldsymbol{x} \leqslant \frac{1}{r_T^2} \int_T |f|^p r\, \mathrm{d}\boldsymbol{x}.$$

Let T be of type 2. On the reference triangle, the following weighted norms on the polynomials $\hat{f}$ of degree k are equivalent, in particular

$$\|\hat{f}\hat{\lambda}_1^{-1/p}\|_{L^p(\hat{T})} \leqslant c\|\hat{f}\hat{\lambda}_1^{1/p}\|_{L^p(\hat{T})}. \tag{9}$$

Hence inequality (8) is proved by

$$\|f\|_{L_{-1}^p(T)}^p = \frac{|T|}{2}\|\hat{f}\,(\hat{\lambda}_1 r_T)^{-1/p}\|_{L^p(\hat{T})}^p$$

$$\leqslant c\,\frac{|T|}{2}\,\frac{1}{r_T^2}\|\hat{f}\,(\hat{\lambda}_1 r_T)^{1/p}\|_{L^p(\hat{T})}^p = c\,\frac{1}{r_T^2}\|f\|_{L_1^p(T)}^p.$$

If T is of type 3, from Eq. (9) with $\hat{\lambda}_1$ replaced by $\hat{\lambda}_2 + \hat{\lambda}_3$,

$$\|f\|_{L_{-1}^p(T)}^p \leqslant c\,\frac{|T|}{2}\|\hat{f}\,((\hat{\lambda}_2 + \hat{\lambda}_3)r_T)^{-1/p}\|_{L^p(\hat{T})}^p$$

$$\leqslant c'\,\frac{|T|}{2}\,\frac{1}{r_T^2}\|\hat{f}\,((\hat{\lambda}_2 + \hat{\lambda}_3)r_T)^{1/p}\|_{L^p(\hat{T})}^p \leqslant c'\,\frac{1}{r_T^2}\|f\|_{L_1^p(T)}^p.$$

$\square$

Lemma 4 *There exists a constant c, such that for every triangle T in $\mathcal{T}_h$ and any polynomial f in $P_k(T)$,*

$$\|\nabla_a f\|_{L_1^p(T)} \leqslant c\,h_T^{-1}\|f\|_{L_1^p(T)}. \tag{10}$$

Proof If T is of type 1, the standard inverse inequality gives

$$\|\nabla_a f\|_{L_1^p(T)} \leqslant \max_{\boldsymbol{x}\in T} r(\boldsymbol{x})^{\frac{1}{p}}\|\nabla_a f\|_{L^p(T)} \leqslant c\Big(\max_{\boldsymbol{x}\in T} r(\boldsymbol{x})\Big)^{\frac{1}{p}} h_T^{-1}\|f\|_{L^p(T)}$$

$$\leqslant c\left(\frac{\max_{\boldsymbol{x}\in T} r(\boldsymbol{x})}{\min_{\boldsymbol{x}\in T} r(\boldsymbol{x})}\right)^{\frac{1}{p}} h_T^{-1}\|f\|_{L_1^p(T)},$$

and the boundedness of the quantity $\max_{\boldsymbol{x}\in T} r(\boldsymbol{x})/\min_{\boldsymbol{x}\in T} r(\boldsymbol{x})$ leads to the desired inequality. If T is of type 2, we have by using the transformation F_T

$$\|\nabla_a f\|_{L_1^p(T)} \leq c\,h_T^{\frac{2}{p}-1} r_T^{\frac{1}{p}}\|(\nabla\hat{f})\hat{\lambda}_1^{\frac{1}{p}}\|_{L^p(\hat{T})},$$

and, by using the equivalence of norms on a finite-dimensional space, we obtain

$$\|\nabla_a f\|_{L_1^p(T)} \leq c\,h_T^{\frac{2}{p}-1} r_T^{\frac{1}{p}}\|\hat{f}\hat{\lambda}_1^{\frac{1}{p}}\|_{L^p(\hat{T})}.$$

Thus going back to T yields

$$\|\nabla_a f\|_{L_1^p(T)} \leq c\,h_T^{-1}\|f\|_{L_1^p(T)},$$

which is the desired result. When T is of type 3, the inequality follows from the same argument as previously, with $\hat{\lambda}_1$ replaced by $\hat{\lambda}_2 + \hat{\lambda}_3$. $\square$

Now we are ready to prove a weighted inverse inequality:

Proposition 2 *There exists a constant c such that for all $\boldsymbol{v}_h$ in $V_{h/2}$,*

$$\|\boldsymbol{v}_h\|^2_{V^1_1(\Omega) \times H^1_1(\Omega)} \leqslant c \sum_{T \in \mathcal{T}_{h/2}} h_T^{-2} \|\boldsymbol{v}_h\|^2_{L^2_1(T)^2}. \tag{11}$$

Proof Let $\boldsymbol{v}_h = (v_r, v_z)$ be in $V_{h/2}$. From Lemmas 1 and 3, for any T in $\mathcal{T}_{h/2}$,

$$\|v_r\|_{L^2_{-1}(T)} \leqslant c h_T^{-1} \|v_r\|_{L^2_1(T)}$$

and from Lemma 4

$$\|\nabla_a v_r\|_{L^2_1(T)} \leqslant c h_T^{-1} \|v_r\|_{L^2_1(T)},$$

and this last inequality holds also for v_z. □

4 Weighted approximation properties

We now prove some properties of the Lagrange interpolation operator and also of some Clément type operators. There are several possible constructions of the Clément operator, we follow here the approach presented in [5, Sect. IX.3] in the Cartesian case. We begin by some technical lemmas which are useful in what follows.

4.1 Preliminary results

The next lemma states a polynomial approximation property which is a weighted extension of a result due to [10] in a more general but unweighted case.

For $\boldsymbol{a}_i$ in Σ_h, $\widetilde{\Delta}_i$ denotes the union of two triangles containing $\boldsymbol{a}_i$ and sharing a common edge, and h_i stands for its diameter.

Lemma 5 *For all p, $1 \leqslant p \leqslant +\infty$, there exists a constant c, independent of h_i, such that for all function in $W^{1,p}_1(\widetilde{\Delta}_i)$,*

$$\inf_{q \in P_0(\widetilde{\Delta}_i)} \left(\|v - q\|_{L^p_1(\widetilde{\Delta}_i)} + h_i |v - q|_{W^{1,p}_1(\widetilde{\Delta}_i)} \right) \leqslant c\, h_i |v|_{W^{1,p}_1(\widetilde{\Delta}_i)}. \tag{12}$$

Proof Let T and T' denote the two triangles which define $\widetilde{\Delta}_i$, and e their common edge. Let h_e denote the diameter of e and $\boldsymbol{m}_e$ its midpoint. There exists a constant λ depending only on the regularity parameter σ, such that $\widetilde{\Delta}_i$ is star shaped with respect to the ball B centered on $\boldsymbol{m}_e$ and with radius $\frac{\lambda h_e}{2}$. The

function : $x \mapsto \hat{x} = 2\frac{x - m_e}{\lambda h_e}$, from $\widetilde{\Delta}_i$ into a region $\hat{\Delta}$ maps the ball B into the unit ball $\hat{B}$. Let $\hat{\varphi}$ be in $\mathcal{D}(\hat{B})$, with $\int_{\hat{B}} \hat{\varphi} \, d\hat{x} = 1$, then the function φ defined by

$$\varphi(x) = \left(\frac{\lambda h_e}{2}\right)^{-2} \hat{\varphi}\left(2\frac{x - m_e}{\lambda h_e}\right),$$

belongs to $\mathcal{D}(B)$ and

$$\int_B \varphi \, dx = 1.$$

Define q as

$$q = \int_B \varphi(y) \, v(y) \, dy.$$

To evaluate the norm of $v - q$ in $L_1^p(\widetilde{\Delta}_i)$, we start with the following Taylor formula: for each $x \in \widetilde{\Delta}_i$, and $y \in B$,

$$v(x) = v(y) + \int_0^1 (x - y) \cdot \nabla_a v(x + s(y - x)) \, ds.$$

Multiplying by $\varphi(y)$ and integrating over B, we obtain

$$v(x) - q = \int_B \int_0^1 \varphi(y)(x - y) \cdot \nabla_a v(x + s(y - x)) \, ds \, dy.$$

