Circular Ones Matrices and the Stable Set Polytope of Quasi-line Graphs

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Abstract. It is a long standing open problem to find an explicit description of the stable set polytope of *claw-free graphs*. Yet more than 20 years after the discovery of a polynomial algorithm for the maximum stable set problem for claw-free graphs, there is even no conjecture at hand today.

Such a conjecture exists for the class of *quasi-line graphs*. This class of graphs is a proper superclass of line graphs and a proper subclass of claw-free graphs for which it is known that not all facets have 0/1 normal vectors. *Ben Rebea's conjecture* states that the stable set polytope of a quasi-line graph is completely described by *clique-family* inequalities. Chudnovsky and Seymour recently provided a decomposition result for claw-free graphs and proved that Ben Rebea's conjecture holds, if the quasi-line graph is not a *fuzzy circular interval graph*.

In this paper, we give a proof of Ben Rebea's conjecture by showing that it also holds for fuzzy circular interval graphs. Our result builds upon an algorithm of Bartholdi, Orlin and Ratliff which is concerned with integer programs defined by circular ones matrices.

1 Introduction

A graph G is *claw-free* if no vertex has three pairwise nonadjacent vertices. Line graphs are claw free and thus the weighted stable set problem for a claw-free graph is a generalization of the weighted matching problem of a graph. While the general stable set problem is NP-complete, it can be solved in polynomial time on a claw-free graph [21, 29] even in the weighted case [22, 23] see also [32]. These algorithms are extensions of Edmonds' [10, 9] matching algorithms.

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The stable set polytope $STAB(G)$ is the convex hull of the characteristic vectors of stable sets of the graph G. The polynomial *equivalence of separation and optimization* for rational polyhedra [16, 26, 18] provides a polynomial time algorithm for the separation problem for $STAB(G)$, if G is claw-free. However, this algorithm is based on the ellipsoid method [19] and no explicit description of a set of inequalities is known that determines $STAB(G)$ in this case. This apparent asymmetry between the algorithmic and the polyhedral status of the stable set problem in claw-free graphs gives rise to the challenging problem of providing a "...decent linear description of $STAB(G)$" [17], which is still open today. In spite of results characterizing the rank-facets [12] (facets with 0/1 normal vectors) of claw-free graphs, or giving a compact lifted formulation for the subclass of distance claw-free graphs [27] , the structure of the general facets for claw-free graphs is still not well understood and even no conjecture is at hand.

The matching problem [9] is a well known example of a combinatorial optimization problem in which the optimization problem on the one hand and the facets on the other hand are well understood. This polytope can be described by a system of inequalities in which the coefficients on the left-hand-side are 0/1. This property of the matching polytope does *not* extend to the polytope $STAB(G)$ associated with a claw-free graph. In fact, Giles and Trotter [14] show that for each positive integer a, there exists a claw-free graph G such that $STAB(G)$ has facets with $a/(a+1)$ normal vectors. Furthermore they show that there exist facets whose normal vectors have up to 3 different coefficients (indeed up to 5 as it is shown in [20]). Perhaps this is one of the reasons why providing a description of $STAB(G)$ is not easy, since 0/1 normal vectors can be interpreted as subsets of the set of nodes, whereas such an interpretation is not immediate if the normal vectors are not 0/1.

A graph is *quasi-line*, if the neighborhood of any vertex partitions into two cliques. The complement of quasi-line graphs are called *near-bipartite*, and a linear description of their stable set polytope has been given in [33]. The class of quasi-line graphs is a proper superclass of line graphs and a proper subclass of the class of claw-free graphs. Interestingly also for this class of graphs there are facets with $a/(a+1)$ normal vectors, for any nonnegative integer a [14], but no facet whose normal vector has more than 2 different coefficients is known for this class. Ben Rebea [28] considered the problem to study $STAB(G)$ for quasi-line graphs. Oriolo [25] formulated a conjecture inspired from his work.

Ben Rebea's Conjecture

Let $\mathcal{F} = \{K_1, \ldots, K_n\}$ be a set of cliques, $1 \le p \le n$ be integral and $r = n \bmod p$. Let $V_{p-1}(\mathcal{F}) \subseteq V(G)$ the set of vertices covered by exactly $(p-1)$ cliques of $\mathcal{F}$ and $V_{\ge p}(\mathcal{F}) \subseteq V(G)$ the set of vertices covered by p or more cliques of $\mathcal{F}$. The inequality

$$(p-r-1) \sum_{v \in V_{p-1}(\mathcal{F})} x(v) + (p-r) \sum_{v \in V_{\ge p}(\mathcal{F})} x(v) \le (p-r) \left\lfloor \frac{n}{p} \right\rfloor \tag{1}$$

is valid for $STAB(G)$ and is called the *clique family inequality* associated with $\mathcal{F}$ and p.

Conjecture 1 (Ben Rebea's conjecture [25]). The stable set polytope of a quasi-line graph $G = (V, E)$ may be described by the following inequalities:

(i) $x(v) \geq 0$ for each $v \in V$
(ii) $\sum_{v \in K} x(v) \leq 1$ for each maximal clique K
(iii) inequalities (1) for each family $\mathcal{F}$ of maximal cliques and each integer p with $|\mathcal{F}| > 2p \geq 4$ and $|\mathcal{F}| \mod p \neq 0$.

In this paper we prove that Ben Rebea's Conjecture holds true. This is done by establishing the conjecture for *fuzzy circular interval graphs*, a class introduced by Chudnovsky and Seymour [6]. This settles the result, since Chudnovsky and Seymour showed that the conjecture holds if G is quasi-line and not a fuzzy circular interval graph. Interestingly, since all the facets are rank for this latter class of graphs, the quasi-line graphs that "produce" non-rank facets are the fuzzy circular interval graphs.

