Uniform Multi-commodity Flow in Wireless Networks with Gaussian Fading Channels

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***Abstract*— Starting with the seminal work of Gupta and Kumar (2000), there have been many interesting results that give information theoretic outer and inner approximations to the rate region for wireless networks. While these bounds are almost tight for geometric random networks, not much is known about their tightness for arbitrary wireless networks. In contrast, Leighton and Rao (1988) established a powerful result that uniform multi-commodity flow (UMCF) is within a factor of $\log n$ of the natural min-cut capacity for any graph (equivalent to a wireline network) of n nodes.**

Our motivation is to obtain a similar simple and general characterization for UMCF (shown to be equivalent to the characterization for a much wider class of traffic models) for any wireless network. In this paper, we apply and extend known results to obtain such characterization for networks with Gaussian fading channels. For channel state information (CSI) only at the receiver, we establish that UMCF is within $\Delta^2 \log n$ factor of information theoretic min-cut capacity for the wireless network, where Δ is the max-degree of a sub-graph induced by the underlying wireless network. For deterministic AWGN channels, we show that UMCF is within *square root* of min-cut bound for any network.

I. Introduction

In their seminal paper [4], Gupta and Kumar considered a wireless network formed by n nodes placed in a unit area (e.g. disc) uniformly at random. Under the protocol model, they showed that the maximal supportable rate per pair of nodes, when n source-destination pairs are chosen randomly, scales as $\Theta(1/\sqrt{n \log n})$. Subsequent to this result, there have been many interesting results that establish information theoretic upper and lower bounds. Some upper bound results are by Leveque and Telatar [10], Xie and Kumar [13], Xue, Xie and Kumar [14], Jovicic, Viswanath and Kulkarni [5]; some lower bound results are by Kulkarni and Viswanath [7], Franceschetti et. al. [3], Xue, Xie and Kumar [14] and Madan and Shah [11]. Note that this is a small illustrative subset of the known results.

Known results lead to a tight characterization of (information theoretic) scaling of capacity for geometric random networks. Though some of these results generalize to arbitrary networks, we do not know of any result that quantifies the "tightness" of upper and lower bounds in terms of the network parameters. Let us mention here that the work of Xue, Xie and Kumar [14] goes in this direction, but relies on strong assumptions on the topology of the underlying graph and focuses on the *transport capacity* of the network. Our goal in this paper is to characterize the uniform multi-commodity flow for wireless networks in terms of cut-properties of the network graph induced by the wireless network. In addition, we aim to obtain bounds that can be *computed efficiently* for any arbitrary wireless network.

We first present the channel and the traffic model. We then present our main results followed by proofs. Finally, we present applications that illustrate our results.

II. Model

A. Channel Model

This is similar to the model in, for example, [5]. We have $V = \{1, \dots, n\}$ wireless nodes with transceiver capabilities located arbitrarily in a plane. Node transmissions happen at discrete times, $t \in \mathbb{Z}+$. Let $X_i(t)$ be the signal transmitted by node i at time $t \in \mathbb{Z}_+$. We assume that each node has a power constraint[1] such that $\limsup_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} |X_i^2(t)| \le P$. Then $Y_i(t)$, the signal received by node i at time t, is given by

$$Y_i(t) = \textstyle\sum_{k \neq i} H_{ik} X_k(t) + Z_i(t), \qquad (1)$$

where $Z_i(t)$ denotes a complex zero mean white Gaussian noise process with independent real and imaginary parts with variance 1/2 such that $Z_i(t)$ are i.i.d. across all i. Let r_{ij} denote the distance between nodes i and j. Let $H_{ik}(t)$ be such that

$$H_{ik}(t) = \sqrt{g(r_{ik})}\hat{H}_{ik}(t),$$

where $\hat{H}_{ik}(t)$ is a stationary and ergodic zero mean complex Gaussian process with independent real and imaginary parts (with variance 1/2). It models channel fluctuations due to frequency flat fading. Also, $g(\cdot)$ is a monotonically decreasing function that models path loss with $g(x) \le 1$ for all $x \ge 0$. We assume that $\mathbb{E}[|\hat{H}_{ik}(t)|^2] = 1$ and that the $\hat{H}_{ik}(t)$'s are independent.

