Journal of Mathematical Imaging and Vision 8, 99–130 (1998)

Analysis of the Superposition of Periodic Layers and Their Moiré Effects through the Algebraic Structure of Their Fourier Spectrum

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Abstract. A new approach is presented for investigating the superposition of any number of periodic structures, and the moiré effects which may result. This approach, which is based on an algebraic analysis of the Fourier-spectrum using concepts from the theory of geometry of numbers, fully explains the properties of the superposition of periodic layers and of their moiré effects. It provides the fundamental notations and tools for investigating, both in the spectral domain and in the image domain, properties of the superposition as a whole (such as periodicity or almost-periodicity), and properties of each of the individual moirés generated in the superposition (such as their profile forms and intensity levels, their singular states, etc.). This new, rather unexpected combination of Fourier theory and geometry of numbers proves very useful, and it offers a profound insight into the structure of the spectrum of the layer superposition and the corresponding properties back in the image domain.

Keywords: superposition of periodic structures, moiré effect, Fourier analysis, geometry of numbers, spectrum support, dense spectrum, discrete spectrum

1. Introduction

The superposition of periodic structures (such as line-gratings, dot-screens, etc.) offers a wide range of interesting properties for exploration: starting from the overall structure of the superposition (which may be periodic or not) and culminating in the interesting and sometimes even spectacular moiré effects which may appear in the superposition. The superposition moiré phenomenon has a vast number of important applications in many different fields [1–5], while in other circumstances (like in the case of color reproduction) it may have an unwanted, adverse effect [6]. It is therefore important to fully understand this phenomenon and its various properties, along with the other global properties of the layer superposition as a whole.

Although classical geometric or algebraic approaches can be used to explain the superposition moiré phenomenon and its geometric properties [7, 8], it has been shown that the best approach for exploring phenomena related to the superposition of *periodic structures* is the spectral approach, which is based on the Fourier theory [9, 10]. Unlike the classical geometric and algebraic methods, this approach enables us to analyze properties not only in the original images and in their superposition but also in their spectral representations, and thus it offers a more profound insight into the problem and provides indispensable tools for exploring it. Moreover, the additional dimension offered by the impulse amplitudes in the spectrum (in addition to their geometric locations) also enables a quantitative analysis of the moiré intensity levels [11], in addition to the qualitative geometric analysis of the moiré, already offered by the earlier approaches.

We start the article with a short review of our Fourier-based approach and its significant advantages. Then, in the main part of the article we further deepen this Fourier approach, by introducing a new algebraic formalism based on the theory of geometry of numbers [12] to describe the structure of the spectrum. Using this new combination of algebraic methods and the Fourier theory we find the rules which determine the positioning of the impulses in the spectrum, and in particular, those impulses which correspond to each moiré effect. We show how this approach fully explains the properties of the superposition as a whole (periodicity or almost-periodicity, the formation of impulse clusters in the spectrum and their significance, etc.), and in particular, the properties of each moiré effect which is generated in the superposition. We proceed as follows: In Section 2 we prepare the ground by reviewing the basic notions of our Fourier-based approach. In Sections 3–6 we develop our new algebraic method, and in Section 7 we show the new insight it offers into the structure of the spectrum of the layer superposition and its eventual moirés. Several illustrative examples are presented in Section 8. Finally, in Section 9 we show how, via the Fourier theory, the algebraic structure of the spectrum relates to properties of the layer superposition and its moirés back in the image domain.

2. The Spectral Approach

The spectral approach is based on the duality between 2D images in the (x, y)-plane and their 2D spectra in the (u, v) frequency plane through the 2D Fourier transform. Let us briefly review here the basic properties of the image types we are concerned with, and the fundamental notions and notations on which our spectral approach is based.

2.1. Properties of our Images and their Spectra

First, we only deal here with monochromatic (black and white) images. In this case each image can be represented in the image domain by a *reflectance* function, which assigns to any point (x, y) of the image a value between 0 and 1 representing its light reflectance: 0 for black (i.e., no reflected light), 1 for white (i.e., full light reflectance), and intermediate values for in-between shades. In the case of transparencies, the reflectance function is replaced by a *transmittance* function defined in a similar way. Since the superposition of black and any other shade always gives black, this suggests a *multiplicative* model for the superposition of monochromatic images. Thus, when m monochromatic images are superposed (for example, by overprinting), the reflectance of the resulting image is given by the *product* of the reflectance functions of the individual images

$$r(x, y) = r_1(x, y)\, r_2(x, y) \ldots r_m(x, y) \qquad (1)$$

According to the Convolution Theorem [13, p. 244] the Fourier transform of the product function is the convolution of the Fourier transforms of the individual functions. Therefore if we denote the Fourier transform of each function by the respective capital letter and the 2D convolution by $**$, the spectrum of the superposition is given by

$$R(u, v) = R_1(u, v) ** R_2(u, v) ** \cdots ** R_m(u, v) \qquad (2)$$

Second, we are basically interested in *periodic* images defined on the continuous (x, y)-plane, such as line-gratings or dot-screens, and their superpositions. This implies that the spectrum of the image on the (u, v)-plane is not a continuous one but rather consists of impulses, corresponding to the frequencies which appear in the Fourier series decomposition of the image [13, p. 204]. In the case of a 1-fold periodic image, such as a line-grating, the spectrum consists of a 1D "comb" of impulses through the origin; in the case of a 2-fold periodic image the spectrum is a 2D "nailbed" of impulses through the origin.

Each impulse in the 2D spectrum is characterized by three main properties: its *label* (which is its index in the Fourier series development); its *geometric location* (or *impulse location*), and its *amplitude* (see Fig. 1). To the geometric location of any impulse is attached a *frequency vector* **f** in the spectrum plane, which connects the spectrum origin to the geometric location of the impulse. This vector can be expressed either by its polar coordinates (f, θ), where θ is the direction of the impulse and f is its distance from the origin (i.e., its frequency in that direction); or by its Cartesian coordinates (f_u, f_v), where f_u and f_v are the horizontal and vertical components of the frequency. In terms of the original image, the *geometric location* of an impulse in the spectrum determines the frequency f and the direction θ of the corresponding periodic component in the image, and the *amplitude* of the impulse represents

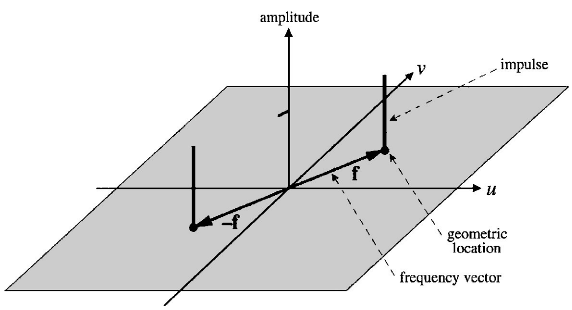


Figure 1. The *geometric location* and *amplitude* of impulses in the 2D spectrum. To each impulse is attached its *frequency vector*, which points to the geometric location of the impulse in the spectrum plane (u, v).

the intensity of that periodic component in the image. (Note that if the original image is not symmetric about the origin, the amplitude of each impulse in the spectrum may also have a non-zero imaginary component).

However, the question of whether or not an impulse in the spectrum represents a *visible* periodic component in the image strongly depends on properties of the human visual system. The fact that the eye cannot distinguish fine details above a certain frequency (i.e., below a certain period) suggests that the human visual system model includes a low-pass filtering stage. This is a bidimensional bell-shaped filter whose form is anisotropic (since it appears that the eye is less sensitive to small details in diagonal directions such as 45° [14]). However, for the sake of simplicity this low-pass filter can be approximated by the *visibility circle,* a circular step-function around the spectrum origin whose radius represents the *cutoff frequency* (i.e., the threshold frequency beyond which fine detail is no longer detected by the eye). Obviously, its radius depends on several factors such as the contrast of the observed details, the viewing distance, light conditions, etc. If the frequencies of the image details are beyond the border of the visibility circle in the spectrum, the eye can no longer see them; but if a strong enough impulse in the spectrum of the image superposition falls inside the visibility circle, then a moiré effect becomes visible in the superposed image. (In fact, the visibility circle has a hole in its center, since very low frequencies cannot be seen, either.)

For the sake of convenience, we may assume that the given images (gratings, grids, etc.) are symmetrically centered about the origin. As a result, we will normally deal with images (and image superpositions) which are *real* and *symmetric*, and whose spectra are consequently also real and symmetric [13, pp. 14–15]. This means that each impulse in the spectrum (except for the DC at the origin) is always accompanied by a twin impulse of an identical amplitude, which is symmetrically located at the other side of the origin as in Fig. 1 (their frequency vectors being $\mathbf{f}$ and $-\mathbf{f}$). If the image is nonsymmetric (but, of course, still real), the amplitudes of the twin impulses at $\mathbf{f}$ and $-\mathbf{f}$ are complex conjugates.

2.2. *The Spectrum Convolution and the Superposition Moirés*

According to the Convolution Theorem (Eqs. (1) and (2)), when m line-gratings are superposed in the image domain, the resulting spectrum is the convolution of their individual spectra. This convolution of combs can be seen as an operation in which frequency vectors from the individual spectra are added vectorially, while the corresponding impulse amplitudes are multiplied. More precisely, each impulse in the spectrum-convolution is generated during the convolution process by the contribution of *one* impulse from *each* individual spectrum: its location is given by the sum of their frequency vectors, and its amplitude is given by the product of their amplitudes. This permits us to introduce an indexing method for denoting each of the impulses of the spectrum-convolution in a unique, unambiguous way. The general impulse in the spectrum-convolution will be denoted the $(k_1, k_2, \ldots, k_m)$-*impulse,* where m is the number of superposed gratings, and each integer k_i is the index (harmonic), within the comb (the Fourier series) of the ith spectrum, of the impulse that this ith spectrum contributed to the impulse in question in the convolution. Using this formal notation we can therefore express the geometric location of the general $(k_1, k_2, \ldots, k_m)$-impulse in the spectrum-convolution by the vectorial sum (or linear combination)

$$\mathbf{f}_{k_1,k_2,\ldots,k_m} = k_1\mathbf{f}_1 + k_2\mathbf{f}_2 + \cdots + k_m\mathbf{f}_m \qquad (3)$$

and its amplitude by:

$$a_{k_1,k_2,\ldots,k_m} = a^{(1)}_{k_1} a^{(2)}_{k_2} \cdots a^{(m)}_{k_m} \qquad (4)$$

where $\mathbf{f}_i$ denotes the frequency vector of the fundamental impulse in the spectrum of the ith grating, and $k_i\mathbf{f}_i$ and $a^{(i)}_{k_i}$ are respectively the frequency vector and the amplitude of the k_ith harmonic impulse in the spectrum of the ith grating.

The vectorial sum of Eq. (3) can also be written in terms of its Cartesian components. If f_i are the frequencies of the m original gratings and θ_i are the angles that they form with the positive horizontal axis, then the coordinates (f_u, f_v) of the $(k_1, k_2, \ldots, k_m)$-impulse in the spectrum-convolution are given by

$$\begin{aligned} f_u &= k_1 f_1 \cos\theta_1 + k_2 f_2 \cos\theta_2 \\ &\quad + \cdots + k_m f_m \cos\theta_m \\ f_v &= k_1 f_1 \sin\theta_1 + k_2 f_2 \sin\theta_2 \\ &\quad + \cdots + k_m f_m \sin\theta_m \end{aligned} \qquad (5)$$

Therefore, the frequency, the period and the angle of the considered impulse (and of the moiré it represents) are given by the length and the direction of the vector

$\mathbf{f}_{k_1,k_2,\ldots,k_m}$ as follows

$$f = \sqrt{f_u^2 + f_v^2} \quad T_M = 1/f \quad \varphi_M = \arctan(f_v/f_u) \tag{6}$$

Note that in the special case of $m = 2$ gratings, when a moiré effect occurs due to the (1,−1)-impulse in the convolution, Eqs. (5) and (6) are reduced to the familiar geometrically obtained formulas of the period and angle of the moiré effect between two gratings [7]

$$\begin{aligned} T_M &= \frac{T_1 T_2}{\sqrt{T_1^2 + T_2^2 - 2T_1T_2\cos\alpha}} \\ \sin\varphi_M &= \frac{T_1 \sin\alpha}{\sqrt{T_1^2 + T_2^2 - 2T_1T_2\cos\alpha}} \end{aligned} \tag{7}$$

(where T_1 and T_2 are the periods of the two original images and α is the angle difference between them, $\theta_2 - \theta_1$). When $T_1 = T_2$ this is further simplified into the well-known formulas [7]

$$T_M = \frac{T}{2\sin(\alpha/2)} \quad \varphi_M = 90^\circ - \alpha/2 \tag{8}$$

Let us now say a word about the notations used for the superposition moirés. We use a notational formalism which provides a systematic means for identifying the various moiré effects. As we have seen, a $(k_1, k_2, \ldots, k_m)$-impulse of the spectrum-convolution which falls close to the spectrum origin, inside the visibility circle, represents a moiré effect in the superposed image (see Fig. 2). We call the m-grating moiré whose fundamental impulse is the $(k_1, k_2, \ldots, k_m)$-impulse in the spectrum-convolution a $(k_1, k_2, \ldots, k_m)$-*moiré*; the highest absolute value in the index-list is called the *order* of the moiré. Note that in the case of doubly periodic images, such as in dot screens, each superposed image contributes two frequency vectors to the spectrum, so that m in Eqs. (3)–(5) above counts each doubly periodic layer as two gratings.

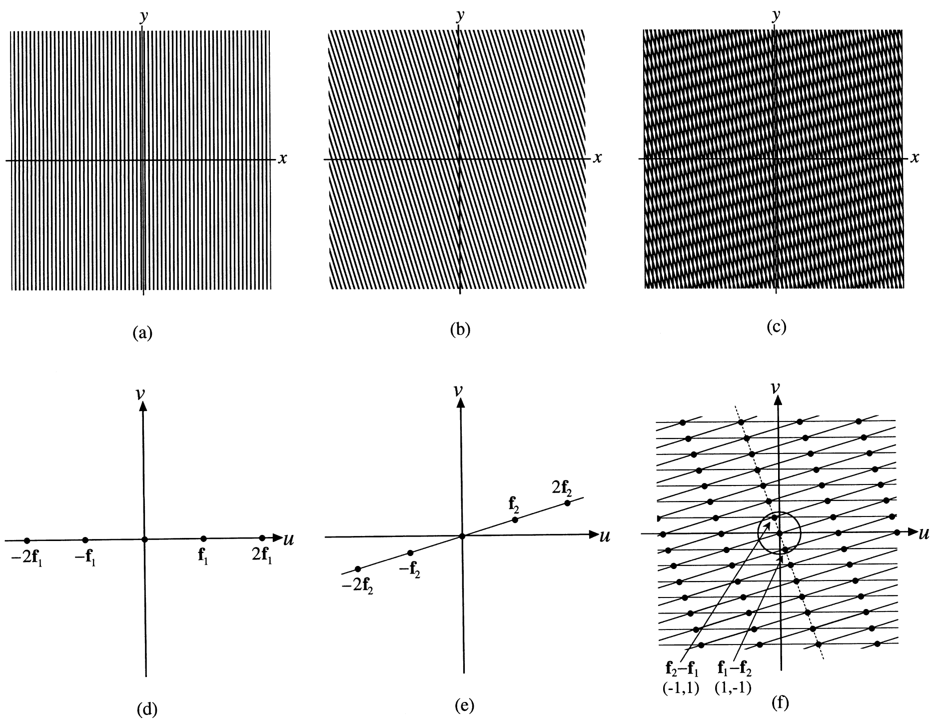


Figure 2. Binary gratings (a) and (b) and their superposition (c) in the image domain; their respective spectra are the infinite impulse-combs shown in (d) and (e) and their convolution (f). Black dots in the spectra indicate the geometric location of the impulses; the line segments connecting them have been added to clarify the geometric relations. Only impulse locations are shown in the spectra, but not their amplitudes. The circle in the center of the spectrum (f) represents the visibility circle. It contains the impulse pair whose frequency vectors are $\mathbf{f}_1 - \mathbf{f}_2$ and $\mathbf{f}_2 - \mathbf{f}_1$ and whose indices are $(1, -1)$ and $(-1, 1)$; this is the fundamental impulse pair of the $(1, -1)$-moiré seen in (c). The dotted line in (f) shows the infinite impulse-comb which represents this moiré.

2.3. Singular States

An interesting special case occurs when impulses of the convolution fall *exactly* on top of the DC impulse at the spectrum origin. This happens for instance in the superposition of 2 identical gratings with an angle difference of 0° or 180° (Fig. 3(a)), or when 3 identical gratings are superposed with angle differences of 120° between each other (Fig. 3(d)). As can be seen from the respective vector diagrams (Figs. 3(c) and (f)), these are limit cases in which the vectorial sum of the frequency vectors is exactly **0**. This means that the moiré frequency is 0 (i.e., its period is infinitely large), and therefore the moiré is not visible. This situation is called a *singular moiré state*; but although the moiré effect in a singular state is not visible, this is a very unstable moiré-free state since any slight deviation in the angle or frequency of any of the superposed layers may cause the new impulses in the spectrum to move slightly off the origin, thus generating a moiré effect with a very significant, visible period (see the center images in Fig. 3).

More formally, we say that a singular moiré state occurs whenever a $(k_1, \ldots, k_m)$-impulse in the spectrum convolution falls exactly on the spectrum origin, i.e., when the frequency-vectors of the m superposed gratings, $\mathbf{f}_1, \ldots, \mathbf{f}_m$, are such that $\sum k_i \mathbf{f}_i = \mathbf{0}$. This implies, of course, that all the impulses of the $(k_1, \ldots, k_m)$-moiré comb fall on the spectrum origin. Furthermore, as it can easily be seen in the spectrum convolution, *any* $(k_1, \ldots, k_m)$-impulse in the spectrum convolution can be made singular by sliding the vector-sum $\sum k_i \mathbf{f}_i$ to the spectrum origin, namely: by appropriately modifying the vectors $\mathbf{f}_1, \ldots, \mathbf{f}_m$ (i.e., the frequencies and angles of the superposed layers). When the $(k_1, \ldots, k_m)$-impulse is located exactly on the spectrum origin we say that the corresponding $(k_1, \ldots, k_m)$-moiré has become singular[1].

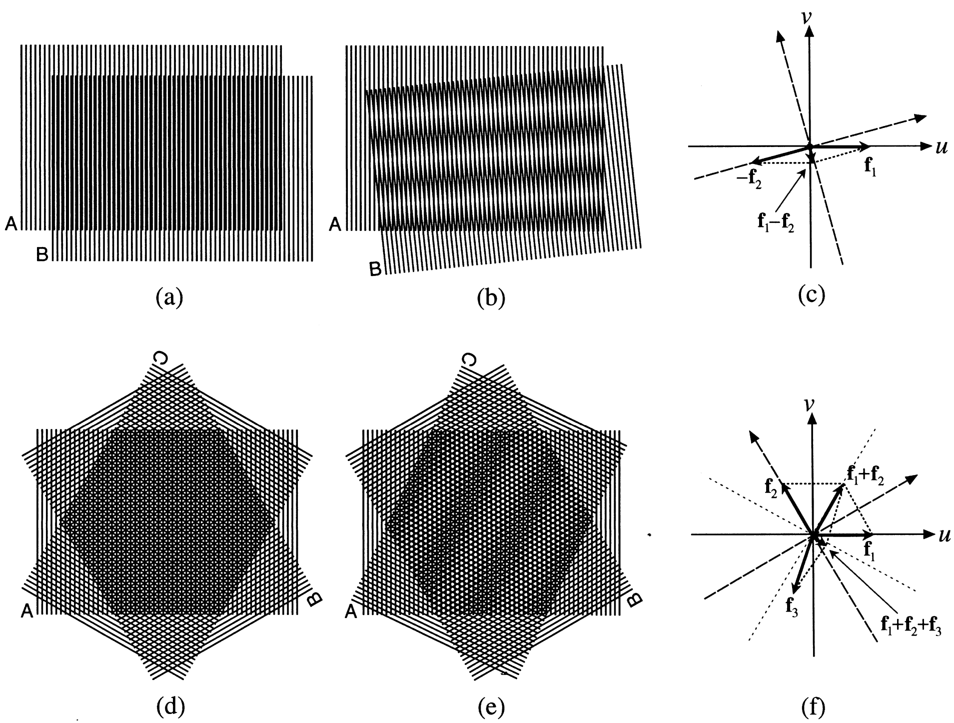


Figure 3. Examples of singular states. First row: (a) the superposition of two identical gratings at an angle difference of 0° gives a singular (unstable) moiré-free state. (b) A small angle or frequency deviation in any of the layers causes the reappearance of the moiré with a very significant visible period. The spectral interpretation of (b) is shown in the vector diagram (c); compare to Fig. 2(f) which shows also impulses of higher orders. Second row: (d) the superposition of three identical gratings with angle differences of 120° gives a singular (unstable) moiré-free state; again, any small angle or frequency deviation may cause the reappearance of the moiré, as shown in (e) and in its vector diagram, (f).

