Turbo Codes with Non-Uniform Constellations

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Abstract—This paper presents parallel concatenated turbo codes that employ a non-uniform constellation to achieve shaping gain. The output signal approximates the Gaussian distribution by using equally likely signals with unequal spacing (a non-uniform constellation). The small distance of points near the center of the constellation may lead to a small overall free distance and thus a high error floor for turbo codes. We avoid this situation by a two-step design procedure, that first creates an interleaver according to the criteria in [1], and then identifies the constituent encoders that maximize the turbo code free distance. Simulation results for 4 bits/sec/Hz show that this use of shaping can offer an improvement of approximately 0.2 dB for turbo codes.

I. INTRODUCTION

This paper presents a method for parallel concatenated trellis coded modulation (PCTCM) with constituent encoders of rate k/n, $k > 1$ that employ a non-uniformly spaced constellation to achieve shaping gain. Information theory establishes that for AWGN shaping gain is obtained if the amplitude of the transmitted output more closely follows a Gaussian distribution.

A typical method to obtain shaping gain is to use a uniformly spaced constellation with different probabilities for each signal point. This idea is investigated in [2]. Other methods include trellis shaping [3], the use of prefix codes [4], or similar, subdivision of the signal constellation into variable-size regions [5]. Recently, shaping was also used in conjunction with multilevel coding/multistage decoding [6]. Trellis shaping requires an additional Viterbi-algorithm at the transmitter but maintains a fixed data rate. The methods in [4], [5] are conceptually simpler but possibly impractical since the non-constant data rate may cause buffer over-run or under-run. Also, they require frequent resynchronization in order to avoid catastrophic error propagation.

Non-uniform signal point constellations (also called multi-resolution modulation) are widely used; for instance, in order to create a signal set for hierarchical (rate-adaptive) transmission [7], [8]. The approach in

[9] uses an asymmetric coded-modulation scheme to optimize trellis coding. In this paper we use the method in [10], [11], to obtain shaping gain for turbo codes. Each constellation point is transmitted with the same probability. However, the distance between the constellation points varies in such a way that the output signal approximately follows a Gaussian distribution. This approach can offer shaping gain of up to 1 dB [12] for high order constellations. Our proposed approach combines the non-uniform constellation [10], [11], [12] with the symbol-interleaved turbo encoder introduced in [13], [1].

The paper is organized as follows. Section II reviews shaping with non-uniform constellations, discusses the peak-to-average power ratio and the constellation labeling. Section III optimizes the turbo encoder. Section IV presents simulation results, and finally Section V concludes the paper.

II. SHAPING WITH A NON-UNIFORM CONSTELLATION

A. Non-uniform constellation construction

Consider N points u_i, for simplicity on a real line. Each point has the same transmission probability $1/N$. To make the output of the transmitter follow approximately a Gaussian distribution, split the Gaussian cumulative distribution function (CDF) into N sections of equal probability [12], i.e., partition the ordinate of the CDF into N equal parts. Choose the points u_i such that

$$\int_{-\infty}^{u_i} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} = p_i, \qquad (1)$$

for $p_i = \frac{2i-1}{2N}$ with $i \in \{1, \dots, N\}$.

B. Peak-to-average power ratio expansion

Table I shows that a non-uniform constellation causes a peak-to-average power ratio (PAPR) expansion [12] compared to uniform ASK. Non-uniform constellation shaping is not the best choice if the maximum (as opposed to the average) transmission power is limited. Instead, different shaping schemes (see Section I) might be more suitable. A simpler solution might also be a uniformly

distributed signal set that has a higher signal to noise ratio (but the same maximum power).

TABLE I
PEAK-TO-AVERAGE POWER RATIO (PAPR) EXPANSION FOR NU-ASK VS. UNIFORM ASK.

4-ASK	0.1 dB
8-ASK	0.7 dB
16-ASK	1.5 dB
32-ASK	2.3 dB
64-ASK	3.1 dB
128-ASK	3.8 dB

On the other hand, applications such as high-speed voiceband modems that suffer increased distortion for points near the perimeter of a QAM constellation, can benefit from the non-uniform constellation, where the points near the center are closer together and the points near the perimeter are further apart [10].

C. Non-uniform 8-PAM constellation labeling

Consider the one-dimensional 8-PAM constellation of Fig. 1. Each pair of constellation points is connected with an edge. The binary symbol error that corresponds to an edge is its edge label [14], [15]. The four distances between neighbor points are denoted by x_1, x_2, x_3 and x_4, with $x_1 < x_2 < x_3 < x_4$. Following the analysis in [15] we can label this constellation by corresponding to each edge between two neighbor points one of three basis vectors e_1, e_2 and e_3, and selecting a point to be labeled 000. Two consecutive edges have to correspond to different vectors.

