BUFFER CONTROLLED FAILURE PRONE TWO-MACHINE TANDEM PRODUCTION SYSTEM

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ABSTRACT. We discuss the dynamics of a controlled failure prone two-machine tandem production system (i.e. two machines separated by one buffer). The production rates of the machines are controlled by the buffer population level. The proposed control is shown to be the optimal policy of a simple convex production control problem taking into account the costly effects due to the starvation and blocking mechanism. In a fluid modeling framework, we explicitly calculate the stationary probability measure governing the buffer population. Analytical and explicit results for the average throughput and its variance are then derived.

1. Introduction

The presence of a buffer stock located between two failure prone machines M_1 and M_2, (see Figure 1) enhances the global "throughput" and the relevant quantitative analysis of this situation is thoroughly exposed and reviewed in numerous contributions from which we non-exhaustively mention [?, ?, ?, ?, ?, ?, ?, ?]. This performance improvement is obviously due to the ability of the buffer population to partly absorb the random interruptions of the production flow delivered by the machines. The presence of a buffer does not however eliminate all interruptions of the production flow even when both machines are potentially able to produce. Indeed starving interruptions of M_2 which arise when the buffer is empty and blocking interruptions of M_1 occurring when the buffer is filled up do actually occur. Besides reducing the overall throughput, the blocking and starving interruptions often generate additional nuisances. To illustrate this point let us mention a few situations:

  i) in fluid installations as those typically encountered in chemical and food industries, overflows and/or dry states of a tank (playing here the role of the buffer) placed between pumps have clearly to be avoided.
  ii) in the Internet which consists of links and buffers in order to transfer data from a source to a destination, data overflows of a buffer in front of a link results in information loss.
  iii) in very high production flows as those arising in tobacco industries, the stopping and the rise to the nominal production regime of the machines, cannot be instantaneous. This can generate large overflow losses as buffer boundaries are reached with maximal production rate. Such overflow losses are in particular likely to be important when uncertainties exists on the actual physical population level.

To reduce the probability of occurrences of starving or blocking states, one obviously can increase the buffer capacity (called H thereafter) but this solution is not always feasible as it usually leads to prohibitive costs (presence of large size installations incompatible with the available layout and creation of large work-in processes). If one is limited to a fixed buffer capacity, an alternative solution is to introduce a control mechanism tuned by the buffer population process $X(t) \in [0, H]$ with the goal of reducing the sojourn times spent in the filled or empty buffer states.

In this paper, we study the production flow dynamics resulting from the introduction of a very elementary control characterized by two inventory thresholds z and Z, $(0 \leq z \leq Z \leq H)$. When the buffer population $X(t)$ approaches one of the buffer boundaries, namely when $X(t) \leq z$ or when $X(t) \geq Z$, an average drift directed opposite to the buffer boundary point is switched on. This drift is created by an ad-hoc reduction of the production rate of M_1 when $X(t) \geq Z$ and similarly by a reduction of the production rate of M_2 when $X(t) \leq z$.

While the possibility of tuning the rate of production installations is not the common rule in actual production systems, let us illustrate how they are naturally realized:

    i) A common engineering response to overflow losses in fluid installations consists in introducing backoffs' replacing hard constraints (here full buffer or empty buffer) by soft constraints (buffer level high or low). This means that the hard buffer constraints are virtually shifted towards the interior of the buffer boundaries by slowing down the corresponding machine before reaching the real boundaries.

    ii) A common response to data overflows of a buffer in the Internet is to run a Transmission Control Protocol TCP which regulates the traffic rate of the source. The TCP controls the transfer rate roughly as follows: During overflow, the buffer sends negative feedback signals to the sources to reduce the sending rate. Otherwise the buffer sends negative feedback signals to the sources to reduce the sending rate [**?**].

    iii) Similar as in i) a common remedy for large buffer overflows when very high production flows are involved consists in reducing in discrete steps the production rate when approaching a hard constraint (i.e. a fully filled buffer or an empty buffer).

    iv) A tuning of the rate of production installations occurs naturally when the flexible behavior of human operators is investigated. Indeed, in an *adaptative production strategy*, the operators move from a production cell with a highly populated upstream buffer to a cell with a low populated downstream buffer and reciprocally. This is precisely the type of dynamics realized when a two-threshold policy as the one studied in this paper is operating. This situation can be viewed as a caricature of a class of more general workforce allocation problems as considered in [**?**].