B. Traffic Model

We now describe the traffic model considered in this paper. We refer the reader to [11] for proofs of results in this section. A rate matrix $\lambda = [\lambda_{ij}] \in \mathbb{R}_+^{n\times n}$ is called *feasible*, if information can be sent (possibly via multiple hops) from node i to node j at rate λ_{ij} for each node pair $(i,j),\ 1 \le i,j \le n$. Let $\Lambda \subseteq \mathbb{R}_+^{n\times n}$ denote the set of all feasible rate matrices. We also call Λ the capacity region. Ideally, we would like to characterize Λ. However, it is hard to obtain a single-letter characterization of Λ that can be evaluated. Hence, we study the scaling of the quantity $\rho^*(\Lambda)$ defined below.

Definition 2.1 ($\rho^*(\Lambda)$): For any feasible $\lambda \in \Lambda$, let $\rho(\lambda) \triangleq \max_i \{\sum_{k=1}^n \lambda_{ik}, \sum_{k=1}^n \lambda_{ki}\}$, $L(x) = \{\lambda \in \mathbb{R}_+^{n\times n} : \rho(\lambda) \le x\}$. Then, define $\rho^*(\Lambda) = \sup\{x \in \mathbb{R}_+ : L(x) \subseteq \Lambda\}$.

[0]Author names appear in the alphabetic order of their last names. The work of the first author was supported by Swiss NSF grant Nr PA002-108976.

[1]For notational simplicity we assume that each node has the same power constraint. The general case, where each node has different maximum average power can be handled using identical techniques.

Thus the quantity $\rho^*(\Lambda)$ is a parametrization of an inner approximation to the capacity region Λ.

Definition 2.2 (Uniform multi-commodity flow): We say that a rate matrix λ is a *uniform multicommodity flow* if $\lambda = f\mathbf{1}$ for some $f \in \mathbb{R}_+$, where $\mathbf{1} \in \mathbb{R}_+^{n\times n}$ is a matrix with all entries equal to 1. We will denote such a flow as $U(f) = f\mathbf{1}$. Let $f^* = \sup\{f \in \mathbb{R}_+ : \ U(f) \text{ is feasible}\}$.
Next, we state equivalence between scaling of maximal UMCF, f^* and that of $\rho^*(\Lambda)$ [11].

Lemma 2.3: In any network, $f^* = \Theta(\rho^*(\Lambda))$.
Hence, we will study scaling of the f^* to determine $\rho^*(\Lambda)$. Finally, we show that Gupta-Kumar capacity scaling model is equivalent to f^*. In setup of Gupta-Kumar [4], n distinct source-destination pairs are chosen at random such that each node is source (destination) for exactly one destination (source) and such pairing is done uniformly at random over all possible pairings. Thus the traffic matrix corresponds to a randomly chosen permutation flow which is defined as follows.

Definition 2.4: Let S_n denote the set of permutation matrices in $\mathbb{R}_+^{n\times n}$. We say that a traffic matrix λ is a *permutation flow* if $\lambda = f\Sigma$ for some $f \in \mathbb{R}_+$ and $\Sigma \in S_n$. We will denote such a flow as $\lambda_\Sigma(f) = f\Sigma$, for $\Sigma \in S_n$.
In light of the above definition, all the previous works study the scaling of $\bar{f}$, where $\bar{f}$ is the maximum value such that for any randomly chosen permutation $\Sigma \in S_n$, the permutation flow $\lambda_\Sigma(\bar{f})$ is feasible with probability at least $1 - 1/n^2$.

Lemma 2.5: If for $\Sigma \in S_n$ chosen uniformly at random, $\lambda_\Sigma(nf)$ is feasible with probability at least $1-n^{-1.5-\alpha}, \alpha > 0$, then there exists a sequence of feasible rate matrices Γ_n such that

$$\|U_n(f) - \Gamma_n\| = O(fn^{-\alpha}) \to 0 \text{ as } n \to \infty,$$

where $\|\cdot\|$ denotes the Frobenius norm for matrices.
The proof of the above lemma can be found in [11]. It implies that for any network, $\bar{f} = \Theta(nf^*)$. We focus our attention on the scaling of f^* in this paper.

III. Main Results

The main result of this paper is the characterization of information theoretic relation between upper and lower bound on UMCF for any general wireless networks. These characterizations for Gaussian channels (with and without random fading) are summarized in Corollaries 3.3 and 3.5. The proofs of the lower bounds are constructive.

A. Deterministic AWGN Channels

First, we consider the case where $\hat{H}_{kj} = 1$ w.p. 1, $\forall k, j = 1, \ldots, n$. This correspond to a standard additive white Gaussian noise (AWGN) channel (no fading). Consider the following two graphs: (1) K_n is the fully connected graph with node set V, and (2) G_r is the graph where each node $i \in V$ is connected to all nodes that are within a distance r of i. Let E_r denote the edge set of G_r. Also, let $r^* = \inf\{r : \ G_r \ \text{ is connected}\}$. Let $\Delta(r)$ be the maximum vertex degree of G_r. We have the following bounds on f^*.