Setting $z = x + s(y - x)$ yields

$$|v(x) - q| \leqslant \int_{\widetilde{\Delta}_i} \left| \int_0^1 \varphi\left(x + s^{-1}(z - x)\right) s^{-1}(x - z) \cdot \nabla_a v(z) s^{-2} \, ds \right| dz, \quad (13)$$

whence, for any x in $\widetilde{\Delta}_i$,

$$|v(x) - q| \leqslant \int_{\widetilde{\Delta}_i} |k(x, z)(x - z) \cdot (\nabla_a v)(z)| \, dz, \quad (14)$$

where

$$k(\pmb{x},\pmb{z}) = \int\limits_0^1 \varphi\left(\pmb{x} + s^{-1}(\pmb{z} - \pmb{x})\right) s^{-3} \mathrm{d}s.$$

Since $\varphi(\pmb{x}+s^{-1}(\pmb{z}-\pmb{x}))$ vanishes when $|\pmb{x}+s^{-1}(\pmb{z}-\pmb{x})-\pmb{m}_e| \geqslant \frac{\lambda h_e}{2}$, and particularly for $s \leqslant (\mu h_e)^{-1}|\pmb{z} - \pmb{x}|$, for a constant μ depending only on σ,

$$\begin{aligned} |k(\pmb{x},\pmb{z})| &\leqslant \|\varphi\|_{L^\infty(B)} \int\limits_{(\mu h_e)^{-1}|\pmb{z}-\pmb{x}|}^1 s^{-3}\,\mathrm{d}s \\ &\leqslant c\,\|\varphi\|_{L^\infty(B)}(\mu h_e)^2 \left||\pmb{x}-\pmb{z}|^{-2} - (\mu h_e)^{-2}\right|. \end{aligned}$$

Using $\|\varphi\|_{L^\infty(B)} = (\frac{\lambda h_e}{2})^{-2}\|\hat{\varphi}\|_{L^\infty(\hat{B})}$, we deduce

$$|k(\pmb{x},\pmb{z})| \leqslant c\left(|\pmb{x}-\pmb{z}|^{-2} + (\mu h_e)^{-2}\right). \tag{15}$$

Let $\tilde{k}$ be the function

$$\tilde{k}(\pmb{z}) = (|\pmb{z} - \pmb{m}_e|^{-2} + (\mu h_e)^{-2})|\pmb{z} - \pmb{m}_e|$$

and deduce from (14) and (15)

$$|(v - q)(\pmb{x})| \leqslant c(\tilde{k} * |\nabla_a v|)(\pmb{x}).$$

We now check that, for a constant c only depending on the regularity parameter σ,

$$r(\pmb{z}) \geqslant c\,r(\pmb{x}). \tag{16}$$

Indeed,

– either the intersection of $\tilde{\Delta}_i$ with Γ_0 is empty. Then, we have

$$r(\pmb{z}) \geqslant \frac{\min_{\pmb{t} \in \tilde{\Delta}_i} r(\pmb{t})}{\max_{\pmb{t} \in \tilde{\Delta}_i} r(\pmb{t})}\,r(\pmb{x}),$$

and the ratio $\min_{\pmb{t} \in \tilde{\Delta}_i} r(\pmb{t})/\max_{\pmb{t} \in \tilde{\Delta}_i} r(\pmb{t})$ is bounded from below by a constant only depending on σ,
– or it is not empty. We note that

$$r(\pmb{z}) = (1 - s)r(\pmb{x}) + s r(\pmb{y}) \geqslant \min\{r(\pmb{x}), r(\pmb{y})\}.$$

Since $r(\boldsymbol{y}) \geqslant \mu h_e$, either $r(\boldsymbol{x}) \leqslant \mu h_e$, so that $r(\boldsymbol{z}) \geqslant r(\boldsymbol{x})$, or $r(\boldsymbol{x}) > \mu h_e$, so that $r(\boldsymbol{z}) \geqslant \mu h_e$ and, since $r(\boldsymbol{x}) \leqslant c h_e$ for a constant c only depending on σ, $r(\boldsymbol{z}) \geqslant \frac{\mu}{c} r(\boldsymbol{x})$.

Thus

$$|(v - q)(\boldsymbol{x})| \, r(\boldsymbol{x})^{\frac{1}{p}} \leqslant c \int\limits_{\widetilde{\Delta}_i} \tilde{k}(\boldsymbol{x} - \boldsymbol{z}) \, |\nabla_a v(\boldsymbol{z})| \, r(\boldsymbol{z})^{\frac{1}{p}} \, d\boldsymbol{z}.$$

Applying Young's inequality yields

$$\|v - q\|_{L_1^p(\widetilde{\Delta}_i)} \leqslant \|\tilde{k}\|_{L^1(\widetilde{\Delta}_i)} \|\nabla_a v\|_{L_1^p(\widetilde{\Delta}_i)^2}.$$

Noting that

$$\|\tilde{k}\|_{L^1(\widetilde{\Delta}_i)} = \int\limits_{\widetilde{\Delta}_i} (|\boldsymbol{z} - \boldsymbol{m}_e|^{-1} + |\boldsymbol{z} - \boldsymbol{m}_e|(\mu h_e)^{-2}) \, d\boldsymbol{z}$$

$$\leqslant c \int\limits_0^{c' h_e} (\varrho^{-1} + (\mu h_e)^{-2}\varrho)\varrho d\varrho = c'' h_e,$$

we derive the first part of the inequality (12).

The second part of (12), i.e., the inequality with the second term in the left-hand side, is obvious. □

The next lemma is an extension of the previous one and the proof is identical to the unweighted case, see [5, Sect. IX, Lemma 3.4].

Let Δ_i denote the union of all elements T in $\mathcal{T}_h$ containing $\boldsymbol{a}_i$.

Lemma 6 *For all p, $1 \leqslant p \leqslant +\infty$, there exists a constant c, independent of h_i, such that, for all functions v in $W_1^{1,p}(\Delta_i)$,*

$$\inf_{q \in P_0(\Delta_i)} \left(\|v - q\|_{L_1^p(\Delta_i)} + h_i |v - q|_{W_1^{1,p}(\Delta_i)} \right) \leqslant c \, h_i \, \|v\|_{W_1^{1,p}(\Delta_i)}. \tag{17}$$

The following lemma is obtained by the same construction as Lemmas 5 and 6. Since the proof is rather long and technical, we only state the result. We refer to [10] for the analogue in the unweighted case.

Lemma 7 *For all p, $1 \leqslant p \leqslant +\infty$, there exists a constant c, independent of h_i, such that, for all functions $v \in W_1^{\ell+1,p}(\Delta_i)$, the following inequality holds:*

$$\inf_{q \in P_\ell(\Delta_i)} \left(\|v - q\|_{L_1^p(\Delta_i)} + h_i |v - q|_{W_1^{1,p}(\Delta_i)} \right) \leqslant c \, h_i^{\ell+1} \, \|v\|_{W_1^{\ell+1,p}(\Delta_i)}. \tag{18}$$

Obviously the results of Lemma 5 to Lemma 7 still hold when replacing Δ_i by an element T of $\mathcal{T}_h$. If we denote by Δ_T the union of all elements of $\mathcal{T}_h$ sharing at least a common vertex with T then these results still hold also.

