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Diamagnetic Currents

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Abstract. We study the diamagnetic surface currents of particles in thermal equilibrium submitted to a constant magnetic field. The current density of independent electrons with Boltzmann (respectively Fermi) statistics has a gaussian (respectively exponential) bound for its fall off into the bulk. For a system of interacting particles at low activity with Boltzmann statistics, the current density is localized near to the boundary and integrable when the two-body potential decays as $|\mathbf{x}|^{-\alpha}, \alpha > 4$, in three dimensions. In all cases, the integral of the current density is independent of the nature of the confining wall and correctly related to the bulk magnetisation. The results hold for hard and soft walls and all field strength. The analysis relies on the Feynman-Kac-Ito representation of the Gibbs state and on specific properties of the Brownian bridge process.

1. Introduction

It is well known that the diamagnetism of a system of charges at thermal equilibrium is a purely quantum mechanical effect. The magnetisation of a body subjected to an uniform magnetic field arises from induced currents localised at its surface. Classically, one argues that these currents should be exactly compensated by the cyclotronic motion of the particles in the bulk [1] and there is no diamagnetism in agreement with van Leuwen's theorem [2]. For quantum mechanical charges, this compensation is not perfect and there is a resulting current density near the boundaries which depends on the nature of the walls enclosing the system. However, the corresponding total magnetisation, a thermodynamical quantity, is independent of boundary effects.

In [3], the existence of the thermodynamic limit of the susceptibility of a system of non-interacting electrons (the Landau susceptibility) is rigorously established for all temperatures and densities. Moreover, the susceptibility is independent of the boundaries for a class of parallelepipeds with Dirichlet conditions, but its relation to

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the surface current is not studied there. In [4], the surface currents for hard and soft walls are calculated in the framework of linear response theory, and shown to have the correct relation to the Landau susceptibility; the analysis is however limited to weak fields. Strong fields are treated in [5], but in a nearly classical approximation.

The purpose of this work is to provide a general and non-perturbative treatment of surface currents in a system of free and interacting particles for soft and hard walls.

Our analysis relies on the Feynman-Kac-Ito representation of the Gibbs state of a particle in an external magnetic field. In Sect. II we briefly recall the formalism as well as some known properties of the Brownian bridge. We also give a proof of the thermodynamic limit of the pressure and of the magnetisation in this setting: the limit is independent of the nature of the boundaries. In Sect. III, we study the surface current of free electrons in a semi-infinite system bounded by a plane wall. This current is localised near to the wall and we obtain a gaussian bound for its fall off into the bulk. The integral of the current density is shown to be independent of the boundaries and to be correctly given by the derivative of the pressure with respect to the external field. The proof of these facts involves some specific and remarkable properties of the Brownian bridge. We first carry the analysis with Boltzmann statistics where the relevant mechanisms in the functional integrals are most easily displayed, and give the necessary modifications for Fermi statistics.

The case of a dilute system of interacting particles with a short range (i.e. integrable) potential is treated in Sect. IV. The Ginibre representation of the reduced density matrix [6] in conjunction with a combinatorial identity for the Mayer factor found in [7] enables us to estimate the low activity expansion of the current. Here we use Boltzmann statistics. If the potential has a finite range (or an exponential fall off) the decay of the current density in the bulk is faster than any inverse power. If the potential behaves as an inverse power $|\mathbf{x}|^{-\alpha}$, $\alpha>3$, the current density still vanishes far away from the wall, and is integrable when $\alpha>4$. In the latter case, its relation to the magnetisation is recovered.

The Ginibre representation in the case of Fermi statistics is not so manageable. For this reason we are not able to treat the case of interacting Fermions. However we can prove similar results for a system of Fermions with impurities (random external potentials). The technique for these is the same as for the above cases and we do not give these results here.

II. The Feynman-Kac-Ito Representation

We consider a quantum particle of charge e and mass m confined by a potential $V(\mathbf{r})$. It is subjected to a constant magnetic field $\mathbf{B}$ in the z direction and has the hamiltonian

$$H=\frac{1}{2m}\left|\mathbf{p}-\frac{e}{c}\,\mathbf{A}(\mathbf{r})\right|^2+V(\mathbf{r}) \tag{2.1}$$

with $\mathbf{B}=\nabla\wedge\mathbf{A}(\mathbf{r})$, and the vector potential $\mathbf{A}(\mathbf{r})=\left(-\frac{B}{2}\,y,\,\frac{B}{2}\,x,\,0\right)$ being given in the symmetric gauge.

Since the cyclotronic motion occurs in the (x, y) plane, we shall treat the system in two dimensions. The contribution of the degree of freedom in the z-direction can be trivially added to the thermodynamic quantities. We set $\hbar = c = 1$ and work with a unit mass $m = 1$ and charge $e = 1$. The two dimensional hamiltonian is

$$H_R = \frac{1}{2}|\mathbf{v}|^2 + V_R(\mathbf{r}) = \frac{1}{2}|\mathbf{p}|^2 - \mathbf{B}\cdot\mathbf{r}\wedge\mathbf{p} + \frac{B^2}{4}|\mathbf{r}|^2 + V_R(\mathbf{r}) \ , \tag{2.2}$$

where $\mathbf{v} = \left(p_x + \frac{B}{2}y, p_y - \frac{B}{2}x\right)$ is the velocity operator.

The confining potential for smooth walls is chosen to be rotationally invariant and of the form

$$V_R(\mathbf{r}) = \Phi(r - R) \ , \quad r = |\mathbf{r}| \ , \tag{2.3}$$

where $\phi(x)$ is a C^1-function on $\mathbb{R}$ with

$$\begin{aligned} &\phi(x) = 0 \ , \quad x \leqq 0 \ , \\ &\phi(x) \geqq \kappa x^2 \quad \text{for} \quad x \geqq x_0 \ , \quad \kappa > 0 \ . \end{aligned} \tag{2.4}$$

We denote $\inf_x \phi(x)$ by $-\phi_0$, $0 \leqq \phi_0 < \infty$.

The hard wall is defined by supplementing $|\mathbf{p}|^2 = -\Delta$ by Dirichlet boundary conditions at $r = R$. The hard wall case will formally correspond to $\phi(x) = \infty$, $x > 0$. Equation (2.2) defines a self-adjoint operator on a suitable domain. Moreover, the kernel of the statistical operator $\exp\{-\beta H_R)$ has a path-integral representation, the Feynman-Kac-Ito formula [8, Theorem 15.5, p. 163]. Using the notation of [8], this kernel is given by

$$(\mathbf{r}|\exp(-\beta H_R)|\tilde{\mathbf{r}}) = \int d\mu_{0\mathbf{r};\tilde{\mathbf{r}}\beta} \exp(F(\boldsymbol{\omega})) \ , \tag{2.5}$$

$$\begin{aligned} F(\boldsymbol{\omega}) &= -i\int_0^\beta \mathbf{A}(\boldsymbol{\omega}(s))\cdot d\boldsymbol{\omega} - \int_0^\beta ds\, V_R(\boldsymbol{\omega}(s)) \\ &= -i\frac{\mathbf{B}}{2}\cdot\int_0^\beta \boldsymbol{\omega}(s)\wedge d\boldsymbol{\omega} - \int_0^\beta ds\, V_R(\boldsymbol{\omega}(s)) \ . \end{aligned} \tag{2.6}$$

$d\mu_{0\mathbf{r};\tilde{\mathbf{r}}\beta}$ is the conditional Wiener measure for two dimensional Brownian paths $\boldsymbol{\omega}(s)$ starting from $\mathbf{r}$ at $s = 0$ and ending in $\tilde{\mathbf{r}}$ at $s = \beta$. The stochastic integral in (2.6) is to be understood in the Ito sense.

It is convenient to introduce the Brownian bridge process $\boldsymbol{\alpha}(s)$, $0 \leqq s \leqq 1$, $\boldsymbol{\alpha}(0) = \boldsymbol{\alpha}(1) = 0$, with covariance

$$\mathbb{E}(\alpha^i(s)\alpha^j(t)) = \int D\boldsymbol{\alpha}\,\alpha^i(s)\alpha^j(t) = \delta_{ij}\, C(s, t) \ , \quad i, j = x, y \ , \tag{2.7}$$

with $C(s, t) = C(t, s)$ and $C(s, t) = s(1 - t)$ for $0 \leqq s \leqq t \leqq 1$. The $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ process are related by

$$\boldsymbol{\omega}(s) = \left(1 - \frac{s}{\beta}\right)\mathbf{r} + \frac{s}{\beta}\tilde{\mathbf{r}} + \sqrt{\beta}\,\boldsymbol{\alpha}\left(\frac{s}{\beta}\right) \ , \tag{2.8}$$

and

$$(\mathbf{r}|\exp(-\beta H_R)|\tilde{\mathbf{r}})=\frac{1}{2\pi\beta}\exp\left(-\frac{|\mathbf{r}-\tilde{\mathbf{r}}|^2}{2\beta}\right)\int D\boldsymbol{\alpha}\exp G(\boldsymbol{\alpha};\mathbf{r},\tilde{\mathbf{r}}) \ , \tag{2.9}$$

with

$$G(\boldsymbol{\alpha};\mathbf{r},\tilde{\mathbf{r}})=\int_0^\beta F\left(\left(1-\frac{s}{\beta}\right)\mathbf{r}+\frac{s}{\beta}\tilde{\mathbf{r}}+\sqrt{\beta}\,\boldsymbol{\alpha}\left(\frac{s}{\beta}\right)\right)ds \ .$$

With (2.8) we can work out the expression (2.6) in terms of $\boldsymbol{\alpha}$. Using $\int_0^1 d\boldsymbol{\alpha}=0$ and $\int_0^1 d\boldsymbol{\alpha}=-\int_0^1 \boldsymbol{\alpha}(s)ds$ (an application of Ito's lemma), we get

$$\begin{aligned} G(\boldsymbol{\alpha};\mathbf{r},\tilde{\mathbf{r}})&=i\,\frac{\mathbf{B}}{2}\cdot(\mathbf{r}\wedge\tilde{\mathbf{r}})+\mathrm{i}\sqrt{\beta}\,\mathbf{B}\cdot(\tilde{\mathbf{r}}-\mathbf{r})\wedge\int_0^1\boldsymbol{\alpha}(s)ds \\ &\quad+i\,\frac{\beta\mathbf{B}}{2}\cdot\int_0^1\boldsymbol{\alpha}(s)\wedge d\boldsymbol{\alpha}-\beta\int_0^1 ds\, V_R(\mathbf{r}-s(\mathbf{r}-\tilde{\mathbf{r}})+\sqrt{\beta}\,\boldsymbol{\alpha}(s)) \ . \end{aligned} \tag{2.10}$$

The hard wall system is also given by (2.9) and (2.10) with the integration in (2.9) restricted to the domain

$$\Delta_R(\mathbf{r},\tilde{\mathbf{r}})=\left\{\boldsymbol{\alpha}\,\middle|\,\sup_{0\leqq s\leqq 1}\ |\mathbf{r}-s(\mathbf{r}-\tilde{\mathbf{r}})+\sqrt{\beta}\,\boldsymbol{\alpha}(s)|\leqq R\right\}$$

and $V_R=0$ in (2.10).

The pressure and density of a system of independent particles with activity z and Boltzmann statistics are

$$p_R=\frac{z}{\beta}\ \frac{\operatorname{Tr}\exp(-\beta H_R)}{\pi R^2} \ , \tag{2.11}$$

$$\varrho_R=z\ \frac{\operatorname{Tr}\exp(-\beta H_R)}{\pi R^2} \ . \tag{2.12}$$

The corresponding average magnetisation is given by

$$m_R=\frac{\partial p_R}{\partial B}=z\ \frac{\operatorname{Tr}\hat{m}\exp(-\beta H_R)}{\pi R^2} \ , \tag{2.13}$$

with $\hat{m}=\frac{1}{2}\mathbf{r}\wedge\mathbf{v}$. The magnetisation at constant density $\varrho=\varrho_R$ is obtained by substituting $z=z(R)$ from (2.12) in (2.13).