Clearly, any rate reduction mechanism will ultimately reduce the average throughput. Therefore increasing the throughput on one hand and reducing the sojourn on the buffer boundaries on the other hand are two competing advantages and an optimal tradeoff has to be found for each specific application. The competing advantages can also be viewed by using a complementary light. Indeed, the two-thresholds control enables in parallel to reduce the variability of the throughput delivered by the TS. While this is obviously a definite advantage - it enables for

example to reduce the optimal hedging level in make-to-stock production systems [**?**, **?**] - one should however not forget that in parallel the average throughput will itself be reduced. Hence the global benefit of the introduction of a two-thresholds policy must be appreciated from case to case.

The above observations suggest that the two thresholds control policy can be viewed as the optimal solution of an associated control problem in which the instantaneous cost criterion includes two terms namely: i) a term which takes its minimum value for half-filled buffer states and ii) a term which is minimized when M_2 reaches its maximal production rate. The relative importance attributed to each term is tuned by a weighting factor and will ultimately the control thresholds z and Z via the gradient of the associated value function. The corresponding Hamilton-Jacobi-Bellmann equation is discussed by using the same lines of arguments as those given in [**?**]. The resulting optimal policy is seen to be of the elementary form as the one considered here. Such a control policy has also been investigated numerically for longer transfer lines in [**?**].

The aim of the present paper is to analyze this simple two threshold control by means of an analytically solvable model. To get analytical results, we make use of a fluid modeling approach which avoids the combinatorial complexity inherent to every Markov chain model with large state spaces. Fluid queues reduce the problem to solve a system of linear partial differential equations (the Chapman-Kolmogorov equations) with their appropriated boundary conditions.

The paper is organized as follows: In section 2 we introduce the controlled tandem system and show that the proposed control is – within a restricted class of controls – optimal. The corresponding Chapmann-Kolmogorov equation governing the buffer population dynamics is given in section 3. The stationary probability measures are derived in section 4. Performance measures and numerical simulations are discussed in section 5. Finally, section 6 is devoted to conclusions and perspectives.

2. The Model

We consider a single product transfer-line composed of two machines in tandem M_1 and M_2 separated by a buffer B with fixed finite capacity $H > 0$ as showed in Figure 1.

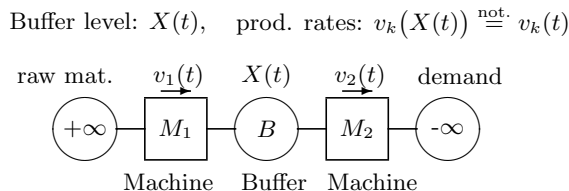Buffer level: $X(t)$,   prod. rates: $v_k\big(X(t)\big) \stackrel{\text{not.}}{=} v_k(t)$

Figure 1. Sketch of a two-stage transfer-line with a permanent supply of raw material and a permanent absorbing demand.

The population level in B at time t is denoted as $X(t) \in \mathbb{R}$. Machine M_1, when operating at time t, can produce continuously with rate $v_1\big(X(t)\big)$. The continuous flow of products is immediately stored in B as long as $X(t) \leq H$, otherwise

$v_1\big(X(t)\big) = 0$ in which case M_1 is blocked. Machine M_2, when operating at time t, can produce continuously with rate $v_2\big(X(t)\big)$ whenever $X(t) > 0$. For $X(t) \leq 0$, M_2 is starved and we set $v_2\big(X(t)\big) = 0$. Starving and blocking mechanisms ensure that $0 \leq X(t) \leq H$. We assume in this paper that the dynamics of the TS is neither influenced by the demand nor by the supply shortage. In other words, M_2 is never blocked and M_1 is never starved.