4.2 Lagrange interpolation operator

We define the Lagrange interpolation operator $\mathcal{I}_h : \mathcal{C}^0(\bar{\Omega}) \to X_h$, where X_h denotes the space of Lagrange finite elements of order k: $\mathcal{I}_h\varphi$ coincides with φ on all nodes of Σ_h. For any T in $\mathcal{T}_h$, we introduce a local interpolation operator $i_T : \mathcal{C}^0(T) \to P_k(T)$, such that for all $\boldsymbol{a}_j$ in $\Sigma_h \cap T$,

$$(i_T\varphi)(\boldsymbol{a}_j) = \varphi(\boldsymbol{a}_j).$$

So, it holds

$$\mathcal{I}_h\varphi_{|T} = i_T\varphi.$$

Moreover this operator maps the functions that vanish on Γ onto $X_h \cap H^1_{1\diamond}(\Omega)$.

The approximation properties of the Lagrange interpolation operator in the framework of weighted Sobolev spaces are proved in [13] (Lemmas 6.1 and 6.2) in the case $p = 2$ (with some restrictions). However, this is not sufficient for our purpose, and we need the more general results stated in the following proposition.

Proposition 3 *For all ℓ, $1 \leqslant \ell \leqslant k + 1$, and for all p, $1 \leqslant p \leqslant +\infty$, such that*

$$\ell > \frac{3}{p} \quad or \quad p = 1, \ell = 3, \tag{19}$$

there exists a constant C, independent of h, such that, for all element T in $\mathcal{T}_h$, the following inequalities hold for all functions $v \in W^{\ell,p}_1(\Omega)$:

$$\|v - \mathcal{I}_h v\|_{L^p_1(T)} \leqslant C h^\ell_T |v|_{W^{\ell,p}_1(T)}, \tag{20}$$

$$|v - \mathcal{I}_h v|_{W^{1,p}_1(T)} \leqslant C h^{\ell-1}_T |v|_{W^{\ell,p}_1(T)}. \tag{21}$$

Proof For any $T \in \mathcal{T}_h$ and for any polynomial p of degree $\ell - 1$,

$$\|u - \mathcal{I}_h u\|_{L^p_1(T)} \leqslant \|u - p\|_{L^p_1(T)} + \|I_h(u - p)\|_{L^p_1(T)}.$$

We consider the second term of this inequality. By going to the reference element, we have

$$\|\mathcal{I}_h(u - p)\|_{L^p_1(T)} \leqslant c\delta h^{2/p}_T \|\hat{\mathcal{I}}(\hat{u} - \hat{p})\chi^{\frac{1}{p}}\|_{L^p(\hat{T})},$$

with δ and χ defined by the type of the triangle T as follows:

$$
\begin{cases}
\delta = (\max_{x \in T} r)^{\frac{1}{p}}, \text{ and } \chi = 1 & \text{if } T \text{ is of type 1,} \\
\delta = h_T^{\frac{1}{p}}, \text{ and } \chi = \hat{\lambda}_1 & \text{if } T \text{ is of type 2,} \\
\delta = h_T^{\frac{1}{p}}, \text{ and } \chi = \hat{\lambda}_2 + \hat{\lambda}_3 & \text{if } T \text{ is of type 3.}
\end{cases}
$$

Let $W_\chi^{\ell,p}(\hat{T})$ be the weighted Sobolev space with weight χ (similarly as for $W_1^{\ell,p}(T)$). The continuous embedding of $W_\chi^{\ell,p}(\hat{T})$ into $\mathcal{C}^0(\overline{\hat{T}})$, which in the first case, derives from the standard Sobolev embedding and in the other two cases from the three-dimensional one, yields

$$
\delta \|\hat{\mathcal{I}}(\hat{u} - \hat{p})\|_{L_\chi^p(\hat{T})} \leqslant \delta \|\hat{u} - \hat{p}\|_{W_\chi^{\ell,p}(\hat{T})},
$$

whence

$$
\|u - \mathcal{I}_h u\|_{L_1^p(T)} \leqslant c \delta h_T^{2/p} \|\hat{u} - \hat{p}\|_{W_\chi^{\ell,p}(\hat{T})}.
$$

An extension of the Bramble–Hilbert lemma to the weighted case can be established by combining the compactness of the embedding $W_\chi^{\ell,p}(\hat{T}) \subset L_\chi^p(\hat{T})$ with the Peetre-Tartar lemma (see [11, Chapt. I, Lemma 2.1]). Using this extension, we obtain

$$
\|u - \mathcal{I}_h u\|_{L_1^p(T)} \leqslant c \delta h_T^{2/p} |\hat{u} - \hat{p}|_{W_\chi^{\ell,p}(\hat{T})}.
$$

Returning back to T leads to (20). Inequality (21) is obtained similarly. $\qquad \square$

4.3 The basic Clément operator

In this section we define a regularization operator Π_h which maps $L_1^2(\Omega)$ into X_h (the space of Lagrange finite elements of order k), and we establish its approximation properties. With each $\boldsymbol{a}_i$ in Σ_h, we associate an arbitrary triangle T_i of $\mathcal{T}_h$ which contains $\boldsymbol{a}_i$. Note that T_i is to be chosen among a finite number of elements (bounded independently of the discretization parameter). Define π_i as the $L_1^2(T_i)$ orthogonal projection operator onto $P_k(T_i)$: For all v in $L_1^1(T_i)$, $\pi_i v$ is in $P_k(T_i)$ and satisfies

$$
\forall q \in P_k(T_i), \quad \int_{T_i} (v - \pi_i v)(\boldsymbol{x}) \, q(\boldsymbol{x}) \, r \, \mathrm{d}\boldsymbol{x} = 0. \tag{22}
$$

We define Π_h as

$$\Pi_h v = \sum_{i=1}^{N_h} (\pi_i v)(\boldsymbol{a}_i)\varphi_i(\boldsymbol{x}), \tag{23}$$

where φ_i is the Lagrange function associated with $\boldsymbol{a}_i$, $1 \leqslant i \leqslant N_h$.

The following lemma states the stability of π_i.

Lemma 8 *For all p, $1 \leqslant p \leqslant +\infty$, there exists a constant c such that, for $1 \leqslant i \leqslant N_h$ and for all functions $v \in L_1^p(T_i)$,*

$$\|\pi_i v\|_{L_1^p(T_i)} \leqslant c\|v\|_{L_1^p(T_i)}. \tag{24}$$

Proof On the reference element $\hat{T}$, we define the projection operator $\hat{\pi}$ such that $\hat{\pi}\hat{v} = \widehat{\pi_i v}$, namely $\hat{\pi}$ satisfies (22) with T_i replaced by $\hat{T}$ and the measure $r\,\mathrm{d}\boldsymbol{x}$ replaced by $\rho_{T_i}(\zeta, \eta)\,d\zeta\,d\eta$ with ρ_{T_i} equal to $r \circ F_{T_i}$.

With the notation introduced in the proof of Proposition 3, the function ρ_{T_i} is equivalent to $\delta^p\chi$. So we derive from Hölder's inequality that, for p' such that $\frac{1}{p} + \frac{1}{p'} = 1$,

$$\|\hat{\pi}\hat{v}\|_{L_\chi^2(\hat{T})}^2 \leqslant c \,\|\hat{v}\|_{L_\chi^p(\hat{T})}\|\hat{\pi}\hat{v}\|_{L_\chi^{p'}(\hat{T})}.$$

Next, we obtain from the equivalence of weighted norms on $P_k(\hat{T})$ for the three weights corresponding to the different values of χ that

$$\|\hat{\pi}\hat{v}\|_{L_\chi^p(\hat{T})} \leq c \,\|\hat{\pi}\hat{v}\|_{L_\chi^2(\hat{T})}, \qquad \|\hat{\pi}\hat{v}\|_{L_\chi^{p'}(\hat{T})} \leq c' \,\|\hat{\pi}\hat{v}\|_{L_\chi^2(\hat{T})}.$$

Combining all this gives

$$\|\hat{\pi}\hat{v}\|_{L_\chi^p(\hat{T})} \leq c \,\|\hat{v}\|_{L_\chi^p(\hat{T})}.$$

Combining the previous line with the use of the transformation F_{T_i} leads to

$$\|\pi_i v\|_{L_1^p(T_i)} \leq c\,\delta \,\|\hat{\pi}\hat{v}\|_{L_\chi^p(\hat{T})} \leq c'\,\delta \,\|\hat{v}\|_{L_\chi^p(\hat{T})} \leq c''\,\|\pi_i v\|_{L_1^p(T_i)},$$

which is the desired result. $\qquad\qquad\square$

The following theorem states the first approximation properties of Π_h.