One way to compare the quality of constellation labelings is by examining their edge profiles. The edge profile of a labeled constellation is defined in [14], [15] to be the list containing the minimum distance associated with each edge label. Table II gives the edge profile for the constellation in Fig. 1. Through exhaustive search, we identified the labeling in Fig. 1 as the labeling with the best edge profile. This labeling has the structure called "symmetric ultacomposite" in [14], [15]. For our constituent encoder
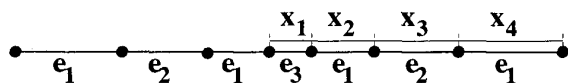

Fig. 1. Non-uniform 8-PAM constellation.

search we used the identified labeling with basis vectors $e_1 = 001$, $e_2 = 011$ and $e_3 = 101$.

TABLE II
EDGE PROFILE FOR THE LABELING OF FIG. 1.

edge label	e_3	e_1	e_2
distance	x_1	x_2	x_3

$e_1 + e_3$	$e_1 + e_2$	$e_1 + e_2 + e_3$	$e_2 + e_3$
$x_1 + x_2$	$x_2 + x_3$	$x_1 + x_2 + x_3$	$x_1 + 2x_2 + x_3$

III. TURBO CODE DESIGN

The employed turbo encoder follows the structure proposed in [1]. Each k/n constituent encoder, for k even, has $k/2$ systematic outputs and $r \geq 1$ parity outputs. The $n = \frac{k}{2} + r$ total output bits of the encoder are mapped to one constellation point. The upper constituent encoder has as systematic outputs the $k/2$ MSB input bits while the lower constituent encoder has as systematic outputs the $k/2$ LSB input bits. Thus the systematic bits are evenly divided between the constituent encoders without puncturing or interleaver constraints. Fig. 2 shows an example of the proposed parallel turbo code structure that employs 16-QAM modulation in connection with rate 4/4 constituent encoders, each with $\frac{k}{2} = 2$ systematic and $r = 2$ parity outputs. The iterative decoder implements the Soft Input Soft Output (SISO) equations appearing in [16].
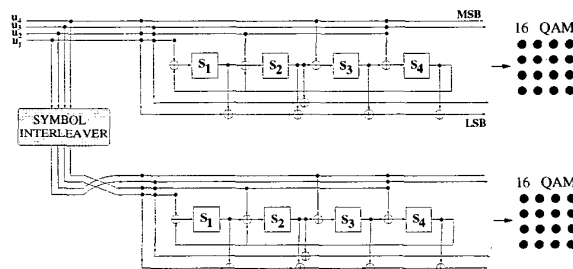

Fig. 2. Two bits/sec/Hz PCTCM turbo code with rate 4/4 constituent encoders.

The turbo encoder design consists of the constituent encoders and the interleaver design. The constituent encoders are optimized for symbol-wise effective distance [13], [1] and use the structure identified in [1].

The role of the interleaver is to lower the error floor, as is called the flattening of the bit error rate curve turbo codes exhibit for moderate to high values of SNR. The error floor depends upon the free distance of the turbo code. The error events with small number of inputs typically determine the free distance, thus the interleaver is designed to avoid them. To this end, we use the semi-random interleaver proposed in [1], which extends Divsalar's spread interleaver to include multiple error events. The spread

interleaver is described by one parameter S, while the extended interleaver is described by three parameters (S, T, X). High parameter values are desirable, because they help the interleaver to avoid more error events and thus achieve a lower error floor [13].

The interleaver performance is closely related to the employed constituent encoders. To take advantage of the semi-random interleaver structure, the constituent encoders must have the following important property: for small input weight error events, the output weight must increase with the length of the error event.

The typical approach for AWGN turbo code design is to select constituent encoders optimized for effective distance, and then design an interleaver specifically tailored to the constituent encoders. However, especially for small interleaver lengths, it is not always easy to identify an interleaver that offers a significant error floor improvement. Moreover, an encoder that has smaller effective distance might lead to a higher free distance turbo code. An appropriate interleaver design might be able to easily avoid the low output weight error events for this constituent encoder.

In this paper we propose a reversed procedure. First, for a specific interleaver length identify a semi-random interleaver that satisfies constraints with as high (S, T, X) parameters ([1], [17]) as possible. Then, among the encoders that have high effective distance identify the one that leads to a higher overall free distance with the selected interleaver.

IV. SIMULATION RESULTS

This section provides simulation results for 4 bits/sec/Hz employing non-uniform 64-QAM= $2 \times$ 8-PAM. The constituent encoders implement a 4/3 code with $r = 1$ parity and $\frac{k}{2} = 2$ systematic outputs, and have $m = 4$ memory elements. The simulated code was identified through computer search using the method in Section III and the edge profile optimal [14] constellation labeling, identified in Section II-C and illustrated in Fig. 3.
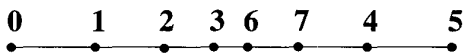
$$\begin{array}{cccccccc} \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{6} & \mathbf{7} & \mathbf{4} & \mathbf{5} \end{array}$$

Fig. 3. Constellation labeling for 64-QAM= $2 \times$ 8-PAM.

