New results about 3D digital lines

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ABSTRACT

The current definition of 3D digital lines[3] , which uses the 2D digital lines of closest integer points (Bresenham's lines) of two projections, has several drawbacks:

- the discrete topology of this 3D digital line notion is not clear,
- its third projection is, generally, not the closest set of points of the third euclidean projection,
- if we consider a family of parallel euclidean lines, we do not know how many combinatorially distinct digital structures will be built by this process,
- and mainly the set of voxels defined in this way is not the set of closest points of the given euclidean line.

And these questions are the simplest ones; many others could be asked: dependence on the choice of the projections, intersections with digital planes, intersections between 3D digital lines,...

This paper gives a new definition of 3D digital lines relying on subgroups of $\mathbb{Z}^3$, whose main advantage over the former one is its ability to convert any practical question into rigorous algebraic terms. It follows previously developed ideas[1] but with a much simpler treatment and new results. In particular, we obtain a complete description of the topology of these lines and a condition for the third projection being a 2D digital line as well as a classification of digital lines of the same direction into classes of equivalent combinatorial structure.

Keywords: Digital Geometry, Digital Lines, Lattices

1 INTRODUCTION

The general idea of our approach will be briefly explained in this paragraph. Our definition is restricted to the discretization of euclidean lines directed by integers vectors (a, b, c) satisfying the following hypotheses:

$$\begin{cases} \gcd(a,b,c) = 1 \\ 0 \le a < b < c \end{cases}$$

The first one, motivated by arithmetical reasons, is not a real restriction: the general case can be easily reduced to it. The second one is much more interesting as it comes from the symmetries of the space $\mathbb{Z}^3$ (or those of the cube). These inequalities describe what is called the *standard simplex*, which is nothing more than the *fundamental domain* of the group of symmetries of the cube. Using this group, the study of 3D digital lines directed by any vector (a,b,c), can be reduced to those directed by vectors belonging to this fundamental domain.

Let us consider the *euclidean plane* (P), normal to (a,b,c), whose equation is

$$ax + by + cz = 0 \tag{P}$$

and the orthogonal projection π of $\mathbb{Z}^3$ to (P):

$$\pi : \mathbb{Z}^3 \rightarrow P$$

It is easy to prove that the image

$$\mathcal{L} = \pi(\mathbb{Z}^3)$$

is a *discrete* and rational lattice.

An important consequence results from $\mathcal{L}$ discreteness. That is: bounded subsets of plane (P) contain only *finite* numbers of points of $\mathcal{L}$. If B is such a bounded set, its inverse image $\pi^{-1}(B)$ is made of a finite number of *fibers* all of which are in one to one correspondance with the subgroup

$$\pi^{-1}(0) = \{k.(a,b,c) \mid k \in \mathbb{Z}\}$$

generated by vector (a,b,c). More precisely for any point $x \in \mathcal{L}$ we know that its fiber $\pi^{-1}(x)$ is equal, within translation, to $\pi^{-1}(0)$.

In this way we reduce the study of 3D digital lines to the study of the 2D lattice $\mathcal{L}$. Lattices (or $\mathbb{Z}$-modules) are structures which are, at the same time, similar and distinct from vector spaces. The reader will find their properties in any algebra treatise.[4]

We are more precisely interested in the study of the subset $\mathfrak{p}$ of $\mathcal{L}$ contained in a fundamental domain of a sub-lattice $\mathcal{S}$ of $\mathcal{L}$. We show hereafter that the parameterization of $\mathcal{L}$ and $\mathfrak{p}$ can be made extremely simple leading to a particularly interesting representation of digital 3D lines. We show also that we can, among others, *read* the topology of the line and recover usual algorithms from this representation.

We will use some results of the theory of numbers[2] and the following usual notations:

- If u and v are two integers we recall that (u,v) is the common abbreviation for $gcd(u,v)$ and $lcm(u,v)$ is their least common multiple.
- The brackets $\left[\frac{u}{v}\right]$ denote the *quotient* of the euclidean division of u by v.
- The curly brackets $\left\{\frac{u}{v}\right\}$ denote the euclidean *remainder* of the euclidean division of u by v.