The following proposition shows that the thermodynamic limit of the pressure and of the magnetisation is independent of the confining potential if the effective area of the region is taken to be πR^2. The proof is given in Appendix A.

Proposition 1. *For the soft wall* (2.4) *or the Dirichlet hard wall,*

$$\lim_{R\to\infty} p_R=p=\frac{zB}{4\pi\beta}\left[\operatorname{sh}\left(\frac{\beta B}{2}\right)\right]^{-1} \ , \tag{2.14}$$

$$\lim_{R\to\infty} m_R=m=\frac{\partial p}{\partial B} \ . \tag{2.15}$$

In fact, p is given by the functional integral

$$p=\frac{z}{\beta}\int \tilde{D}\boldsymbol{\alpha} \ , \tag{2.16}$$

with

$$\tilde{D}\boldsymbol{\alpha}=\frac{1}{2\pi\beta}\exp\left[i\beta\frac{\mathbf{B}}{2}\cdot\int_0^1 \boldsymbol{\alpha}(s)\wedge d\boldsymbol{\alpha}\right]D\boldsymbol{\alpha} \ . \tag{2.17}$$

For latter use, it will be convenient to represent the pressure as a functional integral involving a one dimensional Brownian bridge. This can be done with the help of the formula [8, Chap. V, p. 169]

$$\int D\alpha^y \exp\left(i\beta B\int_0^1 \alpha^x(s)d\alpha^y\right)=\exp\left[-\frac{\beta^2B^2}{2}\left(\int_0^1 \alpha^{x^2}(s)ds-\left(\int_0^1\alpha^x(s)ds\right)^2\right)\right] \ . \tag{2.18}$$

Introducing (2.18) in (2.16) gives $\left(\text{using} \int_0^1 \alpha^y(s)d\alpha^x=-\int_0^1 \alpha^x(s)d\alpha^y\right)$

$$\begin{aligned} p&=\frac{z}{2\pi\beta^2}\int D\alpha^x D\alpha^y \exp\left[i\frac{\beta B}{2}\int_0^1(\alpha^x(s)d\alpha^y-\alpha^y(s)d\alpha^x)\right] \\ &=\frac{z}{2\pi\beta^2}\int D\alpha \exp\left[-\frac{\beta^2B^2}{2}\left(\int_0^1 \alpha^2(s)ds-\left(\int_0^1 \alpha(s)ds\right)^2\right)\right] \ , \end{aligned} \tag{2.19}$$

where here α denotes the one dimensional Brownian bridge.

Note that p is not convex as a function of B. Therefore, we cannot deduce immediately, using the convergence of derivatives of sequences of convex functions, that $m_R=\frac{\partial p_R}{\partial B}$ converges to $\frac{\partial p}{\partial B}$.

III. The Surface Current in a Semi-Infinite System

By a semi-infinite system, we mean one in which the particles are practically confined to the half space of positive x by a potential independent of y and which increases rapidly for negative x. More precisely, the hamiltonian of the semi-infinite system is defined by

$$H^{\mathrm{s.i.}}=\tfrac{1}{2}|\mathbf{v}|^2+U(x) \ , \tag{3.1}$$

where $U(x)=\phi(-x)$, ϕ as in (2.4), and the corresponding one point function

$$\varrho^{(1)}(\mathbf{r};\tilde{\mathbf{r}})=z(\mathbf{r}|\exp(-\beta H^{\mathrm{s.i.}})|\tilde{\mathbf{r}}) \tag{3.2}$$

is given by the functional integral (2.9) with $V_R(\mathbf{r})$ replaced by $U(x)$ in (2.10). Because of translation invariance in the y direction, the current density depends only on x. Moreover, the hamiltonian (3.1) is invariant under the time reversal followed by the space inversion of the y coordinate. Under this transformation, the x component of the current density changes its sign, and thus vanishes. Hence, the current density $j(x)$ is parallel to the wall, and one finds from (3.2) and (2.9) that it is given by the functional integral

$$j(x)=[v_y\varrho^{(1)}(\mathbf{r};\tilde{\mathbf{r}})]_{\mathbf{r}=\tilde{\mathbf{r}}}=\left[\left(-i\frac{\partial}{\partial y}-\frac{B}{2}x\right)\varrho^{(1)}(\mathbf{r},\tilde{\mathbf{r}})\right]_{\mathbf{r}=\tilde{\mathbf{r}}}$$

$$=z\sqrt{\beta}B\int \tilde{D}\boldsymbol{\alpha}\left(\int_0^1\alpha^x(s)ds\right)\exp\left(-\beta\int_0^1 ds\,U(x+\sqrt{\beta}\,\alpha^x(s))\right). \qquad (3.3)$$

The representation (3.3) allows us to estimate the current as a function of the distance to the wall and to relate it to the thermodynamical magnetisation density m. Propositions 2 and 3 below show that $j(x)$ is localised near to the wall. Moreover, its integral is independent of the nature of the wall and has the correct relation to the infinite volume magnetisation.

We note that a similar result to Proposition 2 and its generalizations to non-interacting Fermions and to interacting systems given later can be obtained for particles confined to a circular region with more restrictions on the confining potentials.

The bounds on the decay rate are independent of the radius of the region.

Proposition 2. *For the soft wall* (2.4) *or the hard wall with Dirichlet conditions, there are constants C_1, C_2, C_3, and C_4 independent of β and B such that*

$$|j(x)|\leqq zB\left[\frac{C_1}{\sqrt{\beta}}e^{\beta\phi_0}\exp\left(-C_2\frac{x^2}{\beta}\right)+\frac{C_3}{\beta}|x|\exp(-C_4\beta x^2)\right]. \qquad (3.4)$$

In the case of a hard wall, $\phi_0=C_3=0$.

Proposition 3.

$$-\int_{-\infty}^{+\infty} dx\,j(x)=m\,, \qquad (3.5)$$

where m is given in Proposition 1.

For the proof of Proposition 2, we need the following lemma (see Appendix B).

Lemma 1. *If α is a one dimensional Brownian bridge on* $[0,1]$, *then there exists for each n constants A and C such that*

$$\int_{\sup_s|\alpha(s)|\geqq x}\sup_s|\alpha(s)|^n\leqq A\exp(-Cx^2)\,. \qquad (3.6)$$

Proof of Proposition 2. We split the current density in two contributions $j(x)=j_1(x)+j_2(x)$, where the integral (3.3) is on the set

$$\Delta_x(\boldsymbol{\alpha})=\left\{\boldsymbol{\alpha}\,|\sup_s\sqrt{\beta}|\alpha^x(s)|\leqq\frac{|x|}{2}\right\}$$

for $j_1(x)$ and on the complementary set for $j_2(x)$. For $x\geqq 0$, $j_1(x)$ represents the bulk part of the current. Indeed, if $\boldsymbol{\alpha}$ belongs to $\Delta_x(\boldsymbol{\alpha})$ for $x\geqq 0$, one has

$x+\sqrt{\beta}\,\alpha^x(s)\geqq\frac{x}{2}\geqq 0$, and therefore

$$j_1(x)=z\sqrt{\beta}\,B\int_{\sup_s\sqrt{\beta}|\alpha^x(s)|\leqq\frac{|x|}{2}}\tilde{D}\boldsymbol{\alpha}\left(\int_0^1\alpha^x(s)ds\right)\tag{3.7}$$

does not depend on the confining potential $U(x)$. Obviously $j_1(x)$ vanishes because of the invariance of $\tilde{D}\boldsymbol{\alpha}$ under the change $\boldsymbol{\alpha}\to-\boldsymbol{\alpha}$. For $x<0$, one has $\sqrt{\beta}\,\alpha(s)+x\leqq\frac{x}{2}$, and this implies by (2.4) that for $x\leqq -2x_0$,

$$|j_1(x)|\leqq\frac{z|B|}{2\pi\beta}\,\frac{|x|}{2}\exp\left(-\kappa\beta\,\frac{x^2}{4}\right).\tag{3.8}$$

The non-trivial part of the quantum mechanical current is the quantity $j_2(x)$ which involves the large fluctuations of the Brownian bridge. It follows from Lemma 1 that it has the gaussian bound

$$|j_2(x)|\leqq z\,\frac{|B|e^{\beta\phi_0}}{2\pi\sqrt{\beta}}\int_{\sup_s\sqrt{\beta}|\alpha^x(s)|\geqq\frac{|x|}{2}}D\alpha^x\sup_s|\alpha^x(s)|\leqq\frac{z|B|}{2\pi\sqrt{\beta}}\,e^{\beta\phi_L}A\exp\left(-C\,\frac{x^2}{\beta}\right).\tag{3.9}$$

The combination of (3.8) and (3.9) gives the result of Proposition 2.

We now turn to the proof of Proposition 3.

Proof of Proposition 3. We note from the formula (2.19) that

$$\frac{\partial p}{\partial B}=-\frac{zB}{2\pi}\int D\alpha G(\alpha)\exp\left(-\frac{\beta^2B^2}{2}\,G(\alpha)\right),\tag{3.10}$$

$$G(\alpha)=\int_0^1\alpha^2(s)ds-\left(\int_0^1\alpha(s)ds\right)^2.\tag{3.11}$$

On the other hand, writing the current (3.3) in terms of a one dimensional Brownian bridge α with the help of (2.18) gives

$$j(x)=\frac{zB}{2\pi\sqrt{\beta}}\int D\alpha\left(\int_0^1\alpha(s)ds\right)\exp-\left(\frac{\beta^2B^2}{2}\,G(\alpha)\right)K(\alpha;x),\tag{3.12}$$

$$K(\alpha;x)=\exp\left(-\beta\int_0^1 ds\,U(x+\sqrt{\beta}\,\alpha(s))\right).\tag{3.13}$$

We have to show that the x integral of (3.12) is identical to (3.10). For this we use the following property of the Brownian bridge (proof in Appendix B).

Lemma 2. *For any $s,u\in[0,1]$, define $\gamma(s)=\alpha(\widetilde{s+u})-\alpha(u)$, where $\tilde{v}=v$ [mod 1]. Then for any fixed $u\in[0,1]$, γ is equivalent in probability to α.*

Since $\int_0^1 f(\widetilde{s+u})ds=\int_0^1 f(s)ds$, $u\in[0,1]$ one verifies easily that $G(\gamma)=G(\alpha)$ and $K(\gamma;x)=K(\alpha;x-\sqrt{\beta}\,\alpha(u))$. Applying the lemma to the integral (3.12) gives then

$$j(x)=\frac{zB}{2\pi\sqrt{\beta}}\int D\alpha\exp\left(-\frac{\beta^2B^2}{2}G(\alpha)\right)\cdot\int_0^1 ds(\alpha(s)-\alpha(u))K(\alpha;x-\sqrt{\beta}\,\alpha(u))\ . \tag{3.14}$$

$j(x)$ being independent of u, we integrate both members of (3.14) on $u\in[0,1]$,

$$j(x)=\frac{zB}{2\pi\sqrt{\beta}}\int D\alpha\exp\left(-\frac{\beta^2B^2}{2}G(\alpha)\right)\cdot\int_0^1 du\int_0^1 ds(\alpha(s)-\alpha(u))K(\alpha;x-\sqrt{\beta}\,\alpha(u)) \tag{3.15}$$

$$=\frac{zB}{2\pi\sqrt{\beta}}\int D\alpha\exp\left(-\frac{\beta^2B^2}{2}G(\alpha)\right)\int_0^1 du\int_0^1 ds(\alpha(s)-\alpha(u))\ .$$

$$[K(\alpha;x-\sqrt{\beta}\,\alpha(u))-K(\alpha;x)]\ . \tag{3.16}$$

The subtraction of $K(\alpha,x)$ obviously does not contribute to the integral, since

$$\int_0^1 du\int_0^1 ds(\alpha(s)-\alpha(u))=0\ . \tag{3.17}$$

It follows from the properties of ϕ that

$$K(\alpha;x)=K(\alpha;x-\sqrt{\beta}\,\alpha(u))=1 \quad \text{for}\quad x\geqq a\equiv 2\sqrt{\beta}\sup_s|\alpha(s)| \tag{3.18}$$

and

$$|K(\alpha;x)|\leqq\exp[-\kappa(x+a)^2]\ ,\qquad |K(\alpha;x-\sqrt{\beta}\,\alpha(u))|\leqq\exp[-\kappa(x+a)^2] \quad \text{for}\quad x<-a-x_0\ . \tag{3.19}$$