Theorem 1 *For all integers ℓ, $0 \leqslant \ell \leqslant k + 1$, and for all p, $1 \leqslant p \leqslant +\infty$, there exists a constant C, independent of h_T, such that, for all $T \in \mathcal{T}_h$ and all functions $v \in W_1^{\ell,p}(\Delta_T)$, the following inequalities hold:*

$$\|v - \Pi_h v\|_{L_1^p(T)} \leqslant C\,h_T^\ell |v|_{W_1^{\ell,p}(\Delta_T)} \tag{25}$$

and, when $\ell \geqslant 1$,

$$|v - \Pi_h v|_{W_1^{1,p}(T)} \leqslant C h_T^{\ell-1} |v|_{W_1^{\ell,p}(\Delta_T)}. \tag{26}$$

Proof The proof of (25) is divided into three cases: $\ell = 0$, $0 < \ell \leqslant \frac{3}{p}$ and $\ell > \frac{3}{p}$ (or $\ell = 3$, $p = 1$).

Case $\ell = 0$. It holds

$$\|\Pi_h v\|_{L_1^p(T)} \leqslant \sum_{i=1}^{N_h} \alpha_i \|\pi_i v(\boldsymbol{a}_i)\,\varphi_i\|_{L_1^p(T)}, \tag{27}$$

where α_i is equal to 1 if the intersection of the support of φ_i with T is not empty, and zero otherwise.

Let us fix i, then we can write

$$\|\pi_i v(\boldsymbol{a}_i)\varphi_i\|_{L_1^p(T)} \leqslant \|\pi_i v\|_{L^\infty(T_i)} \|\varphi_i\|_{L_1^p(T)}.$$

If T_i is of type 1, then we obtain from Lemma 2 and a standard inverse inequality

$$\|\pi_i v(\boldsymbol{a}_i)\varphi_i\|_{L_1^p(T)} \leqslant \hat{c} \left(\frac{\max_{x \in T} r}{\min_{x \in T_i} r} \right)^{\frac{1}{p}} h_{T_i}^{-\frac{2}{p}} h_T^{\frac{2}{p}} \|\pi_i v\|_{L_1^p(T_i)}. \tag{28}$$

If T_i is of type 2, it follows from Lemma 2 and the fact that $\max_{\boldsymbol{x} \in T} r(\boldsymbol{x}) \leqslant c\, h_T$ that

$$\|\varphi_i\|_{L_1^p(T)} \leqslant c\, h_T^{\frac{3}{p}}. \tag{29}$$

On the other hand, we have

$$\|\pi_i v\|_{L^\infty(T_i)} = \|\widehat{\pi_i v}\|_{L^\infty(\hat{T})} \leqslant c\, \|\widehat{\pi_i v}(\hat{\lambda}_1)^{\frac{1}{p}}\|_{L^p(\hat{T})},$$

whence

$$\|\pi_i v\|_{L^\infty(T_i)} \leqslant c\, h_{T_i}^{-\frac{3}{p}} \|\pi_i v\|_{L_1^p(T_i)}.$$

Combining this with (29) gives

$$\|(\pi_i v)(\boldsymbol{a}_i)\varphi_i\|_{L_1^p(T)} \leqslant c\, \|\pi_i v\|_{L_1^p(T_i)}. \tag{30}$$

If T_i is of type 3, the same arguments apply and this estimate still holds.

Inserting (28), respectively (30), into (27), we obtain

$$\|\Pi_h v\|_{L_1^p(T)} \leqslant \hat{c} \sum_{i=1}^{N_h} \alpha_i \|\pi_i v\|_{L_1^p(T_i)}.$$

Noting that the number of non zero α_i is bounded only as a function of k, we deduce from Lemma 8 the inequality

$$\|\Pi_h v\|_{L_1^p(T)} \leqslant c\|v\|_{L_1^p(\Delta_T)}. \tag{31}$$

Combining this with a triangle inequality yields (25) when $\ell = 0$.

Case $\ell \leqslant \frac{3}{p}$. We note that for any polynomial $q \in P_k(\Delta_T)$, and for all nodes a_i in T, $\pi_i q$ is equal to q, therefore the restriction of $\Pi_h q$ to T is also equal to q. Hence

$$\|v - \Pi_h v\|_{L_1^p(T)} = \|v - q + \Pi_h(v - q)\|_{L_1^p(T)}$$
$$\leqslant \|v - q\|_{L_1^p(T)} + \|\Pi_h(v - q)\|_{L_1^p(T)}.$$

Using (31), we obtain

$$\|v - \Pi_h v\|_{L_1^p(T)} \leqslant c\|v - q\|_{L_1^p(\Delta_T)}. \tag{32}$$

Combining with the result of Lemma 6, respectively Lemma 7 yields (25) when $\ell \leqslant \frac{3}{p}$.

Case $\ell > \frac{3}{p}$ (or $\ell = 3$, $p = 1$). The functions of $W_1^{\ell,p}(\Delta_T)$ are continuous, therefore we can use the Lagrange interpolation operator $\mathcal{I}_h$. Noting that for all a_i, $\pi_i(\mathcal{I}_h v)$ is equal to $(\mathcal{I}_h v)_{|T_i}$, we have $\Pi_h(\mathcal{I}_h v)$ which is equal to $\mathcal{I}_h v$. Whence

$$\|v - \Pi_h v\|_{L_1^p(T)} \leqslant \|v - \mathcal{I}_h v\|_{L_1^p(T)} + \|\Pi_h(v - \mathcal{I}_h v)\|_{L_1^p(T)}.$$

Using once again (31) leads to

$$\|v - \Pi_h v\|_{L_1^p(T)} \leqslant c\|v - \mathcal{I}_h v\|_{L_1^p(\Delta_T)}.$$

The result follows from Proposition 3.

The proof of (26) is the same as the previous one with obvious modifications (see Lemma 2).   □

Following the same lines we deduce the next statement.

Corollary 1 *For all ℓ, $1 \leqslant \ell \leqslant k + 1$, and for all p, $1 \leqslant p \leqslant +\infty$, there exists a constant c, independent of h, such that, for all elements T in $\mathcal{T}_h$, and all edges*

e of T which are not contained in Γ_0, and for all functions $v \in W_1^{\ell,p}(\Delta_T)$, the following inequality holds:

$$\|v - \Pi_h v\|_{L_1^p(e)} \leqslant c h_T^{\ell - \frac{1}{p}} |v|_{W_1^{\ell,p}(\Delta_T)}. \tag{33}$$

Taking the power p of inequality (25) and summing it over all elements T, we obtain the global result

Corollary 2 *For all ℓ, $1 \leqslant \ell \leqslant k+1$, and for all p, $1 \leqslant p \leqslant +\infty$, there exists a constant c, independent of h, such that, for any function $v \in W_1^{\ell,p}(\Omega)$,*

$$\|v - \Pi_h v\|_{L_1^p(\Omega)} \leqslant c h^{\ell} |v|_{W_1^{\ell,p}(\Omega)}. \tag{34}$$

4.4 Other Clément operators

To take into account boundary conditions, we introduce now a modified operator Π_h^0 which preserves the nullity conditions on the boundary Γ:

$$\Pi_h^0 v = \sum_{i=1, \boldsymbol{a}_i \notin \Gamma}^{N_h} (\pi_i v)(\boldsymbol{a}_i) \varphi_i. \tag{35}$$

The operator Π_h has the same approximation properties given in Theorem 1 and the proof is similar to the unweighted case, see [5, Theorem 3.11].