Theorem 3.1: For a given placement of nodes, under the Gaussian channel model with path loss function $g(\cdot)$, the maximal uniform multi-commodity flow f^* is bounded as follows[2].

$$f^* \leq \min_{S\subset V} \frac{\sum_{i\in S, j\in S^c} \log(1+\sqrt{Pg(r_{ij})})}{|S||S^C|}$$

$$f^* = \Omega\Bigg(\sup_{r\geq r^*,\ \eta\geq 0} \frac{1}{1+\Delta(r)\Delta(r(1+\eta))} \times \min_{S\subset V} \frac{\sum_{i\in S, j\in S^c : r_{ij}\leq r} \log\left(1+\frac{Pg(r_{ij})\Delta(r(1+\eta))}{1+nPg(r(1+\eta))}\right)}{\log n|S||S^C|}\Bigg).$$

Next, we state the explicit relation between the upper and lower bound for f^* under the following mild condition:

Condition 1: Let $P \leq 1$. Further, there exists an $\epsilon > 0$ such that the graph $\hat{G}^\epsilon = (V, \hat{E}^\epsilon)$ is connected, where $\hat{E}^\epsilon = \{(i,j) : \log\left(1+\sqrt{Pg(r_{ij})}\right) \geq n^{-\epsilon/2}\}$.

Definition 3.2: Let $r_1(\epsilon) \geq r^*$ be such that for all $i \in V$, $\sum_{j\in V: r_{ij}\geq r_1(\epsilon)} Pg(r_{ij}) \leq \frac{1}{n^{1+\epsilon}}$.

Corollary 3.3: Let $\epsilon > 0$ be such that Condition 1 is satisfied. Then

$$\Omega\left(\frac{n}{\Delta^3(r_1(\epsilon))\log n}\ \Phi^2\right) \leq f^* \leq \Phi,$$

where $\Phi = \min_{S\subset V} \frac{\sum_{i\in S, j\in S^c : r_{ij}\leq r_1(\epsilon)} \log(1+\sqrt{Pg(r_{ij})})}{|S||S^C|}$.

B. Random Fading

We now derive similar upper and lower bounds for UMCF for Gaussian channel with random fading, as defined in Section II. We make special note of Corollary 3.5 that shows that for *receiver-only CSI* case, our bounds are quite tight for any graph. First, we present general upper bounds.

Theorem 3.4: (1) With channel state information (CSI) only at receivers, f^* is bounded as follows

$$f^* \leq \min_{S\subset V} \frac{\sum_{i\in S, j\in S^c} \mathbb{E}(\log(1+P|H_{ji}|^2))}{|S||S^c|}$$

$$f^* = \Omega\Bigg(\sup_{r\geq r^*,\ \eta\geq 0} \frac{1}{1+\Delta(r)\Delta(r(1+\eta))} \times \min_{S\subset V} \frac{\sum_{i\in S, j\in S^c} \mathbf{1}_{(i,j)\in E_r} \mathbb{E}\log\left(1+\frac{P|H_{ji}|^2\Delta(r(1+\eta))}{1+nPg(r(1+\eta))}\right)}{\log n|S||S^C|}\Bigg)$$

(2) With CSI at both transmitters and receivers, f^* is bounded as follows.

$$f^* \leq \min_{S\subset V} \frac{\sum_{i\in S, j\in S^c} 2\,\mathbb{E}(\log(1+\sqrt{P}\,|H_{ji}|)}{|S||S^c|}$$

The lower bound for the receiver only CSI case is a (weak) lower bound for this case as well.

The following corollary quantifies the gap between the upper and the lower bound for the receiver-only CSI case if the following condition holds.

Condition 2: Let $P \leq 1$. Further, there exists an $\epsilon > 0$ such that the graph $\hat{G}^\epsilon = (V, \hat{E}^\epsilon)$ is connected, where $\hat{E}^\epsilon = \{(i,j) : \mathbb{E}\left[\log\left(1+P|H_{ij}|^2\right)\right] \geq n^{-\epsilon/2}\}$.

[2]For the lower bound, we use the Ω notation in order not to have to write out constants explicitly; we note that the constants are independent of the graph structure and n [8].