2 SIMPLIFICATION OF TRIPLY GENERATED TWO DIMENSIONAL LATTICES.

The main difficulty concerning $\mathbb{Z}$-modules (or lattices or free abelian groups) is that one can find *free families* of vectors whose cardinal is equal to the dimension of the ambient space and which *do not generate* this space. One such example is given by the set $\{(1,0),(3,2)\}$ of $\mathbb{Z}^2$, which is free, has cardinal two and is not generator of $\mathbb{Z}^2$. We can see for instance that vector $(2,1)$ cannot be represented as a linear combination, with *integer* coefficients, of the given vectors.

Let us introduce the following definitions:

- If $V_1, V_2, \ldots, V_i$ are integer vectors of $\mathbb{Z}^3$, $L(V_1, V_2, \ldots, V_i)$ will denote the lattice generated by these vectors. We shall restrict to $i = 2$ and $i = 3$ and say, respectively, in these cases that the lattices are *doubly* or *triply* generated.
- We shall denote any fundamental domain of $L(V_1, V_2)$ by Par (for parallelogram) and by $\mathfrak{p}$ the set of points of $L(V_1, V_2, V_3)$ contained in Par, (keep in mind that the points V_1, V_2 and $V_1 + V_2$ are *not* members of $\mathfrak{p}$).
- We also denote by $\nu(\mathfrak{p})$ the cardinal of $\mathfrak{p}$.

Our goal is parameterize in the most simple way the set $\mathfrak{p}$. Let us start with a 2D version of this problem, that is we suppose vectors V_1, V_2, V_3 are in $\mathbb{Z}^2$.

$V_1 = (x_1, y_1)$ and $V_2 = (x_2, y_2)$ either generate a rank one subgroup, if $\det \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = 0$, or a rank two subgroup of $\mathbb{Z}^2$ otherwise. We suppose this last hypothesis is satisfied in what follows.

If $V_3 = (x_3, y_3)$ is a third vector of $\mathbb{Z}^2$ we also suppose that:

- $\gcd(x_i, y_i) = 1$ for $i = 1, 2, 3$ and that
- All three determinants $det(V_1, V_2)$, $det(V_1, V_3)$ and $det(V_3, V_2)$ are non zero and that the first one is positive.

With these hypotheses the lattice $L(V_1, V_2, V_3)$ is a rank two submodule of $\mathbb{Z}^2$ and we are interested, as explained above, in the set $\mathfrak{p}$ of its points contained in a fundamental domain, Par, of $L(V_1, V_2)$.

Obviously the parallelogram Par and its translations by the vectors $k_1.V_1 + k_2.V_2,\ k_1, k_2 \in \mathbb{Z}$ induce a tiling of $\mathbb{Z}^2$. So any integer point of $\mathbb{Z}^2$ belonging to one such tile has a reduction (homologous point) in Par. Thus, after reduction, the sequence $k.V_3,\ k \in \mathbb{Z}$ gives a subset of the integer points of Par; this inclusion is generally *strict*. In the remaining the following two notations will be used for this reduction modulo Par: either "mod Par" or "mod $\{V_1, V_2\}$".

The following lemma results immediately.

Lemma 2.1. *The set $\mathfrak{p}$ is given by the reduction* modulo Par *of the integer multiples of vector V_3, that is of $\{k.V_3 \mid k \in \mathbb{Z}\}$:*

$$\mathfrak{p} = \{k.V_3 \mid k \in \mathbb{Z}\} \bmod \{V_1, V_2\}$$

It is well known that the number of integer points contained in $Par = card(Par)$, is equal to $\delta = x_1 y_2 - x_2 y_1$. So the cardinal of $\mathfrak{p}$ is bounded by δ. It is also well known that there exists one integer value k such that $k.V_3$ belongs to the lattice generated by V_1 and V_2. This comes from the following observation. As $card(Par)$ is *finite*, there must be two distinct integer values m and n such that $m.V_3$ and $n.V_3$ have the same reduction mod $\{V_1, V_2\}$. Thus for $k = m - n$ we have

$$k.V_3 \equiv 0 \mod \{V_1, V_2\}$$

which is equivalent to this assertion.

