DISCRETE DERIVATION OF RUIJGROK'S AND WU'S NON-LINEAR TWO VELOCITY BOLTZMANN MODEL WITH AN APPLICATION TO TRAFFIC-FLOW MODELLING*

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Abstract. A variant of the Trotter-Kato approximation theorem is used to derive from a space-discrete model the non-linear Boltzmann-like equations introduced by Th.W. Ruijgrok and T.T. Wu. The application include a micro-meso-macro link for cars in traffic.

Key words. Boltzmann-equation, Semigroup approximation, Micro-Macro link, Traffic flow.

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Introduction. The understanding of the collective dynamics of coupled elementary cells forming a complex (e.g., physical, biological and/or socio-ecological) system is a formidable interdisciplinary task. The ubiquity of such cooperative mechanism in various fields generates a strong ongoing research activity in the basic sciences [1, 2, 3] and the applied sciences [4, 5, 6].

The origin of "complex behaviour" is located in the interplay of the "microscopic" (elementary) components of the system and their discrete character which give rise to new collective properties qualitatively different from the microscopic properties. The main steps toward a formal understanding of the collective dynamics are contained in the micro-to-macro paradigm formulated in analogy to the kinetic theory of dilute gases. On the *microscopic* level the evolution of the elementary cells is described either deterministically following Newton-like dynamics or stochastically following the dynamics of a specific particle hopping model. A reduced description is given by the *mesoscopic* Boltzmann-like equation which describes the evolution of the probability distribution of the components in the phase space [7]. When the mean free path between the elementary cells goes to zero, the solution to the Boltzmann equation relaxes to a Maxwellian distribution and the process yields a *macroscopic* description via fluid-dynamic-like equations. This ambitious micro-(meso)-macro program of statistical physics is not at all achieved in general but has contributed to the understanding of micro and macro properties of systems with great practical interest such as granular and self driven many particle systems [4, 8].

In this paper we like to pursue this generic program in a very simple one-dimensional micro-meso context. Here the micro-meso link is realized by a discrete-space approximation to a Boltzmann-type equation introduced in [9] describing the mesoscopic regime. The space-discrete equations are recognized as the (nonlinear) master equations of interacting Markov-processes on a one-dimensional lattice describing the evolution of a stochastic microscopic model (i.e. a particle hopping model).

The micro-meso link is applied to the domain of traffic engineering where it completes the meso-macro link recently derived in [10]. The relevant stochastic microscopic model is derived from phenomenological considerations and pays special attention to the anisotropic character of traffic flow.

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The paper is organized as follows: In section 1 we derive the discrete Ruijgrok-Wu (RW) model from stochastic microscopic considerations and recall the continuous RW model. The solution of the later is given via a representation formula involving a cosine operator function. In section 2 we apply a variant of the Trotter-Kato approximation theorem to derive the space-continuous RW model. In section 3 we give an application in the fields of traffic theory.

1. The models. This section follows closely section 1 of W.A. Rosenkrantz' and L.Z. Bings paper [11]. We consider interacting particles which are spatially distributed over equally spaced cells $C(j) = [jh, (j+1)h[\subset \mathbb{R}$ of length h. Particles can move to the left and to the right with constant velocities $v^{\pm} = \pm 1$. We denote the number of particles in $C(j)$ with speed 1 resp. -1 at time t by $N_h^+(jh,t)$ resp. $N_h^-(jh,t)$.
The particles with speed 1 *migrate* from $C(j)$ to $C(j+1)$ and those with speed -1 from $C(j)$ to $C(j-1)$ both at rate $|v^{\pm}|/h = 1/h$. Other migration rates are zero. More precisely we assume that for a time interval of length $\Delta t < h$ the (non zero) migration probabilities are:

$$\begin{aligned}\tfrac{\Delta t}{h} + o(\Delta t) &= \text{probability that a single particle with speed 1 moves} \\ &\quad\ \text{within } \Delta t \text{ from } C(j) \text{ to } C(j+1) \\ &= \text{probability that a single particle with speed -1 moves} \\ &\quad\ \text{within } \Delta t \text{ from } C(j) \text{ to } C(j-1).\end{aligned}$$

where $o(\Delta t)$ is a quantity verifying $\lim_{\Delta t \downarrow 0} o(\Delta t)/\Delta t = 0$. Without interactions, each particle travels on the set $\{C(j) \mid j \in \mathbb{Z}\}$ which we identify with $I_h = \{jh \in \mathbb{R} \mid 0, \pm 1, \pm 2, ...\}$ according to a continuous time Markov chain with infinitesimal generator matrix $Q_h^i = (Q_h^i(j,k))_{j,k\in\mathbb{Z}}$, $i = \pm$ given by:

$$Q_h^-(j,k) = \begin{cases} 0 & \text{for } |j-k| > 1 \\ 0 & \text{for } k = j+1 \\ 1/h & \text{for } k = j-1 \\ -1/h & \text{for } j = k. \end{cases} \qquad Q_h^+(j,k) = \begin{cases} 0 & \text{for } |j-k| > 1 \\ 1/h & \text{for } k = j+1 \\ 0 & \text{for } k = j-1 \\ -1/h & \text{for } j = k. \end{cases}$$