Corollary 3.5: With CSI only at the receivers and under Condition 2, we have

$$\Omega\left(\frac{\textsf{min-cut}_R}{\Delta^2(r_1(\epsilon))\log n}\right) = f^* = O(\textsf{min-cut}_R),$$

where $\textsf{min-cut}_R = \min_{S\subset V} \frac{\sum_{i\in S, j\in S^c: r_{ij}\le r_1(\epsilon)} \mathbb{E}\log(1+P|H_{ij}|^2)}{|S||S^C|}$.

Let us finally mention that another set of upper bounds can be obtained by slightly modifying the proof of Theorem 3.4 (still relying on the method described in [5]).

Lemma 3.6: (1) If channel state information (CSI) is known only at receivers, then

$$f^* \le \frac{P}{\sum_{i,j} r_{ij}} \sum_{i,j=1}^n r_{ij} g(r_{ij}).$$

(2) If CSI is known at the transmitters and the receivers, then

$$f^* \le \frac{P}{\sum_{i,j} r_{ij}} \sum_{i,j,k=1}^n \min(r_{ij}, r_{kj}) \sqrt{g(r_{ij})g(r_{kj})}.$$

IV. PROOFS

We now present proofs of the results for both deterministic and random fading channels.

A. Deterministic Channels

Proof: [*Theorem 3.1*] We prove the upper and lower bounds separately. We will use the following result by Leighton and Rao [9].

Theorem 4.1: [Essentially Theorem 12, [9]] Consider a graph $G = (V, E)$ with n nodes. For $(i, j) \in E$, let R_{ij} denote the communication rate over this link from i to j. The maximum uniform multi-commodity flow f^* is upper and lower bounded as

$$\Omega\left(\frac{\mathcal{S}}{\log n}\right) = f^* \le \mathcal{S},$$

where $\mathcal{S} = \sup_{R=\{R_{ij}\}} \min_{S\subset V} \frac{\sum_{(i,j)\in E: i\in S, j\in S^c} R_{ij}}{|S||S^c|}$. and the supremum is taken over all possible set of rates $\{R_{ij}\}$ simultaneously achievable with a given communication scheme.

Upper Bound: In order to bound the sum-rate across each given cut, we refer to the proof of the max-flow min-cut lemma in [13], which yields for any $S \subset V$

$$\sum_{i\in S, j\in S^c} R_{ij} \le \sum_{j\in S^c} \log(1 + \mathbb{E}(|\tilde{X}_j|^2)),$$

where $\tilde{X}_j = \sum_{i\in S} \sqrt{g(r_{ji})}\, X_i$. We therefore deduce that

$$\begin{aligned}
&\sum_{i\in S, j\in S^c} R_{ij} \\
&\quad \le \sum_{j\in S^c} \log[1 + \textstyle\sum_{i,k\in S} \sqrt{g(r_{ji})\, g(r_{jk})}\, |\mathbb{E}(X_i \overline{X_k})|] \\
&\quad \le \sum_{j\in S^c} \log[1 + P(\textstyle\sum_{i\in S} \sqrt{g(r_{ji})})^2],
\end{aligned}$$

since $|\mathbb{E}(X_i\overline{X_k})| \le \sqrt{P_i P_k} \le P$. Finally, we obtain

$$\begin{aligned}
\sum_{i\in S, j\in S^c} R_{ij} &\le \sum_{j\in S^c} 2\log(1 + \sqrt{P} \textstyle\sum_{i\in S} \sqrt{g(r_{ji})}) \\
&\le \sum_{i\in S, j\in S^c} 2\log(1 + \sqrt{P g(r_{ji})}).
\end{aligned}$$

i.e. the desired upper bound for f^*.

Lower Bound: To establish the lower bound, we find a transmission scheme for which the multicommodity flow is greater than or equal to that in the lower bound. For $r \ge r^*$, consider graph $G_r = (V, E_r)$ on n nodes as above. We use $\Delta(r(1+\eta))$ to denote the maximum vertex degree of graph $G_{r(1+\eta)}$. Now, consider the following transmission scheme. A node i can transmit to a node j only if $r_{ij} \le r$. Also, when a node i transmits, no node within a distance $r(1+\eta)$ of the receiver can transmit. Thus, when a link $(i, j) \in E_r$ is active, at most $\Delta(r(1 + \eta))$ nodes are constrained to remain silent, i.e., at most $\Delta(r(1+\eta))\Delta(r)$ links are constrained to remain inactive. Hence, the chromatic number of the dual graph is at most $(1 + \Delta(r(1 + \eta))\Delta(r))$. In addition, we assume that the signal transmitted by each node has a Gaussian distribution. Then, subject to the maximum average power constraint, the following average rate from a node i to node j, with $(i, j) \in E_r$, is achievable

$$R_{ij} \ge \frac{\log\left(1 + \frac{Pg(r_{ij})(\Delta(r(1+\eta))}{1+nPg(r(1+\eta))}\right)}{1 + \Delta(r)\Delta(r(1 + \eta))}. \quad (2)$$