Figure 1 illustrates the principle contained in lemma 2.1, showing the tiling and the first multiples of a third vector. The reductions can be placed in any parallelogram.

It can then be deduced that there exists a smallest integer value, still denoted k, such that $k.V_3$ belongs to the lattice generated by V_1 and V_2. It can be proved that this number k is exactly the cardinal of $\mathfrak{p}$.

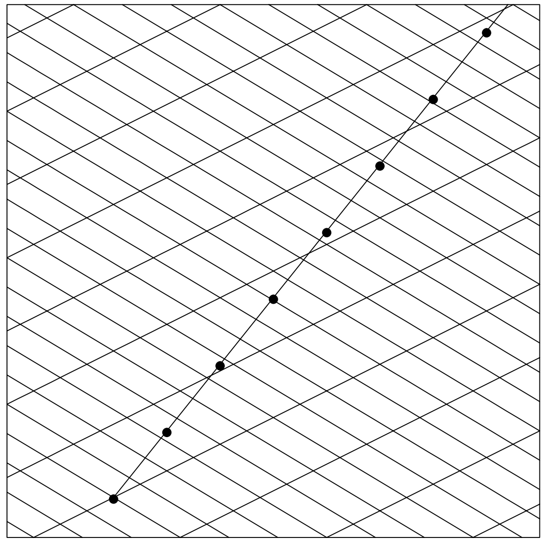

Figure 1: A doubly generated lattice and the multiples of a third vector.

We introduced above the determinant: $\delta = x_1y_2 - x_2y_1$ (which can be supposed to be > 0). Let us also consider the values of the two other determinants reduced modulo δ: $\chi \equiv x_1y_3 - x_3y_1 \pmod{\delta}$ and $\gamma \equiv x_3y_2 - x_2y_3 \pmod{\delta}$.

With a little knowledge from group theory about elements order, we can deduce that

LEMMA 2.2. *The cardinal of* $\mathfrak{p}$, $\nu(\mathfrak{p})$ *can be expressed as*

$$k = \nu(\mathfrak{p}) = lcm\left(\frac{\delta}{(\chi,\delta)}, \frac{\delta}{(\gamma,\delta)}\right).$$

(admitted).

The set $\mathfrak{p}$ can be constructed in $\nu(\mathfrak{p})$ steps, each one involving a mod $\{V_1, V_2\}$ reduction. The components ρ, σ of the mod $\{V_1, V_2\}$ reduction of an arbitrary vector (x, y) can be expressed as:

$$\begin{pmatrix} \rho \\ \sigma \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} - \left[\tfrac{xy_2 - x_2y}{\delta}\right]\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - \left[\tfrac{x_1y - xy_1}{\delta}\right]\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

Even if this formula is fine, we will look for a yet simpler and faster way of generating the set $\mathfrak{p}$. In fact we have the following lemma, where δ, χ, γ are as above.

LEMMA 2.3. *There is an integer* 2×2 *matrix* R *which maps the lattice* $L(V_1, V_2, V_3)$ *bijectively to* $L((\delta, 0), (0, \delta), (\chi, \gamma))$.

Proof. Using the classical identity for euclidean division

$$a = \left[\tfrac{a}{b}\right]b + \left\{\tfrac{a}{b}\right\},$$

the preceding identity can be simplified as follows:

$$\begin{pmatrix} \rho \\ \sigma \end{pmatrix} = \frac{1}{\delta}\left(\left\{\frac{xy_2 - x_2y}{\delta}\right\}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \left\{\frac{x_1y - xy_1}{\delta}\right\}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right)$$

But this can be written in matrix notation as

$$\begin{pmatrix} \rho \\ \sigma \end{pmatrix} = \frac{1}{\delta}\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}\begin{pmatrix} \left\{\frac{xy_2 - x_2y}{\delta}\right\} \\ \left\{\frac{x_1y - xy_1}{\delta}\right\} \end{pmatrix}$$

revealing the rational unimodular matrix

$$U = \frac{1}{\delta}\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$$

We can then transform the situation and map the lattice $L(V_1, V_2, V_3)$ to another lattice with the help of the matrix $R = \delta U^{-1} = \begin{pmatrix} y_2 & -x_2 \\ -y_1 & x_1 \end{pmatrix}$.

