Optimal Feedback Schemes Over Unknown Channels

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Abstract — **Communication over unknown discrete memoryless channels with instantaneous and perfect feedback is considered. For a given set of channels we define a notion of optimal coding schemes in terms of achievable rate and error exponent, and prove the existence of such coding schemes for two families of channels.**

It is well known that the capacity of a discrete memoryless channel (DMC) cannot be increased by means of perfect and instantaneous feedback. However, when perfect feedback is available a significant gain in terms of the error exponent is possible. In 1976, Burnashev [1] computed the maximum achievable error exponent for DMC's with perfect and instantaneous feedback using variable length codes to be

$$E_B(R, Q_{Y|X}) = \left(\max_{x,x'} D(Q_{Y|x}||Q_{Y|x'})\right)\left(1 - \frac{R}{C(Q_{Y|X})}\right) ,$$

where the maximization is over all pairs of input symbols, R is the communication rate and $C(Q_{Y|X})$ the capacity of the channel $Q_{Y|X}$[2]. It is important to note that both the rate and the error exponent are with respect to the expected codeword length.

We study the situation where communication is carried over a DMC with time invariant transition probability matrix $Q_{Y|X}$ that is unknown to both the transmitter and the receiver. However, we make the assumption that transmitter and receiver have the knowledge that $Q_{Y|X}$ belongs to some subset $\mathcal{Q}$ of DMC's.

For sake of clarity, definitions 1, 2 and 3 review the notions of coding scheme, rate and error exponent related to feedback communication. Definition 4 introduces a notion of optimal sequences of coding schemes and is followed by our main result.

Definition 1 *(Coding Scheme).* For any message set $\mathcal{M}$ of size $M \geq 2$, an encoder is a sequence of functions

$$\Phi^M = \{X_n : \mathcal{M} \times \mathcal{Y}^{n-1} \longrightarrow \mathcal{X}\}_{n \geq 1} . \tag{1}$$

For a message m, the symbol x_n to be sent at time n is given by $X_n(m, y_1^{n-1})$ where $y_1^{n-1} = y_1, y_2, \ldots, y_{n-1}$ denotes the received symbols up to time $n-1$.

A decoder $(\Psi^M, U(M))$ consists of a set of functions

$$\Psi^M = \{\psi_n^M : \mathcal{Y}^n \longrightarrow \mathcal{M}\}_{n \geq 1} , \tag{2}$$

and a stopping time $U(M)$ relative to the received symbols $Y_1, Y_2, \ldots$. The decoded message is $\psi^M_{U(M)}(y_1^{U(M)})$. A coding scheme is a tuple $\mathcal{S}^M = (\Phi^M, \Psi^M, U(M))$.

Definition 2 *(Rate).* For a given channel $Q_{Y|X}$, an integer $M \geq 2$ and a coding scheme $\mathcal{S}^M = (\Phi^M, \Psi^M, U(M))$, the average rate is

$$R(\mathcal{S}^M, Q_{Y|X}) = \frac{\ln M}{\mathbb{E}U(M)} \text{ nats per symbol} . \tag{3}$$

The limiting rate for a sequence of coding schemes $\theta = \{\mathcal{S}^M\}_{M \geq 2}$ and a given channel $Q_{Y|X}$ is given by

$$R(\theta, Q_{Y|X}) = \liminf_{M \to \infty} R(\mathcal{S}^M, Q_{Y|X}) . \tag{4}$$

The average error probability, over uniformly chosen messages, given a coding scheme $\mathcal{S}^M$ and a channel $Q_{Y|X}$ is denoted by $\mathbb{P}(\mathcal{E}|Q_{Y|X}, \mathcal{S}^M)$.

Definition 3 *(Error Exponent).* Given a sequence of coding schemes $\theta = \{\mathcal{S}^M\}_{M \geq 2} = \{\Phi^M, \Psi^M, U(M)\}_{M \geq 2}$ the error exponent is

$$E(\theta, Q_{Y|X}) = \liminf_{M \to \infty} -\frac{1}{\mathbb{E}U(M)} \ln \mathbb{P}(\mathcal{E}|Q_{Y|X}, \mathcal{S}^M) . \tag{5}$$

Definition 4 *(Optimal Sequences of Coding Schemes).* Let $\mathcal{Q}$ be a family of DMC's. A set Θ of sequences of coding schemes is said to be optimal for $\mathcal{Q}$ if for any given constant ν with $0 \leq \nu < 1$ there exists $\theta \in \Theta$ such that for any $Q_{Y|X} \in \mathcal{Q}$

$$E(\theta, Q_{Y|X}) = E_B(R(\theta, Q_{Y|X}), Q_{Y|X}) \tag{6}$$

$$\text{and} \quad R(\theta, Q_{Y|X}) \geq \nu C(Q_{Y|X}) . \tag{7}$$

In other words a set Θ of sequence of coding schemes is optimal for a family of channels $\mathcal{Q}$ if for any a priori chosen fraction $0 \leq \nu < 1$ there exists a sequence of coding schemes $\theta \in \Theta$ that simultaneously over $\mathcal{Q}$ satisfies the two following conditions. It achieves a rate at least equal to $\nu C(Q_{Y|X})$ and has a corresponding error exponent equal to the maximum achievable error exponent that could be obtained if the channel statistics were revealed to both the encoder and the decoder.

Theorem. Let L be any constant with $0 \leq L < 1/2$. Let $\mathcal{Q}$ be the family of binary symmetric channels with crossover probability ε with $0 \leq \varepsilon \leq L$. Then there exists a set Θ of optimal sequences of coding schemes for $\mathcal{Q}$.

The same result as above holds if the family $\mathcal{Q}$ represents now the set of Z channels with crossover probability ε such that $0 \leq \varepsilon \leq L$ and with $0 \leq L < 1$.

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References

[1] M. V. Burnashev, "Data transmission over a discrete channel with feedback: Random transmission time" Problems of Information Transmission, vol. 12, number 4, p. 250–265, 1976.

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[2]One can show that at rates above the critical rate $E_B(R, Q_{Y|X})$ exceeds the sphere packing bound and in particular that $\max_{x,x'} D(Q_{Y|x}||Q_{Y|x'}) \geq C(Q_{Y|X})$.