Note that the interference is due to at most n nodes and all the interfering nodes are at least a distance $r(1+\eta)$ away from the receiver. The above simple time-division scheme gives rise to a capacitated graph, for which by Theorem 4.1, the maximum uniform multi-commodity flow is lower bounded as given in the Theorem. This completes the proof of Theorem 3.1. ■

The proof of Corollary 3.3 will utilize the following two inequalities.

Lemma 4.2: (1) Given $x_i \in (0, 1), 1 \le i \le N$,

$$\sum_{i=1}^N \log(1 + \sqrt{x_i}) \le \sqrt{2N}\sqrt{\textstyle\sum_{i=1}^N \log(1 + x_i)}.$$

(2) For any $x \ge 0$, $\alpha \in (0, 1)$, $\frac{1}{\alpha}\log(1 + \alpha x) \ge \log(1 + x)$.

Proof of (1). For any $x \in (0, 1)$, $x/2 \le \log(1 + x) \le x$, so

$$\begin{aligned}
\sum_{i=1}^N \log(1+\sqrt{x_i}) &\le \sum_{i=1}^N \sqrt{x_i} \le \sqrt{N}\sqrt{\textstyle\sum_{i=1}^N x_i} \quad (3) \\
&\le \sqrt{2N}\sqrt{\textstyle\sum_{i=1}^N \log(1 + x_i)},
\end{aligned}$$

where (3) follows from Cauchy-Schwarz inequality.

Proof of (2). Define $f(x) = \frac{1}{\alpha}\log(1 + \alpha x) - \log(1 + x)$. Note that $f'(x) \ge 0$ for $x \ge 0$ and $f(0) = 0$. This completes the proof of (2). ■

Proof: [Corollary 3.3] Consider the upper bound of Theorem 3.1. Let $\epsilon > 0$ be such that Condition 1 is satisfied. Consider any cut defined by (S, S^c). Due to the symmetry of the upper bound, without loss of generality, assume $|S| \le n/2$. Now,

$$\begin{aligned}
&\textsf{Cut}(S, S^c) = \sum_{i\in S, j\in S^c} \log(1 + \sqrt{Pg(r_{ij})}) \\
&\le \sum_{i\in S, j\in S^c: r_{ij}\le r_1(\epsilon)} \log\left(1 + \sqrt{Pg(r_{ij})}\right) + \frac{|S|}{n^{1+\epsilon}}. \quad (4)
\end{aligned}$$

From Lemma 4.2 (1), we have

$$\begin{aligned}
&\sum_{i\in S, j\in S^c: r_{ij}\le r_1(\epsilon)} \log\left(1 + \sqrt{Pg(r_{ij})}\right) \\
&\le \sqrt{\Delta(r_1(\epsilon))|S| \sum_{i\in S, j\in S^c: r_{ij}\le r_1(\epsilon)} \log\left(1 + Pg(r_{ij})\right)}. \quad (5)
\end{aligned}$$

Condition 1 and $\hat{E}^\epsilon \subset E_{r_1(\epsilon)}$ (from definition) imply

$$\frac{\sum_{i\in S, j\in S^c: r_{ij}\le r_1(\epsilon)} \log\left(1 + Pg(r_{ij})\right)}{|S||S^c|} = \Omega\left(\frac{1}{n^{2+\epsilon/2}}\right). \quad (6)$$

From (4), (5), (6) and some manipulations, we obtain that

$$\begin{aligned}
\frac{\textsf{Cut}(S, S^c)}{|S||S^c|} &= O\left(\sqrt{\frac{4\Delta(r_1(\epsilon))}{n}}\right. \\
&\quad \times \left.\sqrt{\frac{\sum_{i\in S, j\in S^c: (i,j)\in E_{r_1(\epsilon)}} \log(1 + Pg(r_{ij})}{|S||S^c|}}\right). \quad (7)
\end{aligned}$$