The quantities $N_h^i(jh,t)$ satisfy the Kolmogorov forward equations:

$$\begin{cases} \partial_t N_h^-(jh,t) &= \sum_k N_h^-(kh,t) Q_h^-(k,j) = A_h N_h^-(jh,t) \\ \partial_t N_h^+(jh,t) &= \sum_k N_h^+(kh,t) Q_h^+(k,j) = -A_h N_h^+((j-1)h,t) \\ N_h^{\pm}(jh,0) &= \text{initial distribution of particles with speed } \pm 1 \end{cases} \tag{1.1}$$

where $A_h f(jh) := \frac{1}{h}\Big[f(jh+h) - f(jh)\Big]$ is the difference operator acting on the Banach space $X_h := \mathcal{C}_0(I_h)$; the space of all functions $f : I_h \to \mathbb{R}$ with $\lim_{|jh|\to\infty} f(jh) = 0$ endowed with the sup norm: $\|f\|_h := \sup_j |f(jh)|$.
In addition to the migration rules we assume that particles *react* as follows: In the small interval of time $[t, t+\Delta t[$ the number of particles in $C(j)$ with speed $+1$ increase due to spontaneous transitions of -1 particles in $C(j)$ to $+1$ particles at rate $\alpha > 0$ by the amount

$$\alpha N_h^-(jh,t)\Delta t + o(\Delta t) \tag{1.2}$$

and decrease by the amount

$$\beta N_h^+(jh,t)\Delta t + o(\Delta t) \tag{1.3}$$

due to spontaneous transitions of $+1$ particles in $C(j)$ to -1 particles at rate $\beta > 0$. Similarly, the number of particles in $C(j)$ with speed -1 decrease by the amount $\alpha N_h^-(jh,t)\Delta t + o(\Delta t)$ and increase by $\beta N_h^+(jh,t)\Delta t + o(\Delta t)$.
Moreover, particles in $C(j)$ of different speed can *collide* thereby giving rise to -1 particles in $C(j)$. This collision rule decrease the number of $+1$ particles and increase the number of -1 particles in $C(j)$ according to:

$$\frac{\mu}{h} N_h^-(jh,t) N_h^+(jh,t)\Delta t + o(\Delta t). \tag{1.4}$$

The term $\frac{\mu}{h}$ reflects the fact that the rate of interactions not only depends on the number of particles in each cell but also on its length i.e. the same number of particles crowded into an interval of smaller length will interact at a proportionally higher rate. Denoting -1 particles by $(-)$ and $+1$ particles by $(+)$, the migration and the interaction (reaction and collision) mechanisms can by summarized as follows:

$$\begin{array}{lll} \text{migration:} & (-) \longrightarrow C(j-1), & (+) \longrightarrow C(j+1), \\ \text{reaction:} & (-) \longrightarrow (+), & (+) \longrightarrow (-), \\ \text{collision:} & (+,-) \longrightarrow (-,-), & \end{array}$$

and the interactions are taken to be of mass action type. This means that the rate of each interaction is proportional to the concentration of each type of particles entering the interaction. Under this assumptions and when $\Delta t \to 0$, the functions $N_h^\pm(jh,t)$ satisfy the nonlinear forward equation:

$$\begin{aligned} \partial_t N_h^-(jh,t) &= A_h N_h^-(jh,t) - \alpha N_h^-(jh,t) + \beta N_h^+(jh,t) + \tfrac{\mu}{h} N_h^-(jh,t) N_h^+(jh,t) \\ \partial_t N_h^+(jh,t) &= -A_h N_h^+((j-1)h,t) + \alpha N_h^-(jh,t) - \beta N_h^+(jh,t) - \tfrac{\mu}{h} N_h^-(jh,t) N_h^+(jh,t) \\ N_h^\pm(jh,0) &= g_h^\pm(jh) \end{aligned} \tag{1.5}$$