In the case where $x = x_3$ and $y = y_3$ the numbers $\left\{\frac{xy_2 - x_2y}{\delta}\right\}$ and $\left\{\frac{x_1y - xy_1}{\delta}\right\}$ become respectively the values χ and γ already introduced. Operator R maps $L(V_1, V_2)$ bijectively to the subgroup $\{(m, n)\delta \mid m, n \in \mathbb{Z}\}$ and the lattice $L(V_1, V_2, V_3)$ to the lattice generated by $(\delta, 0), (0, \delta), (\chi, \gamma)$, which is the assertion of lemma 2.3.

$\square$

The lattice $RL(V_1, V_2, V_3)$ is doubly periodic of periods δ and δ. Through R the set $\mathfrak{p}$ is mapped to the modular sequence

$$(k\chi \ mod \ \delta, \quad k\gamma \ mod \ \delta)$$

which can be written also as

$$\left(\left[\frac{k\chi}{\delta}\right], \left[\frac{k\gamma}{\delta}\right]\right), \quad k = 0, 1, 2, \ldots, \nu(\mathfrak{p}).$$

Thus the complexity in generating the set $\mathfrak{p}$ is reduced to the computation of two modular sequences, which can be done with additions and comparisons only, avoiding divisions. This can even be reduced once more by introducing the particular value of k, say η, for which

$$\eta\chi \equiv (\chi, \delta) \pmod{\delta}$$

.

For this value of k, the other sequence is equal to $\eta\gamma \pmod{\delta}$, that we shall denote ϵ. Obviously the set of points $R\mathfrak{p}$ can be built by the sequence

$$\left(k(\chi, \delta), \left\{\frac{k\epsilon}{\delta}\right\}\right), \quad k = 0, 1, 2, \ldots, \nu(\mathfrak{p})$$

which now needs only one modular computation for each step. Finally, with the former notations, we obtain:

Theorem 2.4. *The set $\mathfrak{p}$ can be built in lcm $\left(\frac{\delta}{(\chi,\delta)}, \frac{\delta}{(\gamma,\delta)}\right)$ computations of a modular arithmetical sequence of type $\left[\frac{k\epsilon}{\delta}\right]$*

3 DEFINITION OF 3D DIGITAL LINES

The lattice $\mathcal{L}$ introduced in 1 is a rational lattice contained in the plane (P) $ax + by + cz = 0$. It is generated by the three vectors $V_1 = \pi(1, 0, 0), V_2 = \pi(0, 1, 0)$ and $V_3 = \pi(0, 0, 1)$, where π is the orthogonal projection on (P). These vectors can be easily written in terms of a, b, c and $\omega^2 = a^2 + b^2 + c^2$:

$$V_1 = \frac{1}{\omega^2}\begin{pmatrix} b^2+c^2 \\ -ab \\ -ac \end{pmatrix} \quad V_2 = \frac{1}{\omega^2}\begin{pmatrix} -ab \\ a^2+c^2 \\ -bc \end{pmatrix} \quad V_3 = \frac{1}{\omega^2}\begin{pmatrix} -ac \\ -bc \\ a^2+b^2 \end{pmatrix}$$

As these vectors are coplanar, we are in a situation almost similar to that of section 2. To reduce it exactly to this case, we just have to clear out the denominator ω^2. The lattice generated by $\omega^2 V_1$ and $\omega^2 V_2$ is a rank two group isomorphic to $L(V_1, V_2)$ of section 2. The only difference between both situations is that vectors V_1, V_2, V_3 now belong to $\mathbb{Z}^3$ instead of $\mathbb{Z}^2$, but all the preceding results go through, with the obvious modifications.