We recall that the upper bound of Theorem 3.1 is

$$\min_{S} \frac{\mathsf{Cut}(S,S^c)}{|S||S^c|}.$$

Now, consider the lower bound of Theorem 3.1. To obtain a specific lower bound, consider choice of $r = r_1(\epsilon)$ and $\eta = 0$. Notice that in the proof of Theorem 3.1, we upper bound the power of interference by $nPg(r(1+\eta))$, which is precisely $I_{ij} = \sum_{k \in V: r_{jk} \geq r_1(\epsilon)} Pg(r_{jk})$ for transmission from i to j. By definition of $r_1(\epsilon)$, we have $I \leq n^{-1-\epsilon} < \delta < 1$ for small enough δ and large enough n. Now, by Lemma 4.2 (2), we have

$$\log\left(1 + \frac{Pg(r_{ij})\Delta(r_1(\epsilon))}{1+I}\right) = \Theta\left(\log(1 + Pg(r_{ij})\Delta(r_1(\epsilon)))\right). \tag{8}$$

Using (8) with the modification of the lower bound of Theorem 3.1 for the choice of $r = r_1(\epsilon)$ and the simple fact that $\Delta(r_1(\epsilon)) \geq 1$, we obtain a new lower bound, say LB, as follows:

$$\begin{aligned} \mathsf{LB} &= \Omega\left(\min_{S \subset V} \frac{\sum_{i \in S, j \in S^c: r_{ij} \leq r_1(\epsilon)} \log(1 + Pg(r_{ij}))}{\Delta(r_1(\epsilon))^2 \log n |S||S^c|}\right) \\ &= \Omega\left(\left(\min_{S \subset V} \frac{\mathsf{Cut}(S,S^c)}{|S||S^c|}\right)^2 \frac{n}{\Delta(r_1(\epsilon))^3}\right), \end{aligned} \tag{9}$$

where (9) follows from (7). This completes the proof. ■

B. Random Fading

Proof: [Theorem 3.4] (1) *Receiver only CSI:*

Upper Bound: Following the steps in the proof of Theorem 2.1 in [5] and using $(1 + \sum_{i=1}^n \alpha_i) \leq \prod_{i=1}^n (1 + \alpha_i)$ for $\alpha_i > 0$, we obtain

$$\begin{aligned} \sum_{i \in S, j \in S^c} R_{ij} &\leq \max_{Q_S \succeq 0, (Q_S)_{ii} \leq P} \mathbb{E}[\log\det(I + H_S Q_S H_S^*)] \\ &\leq \sum_{i \in S, j \in S^c} \mathbb{E}\left(\log(1 + P|H_{ji}|^2)\right), \end{aligned} \tag{10}$$

so the upper bound on f^* follows from Theorem 4.1.

Lower Bound: For the lower bound, we will use the time-division scheme in the proof of Theorem 3.1 which schedules every link in the network for a fraction $\alpha = \frac{1}{1+\Delta(r)\Delta(r(1+\eta))}$ of the time. For any given link that transmits data at a particular time, we treat all other simultaneous transmissions in the network as interference. For the rest of the proof let us focus on the link $(1,2)$ (without any loss in generality). We will show that with receiver only CSI the following rate on link $(1,2)$ is achievable.

$$R_{12} = \alpha \mathbb{E} \log\left(1 + \frac{P|H_{21}|^2 \Delta(r(1+\eta))}{\left(1 + nPg(r(1+\eta))\right)}\right)$$

To show this we will use the following result, which follows directly from Theorem 1 in [6].

Theorem 4.3: Consider a complex scalar channel where the output Y when X is transmitted is given by

$$Y = hX + Z + S,$$

where Z is a complex circularly symmetric Gaussian random variable with unit variance, and S satisfies $\mathbb{E}[S^*S] \leq \hat{P}$. Also, h is zero mean spatially white and i.i.d during each channel use. If X is a complex zero mean circularly symmetric Gaussian random variable with $\mathbb{E}[X^*X] = P$, then

$$I(X;(Y,h)) \geq \mathbb{E}\log\left(1 + \frac{P|h|^2}{1+\hat{P}}\right).$$

We consider a transmission scheme where the signal transmitted over each link, when active, is a complex zero mean white circularly symmetric Gaussian with variance $P\Delta(r(1+\eta))$ (note that this satisfies the average power constraint that each node can use maximum power P.) Moreover, we assume that the transmissions on all links are mutually independent. Let $t_1, t_2, \ldots$ denote times at which link $(1,2)$ is scheduled. Hence, at any such time $t \in \{t_1, t_2, \ldots\}$, the received signal at node 2 is given by

$$Y_2(t) = H_{21}(t)X_1(t) + \sum_{k \neq 1,2} H_{2k}(t)X_k(t) + Z_2(t).$$

Using the mutual independence of transmissions and zero mean property along with the construction of the scheduling scheme,

$$\mathbb{E}\left|\sum_{k \neq 1,2} H_{2k}(t)X_k(t) + Z_2(t)\right|^2 \leq 1 + nPg(r(1+\eta)).$$

From Theorem 4.3,

$$I(X_1(t);(Y_2(t),H_{21}(t))) \geq \mathbb{E}\log\left(1 + \frac{P|H_{21}|^2\Delta(r(1+\eta))}{(1+nPg(r(1+\eta)))}\right). \tag{11}$$

Since the channel is assumed to be i.i.d. during each use, a random coding argument can be used to achieve this rate with a probability of error which goes to zero as the block length goes to infinity.