with $g_h^\pm \in X_h$ some given (positive) initial distribution of the interacting particles. If in addition, we assume the existence of functions $g^\pm \in \mathcal{C}_0^1(\mathbb{R})$ and $\rho^\pm \in \mathcal{C}_0^{2,1}(\mathbb{R}\times\mathbb{R}^+)$ satisfying for all $j \in \mathbb{Z}$ and all $h > 0$:

$$g_h^\pm(jh) = g^\pm(jh) + o(h) \tag{1.6}$$

$$\frac{1}{h} N_h^\pm(jh,t) = \rho^\pm(jh,t) \tag{1.7}$$

then $\rho^-(x,t)$and $\rho^+(x,t)$ satisfy the nonlinear two-velocity Boltzmann equation of Ruijgrok and Wu introduced in [9]:

$$\left\{ \begin{aligned} \partial_t \rho^-(x,t) &= A\rho^-(x,t) - \alpha\rho^-(x,t) + \beta\rho^+(x,t) + \mu\rho^-(x,t)\rho^+(x,t) \\ \partial_t \rho^+(x,t) &= -A\rho^+(x,t) + \alpha\rho^-(x,t) - \beta\rho^+(x,t) - \mu\rho^-(x,t)\rho^+(x,t) \\ \rho^\pm(x,0) &= g^\pm(x) \end{aligned} \right. \tag{1.8}$$

where $A = \frac{\partial}{\partial_x}$ is the differential operator on the Banach space $X := \mathcal{C}_0(\mathbb{R})$ (endowed with the supremum norm $\|f\| := \sup_x |f(x)|$) with domain:

$$\mathcal{D}(A) = \{f \in X \mid f \ \text{ absolutely continuous}, \ f' \in X\}. \tag{1.9}$$

The physical content of the system (1.8) is discussed in the application of section 3 and for the mathematical discussion of the explicit solutions recalled below, we refer to

[9]. Our concern here is to derive the "mesoscopic equations" (1.8) from the discrete equations (1.5) without the assumption eq.(1.7) by showing that for all $x \in \mathbb{R}$ and uniformly for t in compact subsets of $\mathbb{R}^+$ the limit

$$\lim_{h\searrow 0,\, jh\to x} \frac{1}{h} N_h^{\pm}(jh,t) \tag{1.10}$$

exists and that the pointwise defined functions

$$\rho^{\pm}(x,t) := \lim_{h\searrow 0,\, jh\to x} \frac{1}{h} N_h^{\pm}(jh,t) \tag{1.11}$$

solve the equations (1.8).

2. Micro-meso link. Before we derive eq.(1.11) recall that the RW-model (1.8) can be linearized by means of the logarithmic transformation:

$$\begin{cases} \rho^+(x,t) &= \frac{2}{\mu}(\partial_t - A)\Big(\ln\big(u(x,t)\big) + \frac{\beta+\alpha}{2}t - \frac{\beta-\alpha}{2}x\Big) \\ \rho^-(x,t) &= -\frac{2}{\mu}(\partial_t + A)\Big(\ln\big(u(x,t)\big) + \frac{\beta+\alpha}{2}t + \frac{\beta-\alpha}{2}x\Big) \end{cases} \tag{2.1}$$

where the strictly positive function $u := u(x,t) > 0$ satisfies the hyperbolic equation:

$$\partial_t^2 u(x,t) = \big(A^2 + \alpha\beta I\big)u(x,t) \tag{2.2}$$

with I the identity operator. The above linear PDE, equivalent to the telegraphist equation, has to be solved with the initial conditions:

$$u^0(y) = u(y,0) = \exp\Big\{\frac{1}{2}\int^y \mathrm{d}x[\mu(g^-(x)+g^+(x)) + \alpha - \beta]\Big\}, \tag{2.3}$$

$$u_t^0(y) = u_t(y,0) = \frac{1}{2}u(y,0)\big(\mu(g^-(y)-g^+(y)) + \beta + \alpha\big). \tag{2.4}$$

It is well known ([12] Chapt.2.8) that the solution to the above Cauchy problem eqs.(2.2,2.3,2.4) is formally given by:

$$u(x,t) := C(t)u^0(x) + \int_0^t C(s)u_t^0(x)\mathrm{d}s \tag{2.5}$$

where $C(t)$ is the strongly continuous cosine operator function associated to the infinitesimal generator $B = A^2 + \alpha\beta I$ with domain:

$$\mathcal{D}(B) := \{f \in X \mid C(\cdot)f \in \mathcal{C}^2(\mathbb{R}, X)\}. \tag{2.6}$$