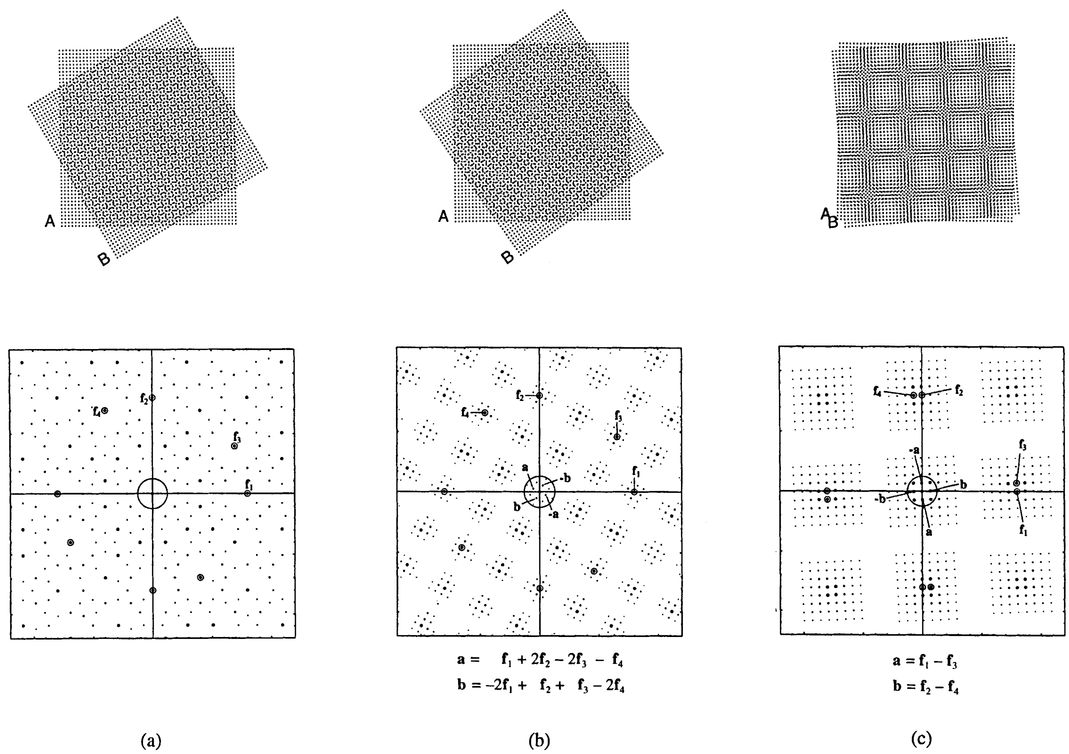


Figure 4. Top: the superposition of two dot-screens with identical frequencies and with an angle difference of: (a) $\alpha = 30°$; (b) $\alpha = 34.5°$; (c) $\alpha = 5°$. Bottom: the corresponding spectra. Only impulse locations are shown in the spectra, but not their amplitudes. Bold points denote the locations of the fundamental impulses of the two original dot-screens. Large points represent convolution impulses of the first order (i.e., (k_1, k_2, k_3, k_4)-impulses with $k_i = 1, 0$, or -1); smaller points represent convolution impulses of higher orders (only impulses of the first few orders are shown). The circle around the spectrum origin represents the visibility circle. Note that while in (a) no significant impulses are located inside the visibility circle, in (b) the spectrum origin is closely surrounded by the impulse-cluster of the second order $(1, 2, -2, -1)$-moiré, and in (c) the spectrum origin is closely surrounded by the impulse-cluster of the first order $(1, 0, -1, 0)$-moiré.

2.4. Impulse Clusters in the Spectrum Convolution; Moiré Extraction

Figure 2(f) shows the spectrum of the superposition of two 1-fold periodic images, namely: the convolution of their original nailbed spectra. Similarly, Fig. 4 shows the spectra of various superpositions of two 2-fold periodic images. As we can see, the spectrum convolution consists of a “forest” of impulses (with real or complex amplitudes, depending on the symmetry properties in the image domain). It has been shown [11] that the occurrence of a moiré phenomenon in the image superposition is associated with the appearance of impulse clusters in the spectrum, as in Figs. 2(f) and 4. In particular it has been shown there that the main cluster, the infinite impulse-cluster which is centered on the spectrum origin and whose fundamental impulse is $(k_1, \ldots, k_m)$, represents in the spectrum the $(k_1, \ldots, k_m)$-moiré effect generated in the superposition. And indeed, by extracting this impulse cluster from the spectrum and taking its inverse Fourier transform, one obtains, back in the image domain, the isolated contribution of the moiré in question to the superposition, i.e., the moiré intensity profile.

2.5. The Advantages of the Spectral Approach

The spectral approach presented above proves very useful in the investigation of superposed periodic layers and their moiré effects. The main advantages of the spectral approach include the following points:

(1) It provides a means for labeling and identifying each of the possible moiré effects in the m-layer superposition individually. Thus, each moiré in the superposition has its own “identity” or index notation: the $(k_1, \ldots, k_m)$-moiré.

(2) The spectrum of the superposition contains all the information about each of the generated moirés: the period and the angle of the moiré are given by the geometric locations of its fundamental impulses, and its intensity profile is given by the amplitudes of its fundamental and higher harmonic impulses (the moiré cluster). This enables a full *quantitative* analysis of each moiré and its intensity levels [11], in addition to the *qualitative* geometric analysis of the moiré, which is already offered by the classical approaches.

(3) Since the spectrum of the superposition contains *simultaneously* all the impulses which may represent moiré effects in the given superposition, it provides an overall, panoramic view of all the different moirés of various orders which are simultaneously present in the same layer superposition [10].

(4) Moreover, this approach permits us to see how changes in the original superposed layers influence the spectrum. This enables us, in particular, to dynamically trace in the spectral domain the development of each of the moirés, and to identify at any moment which of them is visible, singular, or simply irrelevant (beyond the visibility circle).

(5) The spectral approach provides an easy explanation for all multiple-layer moirés, including the more complex cases where the geometric analysis may become too complicated. In our approach all moirés of all orders are treated on an equal basis, and there is no longer any need to deal first with "simple moirés", then with "moirés of moirés", etc. (as, for example, in [15, p. 134], [22, pp. 63–64] or [6, pp. 336–337]).

2.6. Overview of the Following Algebraic Formalization

The numerous advantages of the spectral approach in the analysis of superpositions of periodic layers and their moiré effects clearly show the interest in further deepening this approach. Our aim will be to obtain a full understanding of the spectrum of the superposition, and through it, a better insight into the superposition itself, in the image domain. This will be done in the following sections, using a new algebraic approach, which is based on the theory of geometry of numbers. As we have seen, the spectrum of the superposition of m gratings consists of all the impulses $(k_1, \ldots, k_m)$ where $k_i \in \mathbb{Z}$. This gives, in fact, a mapping of $\mathbb{Z}^m$ (the infinite set of all the integer m-tuples) into the spectrum plane, $\mathbb{R}^2$. The key point of our approach is the algebraic formalization of this mapping using the fundamental relationship given by Eq. (3): we define (in Section 4) the linear transformation $\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}(k_1, \ldots, k_m)$ from $\mathbb{Z}^m$ to $\mathbb{R}^2$ which gives for each $(k_1, \ldots, k_m)$-impulse in the spectrum convolution its geometric location $k_1\mathbf{f}_1 + \cdots + k_m\mathbf{f}_m$ in the (u, v)-plane. The algebraic investigation of the transformation $\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$ proves to be very fruitful: it provides a fundamental explanation of the structure of the spectrum (the impulse "forest" and "clusters"), and in particular it fully explains the clusterization phenomenon and provides a complete identification of all impulses which participate in each of the clusters in the spectrum. Note that throughout this algebraic discussion we ignore the amplitudes of the impulses in the spectrum, and we only concentrate on their indices, their geometric locations, and the relationship between them. Only then, based on the algebraic results obtained, do we reintroduce in Section 9 the impulse amplitudes, and relate the algebraic structure of the spectrum, via the Fourier theory, to properties of the layer superposition and its moirés back in the image domain.

3. The Support of the Spectrum; Modules and Lattices

From the algebraic point of view, the spectrum plane (u, v) is considered as a 2D Euclidean vector space $\mathbb{R}^2$; the geometric location of each impulse is therefore a point (or a vector; we will not distinguish between points and their corresponding vectors) with coordinates (f_u, f_v) in this plane (see Section 2.2 and Fig. 1).

The set of the geometric locations on the (u, v)-plane of all the impulses in a given spectrum (either the spectrum of a single layer or the spectrum of a layer superposition) is called the *support* of that spectrum. It is important to note that the support of a spectrum contains the geometric locations of *all* the impulses in the spectrum, including those whose amplitudes happen to be zero; this ensures that there are no "gaps" or "holes" in the algebraic structure of the support. As a consequence of Eqs. (3) and (4), the support of the spectrum is only determined by the frequencies and angles of the superposed layers, but it is invariant under changes in the profile shape of each layer; such changes do not influence the impulse *locations* in the spectrum, but only their *amplitudes.*

3.1. Modules and Lattices in $\mathbb{R}^n$

Let us define here two algebraic structures which will be used in the following discussions concerning the support of a spectrum.

Definition. Let $\mathbf{v}_1, \ldots, \mathbf{v}_m$ be m arbitrary vectors in $\mathbb{R}^n$. The set of all the points (vectors) in $\mathbb{R}^n$ given by

$$M = \{k_1\mathbf{v}_1 + \cdots + k_m\mathbf{v}_m \mid k_i \in \mathbb{Z}\} \tag{9}$$

(i.e., all the linear combinations of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ with *integer* coefficients) is called a *module* in $\mathbb{R}^n$.

The vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are called *generating vectors* of the module M, but they are not generally a basis, since they are not necessarily linearly independent in $\mathbb{R}^n$ (and in fact, their number m may be even larger than n). The maximum number r of linearly independent (over $\mathbb{R}$) vectors[2] in a module M is called the *rank* of M (denoted: $\text{rank}_{\mathbb{R}}\, M = r$, or simply: $\text{rank}\, M = r$); it is clear that $r \leq m$ and $r \leq n$ [12, p. 44]. We will call the maximum number z of linearly independent vectors over $\mathbb{Z}$ in a module M the *integral rank* of M (denoted: $\text{rank}_{\mathbb{Z}}\, M = z$); it is clear that $r \leq z \leq m$.[3]

Definition. Let $\mathbf{v}_1, \ldots, \mathbf{v}_m$ be m linearly independent (over $\mathbb{R}$) vectors in $\mathbb{R}^n$ (obviously, $m \leq n$). The set of all the points (vectors) in $\mathbb{R}^n$ given by

$$L = \{k_1\mathbf{v}_1 + \cdots + k_m\mathbf{v}_m \mid k_i \in \mathbb{Z}\} \tag{10}$$

is called a *lattice* (or a *dot-lattice*) in $\mathbb{R}^n$ [16].

Clearly, a lattice is a special case of a module, in which the m generating vectors are linearly independent. In this case, the generating vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are, indeed, a *basis* or an *integral basis* (over $\mathbb{Z}$) of the lattice L; and using the notations above we have: $r = z = m$.

While a lattice is always a *discrete* subset of $\mathbb{R}^n$ (meaning that it does not contain arbitrarily close points), a module may be *dense* in $\mathbb{R}^n$ (even though it is not continuous).[4] Consider, for example, the following module in $\mathbb{R}^2$: $M_1 = \{k(1, 0) + l(\sqrt{2}, 0) \mid k, l \in \mathbb{Z}\}$. M_1 is generated by the vectors $(1, 0)$ and $(\sqrt{2}, 0)$, and its integral rank is 2; however, its rank is only 1, since all its members are located within $\mathbb{R}^2$ on a straight line (the x-axis). Moreover, the module M_1 is dense on this line (although it does not fully cover the whole continuous line: for example, $(\frac{1}{2}, 0) \notin M_1$). As a second example, consider the following module in $\mathbb{R}^2$: $M_2 = \{k(1, 0) + l(\frac{1}{2}, \frac{1}{2}) + m(0, 1) \mid k, l, m \in \mathbb{Z}\}$. Although M_2 is generated by three vectors in $\mathbb{R}^2$, the third of them is actually redundant in this case (since it can be obtained as an integral linear combination of the two others), and the module M_2 coincides with a lattice in $\mathbb{R}^2$ having the basis: $(1, 0)$, $(\frac{1}{2}, \frac{1}{2})$.

These two examples can be summarized as follows

M_1: $\text{rank}_{\mathbb{R}}\, M_1 = 1 < \text{rank}_{\mathbb{Z}}\, M_1 = 2$ dense module

M_2: $\text{rank}_{\mathbb{R}}\, M_2 = 2 = \text{rank}_{\mathbb{Z}}\, M_2 = 2$ discrete lattice

In fact, the following general property holds:

Proposition 1. *A module in $\mathbb{R}^n$ is a lattice iff it is discrete* [12, *p.* 44]; *and a module in $\mathbb{R}^n$ is not a lattice iff it is dense in a subgroup of $\mathbb{R}^n$. Moreover, using the notation $r = \text{rank}_{\mathbb{R}}\, M$ and $z = \text{rank}_{\mathbb{Z}}\, M$, a module M is a lattice* (*and therefore discrete*) *iff $z = r$; the module is not a lattice* (*and is dense in a subspace of $\mathbb{R}^n$*) *iff $z > r$.*

It is interesting to note that a module does not necessarily have a basis (over $\mathbb{R}$). For example, we have seen that the module M_1 in the example above is of rank 1; but still, it cannot be generated by a single vector. This means that there exists no basis to M_1. But although a module M does not necessarily have a basis (over $\mathbb{R}$), it does always have an *integral basis* (over $\mathbb{Z}$) which spans it: If the m generating vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ of the module M are linearly independent (over $\mathbb{Z}$), they are themselves an integral basis of M, and $\text{rank}_{\mathbb{Z}}\, M = m$. Otherwise, we take the minimal subset $\mathbf{v}_1, \ldots, \mathbf{v}_z$ from the m generating vectors which still spans the module M; $\mathbf{v}_1, \ldots, \mathbf{v}_z$ are linearly independent over $\mathbb{Z}$ (since otherwise one of them is a linear combination over $\mathbb{Z}$ of the others, and $\mathbf{v}_1, \ldots, \mathbf{v}_z$ is not minimal). Therefore $\mathbf{v}_1, \ldots, \mathbf{v}_z$ are an integral basis (over $\mathbb{Z}$) of M, and their number z is the integral rank of M.

Notation. Let $\mathbf{v}_1, \ldots, \mathbf{v}_m$ be arbitrary vectors in $\mathbb{R}^n$. We denote the *vector space* and the *module* which are spanned (generated) by these vectors by

$$\text{Sp}(\mathbf{v}_1, \ldots, \mathbf{v}_m) = \{k_1\mathbf{v}_1 + \cdots + k_m\mathbf{v}_m \mid k_i \in \mathbb{R}\}$$

$$\text{Md}(\mathbf{v}_1, \ldots, \mathbf{v}_m) = \{k_1\mathbf{v}_1 + \cdots + k_m\mathbf{v}_m \mid k_i \in \mathbb{Z}\}$$

$\text{Sp}(\mathbf{v}_1, \ldots, \mathbf{v}_m)$ is the set of all the linear combinations over $\mathbb{R}$ of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathbb{R}^n$, and $\text{Md}(\mathbf{v}_1, \ldots, \mathbf{v}_m)$ is the set of all their linear combinations over $\mathbb{Z}$. Note that the notations Sp( ) and Md( ) can be also used in the case of an infinite set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots \in \mathbb{R}^n$.

Clearly, $\text{Sp}(\mathbf{v}_1, \ldots, \mathbf{v}_m)$ is a subspace of the vector space $\mathbb{R}^n$, whereas $\text{Md}(\mathbf{v}_1, \ldots, \mathbf{v}_m)$ is a module within this subspace: $\text{Md}(\mathbf{v}_1, \ldots, \mathbf{v}_m) \subset \text{Sp}(\mathbf{v}_1, \ldots, \mathbf{v}_m) \subseteq \mathbb{R}^n$. While $\text{Sp}(\mathbf{v}_1, \ldots, \mathbf{v}_m)$ is continuous and has the cardinality of the continuum, the module $\text{Md}(\mathbf{v}_1, \ldots, \mathbf{v}_m)$ is only a denumerable infinite set which is imbedded within $\text{Sp}(\mathbf{v}_1, \ldots, \mathbf{v}_m)$, and it is either discrete or dense in it. Moreover, we have

$$\text{Sp}(\text{Md}(\mathbf{v}_1, \ldots, \mathbf{v}_m)) = \text{Sp}(\mathbf{v}_1, \ldots, \mathbf{v}_m)$$

This means that $\text{Sp}(\mathbf{v}_1, \ldots, \mathbf{v}_m)$ is the smallest subspace of $\mathbb{R}^n$ which includes the module $\text{Md}(\mathbf{v}_1, \ldots, \mathbf{v}_m)$; we will call it the *continuous extension* of the module. It is clear that "filling the gaps" inside the module $\text{Md}(\mathbf{v}_1, \ldots, \mathbf{v}_m)$ by admitting $k_i \in \mathbb{R}$ rather than $k_i \in \mathbb{Z}$ does not change the number of independent vectors over $\mathbb{R}$, so that we have

$$\text{rank}_{\mathbb{R}} \text{Md}(\mathbf{v}_1, \ldots, \mathbf{v}_m) = \dim \text{Sp}(\mathbf{v}_1, \ldots, \mathbf{v}_m) \quad (11)$$

Using these new terms we can now reformulate results which were obtained earlier in this section:

Since linear independence over $\mathbb{R}$ implies linear independence over $\mathbb{Z}$, it is clear that for any set of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ the maximum number of linear independent vectors over $\mathbb{Z} \geq$ the maximum number of linear independent vectors over $\mathbb{R}$:

$$\text{rank}_{\mathbb{Z}} \text{Md}(\mathbf{v}_1, \ldots, \mathbf{v}_m) \geq \text{rank}_{\mathbb{R}} \text{Md}(\mathbf{v}_1, \ldots, \mathbf{v}_m)$$

and by (11):

$$\text{rank}_{\mathbb{Z}} \text{Md}(\mathbf{v}_1, \ldots, \mathbf{v}_m) \geq \dim \text{Sp}(\mathbf{v}_1, \ldots, \mathbf{v}_m) \quad (12)$$

And furthermore, we can reformulate Proposition 1 as follows:

The module $M = \text{Md}(\mathbf{v}_1, \ldots, \mathbf{v}_m)$ is a lattice (and therefore discrete) *iff* the equality in (12) holds, i.e.,

$$\text{rank}_{\mathbb{Z}} \text{Md}(\mathbf{v}_1, \ldots, \mathbf{v}_m) = \dim \text{Sp}(\mathbf{v}_1, \ldots, \mathbf{v}_m)$$

and conversely, M is not a lattice (and is dense on a subgroup of $\mathbb{R}^n$) *iff* the inequality in (12) holds, i.e.,

$$\text{rank}_{\mathbb{Z}} \text{Md}(\mathbf{v}_1, \ldots, \mathbf{v}_m) > \dim \text{Sp}(\mathbf{v}_1, \ldots, \mathbf{v}_m).$$

3.2. *The Application to the Frequency Spectrum*

Let us now proceed from the general case (with vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathbb{R}^n$) to our particular case of interest, in which $\mathbf{f}_1, \ldots, \mathbf{f}_m \in \mathbb{R}^2$ are frequency vectors in the spectrum plane (u, v). Let us start with some examples:

Example 1. The support of the spectrum of any *periodic* function of two variables $p(x, y)$ is a lattice in $\mathbb{R}^2$, i.e., in the (u, v)-plane; this follows from the decomposition of the periodic function into a Fourier series. If $p(x, y)$ is 2-fold periodic, the support of its spectrum is a 2D lattice. If $p(x, y)$ is 1-fold periodic, like a line-grating, the support of its spectrum is a 1D lattice on a straight line through the origin of the (u, v)-plane. This 1D lattice consists of all the points $k\mathbf{f}$ where $\mathbf{f}$ is the fundamental frequency of $p(x, y)$ and k runs through all integers. Note that all functions with the same period have an identical spectrum support, even when some (or even most) of the impulses in their spectra happen to have a zero amplitude, as in the case of $p(x) = \cos(2\pi x/T)$.