We first show that we can focus our attention on *circular interval graphs* [6] a subclass of fuzzy circular interval graphs. The weighted stable set problem over a circular interval graph may be formulated as a packing problem $\max\{c\,x \mid Ax \leq b,\ x \in \mathbb{Z}^n_{\geq 0}\}$, where $b = \mathbf{1}$ and $A \in \{0,1\}^{m \times n}$ is a *circular ones matrix*, i.e., the columns of A can be permuted in such a way that the ones in each row appear consecutively. Here the last and first entry of a row are also considered to be consecutive. Integer programs of this sort with general right-hand side $b \in \mathbb{Z}^m$ have been studied by Bartholdi, Orlin and Ratliff [3]. From this, we derive a separation algorithm which is based on the computation of a negative cycle, thereby extending a recent result of Gijswijt [13]. We then concentrate on packing problems with right-hand side $b = \alpha\,\mathbf{1}$, where α is an integer. By studying non-redundant cycles leading to separating hyperplanes, we show that each facet of the convex hull of integer feasible solutions to a packing problem of this sort has a normal vector with two consecutive coefficients. Instantiating this result with the case where $\alpha = 1$, we obtain our main result.

Cutting Planes

Before we proceed, we would like to stress some connections of this work to cutting plane theory. An inequality $c\,x \leq \lfloor \delta \rfloor$ is a *Gomory-Chvátal cutting plane* [15, 7] of a polyhedron $P \subseteq \mathbb{R}^n$, if $c \in \mathbb{Z}^n$ is an integral vector and $c\,x \leq \delta$ is valid for P. The *Chvátal closure* P^c of P is the intersection of P with all its Gomory-Chvátal cutting planes. If P is rational, then P^c is a rational polyhedron [30]. The separation problem for P^c is NP-hard [11]. A polytope P has *Chvátal-rank* one, if its Chvátal closure is the integer hull P_I of P. Let $QSTAB(G)$ be the *fractional stable set polytope* of a graph G, i.e., the polytope defined by non-negativity and clique inequalities. It is known [25] that $QSTAB(G)$ does not have Chvátal rank one, if G is a quasi-line graph. A famous example of a polytope of Chvátal-rank one is the fractional matching polytope and thus $QSTAB(G)$, where G is a line graph.

An inequality $cx \leq \delta$ is called a *split cut* [8] of P if there exists an integer vector $\pi \in \mathbb{Z}^n$ and an integer π_0 such that $cx \leq \delta$ is valid for $P \cap \{x \in \mathbb{R}^n \mid \pi x \leq \pi_0\}$ and for $P \cap \{x \in \mathbb{R}^n \mid \pi x \geq \pi_0 + 1\}$. The *split closure* P^s of P is the intersection of P with all its split cuts and this is a rational polyhedron if P itself is rational [8, 2]. The separation problem for the split closure is also NP-hard [4]. A polyhedron $P \subseteq \mathbb{R}^n$ has *split-rank* one, if $P^s = P_I$.

Both cutting plane calculi are simple procedures to derive valid inequalities for the integer hull of a polyhedron. It is easy to see that a clique family inequality is a split cut for $QSTAB(G)$ with $\pi(v) = 1$ if $v \in V_{p-1} \cup V_{\geq p}$, $\pi(v) = 0$ otherwise and $\pi_0 = \lfloor \frac{n}{p} \rfloor$. Thus, while the fractional stable set polytope of a quasi-line graph does not have Chvátal rank one, its split-rank is indeed one.

2 From Circular Interval to Quasi-Line Graphs

A circular interval graph [6] $G = (V, E)$ is defined by the following construction: Take a circle $\mathcal{C}$ and a set of vertices V on the circle. Take a subset of intervals $\mathcal{I}$ of $\mathcal{C}$ and say that $u, v \in V$ are adjacent if $\{u, v\}$ is a subset of one of the intervals.

Any interval used in the construction will correspond to a clique of G. Denote the family of cliques stemming from intervals by $\mathcal{K}_\mathcal{I}$ and the set of all cliques in G by $K(G)$. Without loss of generality, the (intervals) cliques of $\mathcal{K}_\mathcal{I}$ are such that none includes another. Moreover $\mathcal{K}_\mathcal{I} \subseteq K(G)$ and each edge of G is contained in a clique of $\mathcal{K}_\mathcal{I}$. Therefore, if we let $A \in \{0,1\}^{m \times n}$ be the clique vertex incidence matrix of $\mathcal{K}_\mathcal{I}$ and V one can formulate the the (weighted) stable set problem on a circular interval graph as a packing problem

$$\begin{aligned} \max \textstyle\sum_{v \in V} c(v)x(v) & \\ Ax &\leq \mathbf{1} \\ x(v) &\in \{0,1\} \; \forall v \in V \end{aligned}$$

where the matrix A is a circular ones matrix (e.g. using clockwise ordering of the vertices).

Chudnovsky and Seymour [6] also introduced the more general class of *fuzzy circular interval graphs*. A graph G is a fuzzy circular interval if the following conditions hold.

(i) There is a map Φ from V to a circle $\mathcal{C}$.
(ii) There is a set of intervals $\mathcal{I}$ of $\mathcal{C}$, none including another, such that no point of C is the end of more than one interval so that:
   (a) If two vertices u and v are adjacent, then $\Phi(u)$ and $\Phi(v)$ belong to a common interval.
   (b) If two vertices u and v belong to a same interval, which is not an interval with endpoints $\Phi(u)$ and $\Phi(v)$, then they are adjacent.