Corollary 3 *Estimates (25), (26) and (33) still holds with Π_h replaced by Π_h^0, for all functions v in $W_1^{\ell,p}(\Delta_T)$ vanishing on $\Gamma \cap \Delta_T$.*

We need also to introduce two other operators $\widetilde{\Pi}_h$, which maps $V_1^1(\Omega)$ into $X_h \cap V_1^1(\Omega)$ (the space of Lagrange finite elements of order k vanishing at Γ_0), and $\widetilde{\Pi}_h^0$, which maps $V_{1\diamond}^1(\Omega)$ into $X_h \cap V_{1\diamond}^1(\Omega)$ (the space of Lagrange finite elements of order k vanishing at $\Gamma_0 \cup \Gamma$), defined as follows:

$$\widetilde{\Pi}_h v = \sum_{i=1, \boldsymbol{a}_i \notin \Gamma_0}^{N_h} (\pi_i v)(\boldsymbol{a}_i) \varphi_i, \tag{36}$$

$$\widetilde{\Pi}_h^0 v = \sum_{i=1, \boldsymbol{a}_i \notin (\Gamma_0 \cup \Gamma)}^{N_h} (\pi_i v)(\boldsymbol{a}_i) \varphi_i. \tag{37}$$

Since we do not have any application for the approximation properties of these operators for all the spaces $W_1^{\ell,p}(\Omega)$, we restrict ourselves to the case $p = 2$. We state the main result in the following theorem.

Theorem 2 *For all ℓ, $1 \leqslant \ell \leqslant k+1$, there exists a constant c, independent of h, such that, for all elements T in $\mathcal{T}_h$, and for all functions $v \in H_1^\ell(\Delta_T) \cap V_1^1(\Delta_T)$, the following inequality holds:*

$$\left(h_T^{-1} \|v - \tilde{\Pi}_h v\|_{L_1^2(T)} + \|v - \tilde{\Pi}_h v\|_{V_1^1(T)} \right) \leqslant C h_T^{\ell-1} \|v\|_{H_1^\ell(\Delta_T) \cap V_1^1(\Delta_T)}. \quad (38)$$

The same estimate holds with $\tilde{\Pi}_h$ replaced by $\tilde{\Pi}_h^0$ and for all v in $H_1^\ell(\Delta_T) \cap V_{1\diamond}^1(\Delta_T)$.

Proof We consider two cases.

Case $\ell = 1$. We can write

$$\|\tilde{\Pi}_h v\|_{V_1^1(T)} \leqslant \sum_{i=1, \, \boldsymbol{a}_i \notin \Gamma_0}^{N_h} \|\pi_i v\|_{L^\infty(T_i)} \|\varphi_i\|_{V_1^1(T)}.$$

Using (7) and (8) for evaluating $\|\varphi_i\|_{V_1^1(T)}$ and the same arguments as previously for bounding $\|\pi_i v\|_{L^\infty(T_i)}$ according to T_i being of type 1, 2 or 3, we derive

$$\|\tilde{\Pi}_h v\|_{V_1^1(T)} \leqslant c \, \|v\|_{V_1^1(T)},$$

which yields one part of (38). On the other hand, we have

$$\|v - \tilde{\Pi}_h v\|_{L_1^2(T)} \leqslant \|v - \Pi_h v\|_{L_1^2(T)} + \|\Pi_h v - \tilde{\Pi}_h v\|_{L_1^2(T)},$$

and the first term in the right-hand side satisfies the desired estimate, see (25). We also note that the second term vanishes on triangles T of type 1. If T is of type 2 or 3, we derive from (7) that

$$\|\Pi_h v - \tilde{\Pi}_h v\|_{L_1^2(T)} = \sum_{\boldsymbol{a}_i \in \Gamma_0 \cap T} h_T^{\frac{3}{2}} \|\pi_i v\|_{L^\infty(T_i)},$$

where T_i is also of type 2 or 3. If T_i is of type 2 for instance, we derive by the same arguments as in the proof of Lemma 8

$$\|\pi_i v\|_{L^\infty(T_i)} = \|\widehat{\pi_i v}\|_{L^\infty(\hat{T})} \leqslant c \, \|\widehat{\pi_i v}(\hat{\lambda}_1)^{\frac{1}{2}}\|_{L^\infty(\hat{T})}$$
$$\leqslant c \, \|\widehat{\pi_i v}(\hat{\lambda}_1)^{\frac{1}{2}}\|_{L^2(\hat{T})} \leqslant c \, \|\hat{v}(\hat{\lambda}_1)^{\frac{1}{2}}\|_{L^2(\hat{T})}.$$

By applying the Poincaré–Friedrichs inequality to the function $\widehat{\pi_i v}(\hat{\lambda}_1)^{\frac{1}{2}}$ which vanishes on one edge of $\hat{T}$, we obtain

$$\|\pi_i v\|_{L^\infty(T_i)} \leqslant c \left(\|\nabla \hat{v}(\hat{\lambda}_1)^{\frac{1}{2}}\|_{L^2(\hat{T})^2} + \|\hat{v}(\hat{\lambda}_1)^{-\frac{1}{2}}\|_{L^2(\hat{T})} \right).$$

Going back to T_i thus gives

$$\|\pi_i v\|_{L^\infty(T_i)} \leqslant c\, h_{T_i}^{1-\frac{3}{2}} \|v\|_{V_1^1(T_i)}.$$

The same estimate holds when T_i is of type 3, by using similar arguments and applying the Poincaré–Friedrichs inequality to functions in the space

$$\hat{W} = \left\{ \hat{w}(\hat{\lambda}_2 + \hat{\lambda}_3)^{\frac{1}{2}};\ \hat{w} \in H^1(\hat{T}) \right\},$$

see [11, Chap. I, Theorem 2.1]. This concludes the proof of (38).

Case $\ell \geqslant 2$. Since $H_1^\ell(T) \subset C^0(\bar{T})$ and $\tilde{\Pi}_h(\mathcal{I}_h v)$ is equal to $\mathcal{I}_h v$ for all functions v in $H_1^\ell(\Delta_T) \cap V_1^1(\Delta_T)$, we derive from inequality (38) for $\ell = 1$

$$h_T^{-1}\|v - \tilde{\Pi}_h v\|_{L_1^2(T)} + \|v - \tilde{\Pi}_h v\|_{V_1^1(T)} \leqslant c\,\|v - \mathcal{I}_h v\|_{V_1^1(\Delta_T)}.$$

Estimate (38) follows by combining Proposition 3 with a further result proved in [13, Lemma 6.1]. □

To end this section, we give a useful stability property, that we state in the following theorem.