Example 2. Let $r_1(x, y)$ and $r_2(x, y)$ be line gratings, with fundamental frequency vectors $\mathbf{f}_1$ and $\mathbf{f}_2$, respectively, as in Fig. 2. The spectrum of each of them is an impulse comb; and if we superpose (i.e., multiply) $r_1(x, y)$ and $r_2(x, y)$, the spectrum of their superposition is the convolution of these two combs. The support of this spectrum convolution (see Fig. 2(f)) is given by: $\text{Md}(\mathbf{f}_1, \mathbf{f}_2) = \{k_1\mathbf{f}_1 + k_2\mathbf{f}_2 \mid k_i \in \mathbb{Z}\}$, which is a module in the spectrum plane (u, v). If the vectors $\mathbf{f}_1$ and $\mathbf{f}_2$ are linearly independent (over $\mathbb{R}$) in $\mathbb{R}^2$, they are also linearly independent over $\mathbb{Z}$, so that $z = r = 2$, and therefore this module is in fact a lattice of rank 2, as in Fig. 2(f). Otherwise, i.e., if $\mathbf{f}_1$ and $\mathbf{f}_2$ are collinear (=linearly dependent over $\mathbb{R}$), there are two possible cases:

(1) If $\mathbf{f}_1$ and $\mathbf{f}_2$ are also linearly dependent over $\mathbb{Z}$ (so that $z = r = 1$), or in other words if $\mathbf{f}_1$ and $\mathbf{f}_2$ are *commensurable* (i.e., the ratio of their lengths is rational),[5] then $\text{Md}(\mathbf{f}_1, \mathbf{f}_2)$ is a lattice of rank 1 which is located on the line spanned by $\mathbf{f}_1$ and $\mathbf{f}_2$.

(2) If $\mathbf{f}_1$ and $\mathbf{f}_2$ are linearly independent over $\mathbb{Z}$ (so that $z > r$), or in other words if $\mathbf{f}_1$ and $\mathbf{f}_2$ are *incommensurable*, then $\mathrm{Md}(\mathbf{f}_1, \mathbf{f}_2)$ becomes a dense set of points on the line spanned by $\mathbf{f}_1$ and $\mathbf{f}_2$, namely: a module of rank 1 and integral rank of 2.

In the general case, if we superpose m line gratings whose frequency vectors are $\mathbf{f}_i$, then the support of their spectrum convolution is given by the module

$$\mathrm{Md}(\mathbf{f}_1, \ldots, \mathbf{f}_m) = \{k_1\mathbf{f}_1 + \cdots + k_m\mathbf{f}_m \mid k_i \in \mathbb{Z}\} \quad \text{where } \mathbf{f}_i \in \mathbb{R}^2. \tag{13}$$

The rank of this module is obviously $r \leq 2$, since it is imbedded in the 2D spectral plane (u, v), but as for its integral rank z we only know that $r \leq z \leq m$. Therefore, there exist two possible cases: If $z > r$ then the spectrum support $\mathrm{Md}(\mathbf{f}_1, \ldots, \mathbf{f}_m)$ is not a lattice but rather a dense module. But as we will see below, in some cases it may happen that $z = r$, so that the spectrum support $\mathrm{Md}(\mathbf{f}_1, \ldots, \mathbf{f}_m)$ does coincide in the (u, v)-plane with a 2D or 1D lattice, and is discrete.

In the discussion below we will also need the continuous counterpart of $\mathrm{Md}(\mathbf{f}_1, \ldots, \mathbf{f}_m)$, namely

$$\mathrm{Sp}(\mathbf{f}_1, \ldots, \mathbf{f}_m) = \{k_1\mathbf{f}_1 + \cdots + k_m\mathbf{f}_m \mid k_i \in \mathbb{R}\} \quad \text{where } \mathbf{f}_i \in \mathbb{R}^2. \tag{14}$$

It is clear that $\mathrm{Sp}(\mathbf{f}_1, \ldots, \mathbf{f}_m)$ is a subspace of $\mathbb{R}^2$ (it may either coincide with $\mathbb{R}^2$, if $\dim \mathrm{Sp}(\mathbf{f}_1, \ldots, \mathbf{f}_m) = 2$, or be a line through its origin, if $\dim \mathrm{Sp}(\mathbf{f}_1, \ldots, \mathbf{f}_m) = 1$; $\dim \mathrm{Sp}(\mathbf{f}_1, \ldots, \mathbf{f}_m) = 0$ is a degenerate case which occurs when the spectrum only contains the DC impulse and represents a constant image). We therefore have: $\mathrm{Md}(\mathbf{f}_1, \ldots, \mathbf{f}_m) \subset \mathrm{Sp}(\mathbf{f}_1, \ldots, \mathbf{f}_m) \subseteq \mathbb{R}^2$.

4. The Mapping between the Impulse Indices and their Geometric Locations

We return now to the fundamental Eq. (3) which specifies for every $(k_1, \ldots, k_m)$-impulse in the spectrum convolution its impulse location in the (u, v)-plane. Note that throughout the discussion m counts 1-fold periodic layers (gratings) in the superposition, and each 2-fold periodic layer is counted as two 1-fold periodic layers. Let $\mathbf{f}_{k_1,k_2,\ldots,k_m} = k_1\mathbf{f}_1 + \cdots + k_m\mathbf{f}_m$ be a point (vector) in $\mathrm{Md}(\mathbf{f}_1, \ldots, \mathbf{f}_m)$, i.e., the geometric location in the (u, v)-plane of the $(k_1, \ldots, k_m)$-impulse of the spectrum convolution. As we can see, the index-vector $(k_1, \ldots, k_m)$ of this impulse defines a point in $\mathbb{Z}^m$, the lattice of all the points in $\mathbb{R}^m$ with integer coordinates: $\mathbb{Z}^m = \{(k_1, \ldots, k_m) \mid k_i \in \mathbb{Z}\}$. This lattice will henceforth be called the *indices-lattice*. The $(k_1, \ldots, k_m)$-impulse can therefore be represented in two different ways: either by its index-vector $(k_1, \ldots, k_m) \in \mathbb{Z}^m$, or by its geometric location in the (u, v) spectrum plane, $\sum k_i\mathbf{f}_i \in \mathrm{Md}(\mathbf{f}_1, \ldots, \mathbf{f}_m)$.

Moreover, for any given set of frequency vectors $\mathbf{f}_1, \ldots, \mathbf{f}_m \in \mathbb{R}^2$ (i.e., for any given superposition of m gratings) there exists a natural mapping between the indices of the impulses and their geometric locations. This mapping from the indices-lattice $\mathbb{Z}^m$ to the corresponding module (spectrum support) $\mathrm{Md}(\mathbf{f}_1, \ldots, \mathbf{f}_m)$ in the (u, v)-plane is given by the linear transformation (homomorphism) $\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}: \mathbb{Z}^m \to \mathrm{Md}(\mathbf{f}_1, \ldots, \mathbf{f}_m)$ which is defined by

$$\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}(k_1, \ldots, k_m) = k_1\mathbf{f}_1 + \cdots + k_m\mathbf{f}_m \tag{15}$$

We will see below that this transformation is closely related to the moirés generated in the superposition of the m gratings defined by the frequency vectors $\mathbf{f}_1, \ldots, \mathbf{f}_m$. Just as an example, we will see that the transformation $\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$ is singular *iff* the vectors $\mathbf{f}_1, \ldots, \mathbf{f}_m$ represent a singular moiré (see Section 5.1).

Note that although this linear transformation is only defined here for integer coordinates k_i, i.e., between $\mathbb{Z}^m$ and $\mathrm{Md}(\mathbf{f}_1, \ldots, \mathbf{f}_m)$, it has a natural continuous extension to their full enclosing vector spaces $\mathbb{R}^m$ and $\mathrm{Sp}(\mathbf{f}_1, \ldots, \mathbf{f}_m)$: By admitting that $k_i \in \mathbb{R}$ rather than $k_i \in \mathbb{Z}$, $\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$ becomes a continuous linear transformation $\Phi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}: \mathbb{R}^m \to \mathrm{Sp}(\mathbf{f}_1, \ldots, \mathbf{f}_m)$ (where $\mathrm{Sp}(\mathbf{f}_1, \ldots, \mathbf{f}_m) \subseteq \mathbb{R}^2$), which is defined the same way as $\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$ above.

Obviously, each choice of the vectors $\mathbf{f}_1, \ldots, \mathbf{f}_m \in \mathbb{R}^2$ (the fundamental frequency vectors of the m superposed gratings) defines a different linear transformation $\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$, which maps the $(k_1, \ldots, k_m)$-impulse to a different point (geometric location) $k_1\mathbf{f}_1 + \cdots + k_m\mathbf{f}_m$ in the spectrum plane (u, v). We will first consider $\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$ as a function of $(k_1, \ldots, k_m)$ alone, for an arbitrary fixed set of $\mathbf{f}_1, \ldots, \mathbf{f}_m$. Then, in the end of Section 7 below we will consider $\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$ as a function of the frequency vectors $\mathbf{f}_1, \ldots, \mathbf{f}_m$ as well, and we will see what happens in the spectrum when $\mathbf{f}_1, \ldots, \mathbf{f}_m$ are being varied.

5. Some Needed Notions from Linear Algebra

In order to better understand the properties of the discrete linear transformation (15), we will first study its continuous extension $\Phi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$, whose properties can easily be determined using some basic notions from linear algebra. For this end, we will briefly review in the present section the needed algebraic notions concerning vector spaces and linear transformations between them. Then, in the following section we will return to the original discrete transformation (15) and study its properties by considering it as the *restriction* of the continuous transformation $\Phi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$, i.e., with $k_i \in \mathbb{Z}$ rather than $k_i \in \mathbb{R}$.

5.1. The Image and the Kernel of a Linear Transformation

Let Φ be a linear transformation from a vector space V to a vector space W, i.e., $\Phi\colon V \to W$. The *image* of Φ and the *kernel* of Φ are defined as:

$$\operatorname{Im}\Phi = \{\mathbf{w} \in W \mid \mathbf{w} = \Phi(\mathbf{v}), \mathbf{v} \in V\}$$
$$\operatorname{Ker}\Phi = \{\mathbf{v} \in V \mid \Phi(\mathbf{v}) = \mathbf{0}\}$$

Both $\operatorname{Im}\Phi$ and $\operatorname{Ker}\Phi$ are vector spaces (subspaces of W and V, respectively), and moreover, there exists between their dimensions the following relationship [18, p. 318]

$$\dim\operatorname{Ker}\Phi + \dim\operatorname{Im}\Phi = \dim V \tag{16}$$

In the case of our particular transformation $\Phi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$, V and W are respectively $\mathbb{R}^m$ and $\mathbb{R}^2$ and $\operatorname{Im}\Phi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$ is the subspace $\operatorname{Sp}(\mathbf{f}_1,\ldots,\mathbf{f}_m) \subseteq \mathbb{R}^2$. Therefore $\dim V = m$ and $\dim\operatorname{Im}\Phi_{\mathbf{f}_1,\ldots,\mathbf{f}_m} = 2$ (or: $\dim\operatorname{Im}\Phi_{\mathbf{f}_1,\ldots,\mathbf{f}_m} = 1$; the degenerate case of $\dim\operatorname{Im}\Phi_{\mathbf{f}_1,\ldots,\mathbf{f}_m} = 0$, in which the spectrum only contains the DC impulse, can be ignored). Hence we obtain

$$\begin{aligned}\dim\operatorname{Ker}\Phi_{\mathbf{f}_1,\ldots,\mathbf{f}_m} &= \dim V - \dim\operatorname{Im}\Phi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}\\ &= m-2 \quad (\text{or } m-1)\end{aligned} \tag{17}$$

Therefore, when $m \geq 3$ (or respectively: $m \geq 2$) we obtain $\dim\operatorname{Ker}\Phi_{\mathbf{f}_1,\ldots,\mathbf{f}_m} \geq 1$. This means that a non-trivial subspace of $\mathbb{R}^m$ is mapped, under the transformation $\Phi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$, to the location (0, 0) in the (u, v)-plane, i.e., to the spectrum origin. We will see shortly (in Section 6) the significance of this fact.

A linear transformation $\Phi\colon V \to W$ is called *singular* if there exists a non-zero $\mathbf{v} \in V$ that is mapped by Φ to $\mathbf{0} \in W$ (i.e., if $\dim\operatorname{Ker}\Phi \geq 1$). The linear transformation Φ is called *regular* or *non-singular* if the only vector in V which it maps to $\mathbf{0} \in W$ is $\mathbf{0} \in V$, i.e., if $\operatorname{Ker}\Phi = \{\mathbf{0}\}$ (or still in other words, if $\dim\operatorname{Ker}\Phi = 0$).

Note that in our case, the linear transformation $\Phi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$ as well as its discrete counterpart $\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$ of (15) are singular *iff* the vectors $\mathbf{f}_1,\ldots,\mathbf{f}_m$ represent a singular moiré: Since $\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$ is singular *iff* there exist $k_i \in \mathbb{Z}$ not all of them 0 such that $\sum k_i\mathbf{f}_i = \mathbf{0}$, (which also means that the vectors $\mathbf{f}_1,\ldots,\mathbf{f}_m$ are linearly dependent over $\mathbb{Z}$), and hence the vectors $\mathbf{f}_1,\ldots,\mathbf{f}_m$ represent a singular moiré (see Section 2.3).

5.2. Partition of a Vector Space into Equivalence Classes

Let V be a vector space and let U be a subspace in V. For any vector $\mathbf{v} \in V$ we define

$$\mathbf{v} + U = \{\mathbf{v} + \mathbf{u} \mid \mathbf{u} \in U\}$$

$\mathbf{v} + U$ is a copy of the subspace U inside V, parallel to U, which is shifted (translated) from the origin by the vector $\mathbf{v}$. $\mathbf{v} + U$ is called the *equivalence class* (or *coset*) of the vector $\mathbf{v}$ in V modulo U (i.e., with respect to U). The set of all the equivalence classes in V gives a disjoint and exhaustive partition of V (that is: for any $\mathbf{v} \in V$, $\mathbf{v}$ is a member of exactly one equivalence class modulo U). All the vectors within the same equivalence class are called *equivalent modulo* U. An element chosen from an equivalence class is called a *representative* of its equivalence class.

As an illustration, let V be the 3D Euclidean space $\mathbb{R}^3$, and let U be the 2D subspace of $\mathbb{R}^3$ defined by: $U = \{(x, y, z) \mid z = x + y\}$ (i.e., a plane through the origin). Then, each equivalence class (modulo U) in $\mathbb{R}^3$ is a parallel translation of the U plane within $\mathbb{R}^3$: $(x_0, y_0, z_0) + U = \{(x_0, y_0, z_0) + (x, y, z) \mid z = x + y\}$.

Note that the only equivalence class in V which is itself a vector space is $\mathbf{0} + U$ (the equivalence class of the vector $\mathbf{0}$), i.e., the subspace U itself; all the other classes are parallel translations of U within the vector space V, and they do not contain the vector $\mathbf{0}$ (the origin of V). Nevertheless, we will still say that each of the translated equivalence classes has the same dimension as the original, unshifted subspace U: $\dim(\mathbf{v} + U) = \dim U$.

The set of all the equivalence classes modulo U in V is itself a vector space, which is called the *quotient space* and denoted V/U. Between the dimensions of the vector spaces V, U and V/U there exists the following relationship [18, pp. 386–387]

$$\dim U + \dim(V/U) = \dim V \qquad (18)$$

Finally, it is important to note that for each subspace U of V we get a different partition of V into equivalence classes (modulo U). We say that each subspace U of V *induces* a different partition of V into equivalence classes. In the following we will concentrate on one particular partition of V, which has some special properties.

5.3. The Partition of V into Equivalence Classes Induced by Φ

Let Φ be a linear transformation from a vector space V to a vector space W, i.e., $\Phi\colon V \to W$. Since $\operatorname{Ker}\Phi$ is a subspace of V, it follows that $\operatorname{Ker}\Phi$ induces a partition of V into equivalence classes. This particular partition of V has an important property: the whole equivalence class of $\mathbf{0}$ within V, i.e., $\operatorname{Ker}\Phi$, is mapped by Φ into $\mathbf{0} \in W$; and moreover, each of the other equivalence classes within V, $\mathbf{v} + \operatorname{Ker}\Phi$, is mapped by Φ to a single point within $\operatorname{Im}\Phi$: $\Phi(\mathbf{v}+\operatorname{Ker}\Phi) = \Phi(\mathbf{v})+\mathbf{0} = \Phi(\mathbf{v})$. Furthermore, two vectors $\mathbf{a}, \mathbf{b} \in V$ are mapped by Φ to the same point in $\operatorname{Im}\Phi$ *iff* they belong to the same equivalence class of V (modulo $\operatorname{Ker}\Phi$). This means that to each equivalence class in this particular partition of V there belongs one point in $\operatorname{Im}\Phi$, and vice versa; the quotient space $V/\operatorname{Ker}\Phi$ (i.e., the space of all the equivalence classes induced by Φ in V) and the subspace $\operatorname{Im}\Phi$ in W are therefore isomorphic. This is indeed proven by the First Isomorphism theorem [19].

These results can be interpreted, loosely speaking, as a "dimension preservation law" under the linear transformation Φ: Assume, for example, that $\Phi\colon \mathbb{R}^3 \to \mathbb{R}^2$ (where Φ is surjective). Here $\dim\operatorname{Im}\Phi = 2$, and therefore by (16) $\dim\operatorname{Ker}\Phi = 3-2 = 1$. Therefore $\operatorname{Ker}\Phi$ is a 1D subspace of $\mathbb{R}^3$, namely: a certain straight line S through the origin. $\operatorname{Ker}\Phi$ (or simply Φ) induces a disjoint and exhaustive partition of the 3D space $\mathbb{R}^3$ into a 2D set (quotient space) containing all the 1D shifted lines parallel to S (equivalence classes). Each of these 1D lines is "collapsed" by Φ onto a single point in $\operatorname{Im}\Phi$; and hence the 2D set of lines within $\mathbb{R}^3$ is mapped (isomorphically) onto the whole 2D plane $\operatorname{Im}\Phi$. It can be said, loosely speaking, that if 2 dimensions out of the 3 dimensions of $\mathbb{R}^3$ are "used" by Φ to span $\operatorname{Im}\Phi$, then the $3-2 = 1$ remaining dimensions are "invested" in each point of $\operatorname{Im}\Phi$, by mapping onto it a whole 1D portion (shifted line) of $\mathbb{R}^3$. Thus, although the image of Φ is only 2D, each point in it "absorbs" a whole 1D portion of $\mathbb{R}^3$, so that all the 3 dimensions of $\mathbb{R}^3$ have actually been "used" by Φ. The aim of this "naive" description of the "collapsing" process will become clear later, in Sections 6 and 7.

Note that this "collapsing" effect only occurs if Φ is a singular transformation. If Φ is non-singular then $\operatorname{Ker}\Phi = \{\mathbf{0}\}$, and therefore each point of V forms an equivalence class of its own, containing only a single point. Since $\dim\operatorname{Ker}\Phi = 0$ it follows from (16) that $\dim\operatorname{Im}\Phi = \dim V$, and therefore the transformation Φ in this case is an isomorphism which simply maps every single point (equivalence class) of V into a single point in $\operatorname{Im}\Phi$. Obviously, this is only possible when $\dim V \le \dim W$.