In other words, in a fuzzy circular interval graph, adjacencies are completely described by the pair $(\Phi, \mathcal{I})$, except for vertices u and v such that one of the intervals with endpoints $\Phi(u)$ and $\Phi(v)$ belongs to $\mathcal{I}$. For these vertices adjacency is fuzzy. We are particularly interested in *non-empty* cliques arising from endpoints of intervals of $\mathcal{I}$. If $[p, q]$ is an interval of $\mathcal{I}$ such that $\Phi^{-1}(p)$ and $\Phi^{-1}(q)$ are both non-empty, then we call the cliques $(\Phi^{-1}(p), \Phi^{-1}(q))$ a *fuzzy pair.*

Trivially, a circular interval graph is a fuzzy circular interval graph. When is a fuzzy circular interval graph a circular interval graph? The following lemma addresses this question. Say that a graph is C_4*-free* if it does not have an induced subgraph isomorphic to a cordless cycle of length 4. For $X \subseteq V$, we denote by $G[X]$ the subgraph of G *induced by* X.

Lemma 1 (Chudnovsky and Seymour [5]). *Let G be a fuzzy circular interval graph. If for every fuzzy pair of cliques (K_i, K_j), the subgraph $G[K_i \cup K_j]$ is C_4-free, then G is a circular interval graph.*

We sketch a proof of the above lemma. Trivially, if a fuzzy circular interval graph admits a *fuzzy representation* with no fuzzy pairs of cliques, then G is a circular interval graph. Now let $(\Phi, \mathcal{I})$ be a fuzzy representation of G minimizing the number of vertices belonging to fuzzy pairs. Let (K_1, K_2) be a fuzzy pair of cliques with respect to $(\Phi, \mathcal{I})$. Every vertex $v \in K_1$ has a neighbor and a non-neighbor in K_2. Otherwise one could remove v from the fuzzy pair, contradicting the minimality of $(\Phi, \mathcal{I})$. Now the following statement holds true: if $G = (V, E)$ is any graph with $V = V_1 \cup V_2$, V_1 and V_2 cliques and such that every vertex of V_1 (V_2) has a neighbor and a non-neighbor in V_2 (V_1), then there exist an induced $C_4 = \{u_1, u_2, v_2, v_1\}$ with $u_1, u_2 \in V_1$ and $v_1, v_2 \in V_2$.

Theorem 1. *Let F be a facet of $STAB(G)$, where G is a fuzzy circular interval graph. Then F is also a facet of $STAB(G')$, where G' is a circular interval graph obtained from G by removing some edges.*

Proof. Suppose that F is induced by the valid inequality $a\,x \leq \beta$. Trivially, if we remove an edge (u, v) connecting two vertices $u \in K_i$ to $v \in K_j$ of a fuzzy pair of cliques (K_i, K_j), the graph $G \setminus (u, v)$ is still a fuzzy circular interval graph. An edge e is F-critical, if $a\,x \leq \beta$ is not valid for $STAB(G \setminus e)$. If e is not F-critical, then F is also a facet of $STAB(G \setminus e)$. We prove that the removal of *all* non F-critical edges connecting two vertices in different cliques of fuzzy pairs results in a circular interval graph G'. Therefore, since F is a facet of $STAB(G')$, the claim follows.

Suppose G' is not a circular interval graph. Then from Lemma 1, there exists a fuzzy pair of cliques (K_1, K_2) such that the subgraph $G'[K_1 \cup K_2]$ contains a C_4. Say $V(C_4) = \{u_1, u_2, v_1, v_2\}$ with $u_1, u_2 \in K_1$, $v_1, v_2 \in K_2$, $(u_1, v_1), (u_2, v_2) \in E(C_4)$. The edge (u_1, v_1) is F-critical. Hence there exists a set S containing u_1, v_1 such that S violates $a\,x \leq \beta$ and S is stable in $G' \setminus (u_1, v_1)$. Property (ii) above implies that K_1 has no other fuzzy pair than K_2 and thus u_1 and u_2 are adjacent to the same vertices in $G' \setminus K_2$. This implies that $(S \setminus u_1) \cup u_2$ is a stable set. Therefore $a(u_2) < a(u_1)$ (else (u_1, v_1) is not F-critical). Applying the same argument to (u_2, v_2) leads to $a(u_1) < a(u_2)$. Which is a contradiction. □

Fuzzy circular interval graphs are quasi-line graphs. Chudnovsky and Seymour [6] gave a complete characterization of the stable set polytope of a quasi-line graph for the case in which the graph is *not* a fuzzy circular interval graph.

Let $\mathcal{F} = \{K_1, K_2, ..., K_{2n+1}\}$ an odd set of cliques of G. Let $T \subseteq V$ be the set of vertices which are covered by at least two cliques of $\mathcal{F}$. Then the inequality $\sum_{v \in T} x(v) \leq n$ is a valid inequality for $STAB(G)$ and inequalities of this type are called Edmonds' inequalities.

Theorem 2 ([6]). *Let G be a connected quasi-line graph, which is not a fuzzy circular interval graph. Then all non trivial facets of $STAB(G)$ are Edmonds' inequalities.*

Observe that Edmonds' inequalities are special clique family inequalities associated with $\mathcal{F}$ and $p = 2$. Moreover, Theorem 1 implies that, if we prove that all the non-trivial facets of the stable set polytope of a *circular interval graph* are clique family inequalities, then the same holds for the stable set polytope of a *fuzzy* one (cliques of G' are also cliques of G). Therefore, if we combine these facts, we may give a positive answer to the Ben Rebea's Conjecture if we prove that it holds for circular interval graphs. This is what we are going to show in the following sections.