This implies that $[K(\alpha;x-\sqrt{\beta}\,\alpha(u))-K(\alpha;x)]$ is integrable on x and that there exist constants A and B independent of α such that

$$\int_{-\infty}^{+\infty} dx|K(\alpha;x-\sqrt{\beta}\,\alpha(u))-K(\alpha;x)|\leqq A+B\sup_s|\alpha(s)|\ . \tag{3.20}$$

In view of this we are allowed to interchange the dx and the $D\alpha$ integrals when we calculate $\int_{-\infty}^{+\infty} dx j(x)$ from (3.16). We have with (3.18), (3.19) and b large enough $(b>a)$

$$\int_{-\infty}^{+\infty} dx(K(\alpha;x-\sqrt{\beta}\,\alpha(u))-K(\alpha;x))=\int_{-\infty}^{b} dxK(\alpha;x-\sqrt{\beta}\,\alpha(u))-\int_{-\infty}^{b} dxK(\alpha;x)$$

$$=\int_b^{b-\sqrt{\alpha}(u)} dxK(\alpha;x)=-\sqrt{\beta}\,\alpha(u)\ . \tag{3.21}$$

Thus

$$-\int_{-\infty}^{+\infty} dx j(x)=\frac{zB}{2\pi}\int D\alpha \exp\left(-\frac{\beta^2 B^2}{2}G(\alpha)\right)\int_0^1 du\int_0^1 ds(\alpha(s)-\alpha(u))\alpha(u) ,$$

which is identical to (3.10). This concludes the proof of the proposition.

Let us indicate still another representation for m

$$m=-\frac{zB}{2\pi}\int D\alpha \exp\left(-\frac{\beta^2 B^2}{2}G(\alpha)\right)\left(\int_0^1 \alpha(s)ds\right)^2 . \tag{3.22}$$

In view of the lemma, we can replace $\alpha(s)$ by $\gamma(s)$ in the integrand of (3.22), and integrate on $u\in[0,1]$. The equality

$$\int_0^1 du\left(\int_0^1 \gamma(s)ds\right)^2=\int_0^1 \alpha^2(s)ds-\left(\int_0^1 \alpha(s)ds\right)^2$$

shows that (3.22) is identical to (3.10).

Infinite Strip

The current density of a system of particles in an infinite strip in the y-direction is given by the formula (3.3), where the potential $U(x)$ is now confining both for $x>0$ and $x<0$,

$$U(x)\geqq \kappa x^2 , \quad |x|\geqq x_0 . \tag{3.23}$$

To calculate the total current in the strip, we proceed as in Proposition 3, and integrate (3.15) on x. Since now $K(\alpha;x)$ is integrable at $x=\infty$ and $x=-\infty$, we can exchange the dx and the $D\alpha$ integrals in (3.15),

$$\int_{-\infty}^{+\infty} dx j(x)=\frac{zB}{2\pi\sqrt{\beta}}\int D\alpha \exp\left(-\frac{\beta^2 B^2}{2}G(\alpha)\right)\int_0^1 du\int_0^1 ds(\alpha(s)-\alpha(u))\int_{-\infty}^{+\infty} dx K(\alpha;x) , \tag{3.24}$$

which vanishes because of (3.17). This means that the currents flowing along the left and right walls always compensate exactly independently of the shape of $U(x)$.

Fermi Statistics

For systems of non-interacting fermions we can obtain analogous results for low activity since the Fermi quantities can be expanded in power series in z with the corresponding Boltzmann quantities at inverse temperature $n\beta$ as the coefficient of $(-z)^n$. In the rest of this section the Fermi-quantities will be labelled by F. For $0\leqq z<e^{-\beta\phi_0}$, we set

$$p_R^F=\sum_{n=1}^{\infty}\frac{(-z)^{n-1}}{n}p_R^{(n\beta)} , \tag{3.25}$$

$$m_R^F=\sum_{n=1}^{\infty}(-z)^{n-1}m_R^{(n\beta)} , \tag{3.26}$$

$$j^F(x)=\sum_{n=1}^{\infty}(-z)^{n-1}j^{(n\beta)}(x) . \tag{3.27}$$

We can now apply Propositions 1, 2, 3 term by term to obtain exactly the same results as in these propositions except for the statement in Proposition 2 which now becomes

$$|j^F(x)| \leqq |B| \frac{C_1}{\sqrt{\beta}} \left\{ \sum_{n=1}^{\infty} (ze^{\beta\phi_0})^n e^{-C_2 \frac{x^2}{n\beta}} \right\} + |B| C_3 \frac{|x|}{\beta} \frac{ze^{-\beta C_4 x^2}}{1 - ze^{-\beta C_4 x^2}} . \tag{3.28}$$

The decay of the first term is no longer Gaussian but we can obtain for it an exponential bound of the form $|B| \frac{A_1}{\sqrt{\beta}} \exp\left(-A_2 \frac{x}{\sqrt{\beta}}\right)$, where the constants A_1 and A_2 depend on the combination $z \exp \beta\phi_0$. This rate of decay for non-zero temperature should be contrasted with the asymptotic form for the groundstate obtained in [4] in the framework of linear response. The latter has an envelope which decays like x^{-1}.

Finally, we note that by the Schwartz inequality $G(x) > 0$, and therefore by (3.10) for Boltzmann statistics, m has the opposite sign to B [see also (3.22)]. We emphasize that this is only true for Boltzmann statistics since it is well known that for fermions, we have the De Haas-Van Alphen oscillations in the magnetization as a function of B [1].

IV. Interacting Particles

(*a*) *General Setting*. In this section, we study the diamagnetic current of a dilute system of interacting particles in three dimensions with a translation and rotation invariant stable pair potential $w(\mathbf{r})$,

$$\sum_{i<j}^{n} w(\mathbf{r}_i - \mathbf{r}_j) \geqq -bn , \quad 0 \leqq b < \infty . \tag{4.1}$$

We assume the following regularity conditions: $w(\mathbf{r})$ is differentiable with uniformly bounded derivatives

$$|\nabla w(\mathbf{r})| \leqq M , \tag{4.2}$$

and

$$\|w\|_1 = \int d\mathbf{r} |w(\mathbf{r})| < \infty , \quad \|\nabla w\|_1 = \int d\mathbf{r} |\nabla w(\mathbf{r})| < \infty . \tag{4.3}$$

We consider only the Maxwell Boltzmann statistics and use the Ginibre functional integral representation of equilibrium density matrices [6], written in terms of the Brownian bridge,

$$\begin{aligned} ((\mathbf{r})_n | \exp(-\beta H) | (\tilde{\mathbf{r}})_n) = & \prod_{j=1}^{n} \left[\frac{1}{(2\pi\beta)^{3/2}} e^{-\frac{|\mathbf{r}_j - \tilde{\mathbf{r}}_j|^2}{2\beta}} \right] \\ & \cdot \int D(\boldsymbol{\alpha})_n \prod_{j=1}^{n} \exp G(\boldsymbol{\alpha}_j ; \mathbf{r}_j, \tilde{\mathbf{r}}_j) \\ & \cdot \exp\left(-\beta \sum_{i<j}^{n} \int_0^1 ds\, w[(1-s)(\mathbf{r}_i - \mathbf{r}_j) + s(\tilde{\mathbf{r}}_i - \tilde{\mathbf{r}}_j) \right. \\ & \left. + \sqrt{\beta}(\boldsymbol{\alpha}_i(s) - \boldsymbol{\alpha}_j(s))] \right) , \end{aligned} \tag{4.4}$$

where G is as in (2.10) and $D(\boldsymbol{\alpha})_n$ is the measure for n independent three dimensional Brownian bridges.

The Mayer expansion of the grand canonical quantities involve the Mayer factors in the form

$$M((\mathbf{r})_n, (\tilde{\mathbf{r}})_n; (\boldsymbol{\alpha})_n) = \sum_{\gamma \in \Gamma_n} \prod_{(ij) \in \gamma} \left\{ -1 + \exp\left(-\beta \int_0^1 ds\, w[(1-s)(\mathbf{r}_i - \mathbf{r}_j) + s(\tilde{\mathbf{r}}_i - \tilde{\mathbf{r}}_j) + \sqrt{\beta}(\boldsymbol{\alpha}_i(s) - \boldsymbol{\alpha}_j(s))] \right) \right\}, \tag{4.5}$$

Γ_n being the set of connected graphs on $(1, 2, \ldots n)$, and (i, j) runs on all bonds in γ.

The Mayer expansions of the grand canonical pressure and magnetisation density yields in the thermodynamic limit the series

$$\beta p = \sum_{n=1}^{\infty} p_n \,, \tag{4.6}$$

$$m = \sum_{n=1}^{\infty} m_n \,, \tag{4.7}$$

$$p_n = \frac{z^n}{n!} \int d(\mathbf{r})_{n-1} \int \tilde{D}(\boldsymbol{\alpha})_n M(0, (\mathbf{r})_{n-1}; (\boldsymbol{\alpha})_n) \,, \tag{4.8}$$

$$m_n = \frac{z^n}{n!} \int d(\mathbf{r})_{n-1} \int \tilde{D}(\boldsymbol{\alpha})_n \left(\sum_{j=1}^{n} \frac{i}{2} \int_0^1 \boldsymbol{\alpha}_j(s) \wedge d\boldsymbol{\alpha}_j \right) M(0, (\mathbf{r})_{n-1}; (\boldsymbol{\alpha})_n) \,, \tag{4.9}$$

where

$$\begin{aligned} (\mathbf{r})_{n-1} &= (\mathbf{r}_2, \ldots, \mathbf{r}_n) \,, \\ \tilde{D}(\boldsymbol{\alpha})_n &= \tilde{D}\boldsymbol{\alpha}_1 \ldots \tilde{D}\boldsymbol{\alpha}_n \,, \\ \tilde{D}\boldsymbol{\alpha} &= \frac{1}{(2\pi\beta)^{3/2}} \exp\left(i\beta \frac{\mathbf{B}}{2} \cdot \int_0^1 \boldsymbol{\alpha}(s) \wedge d\boldsymbol{\alpha} \right) D\boldsymbol{\alpha} \,, \end{aligned} \tag{4.10}$$

and

$$M((\mathbf{r})_n; (\boldsymbol{\alpha})_n) = M((\mathbf{r})_n, (\tilde{\mathbf{r}})_n; (\boldsymbol{\alpha})_n)|_{(\mathbf{r})_n = (\tilde{\mathbf{r}})_n} \,. \tag{4.11}$$

The thermodynamic limit is proven for each term of the series (4.6) and (4.7) by the same method as in Appendix A. Then the convergence of the series for $\beta \|w\|_1 (2\pi\beta)^{-3/2} e^{\beta b + 1} |z| < 1$ follows from an application of the identity (4.23) below. We do not give the details since a similar application will be presented for the current density.