When M_k is in its operating state, the buffer controlled production rate $v_k\big(X(t)\big)$ can be either zero, v (fast) or rv (slow), where $v > 0$ is the maximal production rate and where $0 < r < 1$ is a fixed dimensionless parameter. Now we introduce a Threshold type production Rate Control (in the following short: TRC) as follows:

$$(1) \qquad \Big(v_1(X(t)), v_2(X(t))\Big) = \left\{ \begin{array}{lll} (rv, v) & \text{when} & Z < X(t) < H \\ (v, v) & \text{when} & z \leq X(t) \leq Z \\ (v, rv) & \text{when} & 0 < X(t) < z \end{array} \right\} (TRC)$$

Therefore, when the buffer level is larger than $Z := H/2 + \Delta$, where $0 < \Delta < H/2$ is a space-parameter, the production of M_1 is less than that of M_2, hence reducing the buffer level. Conversely when the buffer level is below the threshold $z := H/2 - \Delta$, the production rate of M_1 is greater than the rate of M_2, hence increasing the buffer level.

M_1 and M_2 are failure prone giving rise to occasional breakdowns and repairs with random durations. These random events will be modeled by two Markov processes $I_1(t)$ and $I_2(t)$, defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that the resulting failure processes are operation dependent rather than time dependent. This means that machines fail only while processing workpieces. Thus, if Machine M_i, $i = 1$ (resp. $i = 2$) is operational but blocked (resp. starved), it cannot fail (see e.g., [?] p.72). The failure processes are chosen to be Markovian alternating renewal processes taking values in $\{0, 1\}$. They are characterised by the first moments λ_i^{-1} resp. μ_i^{-1} of their exponentially distributed holding times in the states $\{1\}$ resp. $\{0\}$. Here, we assume for simplicity that $\lambda_1^{-1} = \lambda_2^{-1} = \lambda^{-1}$ and $\mu_1^{-1} = \mu_2^{-1} = \mu^{-1}$. Note that other situations can be studied by using the same formalism. In this notation the storage process $X(t)$ reads:

$$(2) \qquad X(t) = X(0) + \int_0^t \big(I_1(s)v_1(X(s)) - I_2(s)v_2(X(s))\big)\mathrm{d}s.$$

Our main goal is to determine the stationary distribution function governing the buffer level $X(t)$. To this aim, we shall consider the mixed discrete-continuous Markov process $\big(X(t), I_1(t), I_2(t)\big)$ and solve the corresponding set of Chapman-Kolmogorov-equations in the stationary state.

2.1. **TRC rule viewed as an optimal policy.** As mentioned in the introduction, the above threshold control can be seen as the optimal policy of an infinite time horizon optimal control problem minimizing a discounted cost criterion. Indeed, suppose that our aim is to control the two possible production rates (slow rv or fast v) of the operational machines in order to produce as much as possible respecting the costly effects of blocking and starvation. To this end we have to find a feedback policy $\big(v_1(X), v_2(X)\big)$ taking values in

$$(3) \qquad \big(v_1(X), v_2(X)\big) \in U := \{(rv, rv); (rv, v); (v, rv); (v, v)\},$$

subject to the constraints $v_1(x) = 0$ for $x \geq H$ and $v_2(x) = 0$ for $x \leq 0$, which minimizes the cost functional $J(x_0, i)$, $x_0 \in [0, H]$, $i \in \{(0,0),(0,1),(1,0),(1,1)\}$ defined by:

$$(4) \qquad J(x_0, i) = \mathbb{E}_{x_0,i}\Big[\int_0^\infty \Big((X(t) - H/2)^2 + \alpha(v - v_2(X(s)))\Big)\mathrm{e}^{-\beta s}\Big].$$