3 Slicing and Separation

Let P be a polytope $P = \{x \in \mathbb{R}^n \mid Ax \leq b,\, x \geq 0\}$, where $A \in \{0,1\}^{m \times n}$ is a circular ones matrix and $b \in \mathbb{Z}^m$ an integral vector. In this section, we consider the separation problem for the integer hull P_I of P:

> Given $x^* \in \mathbb{R}^n$, determine, whether $x^* \in P_I$ and if not, determine an inequality $c\,x \leq \delta$ which is valid for P_I and satisfies $c\,x^* > \delta$.

We present a membership algorithm of Gijswijt [13] and develop it further to retrieve a separating hyperplane. Following Bartholdi, Orlin and Ratliff [3], we consider the unimodular transformation $x = T\,y$, where T is the unimodular matrix

$$T = \begin{pmatrix} 1 & & & & & \\ -1 & 1 & & & & \\ & -1 & 1 & & & \\ & & -1 & & & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & -1 & 1 \end{pmatrix} \tag{2}$$

The problem then reads, separate $y^* = T^{-1}x^*$ from the integer hull Q_I of the polytope Q defined by the system

$$\begin{pmatrix} A \\ -I \end{pmatrix} T\,y \leq \begin{pmatrix} b \\ 0 \end{pmatrix}. \tag{3}$$

In the following we denote the inequality system (3) by $B\,y \leq d$. Let us rewrite the matrix B as $B = (N|v)$, i.e. v is the n-th column of B. Observe that, by construction, v is also the last column of $\begin{pmatrix} A \\ -I \end{pmatrix}$.

Each row of the matrix N has at most one entry which is $+1$ and at most one entry which is -1. All other entries are 0. The matrix N is thus totally unimodular. Thus, whenever $y(n)$ is set to an integer $\beta \in \mathbb{Z}$, the possible values for the variables $y(1), \ldots, y(n-1)$ define an integral polytope $Q_\beta = \{y \in \mathbb{R}^n \mid B\,y \leq d,\ y(n) = \beta\}$. We call this polytope Q_β the *slice* of Q defined by β.

Since T is unimodular, the corresponding slice of the original polyhedron $P \cap \{x \in \mathbb{R}^n \mid \sum_{i=1}^n x(i) = \beta\}$ is an integral polyhedron. From this it is already easy to see that the split-rank of P is one. However, we present a combinatorial separation procedure for the integer hull P_I of P which computes a split cut via the computation of a negative cycle.

If $y^*(n)$ is integral, then y^* lies in Q_I if and only if $y^* \in Q_{y^*(n)}$. Therefore we assume in the following that $y^*(n)$ is not integral and let β be an integer such that $\beta < y^*(n) < \beta + 1$ and let $1 > \mu > 0$ be the real number with $y^*(n) = \beta + 1 - \mu$. Furthermore, let Q_L and Q_R be the left slice Q_β and right slice $Q_{\beta+1}$ respectively. A proof of the next lemma follows from basic convexity.

Lemma 2. *The point y^* lies in Q_I if and only if there exist $y_L \in Q_L$ and $y_R \in Q_R$ such that*

$$y^* = \mu\, y_L + (1-\mu) y_R.$$

In the following we denote by $\overline{y} \in \mathbb{R}^{n-1}$ the vector of the first $n-1$ components of $y \in \mathbb{R}^n$. From the above discussion one has $y^* \in Q_I$ if and only if the following linear constraints have a feasible solution.

$$\begin{aligned} \overline{y^*} &= \overline{y_L} + \overline{y_R} \\ N\overline{y_L} &\leq \mu\, d_L \\ N\overline{y_R} &\leq (1-\mu)\, d_R \end{aligned}\ , \tag{4}$$

where $d_L = d - \beta v$ and $d_R = d - (\beta+1)v$.

Using Farkas' Lemma [31], it follows that equation (4) defines a feasible system, if and only if $\sum_{i=1}^{n-1} \lambda(i) y^*(i) + \mu f_L d_L + (1-\mu) f_R d_R$ is nonnegative, whenever λ, f_L and f_R satisfy

$$\begin{aligned} \lambda + f_L N &= 0 \\ \lambda + f_R N &= 0 \\ f_L, f_R &\geq 0. \end{aligned} \tag{5}$$

Now $\lambda + f_L N = 0$ and $\lambda + f_R N = 0$ is equivalent to $\lambda = -f_L N$ and $f_L N = f_R N$. Thus (4) defines a feasible system, if and only if the optimum value of the following linear program is nonnegative

$$\begin{aligned} &\min -f_L N\, \overline{y^*} + \mu f_L d_L + (1-\mu) f_R d_R \\ &\qquad f_L N = f_R N \\ &\qquad f_L, f_R \geq 0. \end{aligned} \tag{6}$$

Let w be the negative sum of the columns of N. Then (6) is the problem of finding a minimum cost circulation in the directed graph $D = (U, \mathcal{A})$ defined by the edge-node incidence matrix

$$M = \begin{pmatrix} N & w \\ -N & -w \end{pmatrix} \text{ and edge weights } \mu(-N\,\overline{y^*} + d_L), (1-\mu)(-N\,\overline{y^*} + d_R) \tag{7}$$

Thus $y^* \notin Q_I$ if and only if there exists a negative cycle in $D = (U, \mathcal{A})$. The membership problem for Q_I thus reduces to the problem of detecting a negative cycle in D, see [13].