THEOREM 3.1. *There is a 3×3 rational matrix R, which maps the lattice $\mathcal{L}$ bijectively on the sub-lattice $L((c,0),(0,c),(a,b))$ of $\mathbb{Z}^2$. This operator maps π fibers on lines which project on direction (a,b).*

This image $R\mathcal{L}$ is called the simplification (or reduction) of $\mathcal{L}$ and it is denoted by $\hat{\mathcal{L}}$.

In the same way image $R\mathfrak{p}$ is denoted $\hat{\mathfrak{p}}$ and image $RPar$ is denoted by $\hat{Par}$. Of course $\hat{Par} = [0,c[\times[0,c[$ is the new tile of the simplified lattice.

Proof.

The projection π is not *invertible*, but we can still find operators which are almost inverses of it. (The map π being a fibration, such inverses are usually called *sections* of π). A possible section is given by the mapping

$$\begin{cases} V_1 & \to (1,0,0) \\ V_2 & \to (0,1,0) \\ (0,0,1) & \to (0,0,1) \end{cases}$$

As π is the operator defined by

$$\begin{cases} (1,0,0) & \to V_1 \\ (0,1,0) & \to V_2 \\ (0,0,1) & \to V_3, \end{cases}$$

its matrix (still denoted π) is

$$\pi = \frac{1}{a^2+b^2+c^2}\begin{pmatrix} b^2+c^2 & -ab & -ac \\ -ab & a^2+c^2 & -bc \\ -ac & -bc & a^2+b^2 \end{pmatrix}$$

Thus our problem is to find the inverse of the matrix τ:

$$\tau = \frac{1}{a^2+b^2+c^2}\begin{pmatrix} b^2+c^2 & -ab & 0 \\ -ab & a^2+c^2 & 0 \\ -ac & -bc & a^2+b^2+c^2 \end{pmatrix}.$$

A simple computation gives

$$\tau^{-1} = \frac{1}{c^2}\begin{pmatrix} a^2+c^2 & ab & 0 \\ ab & b^2+c^2 & 0 \\ ac & bc & c^2 \end{pmatrix}$$

If we let $R = c.\tau^{-1}$ and evaluate the images of V_1, V_2 and V_3 by R we respectively find the vectors

$$(c,0,0),(0,c,0),(-a,-b,0)$$

.

This proves that the operator R bijectively maps $L(V_1, V_2, V_3)$ on the *integer* lattice $L((c,0,0),$ $(0,c,0),(-a,-b,0))$. We also remark that this last one is the same as the lattice $L((c,0,0),(0,c,0),(a,b,0))$.

□

By definition (cf. 1), points of $\mathfrak{p}$ are the projections of π fibers. In this sense the image $\hat{\mathcal{L}}$ can be seen as the feet of all these fibers. But this strict planar interpretation is not the only one which can be deduced from the preceding computations. What is really lucky is the result of the computation of $R\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{\omega^2}{c}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, because it proves that the lines directed by (a,b) are actually the projections of the *images of the fibers* by operator R, on xOy plane. Besides, as $V_3 = \pi(0,0,1)$, the points of the form $k.RV_3 = k(a,b)$, where $k \in \mathbb{Z}$, represent the sections of these fibers by the horizontal planes $z = cst$.

Moreover the set $\mathfrak{p}$ of points of $L(V_1, V_2, V_3)$ contained in the parallelogram built on V_1 and V_2 is mapped, by R on the set $\hat{\mathfrak{p}}$ of points of the 2D lattice $L((c,0),(0,c),(a,b))$ contained in the square $[0,c[\times[0,c[= \hat{Par}$ which is much simpler to study.

Figure 2 shows the simplification of the lattice $\mathcal{L}$ associated to $a = 9, b = 15$ and $c = 23$. The smallest points are integer points of $\mathbb{Z}^2$, the medium ones are the images of $\mathcal{L}$, while the largest ones belong to the lattice $L((c,0),(0,c))$. Set $\hat{\mathfrak{p}}$ is made of the medium points contained in the square $\hat{Par} = [0,23[\times[0,23[$.