Here, $\beta > 0$ is a discount parameter, $\mathbb{E}_{x_0,i}$ is the expectation operator subject to the initial conditions $X(0) = x_0$ and $(I_1, I_2) = i$ and $\alpha > 0$ is a parameter which controls the trade off between "producing as much as possible" (minimizing the term $b - v_2$) and "avoid the boundaries of the buffer" (minimizing the term $(X - H/2)^2$). Note that so far considerable attention has been given to formulate two stage production/inventory systems with controllable production rates as optimal control problems (see e.g., [?] and especially [?, ?] which are closely connected with the present study). These works however focus on minimizing inventory holding and backordering costs and penalizing the starvation or the blocking mechanism via the cost functional is far less investigated (see however the decomposition method of Hu in [?] where starving costs naturally enters into the cost functional).

Let now $x \mapsto \phi(x, i) = \min_{(v_1,v_2)\in U} J(x, i)$ be the value function defined on the whole line $\mathbb{R}$ of the above minimization problem which is *not* yet subject to the blocking and starving constraints ($v_1(x) = 0$ for $x \geq H$ and $v_2(x) = 0$ for $x \leq 0$). For notational ease we identify the four possible operating states $(I_1, I_2) = (0,0),(0,1),(1,0),(1,1)$ with $i = 1, 2, 3, 4$ respectively. Then $x \mapsto \phi(x, i)$, seen as a function defined on $\mathbb{R}$, is the unique viscosity solution of the HJB dynamic programing equations (see e.g., [?] Chapt. III eq.(9.4)):

$$(5) \quad \begin{pmatrix} \min \alpha(v - v_2) \\ \min \alpha(v - v_2) - v_2\phi_x(x, 2) \\ \min \alpha(v - v_2) + v_1\phi_x(x, 3) \\ \min \alpha(v - v_2) + (v_1 - v_2)\phi_x(x, 4) \end{pmatrix} = A \begin{pmatrix} \phi(x, 1) \\ \phi(x, 2) \\ \phi(x, 3) \\ \phi(x, 4) \end{pmatrix} - (x - \frac{H}{2})^2 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

where the minimum "min" is taken over the couples $(v_1, v_2) \in U$ and where the matrix A is given by:

$$A = \begin{pmatrix} \beta + 2\mu & -\mu & -\mu & 0 \\ -\lambda & \beta + \mu + \lambda & 0 & -\mu \\ -\lambda & 0 & \beta + \mu + \lambda & -\mu \\ 0 & -\lambda & -\lambda & \beta + 2\lambda \end{pmatrix}.$$

Interested in the optimal feedback policy $(v_1(x), v_2(x))$ when both machines are operational at time $t = 0$ (i.e. $i = 4$) we have to investigate the following minimum:

$$(6) \qquad \min_{(v_1,v_2)\in U} \alpha(v - v_2) + (v_1 - v_2)\phi_x(x, 4).$$

The strict convexity of $\phi(x, i)$ considered as a function on $\mathbb{R}$ (see [?] p.149 and also [?] p.380) guarantees the existence of $Z \in \mathbb{R}$ such that:

$$\phi_x(x, 4) \begin{cases} \geq & 0 \quad \text{for } x \geq Z, \\ \leq & 0 \quad \text{for } x \leq Z. \end{cases}$$

Setting $I(v_1, v_2) := \alpha(v - v_2) + (v_1 - v_2)\phi_x(x, 4)$, we see from eq.(6) that:

$$(v_1(x), v_2(x)) = \begin{cases} (rv, v) & \text{minimizes } I(v_1, v_2) \quad \text{for } x \geq Z, \\ (v, v) & \text{minimizes } I(v_1, v_2) \quad \text{for } x \leq Z \text{ and } \alpha + \phi_x(x, 4) \geq 0, \\ (v, rv) & \text{minimizes } I(v_1, v_2) \quad \text{for } x \leq Z \text{ and } \alpha + \phi_x(x, 4) \leq 0. \end{cases}$$