Theorem 3 *There exists a constant c, independent of h, such that, for all elements T in $\mathcal{T}_h$, and for all functions $v \in L_{-1}^2(T)$, the following inequality holds:*

$$\|\tilde{\Pi}_h v\|_{L_{-1}^2(T)} \leqslant c\|v\|_{L_{-1}^2(\Delta_T)}. \tag{39}$$

Proof We have

$$\|\tilde{\Pi}_h v\|_{L_{-1}^2(T)} \leqslant \sum_{i=1,\, \boldsymbol{a}_i \notin \Gamma_0}^{N_h} \|\pi_i v\|_{L^\infty(T_i)} \|\varphi_i\|_{L_{-1}^2(T)}.$$

Combining inequalities (7), (8) and, when T_i is of type 2 or 3 for instance,

$$\|\pi_i v\|_{L^\infty(T_i)} \leqslant h_T^{-\frac{1}{2}} \|v\|_{L_{-1}^2(T_i)}$$

yields (39). □

As a consequence of the previous results, there exist positive constants c and c' and an operator $R_h : V_{1\diamond}^1(\Omega) \times H_{1\diamond}^1(\Omega) \to V_{h/2}$ such that for all triangles T in $\mathcal{T}_h$ and all functions $\boldsymbol{v}$ in $V_{1\diamond}^1(\Omega) \times H_{1\diamond}^1(\Omega)$

$$\|\boldsymbol{v} - R_h\boldsymbol{v}\|_{L_1^2(T)^2} \leqslant c\, h_T \|\boldsymbol{v}\|_{V_1^1(\Delta_T) \times H_1^1(\Delta_T)}, \tag{40}$$

$$\|\boldsymbol{v} - R_h\boldsymbol{v}\|_{V_1^1(T) \times H_1^1(T)} \leqslant c'\|\boldsymbol{v}\|_{V_1^1(\Delta_T) \times H_1^1(\Delta_T)}. \tag{41}$$

5 Inf–Sup condition for the Stokes problem and a priori error estimates

5.1 Axisymmetric inf–sup condition in $(V_1^1(\Omega) \times H_1^1(\Omega)) \times H_1^1(\Omega)$

The following proposition is the weighted version of the theorem presented in [2] about the Cartesian inf–sup condition for P1isoP2/P1 elements. We also adapt it to a regular (but not necessarily uniformly regular) family of triangulations (we refer to [7, Sect. VI.6] for the idea of this extension).

Proposition 4 *There exists a positive constant c independent of h such that*

$$\forall q_h \in Q_h, \quad \sup_{\mathbf{v}_h \in V_{h/2}} \frac{b(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_{V_1^1(\Omega) \times H_1^1(\Omega)}} \geqslant c \left(\sum_{T \in \mathcal{T}_h} h_T^2 |q_h|_{H_1^1(T)}^2 \right)^{1/2} . \quad (42)$$

Proof Let q_h in Q_h be fixed. Let T_k and T_j be two triangles of $\mathcal{T}_h$ with a common side, two common vertices denoted by $\mathbf{x}^k$ and $\mathbf{x}^j$ and let $\mathbf{x}^{kj}$ be their midpoint $\frac{1}{2}(\mathbf{x}^k + \mathbf{x}^j)$ (see Fig. 3). Each element T_k of $\mathcal{T}_h$ being divided into four sub-triangles by joining the midpoints of its edges, let D_k be the union of the three sub-triangles of T_k with one vertex being $\mathbf{x}^{kj}$. Note d_k the weighted measure of D_k:

$$d_k = \int_{D_k} r \, d\mathbf{x}.$$

Define D_j in the same way. To simplify the notation we neglect $d\mathbf{x}$ in the integrals.

We define an element $\mathbf{v}_h$ of $V_{h/2}$, being equal to zero at each vertex of any T in $\mathcal{T}_h$ and on $\Gamma_0 \cup \Gamma$, and equal to arbitrary real vectors $\mathbf{v}_{kj}$ at all midpoints $\mathbf{x}^{kj}$. In what follows and for brevity, we denote by $\sum_{(kj)}$ the sum over all edges with endpoints $\mathbf{x}_k$ and $\mathbf{x}_j$ which are shared by two triangles of $\mathcal{T}_h$. We can also write $\mathbf{v}_h$ as $\sum_{(kj)} \varphi^{kj} \mathbf{v}_{kj}$, where φ^{kj} is a basis function of $V_{h/2}$ which is one at $\mathbf{x}^{kj}$ and zero at the other vertices of triangles of $\mathcal{T}_{h/2}$. The end of the proof is divided into five steps.

Step 1. The norm of $\varphi^{kj} \mathbf{v}_{kj}$ is bounded by

$$\|\varphi^{kj} \mathbf{v}_{kj}\|_{L_1^2(T_k)^2}^2 \leqslant \int_{D_k} r |\varphi^{kj} \mathbf{v}_{kj}|^2 \leqslant |\mathbf{v}_{kj}|^2 d_k.$$

Fig. 3 The two triangles T_k and T_j
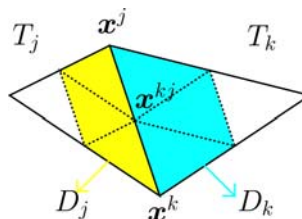
Then

$$\|\boldsymbol{v}_h\|^2_{L^2_1(\Omega)^2} \leqslant \sum_{(kj)} \|\varphi^{kj}\boldsymbol{v}_{kj}\|^2_{L^2_1(\Omega)^2} \leqslant \sum_{(kj)} |\boldsymbol{v}_{kj}|^2 (d_k + d_j).$$

By the inverse inequality (11) and noting h_k the diameter h_{T_k},

$$
\begin{aligned}
\|\boldsymbol{v}_h\|^2_{V^1_1(\Omega)\times H^1_1(\Omega)} &\leqslant c \sum_{T_k \in \mathcal{T}_h} h_k^{-2} \|\boldsymbol{v}_h\|^2_{L^2_1(T_k)^2} \\
&\leqslant c \sum_{T_k \in \mathcal{T}_h} \sum_{T_j \cap T_k = \text{edge}} h_k^{-2} |\boldsymbol{v}_{kj}|^2 d_k \\
&= c \sum_{(kj)} \left(h_k^{-2} d_k + h_j^{-2} d_j \right) |\boldsymbol{v}_{kj}|^2 \\
&\leqslant c\sigma^{-2} \sum_{(kj)} h_{kj}^{-2} |\boldsymbol{v}_{kj}|^2 (d_k + d_j),
\end{aligned}
$$

where $h_{kj} = \max\{h_k, h_j\}$ and the last inequality holds since T_k and T_j have a common edge. We have shown that

$$\|\boldsymbol{v}_h\|^2_{V^1_1(\Omega)\times H^1_1(\Omega)} \leqslant c \sum_{(kj)} h_{kj}^{-2} |\boldsymbol{v}_{kj}|^2 (d_k + d_j). \tag{43}$$

Step 2. The vector field $\nabla_a q_h$ is constant on T_k and it is noted by $(\nabla q)_k$. Let χ_D be the characteristic function of a set D, then by integration by parts

$$b(\varphi^{kj}\boldsymbol{v}_{kj}, q_h) = \int_\Omega r\varphi^{kj}\boldsymbol{v}_{kj} \cdot \nabla_a q_h = \int_\Omega r\varphi^{kj}\boldsymbol{v}_{kj} \cdot ((\nabla q)_k \chi_{D_k} + (\nabla q)_j \chi_{D_j}).$$

Now by considering the cases where T_k is of type 1, 2 or 3 and using [5, Sect. VII, Proposition 2.3], we check that there exist two positive constants α_1 and α_2 only depending on σ and a scalar ρ_{kj} with $\alpha_1 < \rho_{kj} < \alpha_2$, such that

$$\int_{D_k} \varphi^{kj} r \, d\boldsymbol{x} = d_k \rho_{kj}. \tag{44}$$

This implies that

$$\int_\Omega r\varphi^{kj}\boldsymbol{v}_{kj} \cdot (\nabla q)_k \chi_{D_k} = \int_{D_k} r\left((\nabla q)_k \cdot \varphi^{kj}\boldsymbol{v}_{kj}\right) = ((\nabla q)_k \cdot \boldsymbol{v}_{kj}) d_k \rho_{kj},$$

hence

$$b(\boldsymbol{v}_h, q_h) = \sum_{(kj)} \left(\rho_{kj} d_k (\nabla q)_k + \rho_{jk} d_j (\nabla q)_j \right) \cdot \boldsymbol{v}_{kj}. \tag{45}$$