(b) The Semi-Infinite System. We are interested in the current in the semi-infinite system with the wall potential $U(x) = \phi(-x)$ as in Sect. III. Here we assume for simplicity that $\phi(x)$ is positive and monotonically increasing. The corresponding hamiltonian is

$$H = \sum_{i=1}^{n} \left(\frac{1}{2} |\mathbf{v}_i|^2 + U(x_i) \right) + \sum_{i<j}^{n} w(\mathbf{r}_i - \mathbf{r}_0) \,. \tag{4.12}$$

The one point reduced density matrix of this semi-infinite system has the Mayer expansion

$$\varrho(\mathbf{r}_1,\tilde{\mathbf{r}}_1)=\sum_{n=1}^{\infty}\varrho_n(\mathbf{r}_1,\tilde{\mathbf{r}}_1)\ , \tag{4.13}$$

$$\begin{aligned}\varrho_n(\mathbf{r}_1,\tilde{\mathbf{r}}_1)=&\frac{z^n}{(n-1)!}\,\frac{1}{(2\pi\beta)^{3/2}}\,e^{-\frac{|\mathbf{r}_1-\tilde{\mathbf{r}}_1|^2}{2\beta}}\\ &\cdot\int d(\mathbf{r})_{n-1}\int D\boldsymbol{\alpha}_1\int\tilde{D}(\boldsymbol{\alpha})_{n-1}\exp G(\boldsymbol{\alpha}_1;\mathbf{r}_1,\tilde{\mathbf{r}}_1)\\ &\cdot\left[\prod_{j=2}^{n}\exp\left(-\beta\int_0^1 ds\,U(x_j+\sqrt{\beta}\,\alpha_j^x(s))\right)\right]M(\mathbf{r}_1,\tilde{\mathbf{r}}_1,(\mathbf{r})_{n=1};(\boldsymbol{\alpha})_n)\ ,\end{aligned} \tag{4.14}$$

where G is given by (2.10) with $V_R(\mathbf{r})$ replaced by $U(x)$ and

$$M(\mathbf{r}_1,\tilde{\mathbf{r}}_1,(\mathbf{r})_{n-1};(\boldsymbol{\alpha})_n)=M((\mathbf{r})_n,(\tilde{\mathbf{r}})_n;(\boldsymbol{\alpha})_n)|_{(\mathbf{r})_{n-1}=(\tilde{\mathbf{r}})_{n-1}}$$

By symmetry reasons, the only non-vanishing component of the current is along the y direction

$$j(x_1)=v_y\varrho(\mathbf{r}_1,\tilde{\mathbf{r}}_1)|_{\mathbf{r}_1=\tilde{\mathbf{r}}_1}=\left(-i\,\frac{\partial}{\partial y_1}-\frac{B}{2}\,x_1\right)\varrho(\mathbf{r}_1,\tilde{\mathbf{r}}_1)|_{\mathbf{r}_1=\tilde{\mathbf{r}}_1}\ . \tag{4.15}$$

Its Mayer expansion has the form

$$j(x_1)=j^{(1)}(x_1)+j^{(2)}(x_1)=\sum_{n=1}^{\infty}j_n^{(1)}(x_1)+\sum_{n=1}^{\infty}j_n^{(2)}(x_1)\ , \tag{4.16}$$

$$j_n^{(1)}(x_1)=\frac{z^nB\sqrt{\beta}}{(n-1)!}\int d(\mathbf{r})_{n-1}\int\tilde{D}(\boldsymbol{\alpha})_n\left(\int_0^1\alpha_1^x(s)\,ds\right)\prod_{j=1}^{n}K(\alpha_j^x;x_j)\,M((\mathbf{r})_n;(\boldsymbol{\alpha})_n)\ , \tag{4.17}$$

$$j_n^{(2)}(x_1)=-\frac{iz^n}{(n-1)!}\int d(\mathbf{r})_{n-1}\int\tilde{D}(\boldsymbol{\alpha})_n\prod_{j=1}^{n}K(\alpha_j^x;x_j)\,\frac{\partial}{\partial y_1}M(\mathbf{r}_1,\tilde{\mathbf{r}}_1,(\mathbf{r})_{n-1};(\boldsymbol{\alpha})_n)|_{\mathbf{r}_1=\tilde{\mathbf{r}}_1}\ , \tag{4.18}$$

where $K(\alpha;x)$ is as in (3.13). It is useful to remark that the factor $\prod_{j=1}^{n}K(\alpha_j^x;x_j)$ can be replaced by $\prod_{j=1}^{n}K(\alpha_j^x;x_j)-1$ in the formulas (4.17) and (4.18).

The factor $\prod_{j=1}^{n}K(\alpha_j^x;x_j)$ contains the effects of the wall. If it is replaced by 1, we obtain formally the "bulk current"

$$j_n^{(1)\,\mathrm{bulk}}=\frac{z^nB\sqrt{\beta}}{(n-1)!}\int\tilde{D}\boldsymbol{\alpha}_1\left(\int_0^1\alpha_1^x(s)\,ds\right)\left[\int d(\mathbf{r})_{n-1}\int\tilde{D}(\boldsymbol{\alpha})_{n-1}M(0,(\mathbf{r})_{n-1};(\boldsymbol{\alpha})_n)\right]\ , \tag{4.19}$$

$$j_n^{(2)\,\mathrm{bulk}}=-\frac{iz^n}{(n-1)!}\,\frac{\partial}{\partial y_1}\int d(\mathbf{r})_{n-1}\int\tilde{D}(\boldsymbol{\alpha})_nM(\mathbf{r}_1,\tilde{\mathbf{r}}_1,(\mathbf{r})_{n-1};(\boldsymbol{\alpha})_n)|_{\mathbf{r}_1=\tilde{\mathbf{r}}_1}\ . \tag{4.20}$$

Both quantities are equal to zero. The bracket [...] in (4.19) is independent of $\mathbf{r}_1$ because of the translational invariance of $M((\mathbf{r})_n;(\boldsymbol{\alpha})_n)$; moreover it is an even function of $\boldsymbol{\alpha}_1$, since $M(-(\mathbf{r})_n;\ -(\boldsymbol{\alpha})_n)=M((\mathbf{r})_n;(\boldsymbol{\alpha})_n)$. Thus the integrand in (4.19) is an odd function of $\boldsymbol{\alpha}_1$, and the integral vanishes.

By inspection of (4.5), and because of the same invariance properties of M, the quantity

$$\int d(\mathbf{r})_{n-1} \int \tilde{D}(\boldsymbol{\alpha})_n M(\mathbf{r}_1,\tilde{\mathbf{r}}_1,(\mathbf{r})_{n-1};(\boldsymbol{\alpha})_n)|_{x_1=\tilde{x}_1}=f(y_1-\tilde{y}_1)$$

is an even function of $(y_1-\tilde{y}_1)$, therefore the y_1 derivative at $y_1=\tilde{y}_1$ vanishes.

It will be clear from the estimates in paragraph (d) that the above expressions are well defined and the low activity expansions converge for z small enough when the potential satisfies the conditions (4.1) to (4.3).

(c) The Main Propositions. The main tool to handle the low activity series is given by the following remarkable identity [7]: Let V_{ij}, $i,j=1,2,\dots,n$, satisfy the stability condition

$$\sum_{i<j}^{n} V_{ij} \geqq -bn \ , \tag{4.21}$$

and define

$$\begin{gathered} V_n(\sigma)=\sum_{i<j}^{n} V_{ij}\sigma_{ij} \ , \\ \sigma_{ij}=\sigma_{ji} \ , \quad 0\leqq\sigma_{ij}\leqq 1 \ , \quad \sigma=(\sigma_{ij})_{ij=1,\dots,n} \ . \end{gathered} \tag{4.22}$$

To each connected tree graph T on $1,2,\dots,n$ is associated a probability measure $dP_T(\sigma)$ so that

$$\sum_{\gamma\in\Gamma_n}\prod_{(ij)\in\gamma}\{\exp(-V_{ij})-1\}=\sum_{T}\prod_{(ij)\in T}(-V_{ij})\int dP_T(\sigma)\exp(-V_n(\sigma)) \ . \tag{4.23}$$

The measure $dP_T(\sigma)$ is supported on σ such that $V_n(\sigma)$ still satisfies the stability inequality

$$V_n(\sigma)\geqq -bn \ . \tag{4.24}$$

With the help of (4.23), we can show that the current density remains localized in the neighborhood of the wall at sufficiently low activity, and its integral gives the thermodynamic magnetisation (4.7). This is the content of the next two propositions.

Proposition 4. (i) *The series* (4.16) *are absolutely convergent if*

$$\beta\max(\|w\|_1,\|\nabla w\|_1)\frac{e^{\beta b+1}}{(2\pi\beta)^{3/2}}|z|<1 \ . \tag{4.25}$$

(ii) *Moreover, if* $w(\mathbf{r})$ *and* $\nabla w(\mathbf{r})$ *are* $O\left(\frac{1}{|\mathbf{r}|^{\nu+\varepsilon}}\right)$, $\nu\geqq 3$, $\varepsilon>0$, *then there exists* $\delta>0$ *such that*

$$|j(x_1)|=O\left(\frac{1}{|x_1|^{\nu-3+\delta}}\right) \ . \tag{4.26}$$

In particular if $\nu\geqq 4$ *the total current* $\int_{-\infty}^{+\infty} dx_1 j(x_1)$ *is finite.*

(iii) *If $w(\mathbf{r})$ has compact support, one finds a faster decay*

$$|j(x_1)| \leqq C_3 \exp(-C_4 |x_1|^{2/3}) \ . \tag{4.27}$$

Proposition 5. *If* (4.25) *holds and $v \geqq 4$, one has*

$$m = -\int_{-\infty}^{+\infty} dx j(x) \ . \tag{4.28}$$

(d) Proof of Proposition 4. The proof of Proposition 4 relies on the application of the identity (4.23) to (4.17) and (4.18). We treat in detail the term (4.17) and will give later the necessary modifications to handle (4.18).

(α) *Bounds on the Mayer Factors.* We first define

$$\bar{w}(\mathbf{r}, \tilde{\mathbf{r}}; \boldsymbol{\alpha}) = \int_0^1 ds\, w((1-s)\mathbf{r} + s\tilde{\mathbf{r}} + \sqrt{\beta}\, \boldsymbol{\alpha}(s)) \ , \tag{4.29}$$

and apply the formula (4.23) to (4.5) with

$$V_{ij} = \beta \bar{w}(\mathbf{r}_i - \mathbf{r}_j, \tilde{\mathbf{r}}_i - \tilde{\mathbf{r}}_j; \boldsymbol{\alpha}_i - \boldsymbol{\alpha}_j) = V_{ji} \ . \tag{4.30}$$

Since the potential w is stable, it is clear that V_{ij} defined by (4.30) satisfies also the stability condition (4.21).