A separating split cut for y^* is an inequality which is valid for Q_L and Q_R but not valid for y^*. The inequality $f_L N\overline{y} \leq f_L d_L$ is valid for Q_L and the inequality $f_R N\overline{y} \leq f_R d_R$ is valid for Q_R. The corresponding disjunctive inequality (see, e.g., [24]) is the inequality

$$f_L N\overline{y} + c(n)y(n) \leq \delta, \text{ where } c(n) = f_L d_L - f_R d_R \text{ and } \delta = (\beta+1) f_L d_L - \beta f_R d_R. \tag{8}$$

The polytopes Q_L and Q_R are defined by the systems

$$\begin{aligned} y(n) &= \beta \\ N\,\overline{y} + v\,y(n) &\leq d \end{aligned} \qquad \text{and} \qquad \begin{aligned} y(n) &= \beta + 1 \\ N\,\overline{y} + v\,y(n) &\leq d \end{aligned} \tag{9}$$

respectively.

Let $f_{L,0}$ be the number $c(n) - f_L v$. Then the inequality (8) can be derived from the system defining Q_L with the weights $(f_{L,0}, f_L)$. Notice that, if y^* can be separated from Q_I, then $f_{L,0}$ must be positive. This is because y^* violates (8) and satisfies the constraints (9) on the left, where the equality $y(n) = \beta$ in the first line is replaced with $y(n) \geq \beta$. Let $f_{R,0}$ be the number $c(n) - f_R v$. Then the inequality (8) can be derived from the system defining Q_R with the weights $(f_{R,0}, f_R)$. Notice that, if y^* can be separated from Q_I, then $f_{R,0}$ must be negative.

A negative cycle in a graph with m edges and n nodes can be found in time $O(m\,n)$, see, e.g. [1]. Translated back to the original space and to the polyhedron P this gives the following theorem.

Theorem 3. *The separation problem for P_I can be solved in time $O(m\,n)$. Moreover, if $x^* \in P$ and $x^* \notin P_I$ one can compute in $O(m\,n)$ a split cut $c\,x \leq \delta$ which is valid for P_I and separates x^* from P_I together with a negative integer $f_{R,0}$, a positive integer $f_{L,0}$ and a vector f_L, f_R, which is the incidence vector of a simple negative cycle of the directed graph $D = (U, \mathcal{A})$ with edge-node incidence matrix and weights as in (7), such that $c\,x \leq \delta$ is derived with from the systems*

$$\begin{aligned} \mathbf{1}x &\leq \beta \\ Ax &\leq b \\ -x &\leq 0. \end{aligned} \qquad \textit{and} \qquad \begin{aligned} -\mathbf{1}x &\leq -(\beta+1) \\ Ax &\leq b \\ -x &\leq 0, \end{aligned} \tag{10}$$

with the weights $f_{L,0}, f_L$ and $|f_{R,0}|, f_R$ respectively.

The above theorem gives an explicit derivation of the separating hyperplane as a split cut of P. We have the following corollary.

Corollary 1. *The integer hull P_I is the split closure of P.*

4 The Facets of $\mathbf{P_I}$ for the Case $\mathbf{b} = \boldsymbol{\alpha} \cdot \mathbf{1}$

In this section we study the facets of P_I, where $P = \{x \in \mathbb{R}^n \mid Ax \leq b,\, x \geq 0\}$, where A is a circular ones matrix and b is an integer vector of the form $\alpha\,\mathbf{1}$, $\alpha \in \mathbb{N}$. For this, we actually inspect how the facets of the transformed polytope Q described in Section 3 are derived from the systems (9) and apply this derivation to the original system. It will turn out that the facet normal-vectors of P_I have only two integer coefficients, which are in addition consecutive. Since the stable set polytope of a circular interval graph is defined by such a system with $\alpha = 1$, we can later instantiate the results of this section to this special case. We can assume that the rows of A are inclusion-wise maximal.

Let F be a facet of Q_I and let y^* be in the relative interior of F. This facet F is generated by the unique inequality (8), which corresponds to a simple cycle of (6) of weight 0. Furthermore assume that F is not induced by an inequality $y(n) \leq \gamma$ for some $\gamma \in \mathbb{Z}$. Since F is a facet of the convex hull of integer points of two consecutive slices, we can assume that $y^*(n) = \beta + 1/2$ and thus that $\mu = 1/2$ in (6). This allows us to rewrite the objective function of problem (6) as follows:

$$\min(s^* + \tfrac{1}{2}v)\, f_L + (s^* - \tfrac{1}{2}v)\, f_R \tag{11}$$

where s^* is the slack vector

$$s^* = \begin{pmatrix} \alpha\,\mathbf{1} \\ \mathbf{0} \end{pmatrix} - By^* = \begin{pmatrix} \alpha\,\mathbf{1} \\ \mathbf{0} \end{pmatrix} - \begin{pmatrix} A \\ -I \end{pmatrix} x^* \geq \mathbf{0}. \tag{12}$$

The point x^* in (12) is $x^* = T\,y^*$. Notice that x^* satisfies the system $A\,x \leq \alpha\,\mathbf{1}$.

Furthermore, we are interested in the facets of Q_I which are not represented by the system $B\,y \leq d$. If F is such a facet, then one can translate y^* away from Q_I, without changing $y^*(n) = \beta + 1/2$, such that $y^* \notin Q_I$ and $By^* \leq d$ with the property that the facet we are considering is the unique inequality (8), where f_L, f_R is a simple negative cycle in the graph $D = (U, \mathcal{A})$.