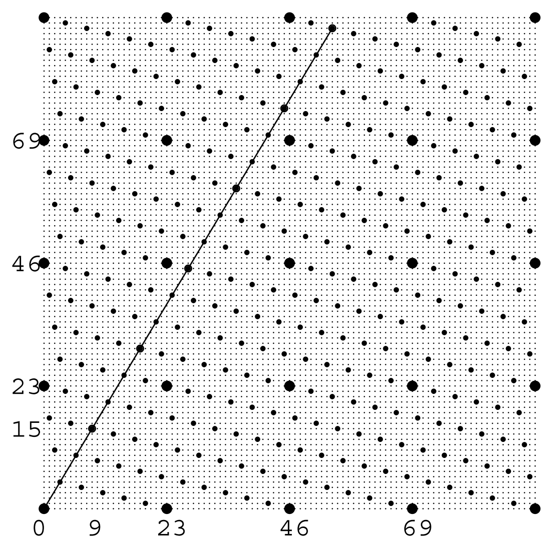

Figure 2: The reduction of the lattice $\mathcal{L}$ associated to a=9, b=15, c=23.

The points located on the line, (of slope 15/9=5/3), are the first multiples of the third vector $(a,b) = (9,15)$. Their reduction modulo $c = 23$ give some of the points of $[0,23[\times[0,23[$.

Finally we obtain the following:

The simplified lattice $\hat{\mathcal{L}}$ *gives the intersection scheme of the euclidean line, directed by* (a,b,c)*, with all the voxels of space.*

Of course the unit cubes of $\mathbb{Z}^3$ are seen as the tiling induced by $\hat{Par} = [0,c[\times[0,c[$.

A closer look at figure 2 reveals this nice interpretation. The line directed by $(9,15,23)$ goes through the origin, then cuts the plane $z = 1$ in the square $[0,1] \times [0,1]$, the plane $z = 2$ in the square $[0,1] \times [1,2]$, the plane $z = 3$ in the square $[1,2] \times [1,2]$,...

This gives a first notion of 3D digital line.

DEFINITION 3.2. *The naive line through the origin, directed by* (a, b, c), *(where* $0 \leq a < b < c, \gcd(a, b, c) = 1$*), is given by the parameterization*

$$\begin{cases} x &= \left[\frac{az}{c}\right] \\ y &= \left[\frac{bz}{c}\right] \end{cases}$$

it is denoted by $\mathcal{D}(a, b, c)$.

We remark immediately that this notion is identical with the usual 3D discrete line built by the double 2D Bresenham algorithm.[3] The set made by the feet of the fibers forming $\mathcal{D}(a, b, c)$ is exactly $\mathfrak{p}$.

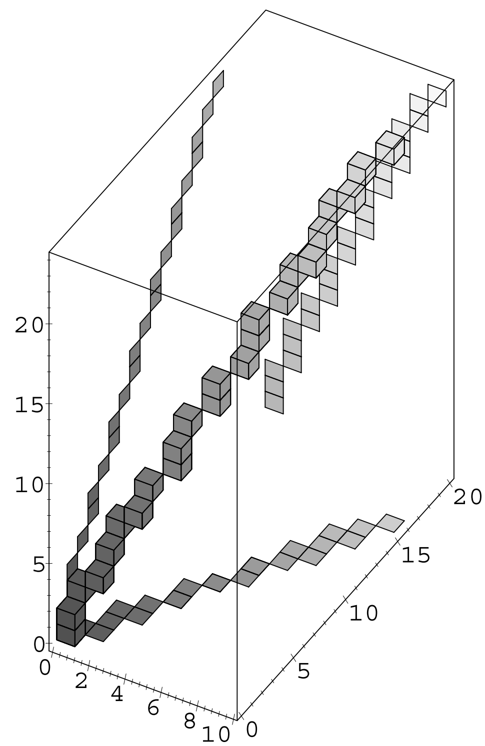

Figure 3: The digital line $\mathcal{D}(9, 15, 23)$.