Combining this with the time-sharing between different sets of links (described above), it follows that

$$R_{12} \geq \alpha \mathbb{E}\log\left(1 + \frac{P|H_{21}|^2\Delta(r(1+\eta))}{1 + nPg(r(1+\eta))}\right).$$

(2) *Full CSI:* The upper bound follows again from the proof of Theorem 2.1 in [5], from which we deduce that

$$\begin{aligned} \sum_{i \in S, j \in S^c} R_{ij} &\leq \mathbb{E}[\max_{Q \succeq 0, Q_{ii} \leq P} \log\det(I + H_S Q_S H_S^*)] \\ &\leq \sum_{j \in S^c} \mathbb{E}[\max_{Q \succeq 0, Q_{ii} \leq P} \log(1 + h_j Q_S h_j^*)], \end{aligned}$$

where h_j is the j^{th} row of H. Since $h_j Q_S h_j^*$ is maximum when $(Q_S)_{ik} \equiv P$ for all $i,k \in S$, we obtain, following the steps of the proof of Theorem 3.1,

$$\begin{aligned} \sum_{i \in S, j \in S^c} R_{ij} &\leq \sum_{j \in S^c} \mathbb{E}\log(1 + P(\sum_{i \in S}|H_{ij}|)^2] \\ &\leq \sum_{j \in S^c} 2\,\mathbb{E}\log(1 + \sqrt{P}\sum_{i \in S}|H_{ij}|)] \\ &\leq \sum_{i \in S, j \in S^c} 2\,\mathbb{E}(\log(1 + \sqrt{P}\,|H_{ij}|), \end{aligned}$$

and the upper bound on f^* follows from Theorem 4.1. ■

Proof: [Corollary 3.5] The general scheme of the proof is the same as that of Corollary 3.3. Using the upper bound in Theorem 3.4, f^* can be further bounded above as

$$\begin{aligned} f^* &\leq \min_S \frac{\sum_{i \in S, j \in S^c: r_{ij} \leq r_1(\epsilon)} \mathbb{E}\left(\log(1 + P|H_{ji}|^2)\right)}{|S||S^c|} + O\left(n^{-2-\epsilon}\right) \\ &= \mathsf{min\text{-}cut}_R + O\left(n^{-2-\epsilon}\right) = O(\mathsf{min\text{-}cut}_R) \end{aligned} \tag{12}$$

where the first inequality follows from Condition 2. Next, we consider the lower bound obtained via the lower bound in Theorem 3.4 and (11). Specifically, we consider the choice of $r = r_1(\epsilon)$ and $\eta = 0$. Notice that in (11), we used the term $nPg(r(1+\eta))$ as a bound on the interference power. However, for our case, the actual intereference is $I_{ij} = \sum_{k \in V: r_{jk} \geq r_1(\epsilon)} Pg(r_{jk})$ for a transmission from i to j. By definition of $r_1(\epsilon)$, we have $I \leq n^{-1-\epsilon} < \delta < 1$ for small enough δ and large enough n. Now, by Lemma 4.2 (2), we have

$$\mathbb{E}\left[\log\left(1+\frac{Pg(r_{ij})\Delta(r_1(\epsilon))}{1+I}\right)\right] = \Theta\left(\log(1+Pg(r_{ij})\Delta(r_1(\epsilon)))\right). \tag{13}$$

Using (13) and (12) along with the lower bound obtained via time-division scheme that led to (11), we obtain that the lower bound LB is bounded below as follows:

$$\begin{aligned} \mathsf{LB} &= \Omega\left(\min_S \frac{\sum_{i\in S, j\in S^c: r_{ij}\le r_1(\epsilon)} \mathbb{E}\left(\log(1+P|H_{ji}|^2)\right)}{|S||S^c|\Delta(r_1(\epsilon))^2\log n}\right) \\ &= \Omega\left(\frac{\text{min-cut}_R}{\Delta(r_1(\epsilon))^2\log n}\right), \end{aligned} \tag{14}$$

where we have used the fact that $\Delta(r_1(\epsilon)) \ge 1$. This completes the proof of the scaling law for the receiver-only CSI case. ■

V. Applications

A. Grid Network - AWGN channels

We illustrate the bounds in Theorem 3.1 for a grid network of n nodes, where nodes are placed at (i,j) for all $i,j = 1,\ldots,\sqrt{n}$. Similar bounds can be obtained for fading channels as well. The minimum value of r^* such that G_{r^*} is connected is $r^* = 1$. We obtain upper bounds and lower bounds to characterize the scaling of f^*. We take $g(r) = e^{-r}$.