In the following we denote $U = \{1, \ldots, n\}$, where node i corresponds to the i-th column of the matrix M in (7). Notice that $\mathcal{A}$ partitions in two classes of arcs $\mathcal{A}_L$ and $\mathcal{A}_R$. The arcs $\mathcal{A}_R$ are simply the reverse of the arcs $\mathcal{A}_L$. $\mathcal{A}_L$ consists of two sets of arcs $\mathcal{S}_L$ and $\mathcal{T}_L$, where $\mathcal{S}_L$ is the set of arcs associated with inequalities $Ax \leq \alpha\,\mathbf{1}$ and $\mathcal{T}_L$ are the arcs stemming from the lower bounds $x \geq 0$. Likewise $\mathcal{A}_R$ can be partitioned into $\mathcal{S}_R$ and $\mathcal{T}_R$. In other words, if we look at the arc-node incidence matrix M in (7), the rows of M appear in the order $\mathcal{S}_L, \mathcal{T}_L, \mathcal{S}_R, \mathcal{T}_R$.

In particular, let a denote a row vector of A. Since A is a circular ones matrix one has $a\,x \leq \alpha \equiv \sum_{h=0}^{p} x(i+h) \leq \alpha$ for some suitable i and p, where computation is modulo n, so $x_n \equiv x_0$, $x_{n+1} \equiv x_1$, etc. It is straightforward to see that $a\,x \leq \alpha$ generates the arcs $(i+p, i-1) \in \mathcal{S}_L$ and $(i-1, i+p) \in \mathcal{S}_R$ of A, see Figure 1. The weights of the two arcs coincide, if $n \notin \{i, i+1, \ldots, i+p\}$ and is exactly the slack $\alpha - \sum_{h=0}^{p} x^*(i+h)$ in this case. Otherwise, the weight of the arc $(i+p, i-1)$ is $\alpha - \sum_{h=0}^{p} x^*(i+h) + 1/2$ and the weight of the arc $(i-1, i+p)$ is $\alpha - \sum_{h=0}^{p} x^*(i+h) - 1/2$.

On the other hand, a lower bound $-x_i \leq 0$ generates the two arcs $(i-1,i) \in \mathcal{T}_L$ and $(i,i-1) \in \mathcal{T}_R$. The weight of both arcs is equal to $x^*(i)$, if $i \neq n$. If $i = n$, the arc $(n-1,n) \in \mathcal{T}_L$ has weight $x^*(n) - 1/2$ and $(n,n-1) \in \mathcal{T}_R$ has weight $x^*(n) + 1/2$.

Since the slacks are non-negative, the arcs whose cost is equal to the corresponding slack minus $\frac{1}{2}$ are the only candidates to have a negative cost. We call those *light* arcs. Consequently we call those arcs whose cost is equal to the slack plus $\frac{1}{2}$ *heavy*. Observe that the light arcs belong to $\mathcal{S}_R \cup \{(n-1,n)\}$.

Lemma 3. *Let $\mathcal{C}$ be a simple negative cycle in D, then the following holds:*

(a) $\mathcal{C}$ contains strictly more light arcs than heavy ones.
(b) An arc of $\mathcal{C}$ in $\mathcal{S}_L$ ($\mathcal{T}_L$) cannot be immediately followed or preceded by an arc in $\mathcal{S}_R$ ($\mathcal{T}_R$).
(c) The cycle $\mathcal{C}$ contains at least one arc of $\mathcal{S}_R$ or contains no arc of $\mathcal{S}_L \cup \mathcal{S}_R$.

Proof. (a) follows from the fact that the slacks are nonnegative. (b) follows from our assumption that the rows of the matrix A are maximal and that $\mathcal{C}$ is simple.

To prove (c) suppose that the contrary holds. It follows that $(n-1,n)$ is in $\mathcal{C}$, because it is the only light arc not in $\mathcal{S}_R$. We must reach $n-1$ on the cycle without using heavy arcs.

Each arc in $\mathcal{S}_L$ with starting node n is heavy. Thus $(n-1,n)$ is followed by $(n,1) \in \mathcal{T}_L$. Suppose that $(n-1,n)$ is followed by a sequence of arcs in $\mathcal{T}_L$ leading to i and let $(i,j) \notin \mathcal{T}_L$ be the arc which follows this sequence. It follows from (b) that $(i,j) \notin \mathcal{T}_R$ and thus that $(i,j) \in \mathcal{S}_L$. Since (i,j) cannot be heavy, we have $1 \leq j < i < n$. This is a contradiction to the fact that $\mathcal{C}$ is simple, since we have a sub-cycle contained in $\mathcal{C}$, defined by (i,j) and $(j,j+1),\dots,(i-1,i)$. □

Lemma 4. *If there exists a simple cycle $\mathcal{C}$ of D with negative cost, then there exists a simple cycle $\mathcal{C}'$ of D with negative cost that does not contain any arc from $\mathcal{S}_L$.*

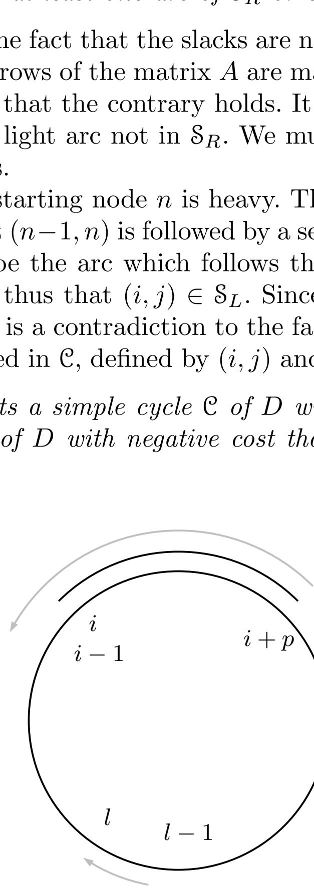

Fig. 1. The incidence vector of a row of A consists of the nodes$\{i,i+1,\dots,i+p\}$ which are consecutive on the cycle in clockwise order. Its corresponding arc in $\mathcal{S}_L$ is the arc $(i+p,i-1)$. The arc $(l-1,l)$ in $\mathcal{T}_L$ corresponds to the lower bound $x(l) \geq 0$