4 THE TOPOLOGY OF $\mathcal{D}(a, b, c)$

4.1 Reading the topology of $\mathcal{D}(a, b, c)$ from $\hat{\mathcal{L}}$

From the previous definition of $\mathcal{D}(a, b, c)$ we can see that the only points of the fiber in the lattice $\hat{\mathcal{L}}$ that generate an x increment are those taken from the points in a vertical strip $[c - a[\times[0, c[$. Similarly the only ones that generate a y increment are taken from an horizontal strip $[0, c[\times[c - b, c[$ and the only points that generate both x and y increments at the same time are those in the common region $[c - a, c[\times[c - b, c[$. Thus the fundamental square $\hat{Par} = [0, c[\times[0, c[$ of lattice $\hat{\mathcal{K}} = L((c, 0), (0, c))$ can be divided into 4 zones that we will denote as (1), (2), (3) and (4). (See figure 4). This gives a partition of $\hat{\mathfrak{p}}$ wich governs the topology of the line $\mathcal{D}(a,b,c)$.

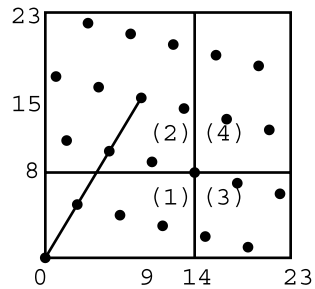

Figure 4: The four zones of interest in $\mathcal{D}(9,15,23)$.

We can see that along a naive 3D digital line of direction (a,b,c) as previously defined there are three types of adjacency only:

- strict *6-adjacency* when two voxels share one common face. 6-adjacency occurs along the z-axis only, *i.e.* shared faces are always parallel to the (xOy) coordinate plane.
- strict *18-adjacency* or *edge-adjacency* when two voxels share one common edge. This type of adjacency occurs along the y-axis, (when the shared edge is parallel to the x axis) or along the x-axis (when the shared edge is parallel to the y axis).
- strict *26-adjacency* or *corner-adjacency* when two voxels share one common vertex. This can occur for two of the eight vertices of a voxel only: the closest and the furthest from the origin.

We immediately deduce the following results from our description of 3D digital lines.

PROPOSITION 4.1. *The number of face-adjacencies in one period of a naive 3D digital line of direction* (a,b,c) *is equal to the number of points of* $\hat{\mathcal{L}}$ *contained in the rectangle* $[0, c-a[\times[0, c-b[$ *(zone (1)).*

PROPOSITION 4.2. *The number of edge-adjacencies along the y-axis in one period of a naive 3D digital line of direction* (a,b,c) *is equal to the number of points of* $\hat{\mathcal{L}}$ *contained in the rectangle* $[0, c[\times[c-b, c[$ *(zone (2)).*

PROPOSITION 4.3. *The number of edge-adjacencies along the x-axis in one period of a naive 3D digital line of direction* (a,b,c) *is equal to the number of points of* $\hat{\mathcal{L}}$ *contained in the rectangle* $[c-a, c[\times[0, c[$ *(zone (3)).*

PROPOSITION 4.4. *The number of corner-adjacencies in one period of a naive 3D digital line of direction* (a,b,c) *is equal to the number of points of* $\hat{\mathcal{L}}$ *contained in the rectangle* $[c-a, c[\times[c-b, c[$ *(zone (4)).*

THEOREM 4.1. *The three projections onto the main planes (xOy) (yOz) and (xOz) of the naive 3D digital line* $\mathcal{D}(a,b,c)$ *are naive 2D digital lines iff* $\hat{\mathfrak{p}} \cap [c-a, c[\times[0, c-b[= \emptyset$.