The (strong) solution to (2.2) is explicitly given by eq.(2.5) via the representation formula (see e.g., [12] p.121):

$$C(t)f = \frac{1}{2}\big[T(t)+T(-t)\big]f + \frac{\alpha\beta}{2}t\int_0^t (t^2-s^2)^{-1/2} I_1\big((t^2-s^2)^{1/2}\big)\big[T(t)+T(-t)\big]f\mathrm{d}s \tag{2.7}$$

for $f \in X$. Therein I_1 is the modified Bessel function and $T = \{T(t) \mid t \in \mathbb{R}\}$ is the $(\mathcal{C}_0)$ group of isometries on X associated to the generator $A = \frac{\partial}{\partial_x}$ given by

$$\big[T(t)f\big](x) = f(x+t). \tag{2.8}$$

The fact that the (strong) solution to eq.(2.2) is given by a strongly continuous cosine operator function $C(t)$ gives us – besides existence, uniqueness and continuous dependance on the initial data – the continuous dependance on A. It is this bonus – exploited in a version of the Trotter-Kato approximation theorem [13] – together with the obvious regularity properties of the explicit solution eq.(2.5) which enables the rigourous derivation of the limit eq.(1.11). To this end we rewrite eq.(2.1) as an abstract inhomogeneous Cauchy problem in the Banach space $Y := X \times X$ equipped with the norm $\|(f_1, f_2)\| = \|f_1\| + \|f_2\|$ and set:

$$\rho(x,t) := \big(\rho^-(x,t), \rho^+(x,t)\big) \tag{2.9}$$
$$\dot\rho(t) := \big(\partial_t\rho^-(x,t), \partial_t\rho^+(x,t)\big) \tag{2.10}$$
$$F\big(\rho(x,t)\big) := \big(-\alpha\rho^-(x,t) + \beta\rho^+(x,t) + \mu\rho^-(x,t)\rho^+(x,t), \quad \alpha\rho^-(x,t) - \beta\rho^+(x,t) - \mu\rho^-(x,t)\rho^+(x,t)\big) \tag{2.11}$$
$$\mathbf{A}\rho(x,t) := \big(A\rho^-(x,t), -A\rho^+(x,t)\big). \tag{2.12}$$

Clearly, $F(Y) \subset Y$ but F is otherwise nonlinear and unbounded. An elementary estimation for arbitrary $\rho, \xi \in Y$ yields:

$$\|F\big(\rho(x,t)\big) - F\big(\xi(x,t)\big)\| \le \big(2\max(\beta,\alpha) + 2\mu\max(\|\xi^+\|, \|\rho^-\|)\big)\|\rho - \xi\| \tag{2.13}$$

establishing that F is locally Lipschitz continuous in the sense that for all ρ, ξ in the set $\{\rho \in Y \mid \|\rho\| \le M\}$ where $M > 0$ is fixed we have:

$$\|F\big(\rho(x,t)\big) - F\big(\xi(x,t)\big)\| \le \big(2\max(\beta,\alpha) + 2\mu M\big)\|\rho - \xi\| =: \tilde M\|\rho - \xi\|. \tag{2.14}$$

Using this notations, eq.(2.1) takes the form of an abstract semilinear Cauchy problem namely:

$$\dot\rho(t) = \mathbf{A}\rho(t) + F\big(\rho(t)\big) \tag{2.15}$$
$$\rho(0) = g(0) = \big(g^-, g^+\big). \tag{2.16}$$

It is clear from eqs.(2.1, 2.5) and the representation formula eq.(2.7) that the initial value problem eqs.(2.15, 2.16) has a strong solution $\rho \in \mathcal{D}(A) \times \mathcal{D}(A)$ whenever g^- and g^+ are sufficiently regular; typically $g^\pm \in \mathcal{C}^2(\mathbb{R})$. Indeed using the representation formula eq.(2.7) it is immediate to check that for all fixed $T > 0$, the following implication hold:

$$(g^-, g^+) \in \mathcal{C}^2(\mathbb{R}) \times \mathcal{C}^2(\mathbb{R}) \Rightarrow (\rho^-(t), \rho^+(t)) \in \mathcal{C}^1(\mathbb{R}) \times \mathcal{C}^1(\mathbb{R}), \ \ t \in [0,T], \tag{2.17}$$
$$\text{and} \ \sup_{0\le t\le T} \Big|\frac{\partial\rho^\pm(x,t)}{\partial x}\Big| < \infty.$$