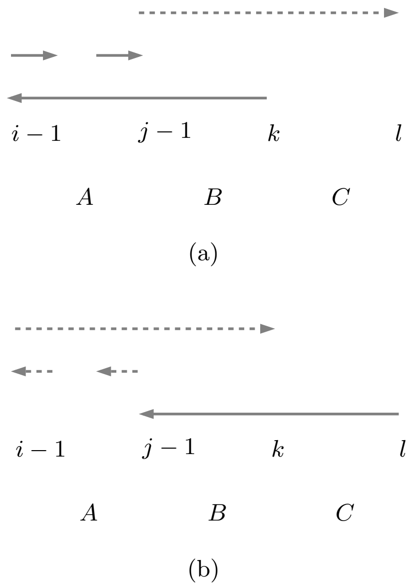

Fig. 2. (a) depicts an arc $(k, i-1) \in \mathcal{S}_L$, followed by arcs in $\mathcal{T}_L$ and the arc $(j-1, l) \in \mathcal{S}_R$. (b) depicts the situation, where the intermediate arcs are in $\mathcal{T}_R$

Proof. Suppose that $\mathcal{C}$ also contains an arc from the set $\mathcal{S}_L$. We know from Lemma 3 that the cycle $\mathcal{C}$ contains at least one arc of $\mathcal{S}_R$. Lemma 3 implies that $\mathcal{C}$ has an arc in $\mathcal{S}_L$, followed by arcs in $\mathcal{T}_L$ or $\mathcal{T}_R$ but not both, followed by an arc in $\mathcal{S}_R$. We first consider the case that the intermediate arcs are all in $\mathcal{T}_L$.

This situation is depicted in Figure 2, (a). The arc in $\mathcal{S}_L$ is $(k, i-1)$. This is followed by the arcs $(i-1, i), \dots, (i-1, j-1)$ in $\mathcal{T}_L$ and the arc $(j-1, l)$ in $\mathcal{S}_R$. Let this be the path $\mathcal{P}_1$. We now show that we can replace this path with the path $\mathcal{P}_2 = (k, k+1), \dots, (l-1, l)$ consisting of arcs in $\mathcal{T}_L$. We proceed as follows. First we show that the weight of this path is at most the weight of the original path, where we ignore the addition of $\pm 1/2$ to the arc-weights. Let light$(\mathcal{P})$ and heavy$(\mathcal{P})$ be the number of light and heavy edges in a path $\mathcal{P}$, respectively. We then show that light$(\mathcal{P}_2)$ – heavy$(\mathcal{P}_2)$ = light$(\mathcal{P}_1)$ – heavy$(\mathcal{P}_1)$, from which we can conclude the claim in this case.

Consider the set of indices $\mathcal{A} = \{i, \dots, j-1\}$, $\mathcal{B} = \{j, \dots, k\}$ and $\mathcal{C} = \{k+1, \dots, l\}$ and the numbers $A = \sum_{\mu \in \mathcal{A}} x^*(\mu)$, $B = \sum_{\mu \in \mathcal{B}} x^*(\mu)$ and $C = \sum_{\mu \in \mathcal{C}} x^*(\mu)$. Ignoring the eventual addition of $\pm 1/2$ to the edge weights, we have that the weight of $\mathcal{P}_2$ is C and that of $\mathcal{P}_1$ is $\alpha - (A+B) + A + \alpha - (B+C)$ and suppose that this is less than C. Then $B + C > \alpha$ which is not possible, since x^* satisfies the constraints $Ax \le \alpha \mathbf{1}$. Thus, if none of the edges in $\mathcal{P}_1$ and $\mathcal{P}_2$ is heavy or light, the weight of $\mathcal{P}_2$ is at most the weight of $\mathcal{P}_1$.

Suppose now that $n \in \mathcal{A}$. Then $\mathcal{P}_1$ contains exactly one heavy edge $(k, i-1)$ and one light edge $(n-1, n)$. The path $\mathcal{P}_2$ contains no heavy or light edge. Suppose that $n \in \mathcal{B}$, then $\mathcal{P}_1$ contains exactly one heavy edge, $(k, i-1)$ and one light edge $(j-1,l)$. $\mathcal{P}_2$ does not contain a heavy or light edge. If $n \in \mathcal{C}$, then $\mathcal{P}_1$ contains exactly one light edge $(j-1,l)$ and no heavy edge. $\mathcal{P}_2$ also contains exactly one light edge $(n-1,n)$. This concludes the claim for the case that an arc of $\mathcal{S}_L$ is followed by arcs of $\mathcal{T}_L$ and an arc of $\mathcal{S}_R$.

The case, where the intermediate arcs belong to $\mathcal{T}_R$ is depicted in Figure 2, (b). The assertion follows by a similar argument. □

Combining Theorem 3 with the above lemma we obtain the following theorem.