5.4. The Application of these Results to our Continuous Case

Let us return back to our continuous linear transformation $\Phi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}\colon \mathbb{R}^m \to \operatorname{Sp}(\mathbf{f}_1, \ldots, \mathbf{f}_m) \subseteq \mathbb{R}^2$. Since $\operatorname{Ker}\Phi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$ is a subspace of $\mathbb{R}^m$ (with dimension $m-2$ or $m-1$), it follows from the above discussion that $\operatorname{Ker}\Phi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$ induces a partition of $\mathbb{R}^m$ into equivalence classes (of dimension $m-2$, or respectively, $m-1$). And furthermore, this partition of $\mathbb{R}^m$ has the following special property: the linear transformation $\Phi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$ maps the whole $m-2$ (or $m-1$) dimensional equivalence class of $\mathbf{0}$ in $\mathbb{R}^m$, $\operatorname{Ker}\Phi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$, into the origin (0, 0) of the (u, v)-plane; and similarly, for every $\mathbf{v} \in \mathbb{R}^m$ the transformation $\Phi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$ maps the whole $m-2$ (or $m-1$) dimensional equivalence class of $\mathbf{v}$ in $\mathbb{R}^m$, $\mathbf{v} + \operatorname{Ker}\Phi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$, into a single point in the (u, v)-plane, $\Phi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}(\mathbf{v})$.

6. The Discrete Mapping Ψ vs. the Continuous Mapping Φ

Let us now return to our original discrete transformation $\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}\colon \mathbb{Z}^m \to \operatorname{Md}(\mathbf{f}_1, \ldots, \mathbf{f}_m)$ given in Eq. (15). Looking now at $\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$ as the *restriction* of $\Phi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$ with $k_i \in \mathbb{Z}$ rather than $k_i \in \mathbb{R}$, we can get a better insight into the properties of the discrete mapping $\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$.

Many of the algebraic notions which have been defined above for vector spaces in the continuous case have a similar counterpart also in the discrete case with $k_i \in \mathbb{Z}$ (see Table 1):

Table 1. Summary of the continuous terms with $k_i \in \mathbb{R}$ and their discrete restrictions with $k_i \in \mathbb{Z}$.

Continuous terms (with $k_i \in \mathbb{R}$)	Equivalent terms in the discrete case (with $k_i \in \mathbb{Z}$)
$\mathbb{R}^m$	$\mathbb{Z}^m$
Vector independence over $\mathbb{R}$	Vector independence over $\mathbb{Z}$
$\Phi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}\colon \mathbb{R}^m \to \mathrm{Sp}(\mathbf{f}_1,\ldots,\mathbf{f}_m)$ (=$\mathbb{R}^2$ or $\mathbb{R}^1$)	$\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}\colon \mathbb{Z}^m \to \mathrm{Md}(\mathbf{f}_1,\ldots,\mathbf{f}_m)$
$\mathrm{Im}\,\Phi_{\mathbf{f}_1,\ldots,\mathbf{f}_m} = \mathrm{Sp}(\mathbf{f}_1,\ldots,\mathbf{f}_m)$	$\mathrm{Im}\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m} = \mathrm{Md}(\mathbf{f}_1,\ldots,\mathbf{f}_m)$
$\mathrm{Ker}\,\Phi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$ is a subspace of $\mathbb{R}^m$	$\mathrm{Ker}\,\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$ is a sub-lattice of $\mathbb{Z}^m$
$\mathbf{v} + \mathrm{Ker}\,\Phi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$	$(n_1,\ldots,n_m) + \mathrm{Ker}\,\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$

Like in the continuous case, we can define for the transformation $\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}\colon \mathbb{Z}^m \to \mathrm{Md}(\mathbf{f}_1,\ldots,\mathbf{f}_m)$ the *image* of $\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$ and the *kernel* of $\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$; $\mathrm{Ker}\,\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$ is a sub-lattice of the indices-lattice $\mathbb{Z}^m$, and $\mathrm{Im}\,\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$ is the module $\mathrm{Md}(\mathbf{f}_1,\ldots,\mathbf{f}_m)$, i.e., the spectrum support in the (u, v)-plane. Furthermore, given a sub-lattice L of $\mathbb{Z}^m$, we can also define the partition of the lattice $\mathbb{Z}^m$ into equivalence classes modulo L. The set of all the equivalence classes $(n_1,\ldots,n_m) + L$ in $\mathbb{Z}^m$ gives a disjoint and exhaustive partition of $\mathbb{Z}^m$, where each equivalence class is a parallel translation of L within $\mathbb{Z}^m$.

Now, if we take as sub-lattice L the kernel of the transformation $\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$, we get a special partition of $\mathbb{Z}^m$ which has the following property, as in the continuous case: the transformation $\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$ maps the whole equivalence class of $\mathbf{0}$ in $\mathbb{Z}^m$, $\mathrm{Ker}\,\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$, into the origin (0, 0) of the (u, v)-plane; and similarly, for every $(n_1,\ldots,n_m) \in \mathbb{Z}^m$ the transformation $\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$ maps the whole equivalence class of $(n_1,\ldots,n_m)$ in $\mathbb{Z}^m$, $(n_1,\ldots,n_m) + \mathrm{Ker}\,\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$, into a single point $\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}(n_1,\ldots,n_m)$ in the module $\mathrm{Md}(\mathbf{f}_1,\ldots,\mathbf{f}_m)$ on the (u, v)-plane.

Furthermore, the equivalent of equality (16) for a discrete linear transformation Ψ between two modules, $\Psi\colon M_1 \to M_2$, is given by

$$\mathrm{rank}_{\mathbb{Z}}\,\mathrm{Ker}\,\Psi + \mathrm{rank}_{\mathbb{Z}}\,\mathrm{Im}\,\Psi = \mathrm{rank}_{\mathbb{Z}}\,M_1 \qquad (19)$$

This can be proved in the same way as the proof of (16) in the continuous case of vector spaces (see for example [18, p. 331 No. 9.23]), by replacing throughout the proof the term "linear independence over $\mathbb{R}$" by the term "linear independence over $\mathbb{Z}$".

Now, in the case of our transformation $\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}\colon \mathbb{Z}^m \to \mathrm{Md}(\mathbf{f}_1,\ldots,\mathbf{f}_m)$, both M_1 and $\mathrm{Ker}\,\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$ are in fact lattices (and hence by Proposition 1 they have $\mathrm{rank}_{\mathbb{Z}} = \mathrm{rank}_{\mathbb{R}}$). We get, therefore, from (19)

$$\mathrm{rank}\,\mathrm{Ker}\,\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m} + \mathrm{rank}_{\mathbb{Z}}\,\mathrm{Md}(\mathbf{f}_1,\ldots,\mathbf{f}_m) = \mathrm{rank}\,\mathbb{Z}^m = m \qquad (20)$$

whereas for the continuous transformation $\Phi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$ we have by (16):

$$\dim\,\mathrm{Ker}\,\Phi_{\mathbf{f}_1,\ldots,\mathbf{f}_m} + \dim\,\mathrm{Sp}(\mathbf{f}_1,\ldots,\mathbf{f}_m) = \dim\mathbb{R}^m = m \qquad (21)$$

(where $\dim\mathrm{Sp}(\mathbf{f}_1,\ldots,\mathbf{f}_m) = 2$ or 1).

It is important to note, however, that the original dimension of $\mathrm{Ker}\,\Phi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$ in $\mathbb{R}^m$ is not necessarily preserved in its restriction to $\mathbb{Z}^m$, $\mathrm{Ker}\,\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$. In fact, since $\mathrm{Ker}\,\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m} \subset \mathrm{Ker}\,\Phi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$, it is clear that

$$\mathrm{rank}\,\mathrm{Ker}\,\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m} \le \dim\mathrm{Ker}\,\Phi_{\mathbf{f}_1,\ldots,\mathbf{f}_m} \qquad (22)$$

However, the equality in (22) does not always hold. This can be illustrated by an example in the 3D case: Let $\mathrm{Ker}\,\Phi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$ be a 2D subspace of $\mathbb{R}^3$ (i.e., a plane through its origin); its restriction to $\mathbb{Z}^3$, $\mathrm{Ker}\,\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$, is the lattice of all points of $\mathbb{Z}^3$ included in $\mathrm{Ker}\,\Phi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$. It is clear that depending on the plane inclinations in the space $\mathbb{R}^3$ $\mathrm{Ker}\,\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$ may have rank $= 2$ (e.g., if the plane $\mathrm{Ker}\,\Phi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$ contains both the x- and y-axes of $\mathbb{R}^3$), rank $= 1$ (e.g., if the plane only contains the x-axis but forms an irrational angle with the y-axis) or rank $= 0$ (if the only integral point in the plane is the origin $(0,\ldots,0)$).

This is, indeed, an important difference between the continuous and the discrete cases. In the continuous case the dimension of $\mathrm{Ker}\,\Phi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$ is automatically determined by equality (16). In the discrete case, however, the rank of $\mathrm{Ker}\,\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$ (the discrete counterpart of $\mathrm{Ker}\,\Phi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$) is only bounded by inequality

(22), but its exact value depends also on other parameters, namely: the inclinations of the continuous subspace $\operatorname{Ker}\Phi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$ within $\mathbb{R}^m$, as shown in the example above. Consequently, the rank of the translated lattice (equivalence class) which "collapses" on each point of $\operatorname{Im}\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m} = \operatorname{Md}(\mathbf{f}_1,\ldots,\mathbf{f}_m)$ in the discrete case may be smaller than the dimension of the full, continuous translated subspace (equivalence class) which "collapses" on each point of $\operatorname{Im}\Phi_{\mathbf{f}_1,\ldots,\mathbf{f}_m} = \operatorname{Sp}(\mathbf{f}_1,\ldots,\mathbf{f}_m)$ in the corresponding continuous case.

The question is, therefore, what happens to the "dimension preservation law" in the discrete case? In fact, the "lost" dimensions of $\operatorname{Ker}\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$ are not really lost, and they are simply taken care of elsewhere, in $\operatorname{Im}\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$. Since the right hand sides of (20) and (21) are equal, (22) implies that

$$\operatorname{rank}_{\mathbb{Z}} \operatorname{Md}(\mathbf{f}_1,\ldots,\mathbf{f}_m) \geq \dim \operatorname{Sp}(\mathbf{f}_1,\ldots,\mathbf{f}_m) \quad (23)$$

(where $\dim \operatorname{Sp}(\mathbf{f}_1,\ldots,\mathbf{f}_m) = 2$ or 1). More precisely, if we note the difference by d, we obtain

$$\dim \operatorname{Ker}\Phi_{\mathbf{f}_1,\ldots,\mathbf{f}_m} - \operatorname{rank}\operatorname{Ker}\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m} = \operatorname{rank}_{\mathbb{Z}} \operatorname{Md}(\mathbf{f}_1,\ldots,\mathbf{f}_m) - \dim \operatorname{Sp}(\mathbf{f}_1,\ldots,\mathbf{f}_m) = d$$

This means that if due to the inclinations of the subspace $\operatorname{Ker}\Phi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$ in $\mathbb{R}^m$ it happens, as in the example above, that $\operatorname{Ker}\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$ cannot attain the full dimension of $\operatorname{Ker}\Phi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$, then this "loss" of d units in rank $\operatorname{Ker}\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$, the first term of (20), is automatically "balanced" by an identical increase in the second term of (20): $\operatorname{rank}_{\mathbb{Z}} \operatorname{Md}(\mathbf{f}_1,\ldots,\mathbf{f}_m)$ is increased by d units with respect to $\dim \operatorname{Sp}(\mathbf{f}_1,\ldots,\mathbf{f}_m)$. This means that the number of independent over $\mathbb{Z}$ vectors which span $\operatorname{Im}\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$ is higher by d than the number of independent over $\mathbb{R}$ vectors which span its enclosing continuous space, $\operatorname{Im}\Phi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$. This situation is illustrated in Examples 3 and 4 of Section 8.

Furthermore, if we only look at the right hand side of the above equation we have

$$\operatorname{rank}_{\mathbb{Z}} \operatorname{Md}(\mathbf{f}_1,\ldots,\mathbf{f}_m) - \dim \operatorname{Sp}(\mathbf{f}_1,\ldots,\mathbf{f}_m) = d \quad (24)$$

or equivalently, by (11)

$$\operatorname{rank}_{\mathbb{Z}} \operatorname{Md}(\mathbf{f}_1,\ldots,\mathbf{f}_m) - \operatorname{rank}_{\mathbb{R}} \operatorname{Md}(\mathbf{f}_1,\ldots,\mathbf{f}_m) = d$$

According to Proposition 1 we obtain, therefore, the following result:

Proposition 2. *The module* $\operatorname{Md}(\mathbf{f}_1,\ldots,\mathbf{f}_m)$, *the support of the spectrum, is a lattice (and therefore discrete) iff* $d = 0$ (*i.e., iff the continuous and discrete dimensions are identical*); *and conversely,* $\operatorname{Md}(\mathbf{f}_1,\ldots,\mathbf{f}_m)$ *is a dense module in* $\operatorname{Im}\Phi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$ *iff* $d > 0$. (*We remember from* (14) *that* $\operatorname{Im}\Phi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$, *i.e.,* $\operatorname{Sp}(\mathbf{f}_1,\ldots,\mathbf{f}_m)$, *can be either the whole* (u, v)*-plane, or a 1D line through its origin*).

As we will see later (in Section 9.1), this important result provides a criterion for the periodicity of the superposition of periodic layers (functions).

Two interesting consequences follow immediately:

(a) The spectrum support of a *non-singular* superposition can be a discrete lattice (meaning that the superposition is periodic; see Section 9.1(b)) only in the case of $m = 2$ gratings (as in Fig. 2). If $m \geq 3$ then $\dim \operatorname{Ker}\Phi = m - \dim \operatorname{Im}\Phi > 0$ (since $\dim \operatorname{Im}\Phi = 2$ or 1), and therefore if rank $\operatorname{Ker}\Psi = 0$ (= non-singular state) then $d = \dim \operatorname{Ker}\Phi - \operatorname{rank}\operatorname{Ker}\Psi > 0$. Therefore for $m \geq 3$ gratings, any non-singular case has a dense spectrum support.

(b) The spectrum support of a *singular* superposition can be a discrete lattice even if $m \geq 3$. This occurs when $d = 0$. In other words, if the spectrum support is 2D this occurs when $\operatorname{rank}_{\mathbb{Z}} \operatorname{Im}\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m} = 2$ and rank $\operatorname{Ker}\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m} = m - 2$, and if the spectrum support is 1D ($\mathbf{f}_1,\ldots,\mathbf{f}_m$ are collinear) this occurs when $\operatorname{rank}_{\mathbb{Z}} \operatorname{Im}\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m} = 1$ and rank $\operatorname{Ker}\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m} = m - 1$.

The various possible cases which may occur in the spectrum support in the superposition of $m = 2,\ldots,6$ gratings are summarized in Table 2. Several illustrative examples are given in Section 8.

7. The Algebraic Interpretation of the Impulse Locations in the Spectrum Support

7.1. The Global Spectrum Support

Using the terminology introduced in the previous sections it now becomes clear that the set of all the impulse locations in the spectrum convolution (the support of the impulse "forest") is in fact the module $\operatorname{Md}(\mathbf{f}_1,\ldots,\mathbf{f}_m)$, i.e., the image of the indices-lattice $\mathbb{Z}^m$ under the transformation $\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$. We have seen that this spectrum support can be either a dense module or a discrete lattice, and we found necessary and sufficient

Table 2. Summary of the algebraic structural properties of the various possible cases for $m = 1, \ldots, 6$ superposed gratings. The interpretation of these properties in terms of the image domain is discussed in Section 9.

m	Frequency vectors	dim Im Φ		rank$_{\mathbb{Z}}$ Im Ψ	Spectrum support	dim Ker Φ	rank Ker Ψ	Sing./ Not	Examples	Remarks
1	$\mathbf{f}_1$	1	=	1	1D-**L**	0	0	**N**	Sec. 3 Ex. 1	(1)
2	$\mathbf{f}_1, \mathbf{f}_2$ coplanar:	2	=	2	2D-**L**	0	0	**N**	Sec. 3 Ex. 2	(2)
	$\mathbf{f}_1, \mathbf{f}_2$ collinear:	1	=	1	1D-**L**	1	1	S	Sec. 3 Ex. 2	(3)
		1	<	2	1D-M	1	0	**N**	Sec. 3 Ex. 2	
3	$\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3$ coplanar:	2	=	2	2D-**L**	1	1	S	Sec. 8 Ex. 2	
		2	<	3	2D-M	1	0	**N**		(4)
	$\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3$ collinear:	1	=	1	1D-**L**	2	2	S	Sec. 8 Ex. 3	
		1	<	2	1D-M	2	1	S	Sec. 8 Ex. 4	
		1	<	3	1D-M	2	0	**N**		
4	$\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \mathbf{f}_4$ coplanar:	2	=	2	2D-**L**	2	2	S	Sec. 8 Ex. 5	
		2	<	3	2D-M	2	1	S		
		2	<	4	2D-M	2	0	**N**		
	$\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \mathbf{f}_4$ collinear:	1	=	1	1D-**L**	3	3	S		
		1	<	2	1D-M	3	2	S		
		1	<	3	1D-M	3	1	S		
		1	<	4	1D-M	3	0	**N**		
5	$\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \mathbf{f}_4, \mathbf{f}_5$ coplanar:	2	=	2	2D-**L**	3	3	S		
		2	<	3	2D-M	3	2	S		
		2	<	4	2D-M	3	1	S	Sec. 8 Ex. 6	(5)
		2	<	5	2D-M	3	0	**N**		
	$\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \mathbf{f}_4, \mathbf{f}_5$ collinear:	1	=	1	1D-**L**	4	4	S		
		1	<	2	1D-M	4	3	S		
		1	<	3	1D-M	4	2	S		
		1	<	4	1D-M	4	1	S		
		1	<	5	1D-M	4	0	**N**		
6	$\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \mathbf{f}_4, \mathbf{f}_5, \mathbf{f}_6$ coplanar:	2	=	2	2D-**L**	4	4	S	Sec. 8 Ex. 7	
		2	<	3	2D-M	4	3	S		
		2	<	4	2D-M	4	2	S	Sec. 8 Ex. 8	(6)
		2	<	5	2D-M	4	1	S		
		2	<	6	2D-M	4	0	**N**		
	$\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \mathbf{f}_4, \mathbf{f}_5, \mathbf{f}_6$ collinear:	1	=	1	1D-**L**	5	5	S		
		1	<	2	1D-M	5	4	S		
		1	<	3	1D-M	5	3	S		
		1	<	4	1D-M	5	2	S		
		1	<	5	1D-M	5	1	S		
		1	<	6	1D-M	5	0	**N**		

Legend: 1D = one dimensional; 2D = two dimensional; L = discrete lattice; M = dense module; S = singular; N = non-singular. By "coplanar" is meant: coplanar but non-collinear.

(*Continued on next page*).

Table 2. (*Continued.*)

Remarks:
1. A single grating; no superposition (and no moiré).
2. This is the only non-singular superposition with a discrete spectrum support.
3. A singular moiré between 2 gratings occurs *iff* $\mathbf{f}_1, \mathbf{f}_2$ are collinear (i.e., $\alpha = 0°$ or $180°$) and commensurable.
4. Note that 2D-M includes also the special case in which the 2D spectrum support is dense in one direction and discrete in the other. For instance, in the case of 3 coplanar frequency vectors this may occur when 2 of the vectors are collinear but incommensurable, while the third vector is oriented in a different direction.
5. To this category belongs the singular superposition of 5 identical gratings with equal angle differences of 72°.
6. To this category belongs the singular superposition of 3 identical screens with angle differences of 30° (or 60°), which is the traditional screen combination used in color printing.

Note that each pair of non-collinear gratings may be counted also as one 2D screen. For example, $m = 4$ corresponds either to 4 superposed gratings or to 2 superposed screens, etc.

conditions for either case. Table 2 gives a systematic summary of the different possible cases in the superposition of $m = 2, \ldots, 6$ gratings (or equivalently, up to three 2-fold periodic layers like dot-screens). The interpretation of the algebraic structure of the spectrum support in terms of the superposition in the image domain will be discussed in Section 9.