By invoking again the strict convexity of $\phi(x, i)$, we conclude that there exists $z \in [-\infty, Z]$ such that:

$$\alpha + \phi_x(x, 4) \begin{cases} \geq & 0 \quad \text{for } x \geq z, \\ \leq & 0 \quad \text{for } x \leq z. \end{cases}$$

Hence, the generic form of the optimal control policy will be:

$$\big(v_1(x), v_2(x)\big) = \begin{cases} (rv, v) & \text{for } x \geq Z \\ (v, v) & \text{for } z \leq x \leq Z \\ (v, rv) & \text{for } x \leq z \end{cases}$$

which is precisely the TRC policy given by eq.(1). Note however that we have not yet taken into account the presence of blocking and starving states due to the finite buffer capacity. This will however be irrelevant provided it exists a range of α values such that the following condition (c) holds:

$$(c) \qquad 0 < z(\alpha) < Z(\alpha) < H.$$

Indeed when (c) is realized the policy is not sensitive to the presence of buffer boundaries. The following technical points addressed in detail in the appendix show clearly that such a range of α-values exists:

a) The functions $z(\alpha)$, $Z(\alpha)$ are well defined.
b) $z(0) = Z(0) = H/2$.
c) $Z(\alpha)$ is constant.
d) $z(\alpha)$ is continuous and decreasing.

Remark. Note that when we "symmetrize" the control by adding to the cost functional eq.(4) the term $\alpha(v - v_1)$ we can replace point $c)$ above by $c')$:

c') $Z(\alpha)$ is continuous and increasing.

The thresholds z and Z of the resulting optimal control would be "symmetric" i.e., of the form $z = H/2 - \Delta(\alpha)$ and $Z + \Delta(\alpha)$ exactly as analysed below.

3. Associated Chapman-Kolmogorov-equations

From the fact that $I_k(t)$, $k = 1, 2$ are alternating Markov renewal processes, $\big(X(t), I_1(t), I_2(t)\big)$ is a mixed discrete-continuous state Markov process in continuous time with state space:

$$(7) \qquad \mathcal{S} = \big\{(x, i, j) \mid 0 \leq x \leq H; \ i, j = 0, 1\big\}.$$

We now adopt the following notations:

(8)

$$
\begin{aligned}
L_{i,j}(t) &= \mathbb{P}\big(X(t) = 0, I_1(t) = i, I_2(t) = j\big), & i, j \in \{0, 1\}, \\
F_{i,j}(x, t) &= \mathbb{P}\big(0 < X(t) \leq x, I_1(t) = i, I_2(t) = j\big), & i, j \in \{0, 1\}, \ x \in]0, z[, \\
z_{i,j}(t) &= \mathbb{P}\big(X(t) = z, I_1(t) = i, I_2(t) = j\big), & i, j \in \{0, 1\}, \\
\overline{F}_{i,j}(x, t) &= \mathbb{P}\big(z < X(t) \leq x, I_1(t) = i, I_2(t) = j\big), & i, j \in \{0, 1\}, \ x \in]z, Z[, \\
Z_{i,j}(t) &= \mathbb{P}\big(X(t) = Z, I_1(t) = i, I_2(t) = j\big), & i, j \in \{0, 1\}, \\
\overline{\overline{F}}_{i,j}(x, t) &= \mathbb{P}\big(Z < X(t) \leq x, I_1(t) = i, I_2(t) = j\big), & i, j \in \{0, 1\}, \ x \in]Z, H[, \\
H_{i,j}(t) &= \mathbb{P}\big(X(t) = H, I_1(t) = i, I_2(t) = j\big), & i, j \in \{0, 1\}.
\end{aligned}
$$