It follows from (4.24) that

$$|M((\mathbf{r})_n; (\boldsymbol{\alpha})_n)| \leqq \beta^{n-1} e^{\beta b n} \sum_T \prod_{(ij) \in T} |\bar{w}(\mathbf{r}_i - \mathbf{r}_j; \boldsymbol{\alpha}_i - \boldsymbol{\alpha}_j)| \ , \tag{4.31}$$

where

$$\bar{w}(\mathbf{r}; \boldsymbol{\alpha}) = \bar{w}(\mathbf{r}, \mathbf{r}; \boldsymbol{\alpha}) = \int_0^1 ds\, w(\mathbf{r} + \sqrt{\beta}\, \boldsymbol{\alpha}(s)) \ . \tag{4.32}$$

Notice that we have obviously

$$\int d\mathbf{r} |\bar{w}(\mathbf{r}; \boldsymbol{\alpha})| \leqq \|w\|_1 \ . \tag{4.33}$$

The derivative of the Mayer factor occurring in (4.18) yields two terms

$$\frac{\partial}{\partial y_1} M((\mathbf{r})_n, (\mathbf{r})_n; (\boldsymbol{\alpha})_n) = \sum_T \prod_{(ij) \in T} (-V_{ij}) \int dP_T(\sigma) \left(-\frac{\partial}{\partial y_1} V_n(\sigma) \right) \exp(-V_n(\sigma)) \tag{4.34}$$

$$+ \sum_T \left[\frac{\partial}{\partial y_1} \prod_{(ij) \in T} (-V_{ij}) \right] \int dP_T(\sigma) \exp(-V_n(\sigma)) \ . \tag{4.35}$$

The structure (4.22) of $V_n(\sigma)$ shows that

$$\frac{\partial}{\partial y_1} V_n(\sigma) = \sum_{j \neq 1}^{n} \left(\frac{\partial}{\partial y_1} V_{1j} \right) \sigma_{1j} \ . \tag{4.36}$$

The assumption (4.2) that the pair potential has a bounded derivative in conjunction with the definitions (4.29), (4.30) implies that

$$\left|\frac{\partial}{\partial y_1} V_{ij}\right| = \left|\int_0^1 ds(1-s)(\nabla_y w)\left[(1-s)(\mathbf{r}_1-\mathbf{r}_j)+s(\tilde{\mathbf{r}}_1-\tilde{\mathbf{r}}_j)+\sqrt{\beta}\,(\boldsymbol{\alpha}_1(s)-\boldsymbol{\alpha}_j(s))\right]\right| \leqq M \ , \tag{4.37}$$

and therefore

$$\left|\frac{\partial}{\partial y_1} V_n(\sigma)\right| \leqq nM \ . \tag{4.38}$$

Using (4.38) and (4.24) we find that

$$\left|\left[\frac{\partial}{\partial y_1} M((\mathbf{r})_n, (\tilde{\mathbf{r}})_n; (\boldsymbol{\alpha})_n)\right]_{(\mathbf{r})_n=(\tilde{\mathbf{r}})_n}\right| \leqq nM\beta^{n-1} e^{\beta bn} \sum_T \prod_{(ij)\in T} |\bar{w}(\mathbf{r}_i-\mathbf{r}_j;\boldsymbol{\alpha}_i-\boldsymbol{\alpha}_j)| \tag{4.39}$$

$$+\beta^{n-1} e^{\beta bn} \sum_T \left|\left[\frac{\partial}{\partial y_1} \prod_{(ij)\in T} \bar{w}(\mathbf{r}_i-\mathbf{r}_j, \tilde{\mathbf{r}}_i-\tilde{\mathbf{r}}_j;\boldsymbol{\alpha}_i-\boldsymbol{\alpha}_j)\right]_{(\mathbf{r})_n=(\tilde{\mathbf{r}})_n}\right| \ . \tag{4.40}$$

(β) *Estimate of the Decay of* $j_n^{(1)}(x_1)$. According to the remark below (4.18), we can replace in (4.17) $\prod_{j=1}^{n} K(\alpha_j^x, x_j)$ by

$$\prod_{j=1}^{n} K(\alpha_j^x, x_j) - \theta(x_1), \ (\theta(x_1)=1, x_1>0; \theta(x_1)=0, x_1<0) \ .$$

Using the bound (4.31) on the Mayer factor, one finds

$$|j_n^{(1)}(x_1)| \leqq \frac{B}{\sqrt{\beta}\,(n-1)!}\left(\frac{ze^{\beta b}}{\sqrt{(2\pi)^3\beta}}\right)^n \prod_T I_{n,T}(x_1) \ , \tag{4.41}$$

with

$$I_{n,T}(x_1) = \int d(\mathbf{r})_{n-1} \int D(\boldsymbol{\alpha})_n \sup_s |\alpha_1^x(s)| \left|\prod_{j=1}^{n} K(\alpha_j^x; x_j) - \theta(x_1)\right| \prod_{(ij)\in T} |\bar{w}(\mathbf{r}_i-\mathbf{r}_j;\boldsymbol{\alpha}_i-\boldsymbol{\alpha}_j)| \ . \tag{4.42}$$

Notice that in the integral $I_{n,T}(x_1)$ we can always relabel the dummy integration variables $(\mathbf{r})_n$, $(\alpha)_n$, in such a way that the tree T is ordered with root 1. The ordered tree has $(n-1)$ bonds denoted by $(j<i)$, $i=2,3,\ldots,n$. To each point $k\neq 1$ in T, we associate the branch going from 1 to k with $|B_k|$ points

$$B_k = \{1, k_2, \ldots, k_{l-1}, k;\ 1<k_2<\ldots<k_{l-1}<k\} \ , \qquad \mathbf{b}_k = \sum_{i\in B_k} \mathbf{r}_i \ . \tag{4.43}$$

A successive change of variable in (4.42) (starting from the maximal points in the tree) leads to

$$I_{n,T}(x_1) = \int d(\mathbf{r})_{n-1} \int D(\boldsymbol{\alpha})_n \sup_s |\alpha_1^x(s)| \left[\prod_{k=1}^{n} K(\alpha_k^x; b_k^x) - \theta(x_1)\right]$$

$$\cdot \prod_{(j<i)\in T}^{n} |\bar{w}(\mathbf{r}_i;\boldsymbol{\alpha}_i-\boldsymbol{\alpha}_j)| \ . \tag{4.44}$$

When x_1 is positive, we use the identity

$$\prod_{j=1}^{n} K_j - 1 = (K_1 - 1)\prod_{j=2}^{n} K_j + \ldots + (K_{n-1} - 1)K_n + (K_n - 1) \ . \tag{4.45}$$

With $|K(\mathbf{r}, \boldsymbol{\alpha}| < 1$ and (4.33), $I_{n,T}$ is estimated as follows:

$$\begin{aligned} I_{n,T}(x_1) &\leqq \sum_{k=1}^{n} \int d(\mathbf{r})_{n-1} \int D(\boldsymbol{\alpha})_n \sup_s |\alpha_1^x(s)|\, |K(\alpha_k^x; b_k^x) - 1| \prod_{(j<i)\in T}^{n} |\bar{w}(\mathbf{r}_i; \boldsymbol{\alpha}_i - \boldsymbol{\alpha}_j)| \\ &\leqq \sum_{k=1}^{n} \|w\|_1^{n-|B_k|} I_k(x_1) \end{aligned} \tag{4.46}$$

$$I_k(x_1) = \int D\boldsymbol{\alpha}_1 \sup_s |\alpha_1^x(s)| \int \prod_{\substack{i\in B_k \\ i\neq 1}} D\boldsymbol{\alpha}_i d\mathbf{r}_i |K(\alpha_k^x; b_k^x) - 1| \prod_{(j<i)\in B_k} |\bar{w}(\mathbf{r}_i; \boldsymbol{\alpha}_i - \boldsymbol{\alpha}_j)| \ . \tag{4.47}$$

An upper bound on the decay of $I_{n,T}(x_1)$ can be obtained from Lemma 3 which is proved in Appendix C.

Lemma 3.

$$|I_{n,T}(x_1)| \leqq n^2 \left[C_1 \|w\|_1^{n-1} \exp\left(-D_1 \frac{x_1^2}{n^2} \right) + C_2 \|w\|_1^{n-2} \int_{|\mathbf{r}|\geqq \frac{x_1}{4n}}^{\infty} d\mathbf{r} |w(\mathbf{r})| \right], \quad x_1 > 0 \ , \tag{4.48}$$

$$|I_{n,T}(x_1)| \leqq C_3 \|w\|_1^{n-1} \exp(-D_2 x_1^2) \ , \quad x_1 < 0 \ . \tag{4.49}$$

The constants $C_1, \ldots, D_1, \ldots$ do not depend on n and $\|w\|_1$.

(γ) *Proof of the Proposition for* $j^{(1)}(x)$

(i) By (4.41) and the fact there are n^{n-2} tree graphs with n points, we have for $x_1 > 0$,

$$|j_n^{(1)}(x_1)| \leqq B\sqrt{\beta}\, \frac{n^n z^n}{(n-1)!} \left[C_1 a^{n-1} \exp\left(-D \frac{x_1^2}{n^2} \right) + C_2 a^{n-2} \int_{|\mathbf{r}|\geqq \frac{x_1}{4n}}^{\infty} d\mathbf{r} |w(\mathbf{r})| \right], \tag{4.50}$$

with

$$a = \frac{e^{\beta b} \|w\|_1}{((2\pi)^3 \beta)^{1/2}} \ ,$$

and for $x_1 < 0$,

$$|j_n^{(1)}(x_1)| \leqq C_1 \frac{n^{n-2}}{(n-1)!} z^n a^{n-1} \exp(-D x_1^2) \ . \tag{4.51}$$

Using Stirling's formula, it is clear that the series (4.17) converges uniformly with respect to x_1 provided that

$$|zea| < 1 \ . \tag{4.52}$$

(ii) To estimate the x_1 dependence of $j^{(1)}(x_1)=\sum_{n=0}^{\infty} j_n^{(1)}(x_1)$, we split this sum in two parts

$$\sum_n j_n^{(1)}(x_1)=\sum_{n<x_1^\gamma} j_n^{(1)}(x_1)+\sum_{n>x_1^\gamma} j_n^{(1)}(x_1) \tag{4.53}$$

for some $\gamma, 0<\gamma<1$.

By (4.50) the first part of the sum (4.53) is majorized by the sum of two terms which are

$$O(\exp(-Dx_1^{2(1-\gamma)})) \tag{4.54}$$

and

$$O\left(\int_{|\mathbf{r}|>\frac{1}{4}x_1^{1-\gamma}} d\mathbf{r}\,|w(\mathbf{r})|\right) . \tag{4.55}$$

The second part of the sum is less than

$$\sum_{n>x_1^\gamma} \frac{n^n}{(n-1)!} z^n[C_1 a^{n-1}+C_2\|w\|_1 a^{n-2}]=O(|zea|^{x_1^\gamma}) . \tag{4.56}$$

Under the assumption on the decay of the potential one has

$$\int_{|\mathbf{r}|>\frac{1}{4}x_1^{1-\gamma}} |w(\mathbf{r})|=O\left(\frac{1}{|x_1|^{\nu-3+\delta}}\right) , \tag{4.57}$$

with $\delta=\varepsilon-\gamma(\nu+\varepsilon-3)$.

Choosing $0<\gamma<1$ such that $\delta>0$, the estimates (4.54), (4.55), (4.56) lead to the bound (4.26) for $j^{(1)}(x_1)$, $x_1>0$. It follows from (4.51) that the bound is gaussian when $x_1<0$.

(iii) If $w(\mathbf{r})$ has compact support, the term (4.55) does not contribute for x_1 large enough, and the optimal choice of γ in (4.54) and (4.56) is $\gamma=2/3$.

(δ) *Proof of the Proposition for* $j^{(2)}(x_1)$. The treatment of $j^{(2)}(x_1)$ is essentially the same as that of $j^{(1)}(x_1)$; $j^{(2)}(x_1)$ is estimated by the sum of two terms coming from (4.39) and (4.40). Comparing with (4.31), it is obvious that the contribution (4.39) can be treated exactly as $j^{(1)}(x_1)$.

By the Leibnitz rule of derivation, the bracket in (4.40) is the sum of at most $(n-1)$ products where one of the $\bar{w}$ has been replaced by its derivative

$$\frac{\partial}{\partial y}\,\bar{w}(\mathbf{r}-\mathbf{r}_k,\tilde{\mathbf{r}}-\tilde{\mathbf{r}}_k\,;\boldsymbol{\alpha}-\boldsymbol{\alpha}_k)\Big|_{\substack{\mathbf{r}=\tilde{\mathbf{r}}\\ \mathbf{r}_k=\tilde{\mathbf{r}}_k}}=\int_0^1 ds(1-s)(\nabla_y w)(\mathbf{r}-\mathbf{r}_k+\sqrt{\beta}(\boldsymbol{\alpha}(s)-\boldsymbol{\alpha}_k(s))) . \tag{4.58}$$

Therefore, these products are of the form

$$\prod_{(ij)\in T} \bar{w}_{(ij)}(\mathbf{r}_i-\mathbf{r}_j\,;\boldsymbol{\alpha}_i-\boldsymbol{\alpha}_j) , \tag{4.59}$$

where one of the $\bar{w}_{(ij)}$ is given by (4.58) and the others are equal to $\bar{w}$.