7.2. *The Individual Impulse Clusters*

We now proceed from the global spectrum support to the support of each of the individual impulse clusters. The cluster of impulse-locations which fall on the spectrum origin when the $(k_1, \ldots, k_m)$-moiré reaches a singular state is simply the image under $\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$ of the lattice $L = \operatorname{Ker}\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$, i.e., $\operatorname{Im} L$. Similarly, the other clusters of impulse-locations which are simultaneously formed in the spectrum plane are the images of the other equivalence classes $(n_1, \ldots, n_m) + L$ in the indices-lattice $\mathbb{Z}^m$. Let us now explain this in more detail; several illustrative examples will be given in the next section.

In Section 2.3 we defined a singular moiré as a configuration of the superposed layers in which the moiré period is infinitely large (i.e., its frequency is zero). More formally, a $(k_1, \ldots, k_m)$-moiré reaches a singular state whenever the location of its fundamental impulse, the $(k_1, \ldots, k_m)$-impulse in the spectrum convolution, coincides with the spectrum origin (0, 0) (i.e., whenever the frequency vectors $\mathbf{f}_1, \ldots, \mathbf{f}_m$ of the superposed layers are such that $\sum k_i \mathbf{f}_i = \mathbf{0}$). We have seen, however, that when a $(k_1, \ldots, k_m)$-moiré reaches a singular state, not only the $(k_1, \ldots, k_m)$-impulse itself falls on the spectrum origin, but rather, a whole infinite impulse-cluster around the spectrum origin. This cluster clearly contains the 1D comb formed by the $(nk_1, \ldots, nk_m)$-impulses with all integer values of n, but in the general case this cluster may contain other impulses, too, and it may be 2D (as in Fig. 4) or even of a higher rank. How can we characterize all the impulses which belong to this cluster (i.e., fall on the spectrum origin)? Using our new terminology, when the frequency vectors $\mathbf{f}_1, \ldots, \mathbf{f}_m$ are such that a $(k_1, \ldots, k_m)$-singular moiré occurs, the linear transformation $\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$ maps to the spectrum origin not only the point $(k_1, \ldots, k_m)$ but the whole sub-lattice $L \subset \mathbb{Z}^m$ induced by $\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$, namely: $L = \operatorname{Ker}\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$. The sub-lattice L corresponds, therefore, to the impulse-cluster which collapses onto the spectrum origin at the $(k_1, \ldots, k_m)$-singular state, and its points (integer m-tuples) are the indices of the impulses of this cluster. This is illustrated in the examples in the next section.

However, as shown in Fig. 4, in the proximity of a $(k_1, \ldots, k_m)$-singular moiré state, apart from the main cluster of the $(k_1, \ldots, k_m)$-impulse which is formed at the spectrum origin, other impulse clusters are also simultaneously formed elsewhere in the spectrum. Let us see now in detail what is the nature of these impulse clusters, and how we can characterize the impulses which belong to each of the clusters. We have seen above that the transformation $\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$ induces a partition of the indices-lattice $\mathbb{Z}^m$ into disjoint and exhaustive equivalence classes $(n_1, \ldots, n_m) + L$, which are translations of the sub-lattice L in $\mathbb{Z}^m$ (the sub-lattice L itself is the equivalence class $\mathbf{0} + L$ which contains all the points of $\mathbb{Z}^m$ that are mapped by $\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$ to the spectrum origin). We have also seen that the transformation $\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$ has the special property that it maps every equivalence class $(n_1, \ldots, n_m) + L$ of the indices-lattice $\mathbb{Z}^m$ into a different *single point* within the spectrum plane. This explains why an infinite (but still denumerable) number of clusters are formed in the spectrum simultaneously with the main cluster of the $(k_1, \ldots, k_m)$-moiré: each of these clusters is simply the image under $\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$ of a different equivalence class $(n_1, \ldots, n_m) + L$ of the indices-lattice $\mathbb{Z}^m$. The indices of the impulses in each of these clusters are therefore a translated replica of the indices of the impulses of L, each of which being incremented by a "cluster representative" $(n_1, \ldots, n_m)$ (see figures in the examples below). The location of each cluster $(n_1, \ldots, n_m) + L$ in the spectrum is given by $\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}(n_1, \ldots, n_m) + \mathbf{0} = \Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}(n_1, \ldots, n_m)$, i.e., it is shifted from the spectrum origin by $\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}(n_1, \ldots, n_m)$.

As for the relationship between the rank of a single cluster and the rank of the whole spectrum support, $\mathrm{Md}(\mathbf{f}_1, \ldots, \mathbf{f}_m)$, we have from (20):

$$\mathrm{rank}\, L + \mathrm{rank}_{\mathbb{Z}}\, \mathrm{Md}(\mathbf{f}_1, \ldots, \mathbf{f}_m) = m \qquad (25)$$

These ranks depend, of course, on the specific choice of the frequency vectors $\mathbf{f}_1, \ldots, \mathbf{f}_m$ of the superposed layers: since the module $\mathrm{Md}(\mathbf{f}_1, \ldots, \mathbf{f}_m)$ is generated by the frequency vectors $\mathbf{f}_1, \ldots, \mathbf{f}_m \in \mathbb{R}^2$, $\mathrm{rank}_{\mathbb{Z}}\, \mathrm{Md}(\mathbf{f}_1, \ldots, \mathbf{f}_m)$ is simply the maximum number of vectors among $\mathbf{f}_1, \ldots, \mathbf{f}_m \in \mathbb{R}^2$ which are still linearly independent over $\mathbb{Z}$. Rank L complements this number to m, the number of superposed gratings, so that it indicates the "redundancy level" of the superposition, i.e., the number of *dependent* vectors (layers), which do not further enrich the spectrum support, but are rather "invested" in its existing points (and hence enrich each of the clusters).

It is interesting to note that for different singular moirés different configurations of clusters are formed in the spectrum (in general, either the assignment of impulses to each cluster or the cluster locations in the spectrum or both may differ). This is due to the fact that for different sub-lattices $\mathrm{Ker}\, \Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$, the indices-lattice $\mathbb{Z}^m$ is partitioned into a different set of equivalence classes.

Finally, let us see what happens in the spectrum when we start moving away from the $(k_1, \ldots, k_m)$-singular state. When we slightly modify one or more of the frequency vectors $\mathbf{f}_1, \ldots, \mathbf{f}_m$ of the superposed layers, each of the clusters in the spectrum starts "spreading out," revealing thus the infinity of points from which it is composed (Figs. 6–12).[6] In particular, the main cluster which spreads out around the spectrum origin enables us to visualize the impulses which correspond to the moiré (which originate from $\mathrm{Ker}\, \Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$ in the singular state of the moiré). Depending on which of the vectors $\mathbf{f}_1, \ldots, \mathbf{f}_m$ have been changed and how, the clusters in the spectrum may be *partially spread-out* (for example, when only one dimension of the cluster has been spread out, and each point still represents an infinity of impulses); or *fully spread-out* (when each point of the cluster represents exactly one single impulse, so that no two impulses in the cluster fall on the same point in the spectrum). It should be noted that although in the examples we have seen previously (Fig. 4) the spread-out moiré clusters in the (u, v)-plane were always 2D or 1D discrete lattices, in the general case each spread-out cluster in the spectrum may also be a dense module. If we denote by $\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}|_L$ the restriction of transformation $\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$ which is only defined between $L \subset \mathbb{Z}^m$ and $\mathrm{Im}\, L \subset \mathrm{Md}(\mathbf{f}_1, \ldots, \mathbf{f}_m)$, then when the moiré cluster is fully spread-out we have $\mathrm{Ker}\, \Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}|_L = \{\mathbf{0}\}$ and therefore from (19): $\mathrm{rank}_{\mathbb{Z}}\, \mathrm{Im}\, L = \mathrm{rank}\, L$. In other words, the integral rank of an individual fully spread-out cluster in the spectrum equals the rank of $L = \mathrm{Ker}\, \Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_m}$ in $\mathbb{Z}^m$. Moreover, according to (25) this means that: $\mathrm{rank}_{\mathbb{Z}}\, \mathrm{Im}\, L = m - \mathrm{rank}_{\mathbb{Z}} \mathrm{Md}(\mathbf{f}_1, \ldots, \mathbf{f}_m)$.

Therefore, when the rank of L is $r > 2$, each fully spread-out cluster within the (u, v)-plane becomes a module with $\mathrm{rank}_{\mathbb{Z}}\, \mathrm{Im}\, L = r > 2$, i.e., a dense module in the 2D spectrum plane. See for example the clusters in Fig. 11 (in Example 7 below), where rank $L = 6 - 2 = 4$. The interpretation of this property of the clusters in terms of the image domain will be discussed in Section 9.2.

Note that even when each of the clusters in itself is a discrete lattice, their intertwined impulses throughout the spectrum are not necessarily located on a common lattice, and their support may be an everywhere dense module.

8. Examples

In this section we present a number of examples to illustrate the above discussion, and to demonstrate the contribution of the algebraic approach developed above to the understanding of the structure of the spectrum support. In particular, these examples illustrate the clusterization phenomenon, and the identification of the impulses which participate in each of the clusters in the spectrum. We start in Example 1 with the simplest possible case, the superposition-moiré between two gratings; in this case the algebraic situation is straightforward, and it is presented rather informally, by way of introduction. Then in Examples 2–4 we present various moiré configurations between 3 gratings, since in the case of 3 gratings all the algebraic structures occur in the 3D space and are therefore easy to understand.

Examples 5–8 illustrate some more interesting cases which occur in higher dimensions. It may be instructive to track each of the examples in the synoptic summary of the different possible cases presented in Table 2.

Example 1. The simplest possible example consists of the superposition of 2 gratings. Let us illustrate this situation with the case of the $(3, -2)$-moiré, a 3rd order moiré which becomes visible when the $(3, -2)$-impulse in the spectrum convolution is located inside the visibility circle, i.e., when the frequency vector $\mathbf{f}_2$ of the second grating is close to $\frac{3}{2}\mathbf{f}_1$ (see the vector diagram in Fig. 5(b)). This $(3, -2)$-impulse is the fundamental impulse of a 1D-cluster through the spectrum origin, which represents the moiré in question; but in the same time other 1D clusters are also formed in the spectrum, in parallel to the main 1D cluster. Note that when $\alpha = 0°$ and the frequency vector $\mathbf{f}_2$ attains exactly the point $\mathbf{f}_2 = \frac{3}{2}\mathbf{f}_1$ each of the 1D clusters collapses into a single point on the u-axis, and in particular, the main cluster collapses into the spectrum origin, so that the $(3, -2)$-moiré becomes singular (and hence invisible in the layer superposition).

Let us analyze this example to illustrate the algebraic discussion of the preceding sections. In this case the indices-lattice (the lattice of all the indices of the impulses obtained in the spectrum convolution) is $\mathbb{Z}^2$, and the linear transformation $\Psi_{\mathbf{f}_1,\mathbf{f}_2}$ which maps each index pair $(k_1, k_2) \in \mathbb{Z}^2$ into the geometric location of the (k_1, k_2)-impulse in the (u, v)-plane is given according to (15) by

$$\Psi_{\mathbf{f}_1,\mathbf{f}_2}(k_1, k_2) = k_1\mathbf{f}_1 + k_2\mathbf{f}_2$$

Figure 5(a) illustrates the indices-lattice $\mathbb{Z}^2$ and its partition into equivalence classes induced by the sub-lattice $(3k, -2k)$. This sub-lattice itself becomes the cluster $n = 0$ of the partition, containing the indices of the fundamental impulse of the $(3, -2)$-moiré and all its harmonics. The indices of this 0th cluster are given by $L = \{(3k, -2k) \mid k \in \mathbb{Z}\}$, and the indices of the nth cluster are given in this case by $(-n, n) + L$. Figure 5(b) shows the image of the transformation $\Psi_{\mathbf{f}_1,\mathbf{f}_2}$ in the (u, v)-plane, i.e., the spectrum support, when the vectors $\mathbf{f}_1, \mathbf{f}_2$ are *almost* in the singular position (α is almost $0°$). When $\mathbf{f}_1, \mathbf{f}_2$ are *exactly* in the singular position, $\Psi_{\mathbf{f}_1,\mathbf{f}_2}$ maps each equivalence class of $\mathbb{Z}^2$ into a single point on the u-axis (the point into which the respective 1D cluster in the spectrum collapses). But as $\mathbf{f}_1, \mathbf{f}_2$ start moving off the singular state, each of these 1D clusters starts spreading out and gives a comb of impulses in the spectrum (as in Fig. 5(b)).

Example 2. (1D clusters on a 2D support in the (u, v)-plane): Consider the $(1, 1, 1)$-singular moiré which occurs between 3 gratings when their frequency vectors are given, in polar coordinates, by: $\mathbf{f}_1 = (0°, 32)$, $\mathbf{f}_2 = (120°, 32)$, $\mathbf{f}_3 = (240°, 32)$, i.e., in Cartesian coordinates: $\mathbf{f}_1 = (32, 0)$, $\mathbf{f}_2 = (-16, 16\sqrt{3})$, $\mathbf{f}_3 = (-16, -16\sqrt{3})$ (see Fig. 6(a)).[7] Since in this case $\mathbf{f}_3$ is a linear combination, both over $\mathbb{Z}$ and over $\mathbb{R}$, of $\mathbf{f}_1$ and $\mathbf{f}_2$ (i.e., $\mathbf{f}_3 = -\mathbf{f}_1 - \mathbf{f}_2$), we have here: $\text{rank}_{\mathbb{Z}}\ \text{Md}(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3) = \dim \text{Sp}(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3) = 2$. This means by (24) that $d = 0$, and the spectrum support, $\text{Md}(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$, is in this singular case a discrete lattice of rank 2 (see Fig. 6(a)). Furthermore, from (25) we learn that each point of this lattice represents a collapsed lattice (cluster) whose rank is: $\text{rank}\, L = 3 - 2 = 1$. And indeed, when the 3 superposed gratings slightly move away from the singular moiré state (i.e., when their frequency vectors $\mathbf{f}_i$ are slightly modified), each of the 1D clusters in the spectrum starts spreading out, and in the image domain a 1D moiré becomes visible in the superposition, as indicated by the low frequencies of the 1D spread-out cluster around the spectrum origin (Fig. 6(b)).

In fact, this explanation already shows how the structural properties of the spectrum support can be determined using (24) and (25). However, in order to illustrate the algebraic discussion of the preceding sections, and particularly, to illustrate the assignment of impulses to each cluster, we will analyze this example in full detail. The linear transformation $\Psi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3}$ is given in this singular case by

$$\begin{aligned}\Psi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3}(k_1, k_2, k_3) &= k_1\mathbf{f}_1 + k_2\mathbf{f}_2 + k_3\mathbf{f}_3 \\ &= k_1(32, 0) + k_2(-16, 16\sqrt{3}) \\ &\quad + k_3(-16, -16\sqrt{3}) \qquad (26)\end{aligned}$$

Let us compare the transformation $\Psi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3}$ itself with its continuous counterpart, $\Phi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3}: \mathbb{R}^3 \to \mathbb{R}^2$. Ker $\Phi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3}$, i.e., the subspace of $\mathbb{R}^3$ which is mapped by $\Phi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3}$ into the origin $(0, 0)$ of the (u, v)-plane, contains all the points $(k_1, k_2, k_3) \in \mathbb{R}^3$ which solve the following set of two linear equations, obtained from (26):

$$\begin{cases} 32k_1 - 16k_2 - 16k_3 = 0 \\ 16\sqrt{3}k_2 - 16\sqrt{3}k_3 = 0 \end{cases}$$

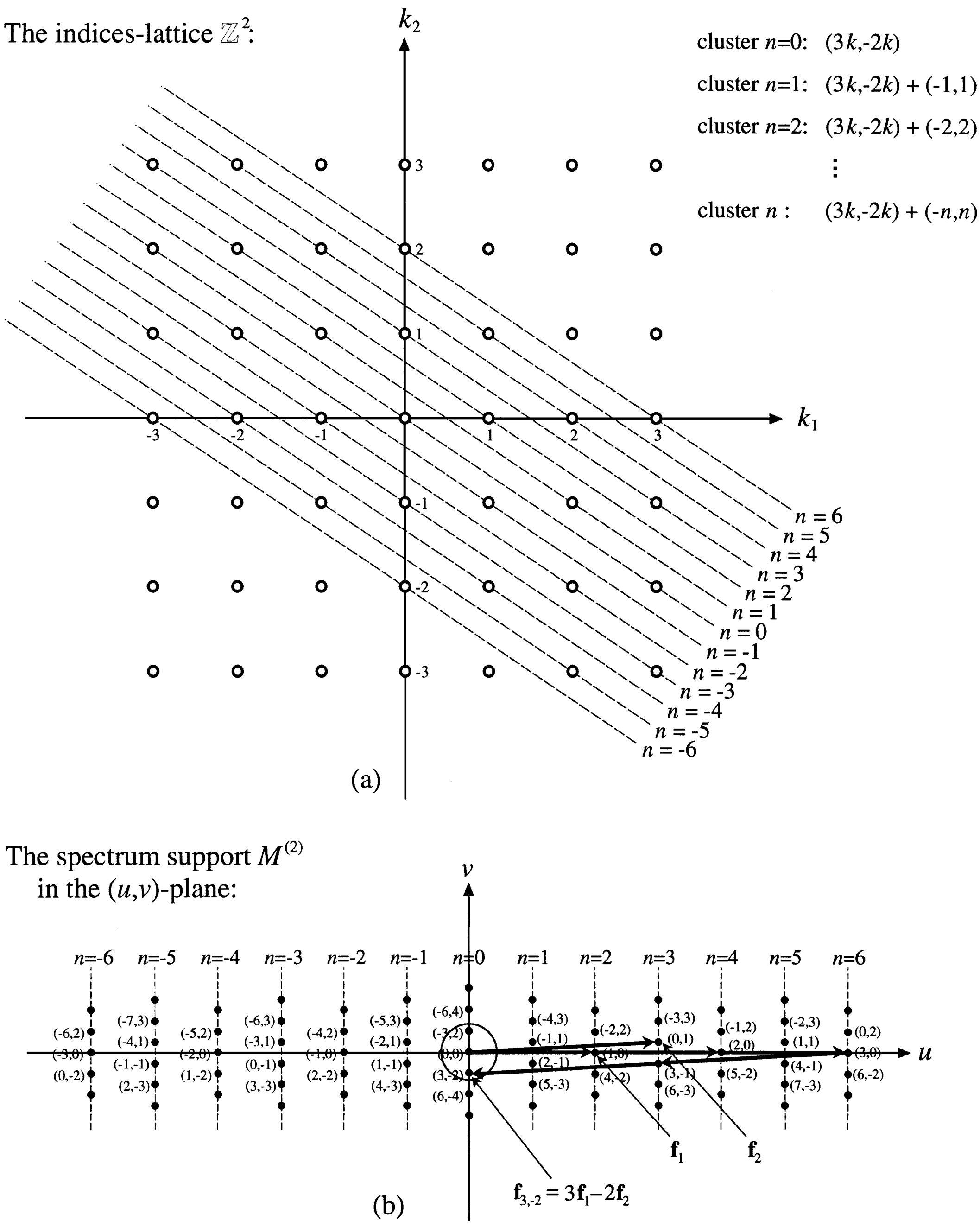


Figure 5. A schematic illustration of the transformation $\Psi_{\mathbf{f}_1,\mathbf{f}_2}(k_1, k_2) = k_1\mathbf{f}_1 + k_2\mathbf{f}_2$ which maps the indices-lattice $\mathbb{Z}^2$ (top) into the (u, v)-spectrum plane (bottom), in the case of a two grating superposition with $f_2 = \frac{3}{2} f_1$ and $\alpha \approx 0°$. (a) Schematic view of the indices-lattice, $\mathbb{Z}^2$. The dashed lines illustrate the $2k_1 + 3k_2 = n$ diagonals (=equivalence classes). (b) The image of the mapping $\Psi_{\mathbf{f}_1,\mathbf{f}_2}$ in the (u, v)-plane, showing the corresponding impulse clusters in the spectrum support, slightly before α reaches 0°; black dots indicate the impulse locations. The nth diagonal in (a) is mapped into the nth comb (1D-cluster) in the (u, v) spectrum (b).