Theorem 4. *Let $P = \{x \in \mathbb{R}^n \mid A\,x \leq \alpha\mathbf{1}, x \geq 0\}$ be a polyhedron, where $A \in \{0,1\}^{m\times n}$ is a circular ones matrix and $\alpha \in \mathbb{N}$ a positive integer. A facet of P_I is of the form*

$$a \sum_{v\in T} x(v) + (a-1) \sum_{v\notin T} x(v) \leq a\,\beta, \tag{13}$$

where $T \subseteq \{1,\dots,n\}$ and $a, \beta \in \mathbb{N}$.

Proof. Theorem 3 implies that a facet which is not induced by $Ax \leq \alpha\,\mathbf{1}$, $x \geq 0$ or $\mathbf{1}\,x \leq \gamma$ is a nonnegative integer combination of the system on the left in (10) with nonnegative weights $f_{L,0}, f_L$. Lemma 4 implies that f_L can be chosen such that the only nonzero $(+1)$ entries of f_L are corresponding to lower bounds $-x(v) \leq 0$. The theorem thus follows with $a = f_{0,L}$ and T set to those variables, whose lower bound inequality does not appear in the derivation. □

5 The Solution to Ben Rebea's Conjecture

Let G be a circular interval graph and let $\mathcal{K}_\mathcal{I}$ the family of cliques stemming from the intervals in the definition of G (see Section 2). Then $QSTAB(G) = \{x \in R^n \mid Ax \leq \mathbf{1}, x \geq 0\}$ where the 0/1 matrix A, corresponding to the cliques $\mathcal{K}_\mathcal{I}$, has the circular ones property. Theorem 4 implies that any facet of $STAB(G)$ is of the form

$$a \sum_{v\in T} x(v) + (a-1) \sum_{v\notin T} x(v) \leq a\cdot\beta \tag{14}$$

We now show that a facet, which is not induced by an inequality of $Ax \leq \mathbf{1}, x \geq 0$ is induced by a clique family inequality associated with some set of cliques $\mathcal{F} \subseteq \mathcal{K}_I$ and some integer p. Recall from Theorem 3 that any facet of this kind can be derived from the system

$$\begin{aligned} -\mathbf{1}x &\leq -(\beta+1)\\ Ax &\leq \mathbf{1}\\ -x &\leq 0, \end{aligned} \tag{15}$$

with weights $|f_{R,0}|, f_R$, where $f_{R,0}$ is a negative integer while f_R is a 0-1 vector. A *root* of F is a stable set, whose characteristic vector belongs to F. In particular, we have that the multiplier $f_R(v)$ associated with a lower bound $-x(v) \leq 0$ must be 0 if v belongs to a root of size $\beta + 1$. If v does not belong to a root of size β or to a root of size $\beta + 1$, then the facet is induced by $x(v) \geq 0$. Thus if $v \notin T$, then v belongs to a root of size $\beta + 1$.

Let $\mathcal{F} = \{K \in \mathcal{K}_I \mid f_R(K) \neq 0\}$ and $p = a + |f_{R,0}|$. The multiplier $|f_{R,0}|$ must satisfy

$$
\begin{array}{ll}
-|f_{R,0}| + |\{K \in \mathcal{F} \mid v \in K\}| = a - 1 & \forall v \notin T \\
-|f_{R,0}| + |\{K \in \mathcal{F} \mid v \in K\}| = a & \forall v \in T,\ v \text{ is in a root of size } \beta + 1 \\
-|f_{R,0}| + |\{K \in \mathcal{F} \mid v \in K\}| \geq a & \forall v \in T,\ v \text{ is not in a root of size } \beta + 1 \\
\quad -|f_{R,0}|(\beta + 1) + |\mathcal{F}| = a\beta &
\end{array}
$$

Observe that $|\mathcal{F}| = (a + |f_{R,0}|)\beta + |f_{R,0}|$ and therefore $r = |\mathcal{F}| \mod p = |f_{R,0}|$. Moreover, any vertex not in T belongs to exactly $p - 1$ cliques from $\mathcal{F}$, while each vertex in T belongs to at least p cliques from $\mathcal{F}$. Therefore, inequality (14) is the clique family inequality associated with $\mathcal{F}$ and p. We may therefore state the following theorem.

Theorem 5. *Let G be a circular interval graph. Then any facet of $STAB(G)$, which is not induced by an inequality of the system $Ax \leq \mathbf{1}$, $x \geq 0$, is a clique family inequality associated with some $\mathcal{F}$ and p such that $|\mathcal{F}| \mod p \neq 0$.*

If we combine this result with Theorem 1, Theorem 2 and we recall that Edmonds' inequalities are also clique family inequalities associated with $|\mathcal{F}|$ odd and $p = 2$, we obtain the following corollary.

Corollary 2. *Let G be a quasi-line graph. Any non-trivial facet of $STAB(G)$ is a clique family inequality associated with some $\mathcal{F}$ and p such that $|\mathcal{F}| \mod p \neq 0$.*

We may assume, without loss of generality, that the cliques in the family $\mathcal{F}$ are maximal [25]. A last lemma is the missing brick to the solution of Ben Rebea's conjecture.

Lemma 5. *Let G be a quasi-line graph and $(\mathcal{F}, p)$ a pair such that*

$$
(p - r - 1) \sum_{v \in V_{p-1}(\mathcal{F})} x(v) + (p - r) \sum_{v \in V_{\geq p}(\mathcal{F})} x(v) \leq (p - r) \left\lfloor \frac{|\mathcal{F}|}{p} \right\rfloor \tag{16}
$$

is a facet of $STAB(G)$. If $|\mathcal{F}| < 2p$, then the inequality (16) *is a clique inequality.*

We may therefore state our main result:

Theorem 6. *Ben Rebea's conjecture holds true.*

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