Step 3. Equation (43) together with (45) gives

$$\max_{v_h \in V_{h/2}} \frac{b(\boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_{V_1^1(\Omega) \times H_1^1(\Omega)}}$$

$$\geq c \max_{(\boldsymbol{v}_{kj})(kj)} \left[\frac{\sum_{(kj)} \left(\rho_{kj} d_k (\nabla q)_k + \rho_{jk} d_j (\nabla q)_j \right) \cdot \boldsymbol{v}_{kj}}{\left(\sum_{(kj)} h_{kj}^{-2} |\boldsymbol{v}_{kj}|^2 (d_k + d_j) \right)^{\frac{1}{2}}} \right]. \tag{46}$$

We now take $\boldsymbol{v}_{kj} = h_{kj}^2 (\rho_{kj} d_k (\nabla q)_k + \rho_{jk} d_j (\nabla q)_j)/(d_k + d_j)$, so we are in the case of equality in the Cauchy–Schwarz inequality

$$\sum_{(kj)} \left[\frac{\rho_{kj} d_k (\nabla q)_k + \rho_{jk} d_j (\nabla q)_j}{h_{kj}^{-1} (d_k + d_j)^{\frac{1}{2}}} \cdot h_{kj}^{-1} \boldsymbol{v}_{kj} (d_k + d_j)^{\frac{1}{2}} \right]$$

$$= \left[\sum_{(kj)} \frac{|\rho_{kj} d_k (\nabla q)_k + \rho_{jk} d_j (\nabla q)_j|^2}{h_{kj}^{-2} (d_k + d_j)} \right]^{\frac{1}{2}} \left[\sum_{(kj)} h_{kj}^{-2} |\boldsymbol{v}_{kj}|^2 (d_k + d_j) \right]^{\frac{1}{2}}. \tag{47}$$

Therefore, from (46) and (47)

$$\max_{v_h \in V_{h/2}} \frac{b(\boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_{V_1^1(\Omega) \times H_1^1(\Omega)}} \geq c \left[\sum_{(kj)} h_{kj}^2 \frac{|\rho_{kj} d_k (\nabla q)_k + \rho_{jk} d_j (\nabla q)_j|^2}{d_k + d_j} \right]^{\frac{1}{2}}. \tag{48}$$

Step 4. Since in T_k the gradient $\nabla_a q_h|_{T_k} = (\nabla q)_k$ is constant, setting $q_k = q_h(\boldsymbol{x}^k)$, etc., leads to

$$(\nabla q)_k \cdot \frac{\boldsymbol{x}^k - \boldsymbol{x}^j}{|\boldsymbol{x}^k - \boldsymbol{x}^j|} = \frac{q_k - q_j}{|\boldsymbol{x}^k - \boldsymbol{x}^j|} \tag{49}$$

and the same holds in T_j. Then

$$\left(\rho_{kj} d_k (\nabla q)_k + \rho_{jk} d_j (\nabla q)_j \right) \cdot \frac{\boldsymbol{x}^k - \boldsymbol{x}^j}{|\boldsymbol{x}^k - \boldsymbol{x}^j|} = \frac{\left(\rho_{kj} d_k + \rho_{jk} d_j \right) \left(q_k - q_j \right)}{|\boldsymbol{x}^k - \boldsymbol{x}^j|}.$$

Note that the ρ_{kj} are larger than α_1. For a vector $\boldsymbol{a}$ and a unit vector $\boldsymbol{e}$, $|\boldsymbol{a}|^2 \geqslant |\boldsymbol{a} \cdot \boldsymbol{e}|^2$. Taking $e = \frac{x^k - x^j}{|x^k - x^j|}$, the square of the previous inequality gives

$$\left(\rho_{kj} d_k (\nabla q)_k + \rho_{jk} d_j (\nabla q)_j \right)^2 \geqslant c \frac{(d_k + d_j)^2 (q_k - q_j)^2}{|x^k - x^j|^2}. \tag{50}$$

Step 5. We note x^l the third vertex of T_k. Equality (49) and the regularity of the family of triangulations allow to write for $m = k$ and l,

$$|(\nabla q)_k|^2 \leqslant c \left[\frac{(q_k - q_l)^2}{|x^k - x^l|^2} + \frac{(q_m - q_j)^2}{|x^m - x^j|^2} \right]. \tag{51}$$

Inequalities (48) and (50) yield

$$\max_{v_h \in V_{h/2}} \frac{b(\boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_{V_1^1(\Omega) \times H_1^1(\Omega)}}$$

$$\geqslant c \left\{ \sum_{T_k \in \mathcal{T}_h} \sum_{(x^k, x^j)}^{\text{edges of } T_k} h_{kj}^2 \frac{(q_k - q_j)^2}{|x^k - x^j|^2} (d_k + d_j) \chi_{\{x^{kj} \notin \Gamma \cup \Gamma_0\}} \right\}^{\frac{1}{2}}.$$

Since for each triangle T in $\mathcal{T}_h$ there is at least one vertex inside Ω, in the previous equation at least two midpoints of the edges are not in $\Gamma \cup \Gamma_0$.

Let t_k denote the weighted measure of the triangle T_k. Since $d_{kj} \geqslant \frac{3}{8} t_k$ (see [9, Appendix A1]), inequality (51) leads to

$$\max_{v_h \in V_{h/2}} \frac{b(\boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_{V_1^1(\Omega) \times H_1^1(\Omega)}} \geqslant c \left(\sum_{T_k \in \mathcal{T}_h} h_k^2 t_k |(\nabla q)_k|^2 \right)^{\frac{1}{2}} = c \left(\sum_{T \in \mathcal{T}_h} h_T^2 \|\nabla_a q_h\|_{L_1^2(T)}^2 \right)^{\frac{1}{2}}.$$

$\square$

5.2 Axisymmetric inf–sup condition in $(V_1^1(\Omega) \times H_1^1(\Omega)) \times L_1^2(\Omega)$

Verfürth in [15] shows the inf–sup condition for the Stokes problem with P1isoP2/P1 elements in the Cartesian case. Verfürth's argument is based on the propositions presented in the previous sections. Now we are going to state and prove the inf–sup theorem for axisymmetric P1isoP2/P1 finite elements.

Theorem 4 (Axisymmetric P1isoP2/P1 inf–sup) *There exists a positive constant c independent of h such that*

$$\forall q_h \in Q_h, \qquad \sup_{\boldsymbol{v}_h \in V_{h/2}} \frac{b(\boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_{V_1^1(\Omega) \times H_1^1(\Omega)}} \geqslant c \|q_h\|_{L_1^2(\Omega)}. \tag{52}$$

Proof Let q_h be in Q_h with $\|q_h\|_{L_1^2(\Omega)} = 1$, and denote η as the quantity $(\sum_{T \in \mathcal{T}_h} h_T^2 |q_h|_{H_1^1(T)}^2)^{1/2}$. Proposition 4 implies

$$\sup_{\mathbf{v}_h \in V_{h/2}} \frac{b(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_{V_1^1(\Omega) \times H_1^1(\Omega)}} \geq c_1 \eta. \tag{53}$$

It follows from the inf–sup condition (5) that there is a $\mathbf{u}$ satisfying

$$\mathrm{div}_a \mathbf{u} + \frac{1}{r} u_r = -q_h \quad \text{and} \quad \|\mathbf{u}\|_{V_1^1(\Omega) \times H_1^1(\Omega)} \leq \beta^{-1} \|q_h\|_{L_1^2(\Omega)} = \beta^{-1}. \tag{54}$$