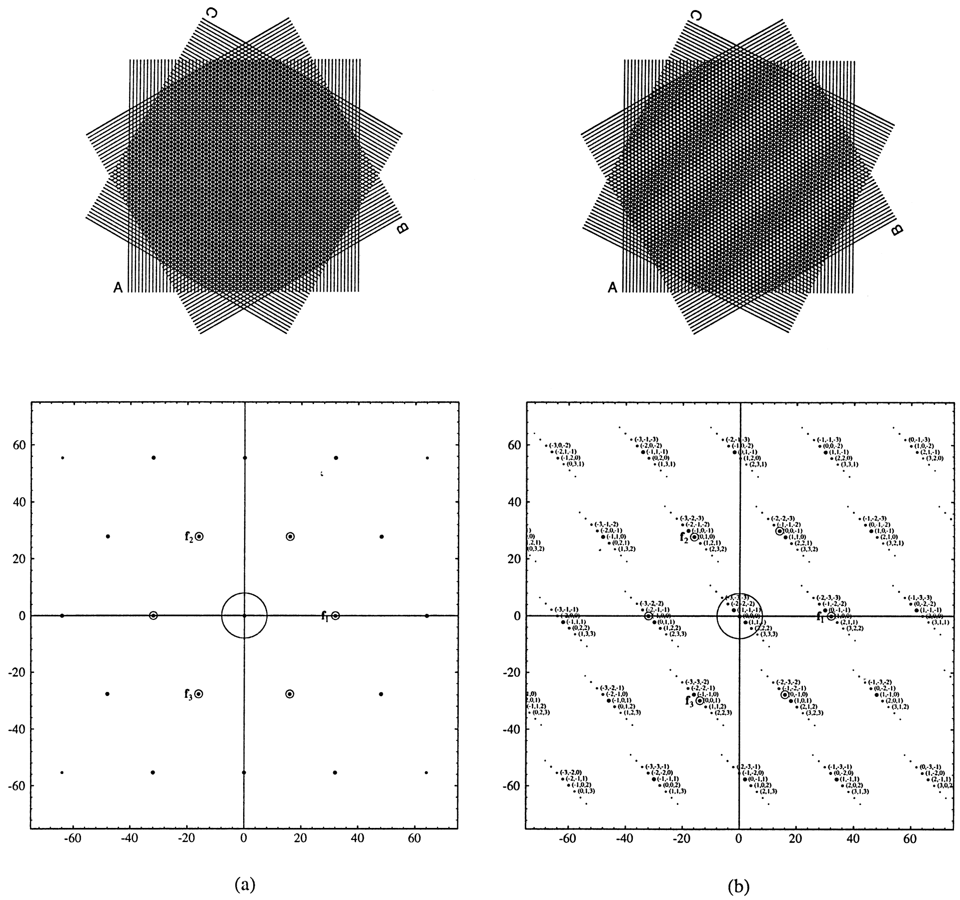


Figure 6. The singular 3-grating superposition of Example 2 (top) and its spectrum support (bottom). (a) Exactly at the singular state: the spectrum support forms here a 2D lattice, each point of which represents a collapsed cluster. (b) Slightly off the singular state: each of the clusters in the spectrum is spread out, clearly demonstrating its 1D nature. Encircled points denote the locations of the fundamental impulses of the 3 original combs. Large points represent convolution impulses of the first order, and smaller points represent convolution impulses of higher orders. Only impulses up to the 5th order are shown.

The solution of this set of equations is: Ker $\Phi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3} = \{(k_1, k_2, k_3) \mid k_1 = k_2 = k_3, k_i \in \mathbb{R}\}$ which means that Ker $\Phi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3}$ is the diagonal line $z = y = x$ of $\mathbb{R}^3$. Therefore, $\mathbb{R}^3$ is partitioned by $\Phi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3}$ into an infinite 2D set of translated lines (1D equivalence classes) parallel to the line $z = y = x$. Since $\dim \text{Ker}\, \Phi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3} = 1$ we have here $\dim \text{Im}\, \Phi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3} = 3 - 1 = 2$, and indeed the continuous transformation $\Phi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3}$ maps into each point of the 2D (u, v)-plane a whole 1D line from this decomposition of $\mathbb{R}^3$.

Returning now to the discrete case where $k_i \in \mathbb{Z}$, it is clear that in this example the lattice $L = \text{Ker}\, \Psi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3} = \text{Ker}\, \Phi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3} \cap \mathbb{Z}^3$ is indeed a lattice of rank 1 on the diagonal line $z = y = x$, given by $L = \{(k_1, k_2, k_3) \mid k_1 = k_2 = k_3, k_i \in \mathbb{Z}\}$, so there is no loss of dimensions in this case. The lattice L consists of the indices of all the impulses of the 1D cluster which collapses, precisely at the singular state, on the origin of the spectrum: $\{\ldots, (-1, -1, -1), (0, 0, 0), (1, 1, 1), \ldots\}$. This cluster can be seen spread-out around the spectrum origin in Fig. 6(b), which shows the spectrum slightly off the singular state.

Each of the other clusters in this spectrum consists of the impulses of one parallel translation of L within $\mathbb{Z}^3$: $(n_1, n_2, n_3) + L = \{(n_1, n_2, n_3) + (k_1, k_2, k_3) \mid k_1 = k_2 = k_3, k_i \in \mathbb{Z}\}$; each of these translated lattices of rank 1 is mapped by $\Psi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3}$ into a single point $\Psi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3}(n_1, n_2, n_3)$ within the (u, v) spectrum plane. For example (see Fig. 6): on top of the fundamental impulse of the first grating, which is the

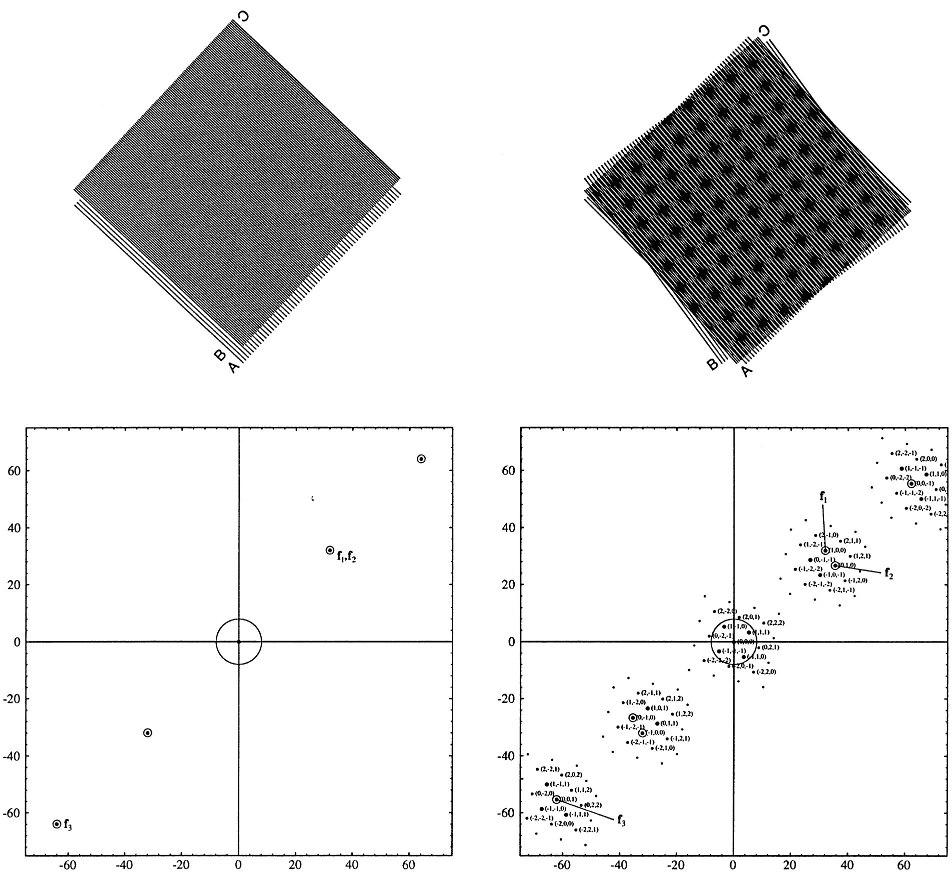
(1, 0, 0)-impulse in the spectrum convolution (located in the (u, v)-plane at $\mathbf{f}_1 = (32, 0)$), collapses the whole 1D cluster $(1, 0, 0) + L$, i.e., $\{\ldots, (0, -1, -1), (1, 0, 0), (2, 1, 1), \ldots\}$. This cluster can be seen spread-out around the impulse $\mathbf{f}_1$ in Fig. 6(b).

Example 3. (2D clusters on a 1D support in the (u, v)-plane): Consider the (1, 1, 1)-singular moiré which occurs between 3 gratings whose frequency vectors are: $\mathbf{f}_1 = \mathbf{f}_2 = (32, 32)$, $\mathbf{f}_3 = (-64, -64)$ (see Fig. 7(a)). In this case the vectors $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3$ in the (u, v)-plane are collinear (linearly dependent over $\mathbb{R}$), and also commensurable (linearly dependent over $\mathbb{Z}$). We have, therefore: $\dim \mathrm{Sp}(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3) = \mathrm{rank}_{\mathbb{Z}} \mathrm{Md}(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3) = 1$, so that the spectrum support, $\mathrm{Md}(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$, is in this singular case a discrete lattice of rank 1. Moreover, from (25) we find that each point of this lattice represents, in fact, a collapsed lattice (cluster) whose rank is: rank $L = 3 - 1 = 2$. And indeed, when the 3 superposed gratings move a little from the singular moiré state, each of the 2D clusters in the spectrum spreads out, and in the image domain a 2D moiré becomes visible in the superposition, as indicated by the low frequencies of the 2D spread-out cluster around the spectrum origin (see Fig. 7(b)).

Let us analyze this example in more detail. In this case the linear transformation $\Psi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3}$ is given by

$$\Psi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3}(k_1, k_2, k_3) = k_1(32, 32) + k_2(32, 32) + k_3(-64, -64)$$

We will consider, again, both this transformation and its continuous counterpart, $\Phi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3}$. The kernel of $\Phi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3}$ is the solution of the equation $32k_1 + 32k_2 -$



(a) (b)

Figure 7. The singular 3-grating superposition of Example 3 (top) and its spectrum support (bottom). (a) Exactly at the singular state: the spectrum support forms here a 1D lattice, each point of which represents a collapsed cluster. (b) Slightly off the singular state: each of the clusters in the spectrum is spread out, clearly demonstrating its 2D nature. Only impulses up to the 3rd order are shown.

$64k_3 = 0$ with $k_i \in \mathbb{R}$, which is the 2D plane $2z = x + y$ in $\mathbb{R}^3$. Therefore $\mathbb{R}^3$ is partitioned by $\Phi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3}$ into an infinite 1D set of translated planes (2D equivalence classes) which are parallel to the plane $2z = x + y$. Since $\dim \operatorname{Ker} \Phi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3} = 2$, we have here $\dim \operatorname{Im} \Phi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3} = 3 - 2 = 1$, and indeed the image of $\Phi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3}$ is the 1D line which is spanned in the (u, v)-plane by $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3$, namely: $v = u$. Therefore the continuous transformation $\Phi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3}$ maps into each point of this line in the (u, v)-plane a whole 2D plane from the decomposition of $\mathbb{R}^3$.

Returning now to the discrete case, it is clear that the kernel L of the discrete mapping $\Psi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3}$ is the 2D lattice: $L = \{(k_1, k_2, k_3) \mid 2k_3 = k_1 + k_2, k_i \in \mathbb{Z}\}$ which is imbedded in the plane $2z = x + y$. Therefore in this case there is no loss of dimensions, and the cluster which falls on the spectrum origin (as well as all the other clusters) is of a 2D nature (see Fig. 7(b), which shows the spread-out clusters in the spectrum slightly off the singular state).[8]

The support of all the collapsed clusters in the spectrum (precisely at the singular state) is the image of the transformation $\Psi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3}$, i.e., the module given by (13):

$$\begin{aligned} \operatorname{Md}(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3) = \{&k_1(32, 32) + k_2(32, 32) \\ &+ k_3(64, 64) \mid k_i \in \mathbb{Z}\} \end{aligned}$$

As we can see, in this case the support of the spectrum forms a lattice of rank 1; and each point of this lattice consists of a whole 2D cluster of impulses, representing one equivalence class (translation of the 2D lattice L) from the partition induced by $\Psi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3}$ in the indices-lattice $\mathbb{Z}^3$.

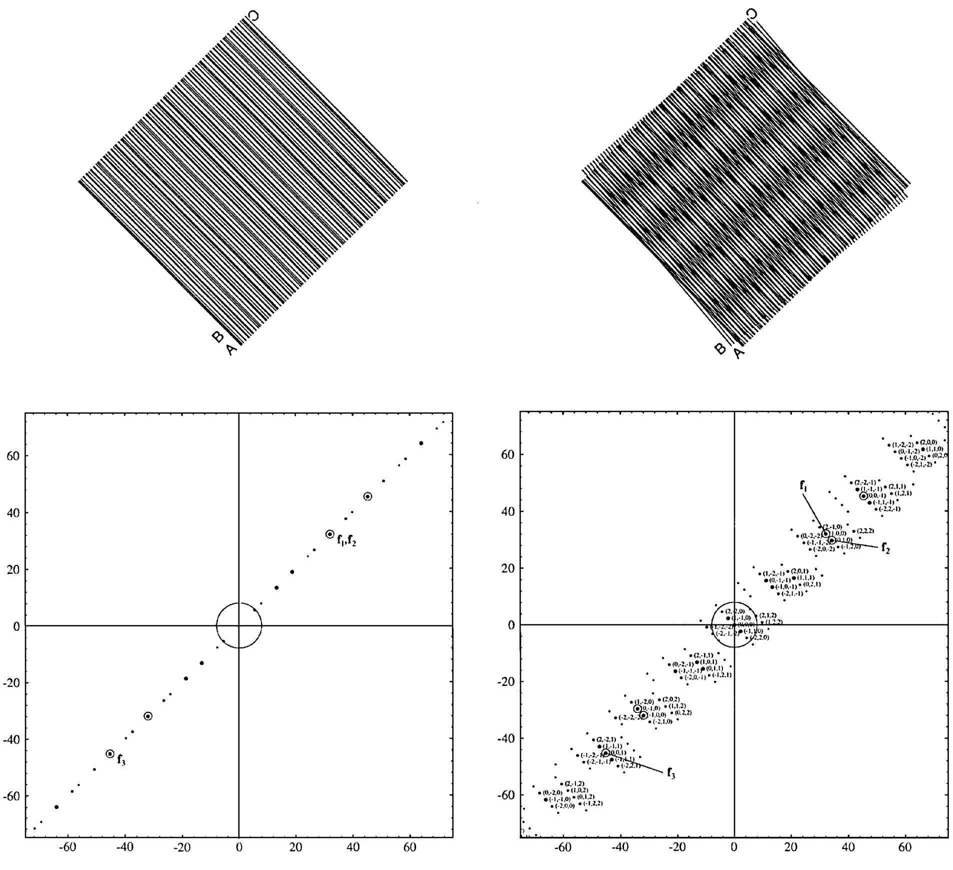
Example 4. (1D clusters on a 1D support in the (u, v)-plane): This example illustrates what happens in a case similar to Example 3 when the lattice $L = \operatorname{Ker} \Psi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3}$ which collapses on the spectrum origin has a lower rank than its continuous counterpart $\operatorname{Ker} \Phi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3}$, due to an irrational inclination of the plane $\operatorname{Ker} \Phi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3}$ within $\mathbb{R}^3$. Consider the singular moiré which occurs between 3 gratings whose frequency vectors are: $\mathbf{f}_1 = \mathbf{f}_2 = (32, 32), \mathbf{f}_3 = q \cdot \mathbf{f}_1$ where q, unlike in Example 3, is an irrational number, say $-\sqrt{2}$ (see Fig. 8(a)). In this case the vectors $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3$ in the (u, v)-plane are linearly dependent over $\mathbb{R}$, but only two of them are linearly dependent over $\mathbb{Z}$. We therefore have: $\dim \operatorname{Sp}(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3) = 1 < \operatorname{rank}_{\mathbb{Z}} \operatorname{Md}(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3) = 2$, so that the spectrum support, $\operatorname{Md}(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$, is in this case a dense module of $\operatorname{rank}_{\mathbb{Z}} = 2$ imbedded on the 1D line $\operatorname{Sp}(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$. Moreover, from (25) we get that each point of this module is, in fact, a collapsed lattice (cluster) whose rank is: $\operatorname{rank} L = 3 - 2 = 1$. And indeed, if the 3 superposed gratings move a little from the singular moiré state, each of the 1D clusters in the spectrum spreads out, and in the image domain a 1D moiré becomes visible in the superposition, as indicated by the low frequencies of the 1D spread-out cluster around the spectrum origin (see Fig. 8(b)).

Let us analyze this example in more detail, comparing it to Example 3. In the present case the linear transformation $\Psi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3}$ is given by

$$\begin{aligned} \Psi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3}(k_1, k_2, k_3) = {}& k_1(32, 32) + k_2(32, 32) \\ &+ k_3(-32\sqrt{2}, -32\sqrt{2}) \end{aligned}$$

The kernel of the continuous counterpart of this transformation, $\Phi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3}$, is the solution of the equation $32k_1 + 32k_2 - 32\sqrt{2}k_3 = 0$ with $k_i \in \mathbb{R}$, which is the 2D plane $\sqrt{2}z = x + y$ in $\mathbb{R}^3$. Therefore $\mathbb{R}^3$ is partitioned by $\Phi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3}$ into an infinite 1D set of translated planes (2D equivalence classes) which are parallel to the plane $\sqrt{2}z = x + y$. The image of $\Phi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3}$ is the 1D line which is spanned in the (u, v)-plane by $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3$, namely: $v = u$. Therefore, like in Example 3, the continuous transformation $\Phi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3}$ maps into each point of this line in the (u, v)-plane a whole 2D plane from the decomposition of $\mathbb{R}^3$.

Let us now return to the discrete case. Although the kernel of the continuous $\Phi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3}$ is still a 2D plane in $\mathbb{R}^3$, as in Example 3, we see that in the present case, due to the irrational inclination of this plane, its discrete restriction to $k_i \in \mathbb{Z}$ (i.e., the lattice $L = \operatorname{Ker} \Psi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3}$ which collapses to the spectrum origin) has a lower rank than 2. In fact, the only points of the indices-lattice $\mathbb{Z}^3$ which fall on the plane $\sqrt{2}z = x + y$ are those for which $z = 0$, so that we get: $L = \operatorname{Ker} \Psi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3} = \{(k_1, k_2, k_3) \mid k_2 = -k_1, k_3 = 0, k_i \in \mathbb{Z}\}$. Therefore, in this case the cluster which falls on the spectrum origin is of rank 1: $L = \{\ldots, (-1, 1, 0), (0, 0, 0), (1, -1, 0), \ldots\}$ (see Fig. 8(b), which shows the spread-out clusters slightly off the singular state). Similarly, a 1D cluster which consists of one parallel translation of L within $\mathbb{Z}^3$ collapses on each point in the spectrum support. For example (see Fig. 8(b)), on the fundamental impulse of the first grating, which is the (1, 0, 0)-impulse in the spectrum convolution, collapses the whole 1D cluster $(1, 0, 0) + L$, i.e., $\{\ldots, (0, 1, 0), (1, 0, 0), (2, -1, 0), \ldots\}$.



(a) (b)

Figure 8. The singular 3-grating superposition of Example 4 (top) and its spectrum support (bottom). (a) Exactly at the singular state: the spectrum support forms here a 1D module (of integral rank 2), each point of which represents a collapsed cluster. (b) Slightly off the singular state: each of the clusters in the spectrum is spread out, clearly demonstrating its 1D nature. Only impulses up to the 3rd order are shown.