One has to estimate $(n-1)$ quantities of the type (4.42) with a product

$$\prod_{(ij)\in T} \bar{w}_{(ij)}(\mathbf{r}_i-\mathbf{r}_j;\boldsymbol{\alpha}_i(s)-\boldsymbol{\alpha}_j(s)) \ .$$

One sees that all the estimates can be performed in the same way provided that $\|w\|_1$ is replaced by $\max(\|w\|_1, \|\nabla w\|_1)$.

(e) *Proof of Proposition 5*. It follows from the proof of Proposition 4 that $\sum_{n=1}^{\infty} |j_n(x)|$ is integrable. So we get by dominated convergence that

$$\int_{-\infty}^{+\infty} dx_1 j(x_1) = \sum_{n=1}^{\infty} \int_{-\infty}^{+\infty} dx_1 j_n(x_1) \ , \tag{4.60}$$

and thus it is sufficient to establish the equality (4.28) term by term.

We first transform the expression (4.9) of m_n by using the integration by parts formula of Gaussian measures [9],

$$\int D\alpha\, \alpha(s) F(\alpha) = \int_0^1 dt \int D\alpha\, C(s,t) \frac{\delta}{\delta\alpha(t)} F(\alpha) \ . \tag{4.61}$$

Here α is the one dimensional brownian bridge with covariance $C(s,t)$ given in (2.7). Since $M(\mathbf{r})_n;(\boldsymbol{\alpha})_n)$ is a symmetric function, m_n can be written in the form

$$m_n = \frac{iz^n}{(n-1)!} \frac{1}{(2\pi\beta)^{3/2}} \int D\boldsymbol{\alpha}_1 \left(\int_0^1 \alpha_1^y(s)\, d\alpha_1^x \right)$$

$$\cdot \exp\left(i\beta B \int_0^1 \alpha_1^y(t)\, d\alpha_1^x \right) \int d(\mathbf{r})_{n-1} \int \tilde{D}(\alpha)_{n-1} M((\mathbf{r})_n;(\boldsymbol{\alpha})_n) \ . \tag{4.62}$$

Performing now the integration by part with respect to the one dimensional brownian bridge α_1^y, we get two contributions

$$m_n = m_n^{(1)} + m_n^{(2)} \ , \tag{4.63}$$

$$m_n^{(1)} = -\frac{\beta B z^n}{(n-1)!} \int d(\mathbf{r})_{n-1} \int \tilde{D}(\boldsymbol{\alpha})_n \int d\alpha_1^x(s) \int d\alpha_1^x(t) C(s,t)\, M((\mathbf{r})_n;(\boldsymbol{\alpha})_n) \ , \tag{4.64}$$

$$m_n^{(2)} = \frac{iz^n}{(n-1)!} \int \tilde{D}(\boldsymbol{\alpha})_n \int_0^1 dt \int d\alpha_1^x(s) C(s,t) \ .$$

$$\cdot \frac{\delta}{\delta\alpha_1^y(t)} \left[\int d(\mathbf{r})_{n-1} M(0,(\mathbf{r})_{n-1};(\boldsymbol{\alpha})_n) \right] \ . \tag{4.65}$$

We will show that

$$\int_{-\infty}^{+\infty} dx_1 j^{(1)}(x_1) = -m^{(1)} \ , \tag{4.66}$$

$$\int_{-\infty}^{+\infty} dx_1 j^{(2)}(x_1) = -m^{(2)} \ . \tag{4.67}$$

We prove the first equality (4.66).

For this, we apply Lemma 2 to (4.17),

$$j_n^{(1)}(x_1)=\frac{z^n B\sqrt{\beta}}{(n-1)!}\int d(\mathbf{r})_{n-1}\int \tilde{D}(\boldsymbol{\alpha})_n\int_0^1 ds(\alpha_1^x(s)-\alpha_1^x(u)) \cdot$$

$$\cdot\prod_{j=1}^n K(\alpha_j^x;x_j-\sqrt{\beta}\,\alpha_j^x(u))M((\mathbf{r}-\sqrt{\beta}\,\boldsymbol{\alpha}(u))_n;(\boldsymbol{\alpha})_n)$$

$$=\frac{z^n B\sqrt{\beta}}{(n-1)!}\int d(\mathbf{r})_{n-1}\int \tilde{D}(\boldsymbol{\alpha})_n\int_0^1 ds(\alpha_1^x(s)-\alpha_1^x(u) \cdot$$

$$\cdot K(\alpha_1^x;x_1-\sqrt{\beta}\,\alpha_1^x(u))\prod_{j=2}^n K(\alpha_j^x;x_1+x_j-\sqrt{\beta}\,\alpha_1^x(u)) \cdot$$

$$\cdot M(0,(\mathbf{r})_{n-1};(\boldsymbol{\alpha})_n) \ . \tag{4.68}$$

The expression (4.68) results from the change of integration variables $\mathbf{r}_j\to\mathbf{r}_j+\mathbf{r}_1+\sqrt{\beta}\,(\boldsymbol{\alpha}_j(u)-\boldsymbol{\alpha}_1(u))$ for $j=2,\ldots,n$ and from the translation invariance of the Mayer factor (4.11). We can now proceed exactly as in the proof of Proposition 3. After having integrated (4.68) on u, we can subtract the same quantity with $\sqrt{\beta}\,\alpha_1^x(u)$ set equal to zero in the arguments of the K as in (3.16). Exchanging the order of integration, the x_1 integral of this difference for fixed $((x)_{n-1};(\alpha^x)_n)$ gives the same result as (3.21). Therefore, we get

$$\int_{-\infty}^{+\infty} dx_1 j_n^{(1)}(x_1)=\frac{z^n B\beta}{(n-1)!}\int d(\mathbf{r})_{n-1}\int \tilde{D}(\boldsymbol{\alpha})_n G(\alpha_1^x)M(0,(\mathbf{r})_{n-1};(\boldsymbol{\alpha})_n) \ , \tag{4.69}$$

where $G(\alpha_1^x)$ is defined as in (3.11).

One easily checks that

$$G(\alpha)=\int d\alpha(s)\int d\alpha(t)C(s,t) \ , \tag{4.70}$$

and this shows that (4.69) and (4.64) are identical (up to the sign).

We now establish (4.67). We first give the two following useful identies which are easily derived from the expression (4.5) of the Mayer factor

$$\sqrt{\beta}\,\frac{\partial}{\partial y_1}M(\mathbf{r}_1,\tilde{\mathbf{r}}_1,(\mathbf{r})_{n-1};(\boldsymbol{\alpha})_n)\Big|_{\mathbf{r}_1=\tilde{\mathbf{r}}_1}=\int_0^1 ds(1-s)\frac{\delta}{\delta\alpha_1^y(s)}M(\mathbf{r})_n;(\boldsymbol{\alpha})_n) \ , \tag{4.71}$$

$$\sqrt{\beta}\,\frac{\partial}{\partial y_1}M((\mathbf{r})_n;(\boldsymbol{\alpha})_n)=\int_0^1 ds\frac{\delta}{\delta\alpha_1^y(s)}M((\mathbf{r})_n;(\boldsymbol{\alpha})_n) \ . \tag{4.72}$$

One deduces from (4.72) and the translation invariance of $M((\mathbf{r})_n,(\boldsymbol{\alpha})_n)$ that

$$\int_0^1 ds\frac{\delta}{\delta\alpha_1^y(s)}\int d(y)_{n-1}M(\mathbf{r})_n;(\boldsymbol{\alpha})_n)=0 \ . \tag{4.73}$$

We now apply Lemma 2 to (4.18) setting $\gamma(s)=\alpha(\widetilde{s+u})-\alpha(u)$. Using (4.71) we get

$$j_n^{(2)}(x_1)=-\frac{iz^n}{\sqrt{\beta}(n-1)!}\int d(\mathbf{r})_{n-1}\int \tilde{D}(\boldsymbol{\alpha})_n \prod_{j=1}^{n} K(\gamma_j^x, x_j)$$
$$\cdot\int_0^1 ds(1-s)\frac{\delta}{\delta\gamma_1^y(s)} M((\mathbf{r})_n;(\gamma)_n) \; . \tag{4.74}$$

We remark that

$$\frac{\delta}{\delta\gamma_1^y(s)} M((\mathbf{r})_n;(\gamma)_n)=\frac{\delta}{\delta\alpha_1^y(\widetilde{s+u})} M((\mathbf{r})_n;(\gamma)_n)$$
$$+\delta(\widetilde{s+u}-u)\int_0^1 ds\frac{\delta}{\delta\gamma_1^y(s)} M((\mathbf{r})_n;(\gamma)_n) \; . \tag{4.75}$$

When we use (4.75), (4.73) and the translation invariance we find

$$j_n^{(2)}(x_1)=-\frac{iz^n}{\sqrt{\beta}(n-1)!}\int d(\mathbf{r})_{n-1}\int \tilde{D}(\boldsymbol{\alpha})_n \prod_{j=1}^{n} K(\alpha_j^x, x_j-\sqrt{\beta}\,\alpha_j^x(u))$$
$$\cdot\int_0^1 ds(1-s)\frac{\delta}{\delta\alpha_1^y(\widetilde{s+u})} M((\mathbf{r}-\sqrt{\beta}\,\boldsymbol{\alpha}(u))_n;(\boldsymbol{\alpha})_n)$$
$$=-\frac{iz^n}{\sqrt{\beta}(n-1)!}\int d(\mathbf{r})_{n-1}\int \tilde{D}(\boldsymbol{\alpha})_n K(\alpha_1^x;x_1-\sqrt{\beta}\,\alpha_1^x(u))$$
$$\cdot\prod_{j=2}^{n} K(\alpha_j^x;x_1+x_j-\sqrt{\beta}\,\alpha_1^x(u))\int_0^1 ds(1-s)\frac{\delta}{\delta\alpha_1^y(\widetilde{s+u})} M(0,(\mathbf{r})_{n-1};(\boldsymbol{\alpha})_n) \; . \tag{4.76}$$

This last expression results of the same change of variable which leads to (4.68). The expression (4.76) vanishes when it is integrated on u and $\sqrt{\beta}\,\alpha_1^x(u)$ is set equal to zero in the arguments of K. This is because (4.73) implies

$$\int d(y)_{n-1}\int_0^1 du\frac{\delta}{\delta\alpha_1^y(\widetilde{s+u})} M(0,(\mathbf{r})_{n-1};(\boldsymbol{\alpha})_n)=0 \; . \tag{4.77}$$

Then, proceeding as in the proof of (4.69), we get

$$\int_{-\infty}^{+\infty} dx_1 j_n^{(2)}(x_1)=\frac{iz^n}{(n-1)!}\int \tilde{D}(\boldsymbol{\alpha})_n\int_0^1 du\,\alpha_1^x(u)\int_0^1 ds(1-s)f(\widetilde{s+u};(\boldsymbol{\alpha})_n) \; , \tag{4.78}$$

where

$$f(s,(\boldsymbol{\alpha})_n)=\frac{\delta}{\delta\alpha_1^y(s)}\int d(\mathbf{r})_{n-1} M(0,(\mathbf{r})_{n-1};(\boldsymbol{\alpha})_n) \; , \tag{4.79}$$

and from (4.73)

$$\int_0^1 ds f(s;(\boldsymbol{\alpha})_n)=0 \; . \tag{4.80}$$

The equality of (4.78) with (4.65) (up to the sign) follows immediately from Lemma 4 (Appendix D).

Lemma 4. *Let $f(s)$ be a continuous function on* $[0,1]$ *such that* $\int_0^1 ds\, f(s)=0$, *and* α *a one dimensional Brownian bridge, then*

$$\int_0^1 du\,\alpha(u)\int_0^1 ds\,s f(\widetilde{s+u}) = -\int_0^1 dt\int_0^1 ds\,\alpha(s)\left(\frac{\partial}{\partial s}C(s,t)\right)f(t)$$
$$=\int_0^1 dt\int d\alpha(s)C(s,t)f(t)\ . \tag{4.81}$$

This concludes the proof of Proposition 5.