Every linear convolutional encoder has a state space description $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$ (see [18], [19]) as follows:

$$s_{j+1} = s_j \mathbf{A} + u_j \mathbf{B}$$
$$x_j = s_j \mathbf{C} + u_j \mathbf{D}$$

where s_j is the state vector, x_j is the output vector and u_j is the input vector. The simulated code can be described in octal notation by the polynomials: $\{035, 01, 05, 011, 013, 01, 01\}$, which refer to the feedback polynomial f, the $k = 4$ rows $\{\mathbf{b}_1 \ldots \mathbf{b}_k\}$ of matrix $\mathbf{B}$, the $r = 1$ columns $\{\mathbf{c}_1 \ldots \mathbf{c}_r\}$ of matrix $\mathbf{C}$ and the $r = 1$ columns $\{\mathbf{d}_1 \ldots \mathbf{d}_r\}$ of matrix $\mathbf{D}$ that correspond to the parity outputs. Matrix $\mathbf{A}$ is the companion matrix of the feedback polynomial, as is described in [20], [21].

Fig. 4 shows that for a uniform constellation and the symbol interleaved system in [1] the performance at BER 10^{-6} is within 0.7 dB of the constrained capacity. For a uniform 64-QAM constellation and 4 bits/sec/Hz the constrained capacity is at 6.62 dB. This system employs an (30-0-0) interleaver of length $4,096$ symbols (input block size in bits: $4,096 \times 4$).

When using the non-uniform constellation the symbol interleaved system converges 0.2 dB earlier. This system employs an (20-5-0) interleaver of length $4,096$ symbols (input block size in bits: $4,096 \times 4$). The capacity, calculated with the methods described in [12], is 6.26 dB for the non-uniform constellation. Thus the gain when using the non-uniform constellation for 64-QAM is approximately 0.36 dB.
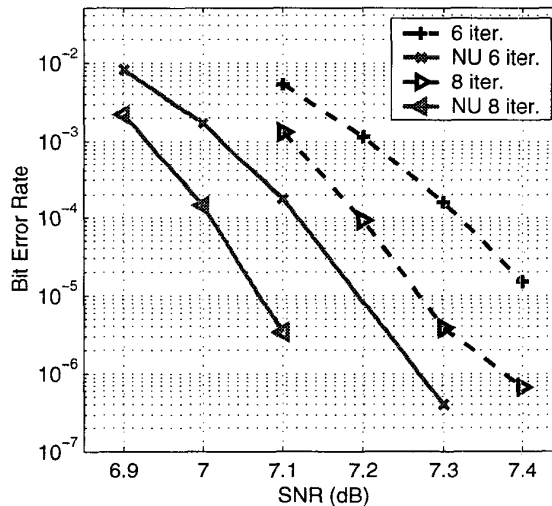

Fig. 4. Effect of non-uniform shaping for 64-QAM= $2 \times$ 8-PAM. Interleaver length 4,096 symbols.

The small distance of points near the center of the non-uniform constellation may lead to small overall free dis-

tance and thus a higher error floor. The proposed design procedure helps avoid this problem. Fig. 5 compares the performance of the encoder identified using the method in Section III with the performance of the encoder: {035, 010, 011, 012, 016, 012, 01} that is only optimized for symbol-wise effective distance.

It is worth noting that the second encoder has a much higher error floor, although it employs the same (20-5-0) interleaver. Also, both the encoders have the same symbol-wise effective free distance. The difference in performance is due to the additional good properties the first encoder has, i.e. the error events output weight increases with the error events length, for the error events with small input weight.
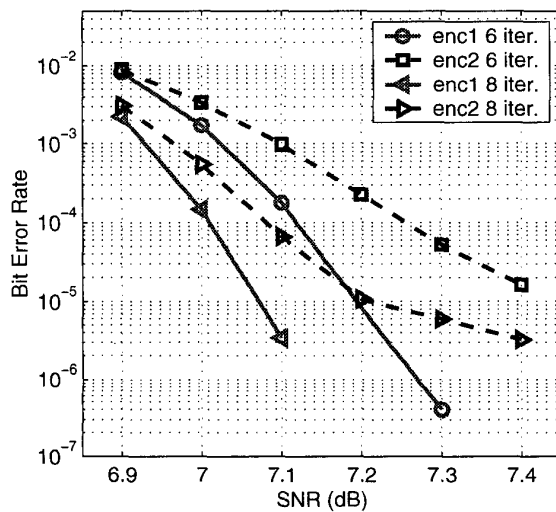

Fig. 5. Comparison of two different encoders for 64-QAM= 2 × 8-PAM.

V. Conclusions

This paper examined the performance of a symbol interleaved parallel concatenated turbo code that employs a non-uniform constellation to achieve shaping gain. The small distance of points near the center of the non-uniform constellation may lead to small overall free distance for the turbo code, and thus to a higher error floor. A turbo encoder design procedure that helps avoid the high error floor, first identifies an interleaver according to the criteria in [1], and then selects the constituent encoders that maximize the free distance when coupled with this interleaver. To take advantage of the interleaver structure,

the constituent encoders must have the property that the output weight increases with the error event's length, for small input weight error events. Simulation results for 4 bits/sec/Hz show that the use of shaping can offer an improvement of approximately 0.2 dB for turbo codes.

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