The support of all these collapsed clusters in the spectrum (precisely at the singular state) is the image of the transformation $\Psi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3}$, i.e., the module of $\text{rank}_\mathbb{Z} = 2$ given by (13)

$$\text{Md}(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3) = \{k_1(32, 32) + k_2(32, 32) + k_3(-32\sqrt{2}, -32\sqrt{2}) \mid k_i \in \mathbb{Z}\}$$

This module is imbedded in the image of the continuous transformation $\Phi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3}$ in the (u, v)-plane, which is the same 1D line as in Example 3:

$$\text{Sp}(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3) = \{(u, v) \in \mathbb{R}^2 \mid v = u\}.$$

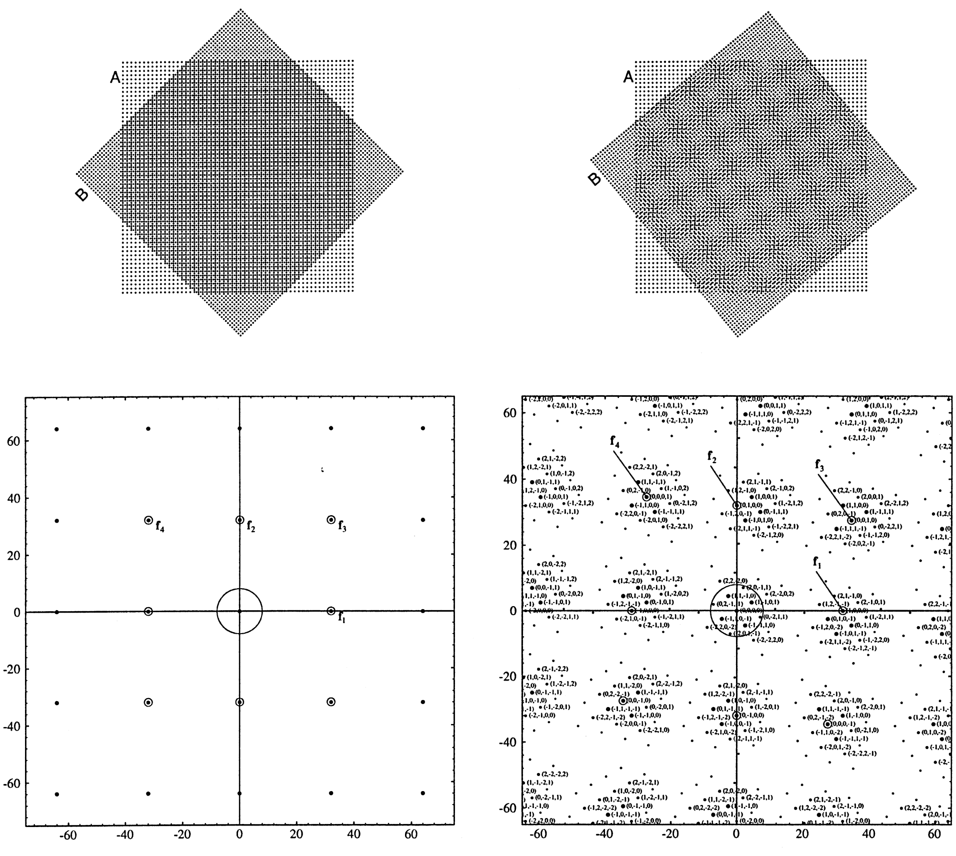
In this example we therefore have: $\dim \text{Sp}(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3) = 1 < \text{rank}_\mathbb{Z} \text{Md}(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3) = 2$. This means that in this case the support of the spectrum forms a dense module of $\text{rank}_\mathbb{Z} = 2$ which is imbedded on the 1D line $v = u$; and each point of this module consists of a whole 1D cluster, representing one equivalence class (translation of the 1D lattice L) from the partition induced by $\Psi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3}$ in the indices-lattice $\mathbb{Z}^3$.

As we can see in this example, the “loss” of one dimension in the discrete $\text{Ker}\,\Psi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3}$ due to an irrational inclination of the 2D plane $\text{Ker}\,\Phi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3}$ in $\mathbb{R}^3$ (i.e., the loss of one dimension in each cluster) is “compensated” in the image of $\Psi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3}$ in the (u, v)-plane by an increment of 1 in the integral rank of this module: whereas in Example 3 $\text{Im}\,\Psi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3}$ was a module of $\text{rank}_\mathbb{Z} = 1$ imbedded on the 1D line $\text{Im}\,\Phi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3}$ (namely a lattice of rank 1), in the present case $\text{Im}\,\Psi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3}$ is a dense module of $\text{rank}_\mathbb{Z} = 2$ which is imbedded on the same line $\text{Im}\,\Phi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3}$. (Note that the continuous $\text{Ker}\,\Phi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3}$

and Im $\Phi_{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3}$ have both the same dimensions as in Example 3; only the dimensions of their discrete counterparts have changed).

Example 5. (2D clusters on a 2D support in the (u, v)-plane): Consider the $(2, 0, -1, 1)$-singular moiré which occurs between 2 screens (or 4 gratings) when their frequency vectors are given by: $\mathbf{f}_1 = (32, 0), \mathbf{f}_2 = (0, 32), \mathbf{f}_3 = (32, 32)$ and $\mathbf{f}_4 = (-32, 32)$ (see Fig. 9(a)). It is easy to see that in this case $\mathbf{f}_3$, $\mathbf{f}_4$ are linear combinations, both over $\mathbb{Z}$ and $\mathbb{R}$, of $\mathbf{f}_1$, $\mathbf{f}_2$ (namely: $\mathbf{f}_3 = \mathbf{f}_1 + \mathbf{f}_2, \mathbf{f}_4 = \mathbf{f}_2 - \mathbf{f}_1$), while $\mathbf{f}_1$ and $\mathbf{f}_2$ are independent. Therefore, we have here: $\mathrm{rank}_{\mathbb{Z}}\ \mathrm{Md}(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \mathbf{f}_4) = \dim \mathrm{Sp}(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \mathbf{f}_4) = 2$. This means that the spectrum support, $\mathrm{Md}(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \mathbf{f}_4)$, is in this singular case a discrete lattice of rank 2. And furthermore, from (25) we get that each point of this lattice represents, in fact, a collapsed lattice (cluster) whose rank is: rank $L = 4 - 2 = 2$.

It should be noted that in general it is not always practical to find arithmetically the Cartesian coordinates of the frequency vectors $\mathbf{f}_i$ and to determine the lattice L. In such cases, a computer program which calculates the comb convolutions in the spectral domain can be helpful. Given the polar coordinates of the frequency vectors $\mathbf{f}_i$ (i.e., the frequencies and the directions of each superposed layer) this program calculates the spectral convolution (up to a specified number of harmonics on each impulse comb), using the rules of comb convolution (Eqs. (3) and (4)). The resulting impulse configuration (= spectrum support) is graphically displayed in the (u, v)-spectrum, showing the location (and optionally also the index) of each impulse. This is

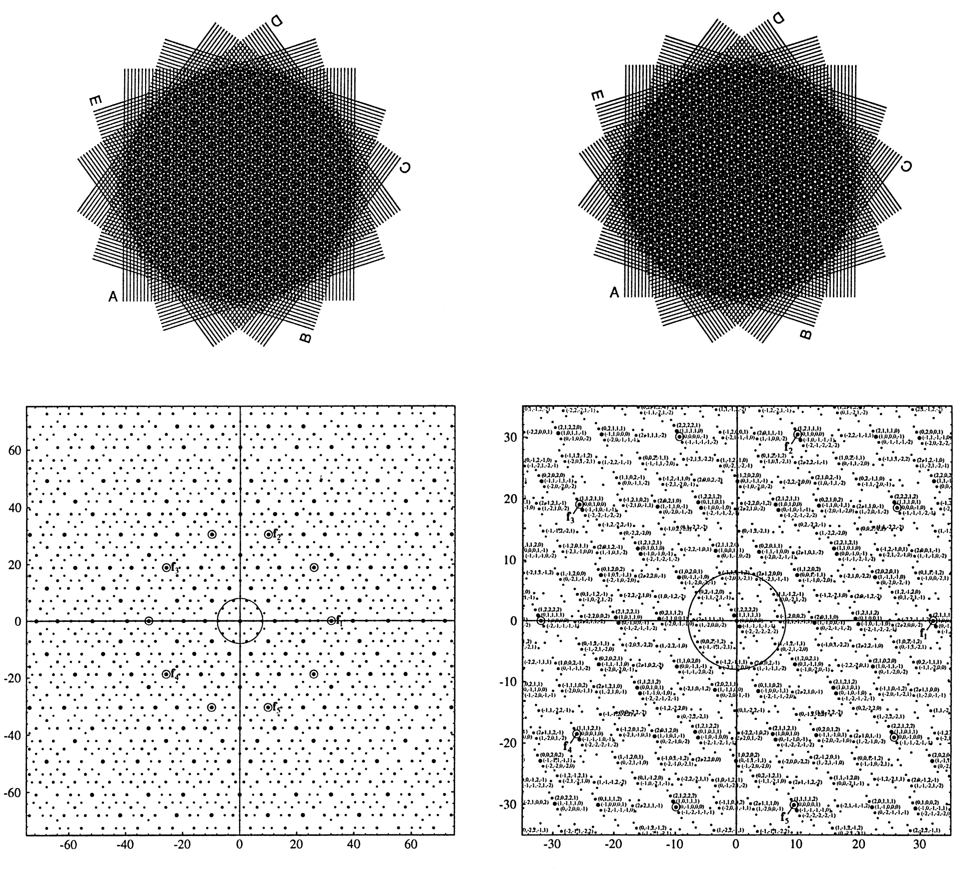


(a) (b)

Figure 9. The singular 4-grating superposition of Example 5 (top) and its spectrum support (bottom). (a) Exactly at the singular state: the spectrum support forms here a 2D lattice, each point of which represents a collapsed cluster. (b) Slightly off the singular state: each of the clusters is spread out, clearly demonstrating its 2D nature. Only impulses up to the 3rd order are shown.

how the figures illustrating the examples of this section have been prepared. This method is useful both for getting a general overview of the spectrum support, and for determining the indices of any particular impulses in the spectrum. This is demonstrated in the following example:

Example 6. (1D clusters on a dense 2D support in the (u, v)-plane): Consider the $(1, 1, 1, 1, 1)$-singular moiré which occurs in the superposition of 5 gratings with identical frequencies, and angle differences of $360°/5 = 72°$. In this case the arithmetic calculation of the Cartesian coordinates is more tricky (the values $\sin 72° = \frac{1}{4}\sqrt{10 + 2\sqrt{5}}$ and $\cos 72° = \frac{1}{4}(\sqrt{5} - 1)$ can be obtained from the radiuses of the circumscribed and the inscribed circles in a regular polygon [17, Vol. 7, p. 221, "Polygon"]). However, the spectrum support obtained by computer immediately gives us an insight into the nature of this case. Figure 10(a) shows the spectrum support exactly at the specified singular configuration (taking the frequency of each layer to be 32). Visibly, this spectrum support is *not* a discrete lattice, but rather an everywhere dense module on the (u, v)-plane. In order to visually identify the individual impulses belonging to each of the collapsed clusters in the singular state, we move slightly off the singular state (by modifying the values of one or more of the frequency vectors) so that the impulse clusters in the spectrum become fully spread out (see Fig. 10(b)). As we can see here, each cluster is only of rank 1; this implies (according to (25)) that $\text{rank}_{\mathbb{Z}}\, \text{Im}\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_5} = 5 - 1 = 4$, and since $\dim \text{Im}\Phi_{\mathbf{f}_1,\ldots,\mathbf{f}_5}$ is obviously only 2, it follows, indeed, that the spectrum support of this singular state is everywhere dense on the (u, v)-plane (see Table 2).



(a) (b)

Figure 10. The singular 5-grating superposition of Example 6 (top) and its spectrum support (bottom). (a) Exactly at the singular state: the spectrum support forms here an everywhere dense 2D module, each point of which represents a collapsed cluster. The spectrum in (b) shows an enlarged view of the central part of the spectrum (a), slightly off the singular state: each of the clusters in the spectrum is spread out, clearly demonstrating its 1D nature. Only impulses up to the 3rd order are shown.

Example 7. (2D dense clusters on a discrete 2D support in the (u, v)-plane): Consider the singular rational superposition of 3 dot-screens, whose frequency vectors are given by: $\mathbf{f}_1 = (32, 0)$, $\mathbf{f}_2 = (0, 32)$, $\mathbf{f}_3 = (\frac{4}{5} \cdot 32, \frac{3}{5} \cdot 32)$, $\mathbf{f}_4 = (-\frac{3}{5} \cdot 32, \frac{4}{5} \cdot 32)$, $\mathbf{f}_5 = (\frac{4}{5} \cdot 32, -\frac{2}{5} \cdot 32)$ and $\mathbf{f}_6 = (\frac{2}{5} \cdot 32, \frac{4}{5} \cdot 32)$. The linear transformation $\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_6}$ is given here by

$$\begin{aligned}\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_6}&(k_1, k_2, k_3, k_4, k_5, k_6)\\ &= k_1(32, 0) + k_2(0, 32)\\ &\quad + k_3\left(\tfrac{4}{5} \cdot 32, \tfrac{3}{5} \cdot 32\right) + k_4\left(-\tfrac{3}{5} \cdot 32, \tfrac{4}{5} \cdot 32\right)\\ &\quad + k_5\left(\tfrac{4}{5} \cdot 32, -\tfrac{2}{5} \cdot 32\right) + k_6\left(\tfrac{2}{5} \cdot 32, \tfrac{4}{5} \cdot 32\right)\end{aligned}$$

In order to find Ker $\Phi_{\mathbf{f}_1,\ldots,\mathbf{f}_6}$ for the continuous case, we have to solve the following set of two linear equations for $k_1, \ldots, k_6 \in \mathbb{R}$:

$$\begin{cases} 32k_1 + \frac{4}{5} \cdot 32k_3 - \frac{3}{5} \cdot 32k_4 + \frac{4}{5} \cdot 32k_5 + \frac{2}{5} \cdot 32k_6 = 0 \\ 32k_2 + \frac{3}{5} \cdot 32k_3 + \frac{4}{5} \cdot 32k_4 - \frac{2}{5} \cdot 32k_5 + \frac{4}{5} \cdot 32k_6 = 0 \end{cases}$$

The solution of this set of equations is: $\{(k_1, k_2, k_3, k_4, k_5, k_6) \mid 2k_5 = -2k_1 + k_2 - k_3 + 2k_4,\ 2k_6 = -k_1 - 2k_2 - 2k_3 - k_4,\ k_i \in \mathbb{R}\}$. This is clearly a 4D volume (with 4 free variables) in the 6D space $\mathbb{R}^6$. Furthermore, the lattice L, which is the discrete solution for $\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_6}$ (i.e., with $k_i \in \mathbb{Z}$), is also a 4D lattice imbedded in this volume; this means that there is no loss of dimensions, so that the spectrum support is indeed a discrete lattice, and each cluster in the 2D spectrum is a dense module with $\text{rank}_{\mathbb{Z}} = 4$.

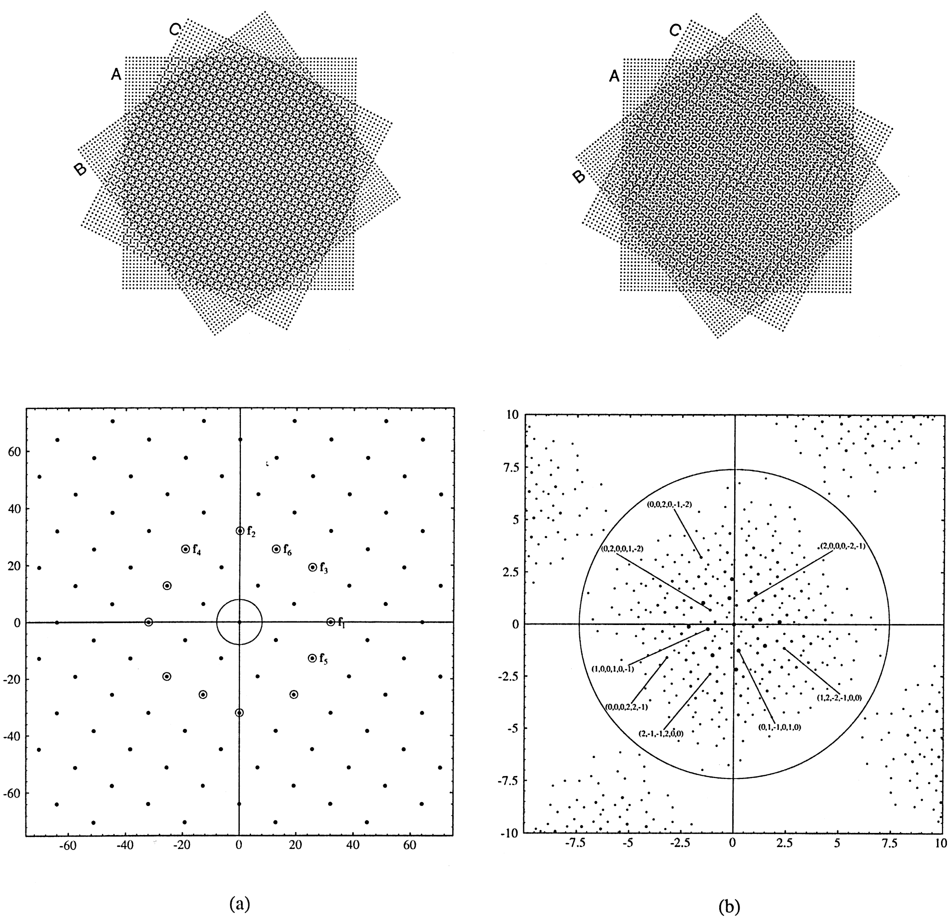
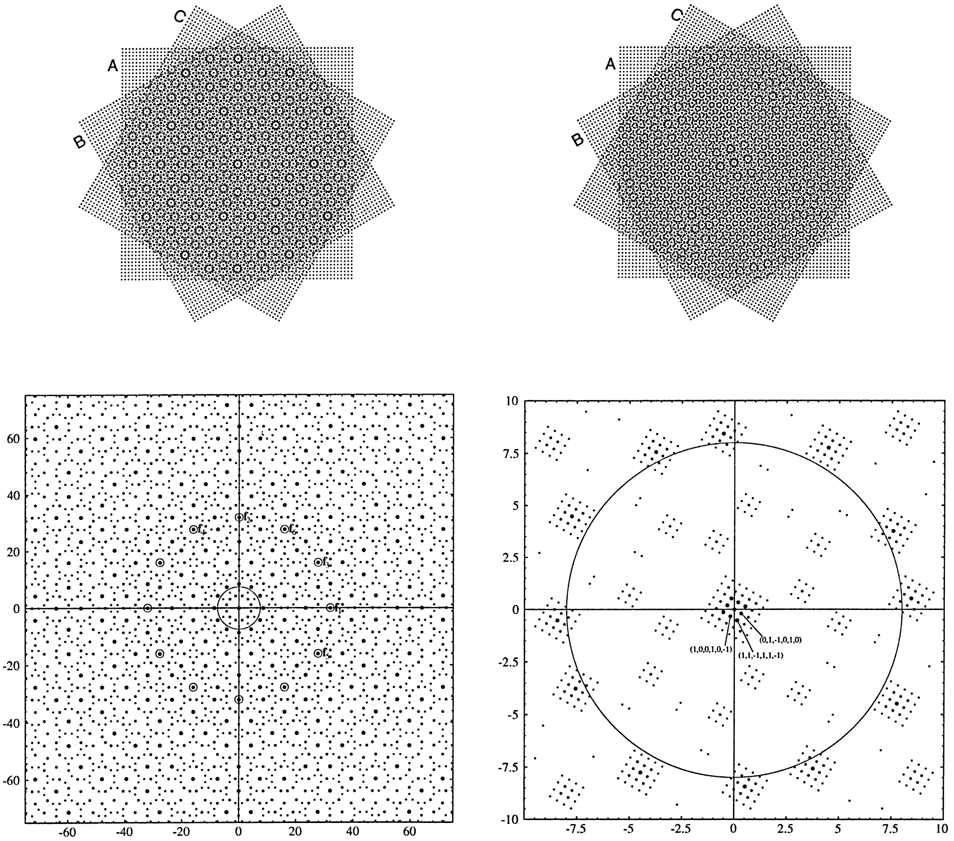


Figure 11. The singular 3-screen superposition of Example 7 (top) and its spectrum support (bottom). (a) Exactly at the singular state: the spectrum support forms here a 2D lattice, each point of which represents a collapsed cluster. The spectrum in (b) shows an enlarged view of the central part of the spectrum (a), showing the spread-out main cluster slightly off the singular state: this cluster forms in the (u, v)-plane a dense 2D module. Only impulses up to the 3rd order are shown.