Appendix A

With (2.9) and (2.10) and the notation (2.17), we have

$$p_R=\frac{z}{\beta\pi R^2}\int d\mathbf{r}(\mathbf{r}|\exp(-\beta H_R)|\mathbf{r})$$
$$=\frac{1}{\pi R^2}\int d\mathbf{r}\int \tilde{D}\boldsymbol{\alpha}\exp\left[-\beta\int_0^1 ds\,V_R(\mathbf{r}+\sqrt{\beta}\,\boldsymbol{\alpha}(s))\right] \tag{A.1}$$

and

$$m_R=\frac{1}{\pi R^2}\int d\mathbf{r}\int \tilde{D}\boldsymbol{\alpha}\left(\frac{iz}{2}\int_0^1 \boldsymbol{\alpha}(s)\wedge d\boldsymbol{\alpha}\right)\exp\left[-\beta\int_0^1 ds\,V_R(\mathbf{r}+\sqrt{\beta}\,\boldsymbol{\alpha}(s))\right]\ . \tag{A.2}$$

We will show that for any functional $F(\boldsymbol{\alpha})$ which is square integrable,

$$\|F\|_2^2=\int D\boldsymbol{\alpha}|F(\boldsymbol{\alpha})|^2<\infty\ , \tag{A.3}$$

one has

$$\lim_{R\to\infty}\frac{1}{\pi R^2}\int d\mathbf{r}\int \tilde{D}\boldsymbol{\alpha}F(\boldsymbol{\alpha})\exp\left[-\beta\int_0^1 ds\,V_R(\mathbf{r}+\sqrt{\beta}\,\boldsymbol{\alpha}(s))\right]=\int \tilde{D}\boldsymbol{\alpha}F(\boldsymbol{\alpha}) \tag{A.4}$$

Recalling (2.16), the result of Proposition 1 follows with the choices $F(\boldsymbol{\alpha})=\frac{z}{\beta}$ for (A.1) and $F(\boldsymbol{\alpha})=\frac{iz}{2}\int_0^1 \boldsymbol{\alpha}(s)\wedge d\boldsymbol{\alpha}$ for (A.2). Notice that

$$\int D\boldsymbol{\alpha}\left(\int_0^1 \boldsymbol{\alpha}(s)\wedge d\boldsymbol{\alpha}\right)^2=4\left(\int_0^1 ds\,C(s,s)-\int_0^1 ds\int_0^1 dt\,C(s,t)\right)=\frac{1}{2}<\infty\ .$$

To prove (A.4) we define for a fixed R the set

$$\Delta_R(\mathbf{r})=\left\{\boldsymbol{\alpha}\,\middle|\,\sup_{0\leqq s\leqq 1}|\mathbf{r}+\sqrt{\beta}\,\boldsymbol{\alpha}(s)|\leqq R\right\}\ ,$$

and decompose the integral in (A.4) as the sum of the quantities

$$I_R^1=\frac{1}{\pi R^2}\int_{r\leqq R} d\mathbf{r}\int_{\Delta_R(\mathbf{r})}\tilde{D}\boldsymbol{\alpha}F(\boldsymbol{\alpha})\ ,$$

$$I_R^2=\frac{1}{\pi R^2}\int_{r\leqq R} d\mathbf{r}\int_{\mathscr{C}(\Delta_R(\mathbf{r}))}\tilde{D}\boldsymbol{\alpha}F(\boldsymbol{\alpha})\exp\left[-\beta\int_0^1 ds\, V_R(\mathbf{r}+\sqrt{\beta}\,\boldsymbol{\alpha}(s))\right]\ ,$$

$$I_R^3=\frac{1}{\pi R^2}\int_{r\geqq R} d\mathbf{r}\int \tilde{D}\boldsymbol{\alpha}F(\boldsymbol{\alpha})\exp\left[-\beta\int_0^1 ds\, V_R(\mathbf{r}+\sqrt{\beta}\,\boldsymbol{\alpha}(s))\right]\ .$$

We show that

$$\lim_{R\to\infty} I_R^1=\int \tilde{D}\boldsymbol{\alpha}F(\boldsymbol{\alpha})\ , \tag{A.5}$$

$$\lim_{R\to\infty} I_R^2=0\ , \tag{A.6}$$

$$\lim_{R\to\infty} I_R^3=0\ . \tag{A.7}$$

We consider first the limit in (A.5),

$$\begin{aligned}|I_R^1-\int \tilde{D}\boldsymbol{\alpha}F(\boldsymbol{\alpha})|&=\frac{1}{\pi R^2}\left|\int_{r\leqq R} d\mathbf{r}\int_{\mathscr{C}(\Delta_R(\mathbf{r}))}\tilde{D}\boldsymbol{\alpha}F(\boldsymbol{\alpha})\right|\\ &\leqq\frac{1}{2\pi^2\beta R^2}\int_{r\leqq R} d\mathbf{r}\int_{\mathscr{C}(\Delta_R(\mathbf{r}))}D\boldsymbol{\alpha}|F(\boldsymbol{\alpha})|\leqq\frac{\|F\|_2}{2\pi^2\beta R^2}\int_{r\leqq R} d\mathbf{r}\left[\int_{\mathscr{C}(\Delta_R(\mathbf{r}))}D\boldsymbol{\alpha}\right]^{1/2}\\ &\leqq\frac{4\|F\|_2}{\pi\beta R^2}\left(\int_0^R \exp\left(-\frac{(R-r)^2}{2\beta}\right)r dr\right)\ ,\end{aligned} \tag{A.8}$$

which tends to zero as $R\to\infty$.

In the second line, we have used Schwartz inequality, while the last one results from the estimate:

$$\int_{\mathscr{C}(E)} D\boldsymbol{\alpha}\leqq 4\exp\left(-\frac{d^2(\mathbf{r},\partial\mathscr{B})}{\beta}\right)\ , \tag{A.9}$$

$$E=\left\{\boldsymbol{\alpha}|\mathbf{r}+\sqrt{\beta}\,\boldsymbol{\alpha}(s)\in\mathscr{B},\ 0\leqq s\leqq 1\right\}\ ,$$

where $\mathscr{B}$ is any region in $\mathbb{R}^2$ with regular boundaries (cf. [10]).

To prove (A.6), we note that if $\boldsymbol{\alpha}\in\mathscr{C}(\Delta_R(\mathbf{r}))$, one has

$$\exp\left[-\beta\int_0^1 ds\, V_R(\mathbf{r}+\sqrt{\beta}\,\boldsymbol{\alpha}(s))]\leqq\exp(\beta\Phi_0)\right]\equiv M\ ,$$

and thus

$$|I_R^2|\leqq\frac{M}{2\pi^2\beta R^2}\int_{r\leqq R} d\mathbf{r}\int_{\mathscr{C}(\Delta_R(\mathbf{r}))}D\boldsymbol{\alpha}|F(\boldsymbol{\alpha})|\ .$$

This is estimated as in (A.8) and vanishes as $R\to\infty$.

We now estimate the integral I_R^3:

$$|I_R^3| \leqq \frac{1}{2\pi^2\beta} \int_{r \geqq 1} d\mathbf{r} \int D\boldsymbol{\alpha} |F(\boldsymbol{\alpha})| \exp\left[-\beta \int_0^1 ds\, V_R\left(R\left(\mathbf{r} + \sqrt{\beta}\, \frac{\boldsymbol{\alpha}(s)}{R}\right)\right)\right] \tag{A.10}$$

$$= \frac{1}{2\pi^2\beta} \int_{1 \leqq r \leqq 1+\varepsilon} d\mathbf{r} \int D\boldsymbol{\alpha} |F(\boldsymbol{\alpha})| K_R(\boldsymbol{\alpha}, \mathbf{r}) + \frac{1}{2\pi^2\beta} \int_{r \geqq 1+\varepsilon} d\mathbf{r} \int D\boldsymbol{\alpha} |F(\boldsymbol{\alpha})| K_R(\boldsymbol{\alpha}, \mathbf{r}) , \tag{A.11}$$

where

$$K_R(\boldsymbol{\alpha}, \mathbf{r}) = \exp\left[-\beta \int_0^1 ds\, V_R\left(R\left(\mathbf{r} + \sqrt{\beta}\, \frac{\boldsymbol{\alpha}(s)}{R}\right)\right)\right] .$$

The first contribution to (A.11) is majorized uniformly with respect to R by

$$\frac{M}{2\pi^2\beta} \int_{1 \leqq r \leqq 1+\varepsilon} d\mathbf{r} \int D\boldsymbol{\alpha} |F(\boldsymbol{\alpha})| \leqq \frac{M}{2\pi^2\beta} (2\varepsilon + \varepsilon^2) \int D\boldsymbol{\alpha} |F(\boldsymbol{\alpha})| . \tag{A.12}$$

We split the second contribution to (A.11) in two parts $I_R^4 + I_R^5$,

$$I_R^4 = \frac{1}{2\pi^2\beta} \int_{r \geqq 1+\varepsilon} d\mathbf{r} \int_{S_R(\mathbf{r})} D\boldsymbol{\alpha} |F(\boldsymbol{\alpha})| K_R(\boldsymbol{\alpha}, \mathbf{r}) ,$$

$$I_R^5 = \frac{1}{2\pi^2\beta} \int_{r \geqq 1+\varepsilon} d\mathbf{r} \int_{\mathscr{C}(S_R(\mathbf{r}))} D\boldsymbol{\alpha} |F(\boldsymbol{\alpha})| K_R(\boldsymbol{\alpha}, \mathbf{r}) ,$$

where the $\boldsymbol{\alpha}$ integration in I_R^4 is restricted to the set

$$S_R(\mathbf{r}) = \left\{\boldsymbol{\alpha} \,|\, \sqrt{\beta} \sup_s |\boldsymbol{\alpha}(s)| \geqq \frac{R(r-1)}{2}\right\} ,$$

and in I_R^5 to the complementary set. One gets

$$|I_R^4| \leqq \frac{M}{2\pi^2\beta} \int_{r \geqq 1+\varepsilon} d\mathbf{r} \int_{S_R(\mathbf{r})} D\boldsymbol{\alpha} |F(\boldsymbol{\alpha})| \leqq \frac{M}{2\pi^2\beta} \|F\|_2 \int_{r \geqq 1+\varepsilon} d\mathbf{r} \left[\int_{S_R(\mathbf{r})} D\boldsymbol{\alpha}\right]^{1/2}$$

$$\leqq \frac{2M}{\pi^2\beta} \|F\|_2 \int_{r \geqq 1+\varepsilon} d\mathbf{r} \exp\left(-\frac{R^2(r-1)^2}{8}\right) , \tag{A.13}$$

where the last estimate follows again from (A.9).

In $\mathscr{C}(S_R(\mathbf{r}))$, we have

$$r \leqq \left|\mathbf{r} + \sqrt{\beta}\, \frac{\boldsymbol{\alpha}(s)}{R}\right| + \frac{\sqrt{\beta}}{R} |\boldsymbol{\alpha}(s)| \leqq \left|\mathbf{r} + \sqrt{\beta}\, \frac{\boldsymbol{\alpha}(s)}{R}\right| + \frac{r-1}{2} ,$$

hence

$$\left|\mathbf{r} + \sqrt{\beta}\, \frac{\boldsymbol{\alpha}(s)}{R}\right| - 1 \geqq \frac{r-1}{2} \geqq \frac{\varepsilon}{2} . \tag{A.14}$$

This implies with (2.4) that for $\frac{R\varepsilon}{2} \geqq x_0$,

$$V_R\left(R\left(\mathbf{r}+\sqrt{\beta}\,\frac{\boldsymbol{\alpha}(s)}{R}\right)\right)=\Phi\left(R\left|\mathbf{r}+\sqrt{\beta}\,\frac{\boldsymbol{\alpha}(s)}{R}\right|-1\right)\geqq\frac{\kappa}{4}\,(r-1)^2R^2\;,$$

$$|I_R^5| \leqq \frac{1}{2\pi^2\beta} \int_{r\geqq 1+\varepsilon} d\mathbf{r} \exp\left(-\frac{\kappa}{4}\,(r-1)^2R^2\right) \int D\boldsymbol{\alpha}\,|F(\boldsymbol{\alpha})|\;. \tag{A.15}$$

Letting first $R\to\infty$ and then $\varepsilon\to 0$, we conclude from (A.12, A.15) that I_R^3 vanishes.