4.2 Combinatorially distinct 3D digital lines

Definition 3.2 shows that our approach contains the former classical digital lines, but also many others that we can build by an extension of the previous notion. This situation is similar to that of 2D lines.[5] Up to this point we have defined and built 3D digital lines from the points of $\hat{\mathcal{L}}$ contained in one of the fundamental domains of $\hat{\mathcal{K}} = L((c,0),(0,c))$. In fact we can also consider the collections of fibers whose feet are contained in other fundamental domains B of $\hat{\mathcal{K}}$. We can prove that each time $B \cap \hat{\mathcal{L}}$ is *8-connected* for the topology of the *minimal basis* of $\hat{\mathcal{L}}$ then $\pi^{-1}(B)$ is a valuable notion of a 3D digital line.

We can extend the idea even further and consider domains over $\hat{\mathcal{L}}$ other than fundamental domains of $\hat{\mathcal{K}}$. An especially interesting case consists of fundamental domains of lattices $\hat{\mathcal{T}}(s,t)$ generated from integer affine translations of $\hat{\mathcal{K}}$. As the fundamental domain $\hat{Par}$ of $\hat{\mathcal{K}}$ contains c^2 integer points, there exists c^2 such lattices $\hat{\mathcal{T}}(s,t) \quad (s,t) \in [0..c[\times[0..c[$. For each possible integer translation of vector (s,t) of $\hat{Par}$ it is possible to build a new 3D digital line leading to c^2 different digital lines of direction (a,b,c). Actually these c^2 different lines can be grouped into c classes of c digital lines having an equivalent structure within integer 3D translation. Thus given an integer direction (a,b,c) there exist c combinatorially distinct possible structures of the corresponding digital line. This is due to the fact that any fundamental domain of $\hat{\mathcal{L}}$ has an integer area of c and tiles $\hat{Par}$ into c subtiles.

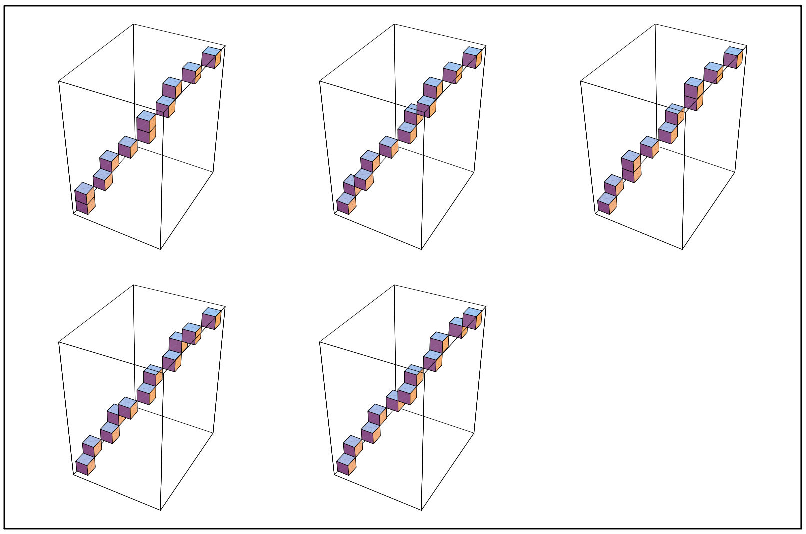

Figure 5: Combinatorially distinct 3D digital lines directed by (3,4,5)

Figure 5 shows the 5 combinatorially distinct possible structures of the digital line of direction (3,4,5). Two periods are represented in each example.

5 CONCLUSION

We have presented in this paper an original approach to the study of digital 3D lines which extends considerably the classical notion based on Bresenham 2D lines. We have shown that the projection of $\mathbb{Z}^3$ onto an euclidean plane of rational direction yields a planar lattice, of which the parameterization can be extremely simplified. Simple geometric properties of this lattice are in direct correspondence with the topological structure of the associated 3D digital lines, leading to an interesting definition of such discrete objects and thus reducing their study from dimension 3 to dimension 2. Such an analysis proves to be particularly powerful, yielding new interesting results such as the classification of lines of a given direction into classes of combinatorially distinct structure. It could also certainly lead to a solution to the problem of the closest digital connected set to an euclidean line, though this is still under study.

6 REFERENCES

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