(a) (b)

Figure 12. The singular 3-screen superposition (top) and the spectrum support (bottom) of Example 8: the traditional 3-screen combination used for color printing. (a) Exactly at the singular state: the spectrum support forms here an everywhere dense 2D module, each point of which represents a collapsed cluster. The spectrum in (b) shows an enlarged view of the central part of the spectrum (a), slightly off the singular state: each of the clusters is spread out, clearly demonstrating its 2D lattice structure. Only impulses up to the 3rd order are shown.

If we closely look at the points (impulses) of the main cluster around the origin (for example, all the cluster impulses up to order 2), we can see that in this case there occur simultaneously several different moirés (isocentric moirés, i.e., moirés which have the same singular state): first, we have a 3-screen moiré, spanned by the $(0, 1, -1, 0, 1, 0)$-impulse and its orthogonal counterpart, $(1, 0, 0, 1, 0, -1)$; and then we have a 2-screen moiré between each of the 3 screen pairs: a moiré spanned by the $(1, 2, -2, -1, 0, 0)$-impulse and its orthogonal counterpart; a moiré spanned by the $(2, 0, 0, 0, -2, -1)$-impulse and its orthogonal counterpart; and a moiré spanned by the $(0, 0, 2, 0, -1, -2)$-impulse and its orthogonal counterpart. To each of these moirés belongs a 2D sub-cluster (sub-lattice of the 4D lattice L); obviously, all of them collapse together onto the spectrum origin at the singular state.

Example 8. (2D discrete clusters on a dense 2D support in the (u, v)-plane): Consider the singular superposition of 3 screens with identical frequencies and equal angle differences of 30° (this is the conventional screen combination traditionally used in color printing; see Fig. 12(a)). It is interesting to note that this superposition manifests a 12-fold symmetry, which is clearly seen both in the image domain and in the spectrum. And yet, whenever the 3 superposed screens move slightly off the singular state, the generated moiré is two-dimensional and it only presents a 4-fold symmetry.

The algebraic analysis of this example provides the explanation of this phenomenon: the cluster which is collapsed on the spectrum origin in this singular superposition is indeed a 2D lattice (see the spread-out cluster in Fig. 12(b)).

In this example we have: $\mathbf{f}_1 = (32, 0)$, $\mathbf{f}_2 = (0, 32)$, $\mathbf{f}_3 = (16\sqrt{3}, 16)$, $\mathbf{f}_4 = (-16, 16\sqrt{3})$, $\mathbf{f}_5 = (16\sqrt{3}, -16)$ and $\mathbf{f}_6 = (16, 16\sqrt{3})$. Therefore the linear transformation $\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_6}$ is given in this case by

$$\begin{aligned}\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_6}&(k_1, k_2, k_3, k_4, k_5, k_6)\\ &= k_1(32, 0) + k_2(0, 32) + k_3(16\sqrt{3}, 16)\\ &\quad + k_4(-16, 16\sqrt{3}) + k_5(16\sqrt{3}, -16)\\ &\quad + k_6(16, 16\sqrt{3})\end{aligned}$$

In order to find Ker $\Phi_{\mathbf{f}_1,\ldots,\mathbf{f}_6}$ for the continuous case, we have to solve the following set of two linear equations for $k_1, \ldots, k_6 \in \mathbb{R}$:

$$\begin{cases} 32k_1 + 16\sqrt{3}k_3 - 16k_4 + 16\sqrt{3}k_5 + 16k_6 = 0\\ 32k_2 + 16k_3 + 16\sqrt{3}k_4 - 16k_5 + 16\sqrt{3}k_6 = 0\end{cases}$$

The solution of this set of equations is: $\{(k_1, k_2, k_3, k_4, k_5, k_6) \mid 2k_5 = -\sqrt{3}k_1 + k_2 - k_3 + \sqrt{3}k_4,\ 2k_6 = -k_1 - \sqrt{3}k_2 - \sqrt{3}k_3 - k_4,\ k_i \in \mathbb{R}\}$. This is clearly a 4D volume (having 4 free variables) in the 6D space $\mathbb{R}^6$. However, the lattice L, which is the discrete solution for $\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_6}$ (i.e., with $k_i \in \mathbb{Z}$), is *not* a 4D lattice but rather only a 2D lattice (since for k_5 and k_6 to be integers it is required that $k_3 = -k_5$ and $k_4 = -k_6$ in order that all the roots be cancelled out). This means that in this case there is a loss of 2 dimensions in $L = \text{Ker}\,\Psi_{\mathbf{f}_1,\ldots,\mathbf{f}_6}$ with respect to Ker $\Phi_{\mathbf{f}_1,\ldots,\mathbf{f}_6}$, and therefore the spectrum support of this singular case is an everywhere dense module. And furthermore, according to (25) we see that each point of this module (at the singular state) represents, in fact, a collapsed lattice (cluster) of a 2D nature: rank $L = 6 - 4 = 2$.

9. The Interpretation of the Spectrum Structure Back in the Image Domain

In the previous sections we analyzed the properties of the spectrum convolution (i.e., the spectrum of the layer superposition) from a pure algebraic point of view, concentrating only on the spectrum support, and ignoring the impulse amplitudes. Let us now "augment" these algebraic foundations by reintroducing the impulse amplitudes on top of their geometric locations in the spectrum. We will see how both the structural and the amplitude properties of the spectrum are related to properties of the layer superposition and its moiré effects in the image domain.

9.1. *The Image Domain Interpretation of the Global Structure of the Spectrum Support*

As we have seen in Eq. (4) the amplitude of the $(k_1, \ldots, k_m)$-impulse in the spectrum convolution is a product of the amplitudes of the individual impulses contributed by the spectrum of each of the layers. By reintroducing the amplitude values of the spectrum impulses on top of their geometric locations, we get again a full description of the spectrum. This permits us to use the Fourier theory to transform the structural results we have algebraically obtained in the spectral domain back into the image domain as well. We start by considering the structure of the global spectrum support and interpreting its influence on the image domain. The structure of the individual impulse clusters and its image domain interpretation will be discussed in Section 9.2.

As we have seen in Table 2, the spectrum convolution (i.e., the spectrum of the layer superposition) can have four different types of spectrum support, which are denoted in the table by 2D-L, 2D-M, 1D-L and 1D-M. These four types are the four possible combinations of two basic and independent properties of the spectrum support: (a) it can be either 2D or 1D; (b) it can be either a discrete lattice or a dense module. Let us see now what is the image domain interpretation of each of these two basic independent properties.

(a) Clearly, a 2D spectrum support indicates that the image superposition is indeed of a 2D nature. A 1D spectrum support in the (u, v)-plane means that all the "action" in the image domain takes place only in one direction, while in the perpendicular direction the image is constant. This happens in a grating superposition where all the original gratings are parallel (their frequency vectors are collinear); this is in fact a case of one-dimensional nature which is artificially extended to the 2D (x, y) image plane.

(b) The support of the spectrum convolution is a discrete lattice *iff* the layer superposition in the image domain is a periodic function (either 1D or 2D).

Table 3. The four possible spectrum support types and their interpretation in the image domain.

Spectrum support	Superposition in the image domain	Examples
2D-L	2-fold periodic	Sec. 8 Ex. 2
1D-L	1-fold periodic	Sec. 8 Ex. 3
2D-M	2-fold almost-periodic*	Sec. 8 Ex. 8
1D-M	1-fold almost-periodic	Sec. 8 Ex. 4

*Note that the case of 2D-M includes also a special case in which the 2D spectrum is dense in one direction and discrete in the other; this case corresponds in the image domain to a 2D function which is almost-periodic in one direction and periodic in the other. This hybrid case may occur, for instance, in the superposition of 3 gratings, 2 of which have the same direction, but with incommensurable frequencies.

This follows from the decomposition of the periodic function into a Fourier series. When the spectrum support is a dense module, it is clear that the layer superposition is not periodic; but on the other hand the spectrum is still impulsive and not continuous, meaning that the superposition is not aperiodic, either. In fact, such cases belong to an intermediate class of functions which is known as *almost-periodic* functions [20]; a spectrum formed by a dense module of impulses represents a generalized Fourier series expansion which belongs to an almost-periodic function. This means that in such cases the layer superposition back in the image domain is an almost-periodic function.

The four possible types of spectrum support and their interpretations in the image domain are summarized in Table 3.

We can now reformulate Proposition 2 as a criterion for the periodicity of the superposition of periodic layers (functions):

Proposition 3. *The superposition of m gratings (or $m/2$ 2D dot-screens, etc.) is periodic iff* $\mathrm{rank}_{\mathbb{Z}}\,\mathrm{Md}(\mathbf{f}_1,\ldots,\mathbf{f}_m) = \dim \mathrm{Sp}(\mathbf{f}_1,\ldots,\mathbf{f}_m)$. *The superposition is almost-periodic iff* $\mathrm{rank}_{\mathbb{Z}}\,\mathrm{Md}(\mathbf{f}_1,\ldots,\mathbf{f}_m) > \dim \mathrm{Sp}(\mathbf{f}_1,\ldots,\mathbf{f}_m)$. *(Note that the case of '$<$' is impossible. This means that the two conditions above are exhaustive; and indeed, the superposition of periodic functions is either periodic or almost-periodic).*

9.2. *The Image Domain Interpretation of the Clusters in the Spectrum*

We have already mentioned that the main cluster in the spectrum (the impulse cluster which is centered on the spectrum origin) is the Fourier transform of the isolated (extracted) moiré. When this cluster is slightly spread-out and its fundamental impulses are located within the visibility circle, the corresponding moiré may be clearly visible in the image domain (if the amplitudes are not too weak); but when the singular state is reached and the cluster impulses collapse onto the DC, the moiré in the image domain gets an infinite period and disappears.

However, the support of this cluster in the (u, v)-plane is not necessarily a discrete lattice, and in the more general case it can even be a dense module (like in Example 7 of Section 8). In fact, here too there exist four different cases, whose interpretation back in the image domain is summarized in Table 4.

It is interesting to note that the algebraic structure of the moiré cluster is not necessarily the same as the algebraic structure of the overall spectrum of the superposition. The moiré cluster may have a lower dimension (a 1D cluster imbedded in an overall 2D spectrum, as in Example 2 in Section 8) or a simpler structure (a 2D-L cluster within a 2D-M spectrum, such as in Figs. 4(b) and (c), or in Example 8 of Section 8), or even both (a 1D-L cluster within a 2D-M spectrum, as in Example 6 of Section 8).[9] This means, back in the image domain, that even when the overall superposition of the periodic layers is not periodic but rather almost-periodic, a moiré generated in this superposition may still be periodic. This is, in fact, a very common situation, which is clearly illustrated, for instance, in the superposition of Fig. 12(b): although the overall superposition is not periodic (notice the micro-structure!), the intensity profile of the isolated moiré is indeed periodic.

Our algebraic approach provides also information regarding the other clusters which are formed in the

Table 4. The four possible cluster support types and their interpretation in the image domain.

Main cluster	Extracted moiré in the image domain	Examples
2D-L	2-fold periodic	Sec. 8 Ex. 5
1D-L	1-fold periodic	Sec. 8 Ex. 2
2D-M	2-fold almost-periodic	Sec. 8 Ex. 7
1D-M	1-fold almost-periodic	

spectrum simultaneously with the main cluster: the support of each of these clusters in the (u, v)-plane is simply a shifted replica of the support of the main cluster. However, the impulse amplitudes within each cluster are calculated according to Eq. (4), meaning that the impulse amplitudes in the shifted clusters are *not* simply shifted replicas of the impulse amplitudes in the main cluster. These clusters contribute higher frequencies to the global structure of the spectrum support, and in terms of the image domain, they take part in the generation of micro-structure details in the superposition.

9.3. The Amplitude of Compound Impulses in the Singular States

As the superposed layers gradually approach a singular state, the spectrum undergoes an "inverse playback" of the cluster spreading-out process. The singular state itself is the limit case where each of the spread-out impulse clusters collapses down into a single *compound impulse*. The question is, what happens to the amplitudes of the impulses of each cluster at the limit point when the singular state is attained, and each of the impulse clusters fuses down into a single compound impulse in the spectrum?

Since the spectrum of the superposed layers is always the convolution of the individual spectra (by the Convolution Theorem), the answer to this question follows from the properties of convolution. The convolution of a function $f(x, y)$ with an impulsive function such as a comb is simply a sum of replicas of $f(x, y)$, which are copied on top of each impulse of the comb [21, pp. 295–296]; so if an overlapping occurs between several replicas, the overlap behaves *additively*. Since in our case, $f(x, y)$, too, is impulsive, we obtain the following result:

When an impulse cluster collapses down into a single impulse, the amplitude of the resulting compound impulse is the sum of the individual amplitudes of all the collapsed impulses[10].

10. Summary

The superposition of periodic layers (such as line-gratings, dot-screens etc.) and the phenomena related thereto, such as the superposition moiré effects, can be fully explained by analyzing the Fourier spectrum of the superposition. In the present article we provide a solid algebraic foundation for the analysis of the Fourier spectrum of the superposed layers and their moiré effects. We introduce an algebraic formalization of the structure of the Fourier spectrum, based on the theory of geometry of numbers. The key point is the fundamental relationship between the index $(k_1, \dots, k_m)$ of each impulse and its geometric location in the spectrum, which is given by the transformation $\Psi_{\mathbf{f}_1,\dots,\mathbf{f}_m}(k_1, \dots, k_m) = k_1\mathbf{f}_1 + \cdots + k_m\mathbf{f}_m$. By analyzing this algebraic relationship we acquire a full understanding of the structural properties of the spectrum of the superposition. These spectral domain properties are then reflected back to the layer superposition in the image domain and interpreted there by means of the Fourier theory.

Using this new approach we show that the spectrum support can be either a discrete lattice or a dense module; in the first case the layer superposition in the image domain is periodic, while in the second case it is almost-periodic. Furthermore, we obtain a criterion for the periodicity of the superposition of any number of periodic layers: the superposition is periodic *iff* the continuous and the discrete dimensions of the spectrum support are equal; otherwise the superposition is almost-periodic. We also show that a singular case occurs in the superposition *iff* the frequency vectors $\mathbf{f}_1, \dots, \mathbf{f}_m$ of the superposed layers are linearly dependent over $\mathbb{Z}$, i.e., *iff* $\text{rank}_{\mathbb{Z}}\, \text{Md}(\mathbf{f}_1, \dots, \mathbf{f}_m) < m$. When more than two gratings are superposed, the spectrum support can be a discrete lattice (and hence the superposition can be periodic) only if the superposition is singular; but singular superpositions may have either a discrete or a dense support (and hence be either periodic or almost-periodic).

Other important results concern the formation of impulse clusters in the spectrum of the superposition. We show that this clusterization of the spectrum support reflects the partition of the lattice of the impulse indices (i.e., $\mathbb{Z}^m$) into equivalence classes, which is induced by $\Psi_{\mathbf{f}_1,\dots,\mathbf{f}_m}$ (or simply, by the frequency vectors $\mathbf{f}_1, \dots, \mathbf{f}_m$ which define the layer superposition). The main impulse cluster which is centered on the spectrum origin is the spectral representation of a moiré effect in the image superposition, and the other clusters are simply translated replicas (in terms of impulse indices and impulse locations) of this cluster. When the layer superposition is singular, each of the clusters is collapsed down into a single point in the spectrum, but when moving a little out of the singular state (by slightly modifying the frequency vectors $\mathbf{f}_1, \dots, \mathbf{f}_m$ of the superposed layers), each of the clusters in the spectrum starts "spreading out", thus revealing its internal structure. Several examples of superposed periodic layers are provided to illustrate our results, both in the spectrum and in the image domains.

Acknowledgment

We would like to thank Dr. J.-P. Revéillès for reading an early version of the manuscript and for his helpful comments.

Notes

1. It should be noted however that not every singular state is necessarily moiré-free: although the $(k_1, \ldots, k_m)$-moiré itself is not visible in its singular state, other impulses may be present in the same time within the visibility circle and cause other moirés to be visible.
2. Vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ in $\mathbb{R}^n$ are called *linearly independent* over $\mathbb{R}$ (or over $\mathbb{Z}$, etc.) if $t_1\mathbf{v}_1 + \cdots + t_m\mathbf{v}_m = \mathbf{0}$ with $t_i \in \mathbb{R}$ (respectively, $t_i \in \mathbb{Z}$) implies that $t_1 = \cdots = t_m = 0$.
3. Note that $z < r$ is impossible, since linear independence over $\mathbb{R}$ implies linear independence over $\mathbb{Z}$ (and linear dependence over $\mathbb{Z}$ implies linear dependence over $\mathbb{R}$).
4. Formally, a subset D of $\mathbb{R}^n$ is called *discrete* if there exists a number $d > 0$ such that for any points $a, b \in D$ the distance between a and b is larger than d. A subset S of $\mathbb{R}^n$ is called *dense* or *everywhere dense* in $\mathbb{R}^n$ if $[S] = \mathbb{R}^n$, where $[S]$ denotes the closure of S, i.e., the set containing S and all its limit points [17, Vol. 3, p. 434]. Examples: (1) The set of all integer numbers is discrete. (2) Both the set of all rational numbers and the set of all irrational numbers are dense in $\mathbb{R}$, although none of them is continuous in $\mathbb{R}$.
5. Two vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$ (or real numbers in $\mathbb{R}$) are called *commensurable* if there exist non-zero integers m, n such that $\mathbf{v}_2 = (m/n)\mathbf{v}_1$. This means that both $\mathbf{v}_1$ and $\mathbf{v}_2$ can be measured as integer multiples of the same length unit, say $(1/n)\mathbf{v}_1$. More generally, k vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in $\mathbb{R}^n$ (or real numbers in $\mathbb{R}$) are called *commensurable* if they are linearly dependent over $\mathbb{Q}$ (the set of all rational numbers); note that this is identical to linear dependence over $\mathbb{Z}$. And conversely, vectors (or real numbers) which are linearly independent over $\mathbb{Q}$ (or over $\mathbb{Z}$) are called *incommensurable* [17, Vol. 7, p. 436]. Note that $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are commensurable *iff* $\text{rank}_\mathbb{Z}\, \text{Md}(\mathbf{v}_1, \ldots, \mathbf{v}_k) < k$; they are incommensurable *iff* $\text{rank}_\mathbb{Z}\, \text{Md}(\mathbf{v}_1, \ldots, \mathbf{v}_k) = k$.
6. Formally speaking, when the superposition moves out of the singular state Ker Ψ becomes $\{\mathbf{0}\}$, so that each point in $\mathbb{Z}^m$ becomes its own one-member equivalence class. Therefore the spread-out clusters no longer correspond to the current equivalence classes. However, we will still consider the "spread-out clusters" in the spectrum to be traces of the clusters of the singular state that we have just left, and we will continue to call them "clusters" in this sense.
7. Note that the choice of $\mathbf{f}_1 = (32, 0)$ for the first layer is arbitrary, and for any other choice, $\mathbf{f}_2$ and $\mathbf{f}_3$ could be adapted accordingly. However, for the sake of consistency and to facilitate comparisons between the spectra we will use the same convention in most of the following examples, too.
8. Note that $(1, 1, 1)$ and $(1, -1, 0)$ form a basis of the 2D lattice L (see Fig. 7(b)). This means that in the superposition of this example not only the 3-layer $(1, 1, 1)$-moiré is singular, but also the 2-layer $(1, -1, 0)$-moiré. And indeed, slightly out of the singular state both of them are simultaneously visible.
9. Obviously, since the moiré cluster is a subset of the overall spectrum, its structure can never be *more* complex than that of the overall spectrum.
10. This *additive* behaviour of the impulse amplitudes should not be confused with the *multiplicative* behaviour of the individual impulse amplitudes in the convolution process: Each individual impulse amplitude in the convolution process is obtained by Eq. (4) as a *product*; but if several impulses thus obtained happen to fall on the same geometric location, their individual amplitudes are then *summed*.

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