Upper Bound. We have

$$\begin{aligned} &\min_{S\subset V} \frac{\sum_{i\in S, j\in S^c} 2\log(1+\sqrt{Pg(r_{ij})})}{|S||S^C|} \le 2\log(1+\sqrt{Pg(r)}) \\ &\quad + \min_{S\subset V} \frac{\sum_{i\in S, j\in S^c, (i,j)\in E_r} 2\log(1+\sqrt{Pg(r_{ij})})}{|S||S^C|}. \end{aligned}$$

Take $r = \delta\log n$, where $\delta > 0$. Then $2\log(1+\sqrt{Pg(r)}) = \log(1+\sqrt{P}e^{-r/2}) = O(\log(1+\epsilon))$ for any $\epsilon > 0$. Now consider a cut that such that all nodes with $i < \sqrt{n}/2$ are in S. Then

$$\begin{aligned} &\textstyle\sum_{i\in S, j\in S^c, (i,j)\in E_r} \log(1+\sqrt{Pg(r_{ij})}) \\ &\le 2\sqrt{n}\log(1+\tfrac{P}{e^{1/2}}) + 4\sqrt{n}(\delta\log n)^3\log(1+\tfrac{P}{e}). \end{aligned}$$

Hence, the upper bound in Theorem 3.1 gives $f^* = O(\frac{(\log n)^3}{n^{1.5}})$.

Lower Bound. Consider the lower bound in Theorem 3.1. Take $r = r^* = 1$ and $r(1+\eta) = (1+\delta)\log n$, for some $\delta > 0$. Then

$$\Delta(r^*) = 4, \quad \Delta(n) \le 4(1+\delta)^2(\log n)^2,$$

$$\frac{Pe^{-r^*}}{1+Pne^{-(1+\delta)\log n}} \to Pe^{-r^*}, \quad \text{as} \quad n\to\infty,$$

$$\text{and} \quad \min_{S\subset V} \frac{\sum_{i\in S, j\in S^c: r_{ij}\le r^*}}{\log n|S||S^c|} = \Omega\left(\frac{1}{n^{1.5}\log n}\right).$$

Using the above relations in the lower bound in Theorem 3.1, we get $f^* = \Omega(\frac{1}{n^{1.5}(\log n)^3})$.

B. Geometric Random Network - Random Fading

Let us consider a geometric random network (i.e. with nodes are located at the vertices of a geometric random graph), with random fading and receiver CSI only. A geometric random graph has a constant node density: n nodes are placed uniformly at random in a torus of area n. Using Lemma 3.6, it is possible to infer that f^* is upper bounded as $O(1/n^{3/2})$ with probability $1-\epsilon$ for any fixed $\epsilon > 0$, but the proof is omitted here due to space constraints.

C. Computation for Generic Wireless Networks

All the bounds obtained in the paper involve computation of UMCF over graphs (not networks), which is done through solving a linear program, therefore in polynomial time. Again, due to space constraints, we do not provide more details here.

VI. Conclusion

In this paper, we obtained bounds on uniform multicommodity flow in a wireless network in terms of the cut properties of the underlying graphs. The bounds are applicable to any arbitrary wireless network.

The min-cut capacity is a fundamental entity in wireless networks; many flavors of the min-cut max-flow result are known (see e.g. [2, Chapter 14], [5], [14]). Analogous to [8], an implication of our results is that we can now compute bounds on the min-cut capacity (which is hard to compute in general) of graphs induced by a wireless network, using linear programming.

A. Open Questions

This paper naturally gives rise to a couple of questions.

- What are the implications of the fact that the min-cut capacity of a wireless network can be computed efficiently? For example, in [9], it was shown that the computation of the min-cut of a graph is crucial for many different engineering problems.
- Our proof techniques are very simple. One would imagine that it is possible to tighten the bounds in this paper using more complicated arguments. One direction worth exploring is to design more sophisticated achievable schemes.

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