This implies $|b(\mathbf{u}, q_h)| = \|q_h\|_{L_1^2(\Omega)}^2 = 1$. On the other hand, taking $\mathbf{u}_h = R_h \mathbf{u}$ and using (40) yields

$$\sum_{T \in \mathcal{T}_h} h_T^{-2} \|\mathbf{u}_h - \mathbf{u}\|_{L_1^2(T)}^2 \leq c^2 \sum_{T \in \mathcal{T}_h} \|\mathbf{u}\|_{V_1^1(\Delta_T) \times H_1^1(\Delta_T)}^2$$
$$\leq c'^2 \|\mathbf{u}\|_{V_1^1(\Omega) \times H_1^1(\Omega)}^2 \leq \beta^{-2} c'^2.$$

By integration by parts together with the Schwarz inequality, we derive that

$$|b(\mathbf{u}_h - \mathbf{u}, q_h)| = \left| \sum_{T \in \mathcal{T}_h} \int_T (\mathbf{u}_h - \mathbf{u}) \nabla_a q_h \, r \mathrm{d}x \right|$$
$$\leq \left(\sum_{T \in \mathcal{T}_h} h_T^{-2} \|\mathbf{u} - \mathbf{u}_h\|_{L_1^2(T)}^2 \right)^{1/2} \left(\sum_{T \in \mathcal{T}_h} h_T^2 \|\nabla_a q_h\|_{L_1^2(T)}^2 \right)^{1/2}$$
$$\leq c \eta. \tag{55}$$

Using (54), (41) and (55) yields

$$\sup_{\mathbf{v}_h \in V_{h/2}} \frac{b(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_{V_1^1(\Omega) \times H_1^1(\Omega)}}$$
$$\geq \frac{b(\mathbf{u}, q_h) - |b(\mathbf{u}_h - \mathbf{u}, q_h)|}{\|\mathbf{u}\|_{V_1^1(\Omega) \times H_1^1(\Omega)} + \|\mathbf{u}_h - \mathbf{u}\|_{V_1^1(\Omega) \times H_1^1(\Omega)}}$$
$$\geq c \left(b(\mathbf{u}, q_h) - |b(\mathbf{u}_h - \mathbf{u}, q_h)| \right) \geq c \left(1 - c' \eta \right) = c_2 - c_3 \eta. \tag{56}$$

Inequalities (53) and (56) imply

$$\sup_{\mathbf{v}_h \in V_{h/2}} \frac{b(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_{V_1^1(\Omega) \times H_1^1(\Omega)}} \geqslant \max\{c_1\eta, c_2 - c_3\eta\}$$

$$\geqslant \min_{t \geqslant 0} \max\{c_1 t, c_2 - c_3 t\} = \frac{c_1 c_2}{c_1 + c_3}.$$

This ends the proof in the case $\|q_h\|_{L_1^2(\Omega)} = 1$. Otherwise, if q_h is different from zero, take $\tilde{q}_h = q_h / \|q_h\|_{L_1^2(\Omega)}$, which concludes the proof.   □

It is also possible to replace $V_{h/2}$ by piecewise quadratic functions on $\mathcal{T}_h$, i.e.,

$$\tilde{V}_h = \left\{ \mathbf{v}_h \in \mathcal{C}^0(\Omega)^2 \ : \ \mathbf{v}_h|_\Gamma = 0, \ v_{h,r}|_{\Gamma_0} = 0; \ \forall T \in \mathcal{T}_h \ \mathbf{v}_h|_T \in P_2(T)^2 \right\}.$$

The degrees of freedom of this space are exactly the same as for $V_{h/2}$, so by the same arguments as previously, the inf–sup condition (52) still holds with $V_{h/2}$ replaced by $\tilde{V}_h$.

5.3 Existence, uniqueness and a priori error estimates

The spaces $V_{1\diamond}^1(\Omega) \times H_{1\diamond}^1(\Omega)$ equipped with $\|\cdot\|_{V_1^1(\Omega) \times H_1^1(\Omega)}$ and $L_{1,0}^2(\Omega)$ with $\|\cdot\|_{L_1^2(\Omega)}$ are Hilbert spaces. In fact (see Proposition 1 and [4, Sect. II.2]) they are isomorphic to subspaces of $H^1(\check{\Omega})^3$ and $L^2(\check{\Omega})$ respectively. The bilinear form $a(\cdot, \cdot)$ is elliptic (property derived from $\breve{a}(\cdot, \cdot)$), and the bilinear form $b(\cdot, \cdot)$ satisfies by Theorem 4 the inf–sup condition. Hence the abstract results of Babuška [1], Brezzi [6] (see also [7, Sect. II.2.2] and [11, Sect. II.1]) yield the well-posedness of the discrete Stokes problem (6).

Theorem 5 *Problem* (6) *has a unique solution* $(\mathbf{u}_h, p_h)$ *in* $V_{h/2} \times Q_h$. *Furthermore, if* $\mathbf{u}$ *is in* $H_1^{s+1}(\Omega)^2$ *and* p *in* $H_1^s(\Omega)$, $0 < s \leqslant 1$, *then there exists a constant* C *such that*

$$\|\mathbf{u} - \mathbf{u}_h\|_{V_1^1(\Omega) \times H_1^1(\Omega)} + \|p - p_h\|_{L_1^2(\Omega)} \leqslant Ch^s \left(\|v\|_{H_1^{s+1}(\Omega)^2} + \|p\|_{H_1^s(\Omega)} \right). \quad (57)$$

Proof As pointed out in Eq. (4), the operator $a(\cdot, \cdot)$ can be expressed by the three-dimensional operator $\breve{a}(\cdot, \cdot)$, which is coercive (see for example [4, Sect. II.2]). In Sect. 5.2 the inf–sup condition (52) is proved, hence Theorem 1.1 in [11, Sect. II.1] yields

$$\|\mathbf{u} - \mathbf{u}_h\|_{V_1^1(\Omega) \times H_1^1(\Omega)} + \|p - p_h\|_{L_1^2(\Omega)}$$

$$\leqslant C \left(\inf_{\mathbf{v}_h \in V_{h/2}} \|\mathbf{u} - \mathbf{v}_h\|_{V_1^1(\Omega) \times H_1^1(\Omega)} + \inf_{q_h \in Q_h} \|p - q_h\|_{L_1^2(\Omega)} \right).$$

In the case $s = 1$, Mercier and Raugel [13] in Theorem 4.4 show that the space of functions in $H_1^2(\Omega)$ vanishing on Γ_0 is included in $V_1^1(\Omega)$. Theorems 1 and 2 and Corollary 3 lead to

$$\inf_{\boldsymbol{v}_h \in V_h} \|\boldsymbol{u} - \boldsymbol{v}_h\|_{V_1^1(\Omega) \times H_1^1(\Omega)} \leqslant C\,h \|v\|_{H_1^2(\Omega)^2}$$

and

$$\inf_{q_h \in Q_h} \|p - q_h\|_{L_1^2(\Omega)} \leqslant C\,h \|p\|_{H_1^1(\Omega)},$$

which proves (57). Extending this result to the case $0 < s < 1$ follows from an interpolation argument (note that the inequality $\|v\|_{V_1^1(\Omega)} \leqslant c\,\|v\|_{H_1^{s+1}(\Omega)}$ is valid for all $s > 0$ and for any function v in $H_1^{s+1}(\Omega) \cap V_1^1(\Omega)$, see [3, Chap. I, Appendix]).   □

In particular, Bernardi et al. in [4, Sect. IX.1] show that if Ω is convex and the angles between Γ and Γ_0, are not too large (for example less than $\frac{3}{4}\pi$ is enough), and if $\boldsymbol{f}$ is in $L_1^2(\Omega)^2$, then $\boldsymbol{u}$ is in $H_1^2(\Omega)^2$ and p is in $H_1^1(\Omega)$ (in fact they show that $\boldsymbol{u}$ is even more regular than that). So the error behaves like $c\,h$ at least when these conditions on the geometry of Ω are satisfied.

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