In the hard wall case all the $\boldsymbol{\alpha}$-integrals are restricted to $\Delta_R(\mathbf{r})$. Thus I_R^2 does not contribute and the $\boldsymbol{\alpha}$ integral in (A.10) has to be carried on the set

$$\Delta_R(R\mathbf{r})=\left\{\boldsymbol{\alpha}\,|\sup_s\left|\mathbf{r}+\sqrt{\beta}\,\frac{\boldsymbol{\alpha}(s)}{R}\right|\leqq 1\right\}\;.$$

Since by (A.14) $\Delta_R(R\mathbf{r})\subset S_R(\mathbf{r})$, the proof that I_R^3 vanishes follows from the estimates (A.12) and (A.13).

Appendix B

Proof of Lemma 1. Define

$$M(x)=\int_{-\sup_s|\alpha(s)|<x} D\alpha\;.$$

The Brownian bridge is equivalent to $b(s)-sb(1)$ on $[0,1]$ where b is the one-dimensional Brownian motion. Therefore since

$$2\sup_{0\leqq s\leqq 1}|b(s)|\geqq \sup_{0\leqq s\leqq 1}|b(s)-sb(s)|$$

we have for $x<0$ (cf. [6, p. 268])

$$M(x)\leqq \mathbb{E}\left(\sup_{0\leqq s\leqq 1}|b(s)|>-\frac{x}{2}\right)\leqq Ae^{-Cx^2}\;.$$

For $x>0$

$$\int_{\sup_s|\alpha(s)|>x} D\alpha\,\sup_s|\alpha(s)|^n=\int_{-\infty}^{-x}(-y)^n M(dy)$$
$$=x^nM(-x)+\int_{-\infty}^{-x} n(-y)^{n-1}M(y)dy\leqq Ae^{-Cx^2}\;.$$

Proof of Lemma 2. Since $\gamma(s)$ is again a Gaussian process with zero mean it is sufficient to check that

$$\mathbb{E}(\gamma(s)\gamma(t))=\mathbb{E}(\alpha(s)\alpha(t))$$

This can be checked by considering the three cases: (i) $s+u\in[0,1)$, $t+u\in[0,1)$; (ii) $s+u\in[1,2)$, $t+u\in[0,1)$; (iii) $s+u\in[1,2)$, $t+u\in[1,2)$. For instance, in case (ii) $\widetilde{s+u}=s+u-1<u\leqq t+u$.

Therefore

$$\begin{aligned}\mathbb{E}(\gamma(s)\gamma(t)) &= \mathbb{E}(\alpha(s+u-1)\alpha(t+u)-\alpha(u)\alpha(t+u)-\alpha(s+u-1)\alpha(u)+\alpha(u)\alpha(u))\\ &=(s+u-1)(1-t-u)-u(1-t-u)-(s+u-1)(1-u)+u(1-u)\\ &=t(1-s)=\mathbb{E}(\alpha(s)\alpha(t)) \ .\end{aligned}$$

Appendix C

When x_1 is positive, we estimate each term of the sum (4.46). For $k>1$, we split the configurational integral in (4.47) over two sets

$$\begin{aligned}\Delta_1 &= \left\{\mathbf{r}_i \,|\, |\mathbf{r}_i| \leqq \frac{x_1}{2(|B_k|-1)} \quad \text{for all} \quad i\in B_k,\ i\neq 1\right\}\\ \Delta_2 &= \mathscr{C}(\Delta_1)\end{aligned} \tag{C.1}$$

We call $I_k^{(1)}$ and $I_k^{(2)}$ the corresponding integrals. We note that in $\Delta_1 : b_k^x \geqq \frac{x_1}{2}$; since $U(x)$ is positive monotonuous decreasing, this implies

$$|K(\alpha_k^x;n_k^x)-1| \leqq \left|K\left(\alpha_k^x;\frac{x_1}{2}\right)-1\right| \leqq 1 \tag{C.2}$$

and thus

$$\begin{aligned}|I_k^{(1)}| &\leqq \|w\|_1^{|B_k|-1} \int D\boldsymbol{\alpha}_1 \sup_s |\alpha_1^x(s)| \int D\boldsymbol{\alpha} \left|K\left(\alpha^x;\frac{x_1}{2}\right)-1\right|\\ &\leqq \|w\|_1^{|B_k|-1}\left(\int D\boldsymbol{\alpha}_1 \sup_s |\alpha_1^x(s)|\right) \int_{\sup_s \sqrt{\beta}|\boldsymbol{\alpha}(s)| \geqq \frac{x_1}{4}} D\boldsymbol{\alpha}\\ &\leqq C\|w\|_1^{|B_k|-1} e^{-Dx_1^2}\end{aligned} \tag{C.3}$$

The second inequality follows from the fact that $\left|K\left(\alpha^x;\frac{x_1}{2}\right)-1\right|=0$ when $\sup_s \sqrt{\beta}|\boldsymbol{\alpha}(s)| < \frac{x_1}{4}$. The last bound (C.3) is a consequence of Lemma 1.

Since Δ_2 is contained in the union of the $|B_k|-1$ sets in which at least one of the variables r_i ($i\in B_k$, $i\neq 1$) is larger than $\frac{x_1}{2(|B_k|-1)} \geqq \frac{x_1}{2n}$, we get

$$\begin{aligned}I_k^{(2)} &\leqq (|B_k|-2)\,\|w\|_1^{|B_k|-2} \int D\boldsymbol{\alpha}_1 \sup_s |\alpha_1^x(s)|\\ &\quad \cdot \int D\boldsymbol{\alpha} D\boldsymbol{\alpha}' \int_{|\mathbf{r}| \geqq \frac{x_1}{2n}} d\mathbf{r}\, |\bar{w}(\mathbf{r};\boldsymbol{\alpha}-\boldsymbol{\alpha}')|\\ &\quad + \|w\|_1^{|B_k|-2} \int D\boldsymbol{\alpha}_1 \sup_s |\alpha_1^x(s)| \int_{|\mathbf{r}| \geqq \frac{x_1}{2n}} D\boldsymbol{\alpha} \int d\mathbf{r}\, |\bar{w}(\mathbf{r};\boldsymbol{\alpha}-\boldsymbol{\alpha}_1)| \ .\end{aligned} \tag{C.4}$$

The functional integral which occurs in the first part of (C.4) is estimated as follows. If we restrict the integration to the domain

$$\Gamma_{x_1}=\left\{\boldsymbol{\alpha},\boldsymbol{\alpha}' \mid \sup_s \sqrt{\beta}\,|\boldsymbol{\alpha}(s)-\boldsymbol{\alpha}'(s)|\leqq\frac{x_1}{4n}\right\}, \tag{C.5}$$

we get

$$\begin{aligned}&\int_{\Gamma_{x_1}} D\boldsymbol{\alpha}D\boldsymbol{\alpha}' \int_{|\mathbf{r}|\geqq\frac{x_1}{2n}} d\mathbf{r}\,|\bar{w}(\mathbf{r};\boldsymbol{\alpha}-\boldsymbol{\alpha}')|\\ &\leqq\int_0^1 ds \int_{\Gamma_{x_1}} D\boldsymbol{\alpha}D\boldsymbol{\alpha}' \int_{|\mathbf{r}-\sqrt{\beta}(\boldsymbol{\alpha}(s)-\boldsymbol{\alpha}'(s))|\geqq\frac{x_1}{2n}} d\mathbf{r}\,|w(\mathbf{r})|\leqq \int_{|\mathbf{r}|\geqq\frac{x_1}{4}} d\mathbf{r}\,|w(\mathbf{r})|\;.\end{aligned} \tag{C.6}$$

On the complement of Γ_{x_2}, we find with the help of Lemma 1 that

$$\begin{aligned}\int_{\mathscr{C}(\Gamma_{x_1})} D\boldsymbol{\alpha}D\boldsymbol{\alpha}' \int_{|\mathbf{r}|\geqq\frac{x_1}{2n}} d\mathbf{r}\,|\bar{w}(\mathbf{r};\boldsymbol{\alpha}-\boldsymbol{\alpha}')|&\leqq 2\emptyset w\emptyset_1 \int_{\sup_s \sqrt{\beta}|\boldsymbol{\alpha}(s)|\geqq\frac{x_1}{8n}} D\boldsymbol{\alpha}\\ &\leqq C_1\emptyset w\emptyset_1\, e^{-D\frac{x_1^2}{n^2}}\;.\end{aligned} \tag{C.7}$$

The second term of (C.4) is bounded in the same way. Finally one sees that the bound (C.3) applies also to $I_1(x_1)$. When the estimates (C.3), (C.6) and (C.7) are introduced in (4.47) and (4.46), one gets the result of the lemma for $x_1>0$.

When x_1 is negative, we can majorize (4.42) by

$$|I_{n,T}(x_1)|\leqq C\|w\|_1^{n-1}\int D\boldsymbol{\alpha}_1 \sup_s |\alpha_1^x(s)|\exp\left[-\beta\int_0^1 ds\,U(x_1+\sqrt{\beta}\,\alpha_1^x(s))\right], \tag{C.8}$$

which can be estimated as in the non-interacting case with the gaussian bound (4.49).

Appendix D

One has

$$\begin{aligned}\int_0^1 du\,\alpha(u)\int_0^1 ds\,s f(\widetilde{s+u})&=\int_0^1 du\,\alpha(u)\int_0^{1-u} ds\,s f(s+u)\\ &\quad+\int_0^1 du\,\alpha(u)\int_{1-u}^1 ds\,s f(s+u-1)\\ &=\int_0^1 du\,\alpha(u)\int_0^1 ds(s-u)f(s)\\ &\quad+\int_0^1 du\,\alpha(u)\int_0^u ds\,f(s)\;.\end{aligned}$$

Since $\int_0^1 f(s)ds=0$ and integrating the second term by parts, one finds

$$\int_0^1 du\alpha(u)\int_0^1 dssf(\widetilde{s+u})=\int_0^1 dssf(s)\int_0^1 du\alpha(u)-\int_0^1 duf(u)\int_0^u ds\alpha(s)$$

$$=-\int_0^1 dt\int_0^1 ds\alpha(s)\frac{\partial}{\partial s}C(s,t)f(t)\ .$$

The last equality in Lemma 4 follows from Ito's lemma.

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References

1. Peierls, R.: Surprises in theoretical physics. Princeton, NS: Princeton University Press 1979
2. Leeuwen, H. van: J. Phys. **2**, 361 (1921)
3. Angelescu, N., Nenciu, G., Bundaru, M.: Commun. Math. Phys. **42** (1975)
4. Jancovici, B.: Physica **101**A, 324 (1980)
5. Alastuey, A., Jancovici, B.: Physica **102**A, 327 (1980)
6. Ginibre, J.: J. Math. Phys. **6**, 238, 252, 1432 (1965)
7. Brydges, D.: Proceedings of 1984 Les Houches Summer School. Ed. K. Osterwalder (ed.). Amsterdam: North-Holland; J. State Phys. **42**, 425 (1986)
8. Simon, B.: Functional integration and quantum physics. New York: Academic Press 1979
9. Glimm, J., Jaffe, A.: Quantum physics. Berlin, Heidelberg, New York: Springer 1981
10. den Berg, M. van: J. Math. Phys. **22**, 2452 (1981)

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