Clearly the strong solution u is also a mild one i.e., u is continues and satisfy the integral equation (see e.g., [14] p.183):

$$\rho(t) = G(t)\rho(0) + \int_0^t G(t-s)F(\rho(s))\mathrm{d}s \tag{2.18}$$

where the (C_0) contraction semigroup $G(t)$ is given by:

$$G(t)\big(f_1(x), f_2(x)\big) = \big(T(t)f_1(x), T(t)f_2(x)\big) = \big(f_1(x+t), f_2(x+t)\big).$$

Similarly, equation (1.5) takes the form:

$$\dot{\rho}_h(t) = \mathbf{A}_h \rho_h(t) + F\big(\rho_h(t)\big) \tag{2.19}$$
$$\rho_h(0) = g_h(0) \tag{2.20}$$

where

$$\begin{aligned} \rho_h(jh,t) &:= \big(h^{-1}N_h^-(jh,t), h^{-1}N_h^+(jh,t)\big) \\ \mathbf{A}_h\rho_h(t) &:= \big(A_h h^{-1}N_h^-(jh,t), -A_h h^{-1}N_h^+(jh,t)\big). \end{aligned}$$

Note that the Cauchy problem eqs.(2.19, 2.20) has to be solved in the Banach space $Y_h := X_h \times X_h$ equipped with the norm $\|(f_1,f_2)\|_h = \|f_1\|_h + \|f_2\|_h$. The existence of a unique mild solution ρ_h of (2.19, 2.20) on some maximal interval $[0,T[$ where $T>0$ depends on the Lipschitz constant $\widetilde{M}$ and on the initial conditions $g^{\pm}$ relies on the local Lipschitz property established in (2.14) (see e.g., Thm. 1.4 in [14]). For $t \in [0,T[$ this solution satisfies:

$$\rho_h(t) = G_h(t)\rho_h(0) + \int_0^t G_h(t-s)F(\rho_h(s))\mathrm{d}s \tag{2.21}$$

where $G_h(t)$ is the (C_0) contraction semigroup on Y_h given by:

$$G_h(t)\big(f_1(jh), f_2(jh)\big) = \big(T_h(t)f_1(jh), T_h(t)f_2(jh)\big).$$

Here $T_h(t) = \exp(tA_h)$ means the (C_0) contraction semigroup generated by the finite difference operator A_h (with domain $\mathcal{D}(A_h) = X_h$) explicitly given by (see e.g., [12] p.23):

$$T_h(t)f(j) = e^{-t/h}\sum_{k=0}^{\infty}\frac{(t/h)^k}{k!}f(j+hk). \tag{2.22}$$

Following [11] we define the bounded linear mappings

$$\begin{array}{lllll} P_h: & Y & \to & Y_h & \\ & (\rho^-,\rho^+) & \mapsto & P_h(\rho^-,\rho^+): & \mathbb{R} \to \mathbb{R}\times\mathbb{R} \\ & & & & x \mapsto (\rho^-(jh),\rho^+(jh)) \ \text{ for } \ jh \le x < (j+1)h \end{array}$$

joining the evident properties:

1) $\|P_h f\|_h \le \|f\|$
2) $\lim_{h\to 0}\|P_h f\|_h = \|f\|$
3) for any $f_h \in Y_h$ there exists a $f \in Y$ such that $f_h = P_h f$ and $\|f_h\|_h \le \|f\|$.

It follows (by definition) that the sequence of Banach spaces Y_h with the sequence of bounded linear maps P_h approximate the Banach space Y for $h \to 0$.
The convergence of the "linear" part $G_h(t)\rho_h(0)$ of ρ_h given in the integral representation (2.21) to $G(t)\rho(0)$ follows from a variant of the Trotter-Kato approximation theorem due to Kurtz (see e.g., [15] Thm 2.6). The theorem assures that for every fixed $s \in [0,\infty[$ and for every $\rho \in \mathcal{D}(A)\times\mathcal{D}(A) \subset \mathcal{C}_0^2(\mathbb{R})\times\mathcal{C}_0^2(\mathbb{R})$ we have:

$$\lim_{h\searrow 0}\sup_{0\le t\le s}\|G_h(t)P_h\rho - P_h G(t)\rho\|_h = 0. \tag{2.23}$$

In particular for the initial condition $\rho(0)$ we have:

$$P_h G(t)\rho(0) = G_h(t)P_h\rho(0) + o(h). \tag{2.24}$$

To treat the inhomogeneous part within the time horizon $T > 0$, we apply P_h to the integral representation eq.(2.18) and use eq.(2.24) to obtain:

$$\begin{aligned}P_h\rho(t) &= P_hG(t)\rho(0) + \int_0^t P_hG(t-s)F(\rho(s))\mathrm{d}s \\ &= G_h(t)P_h\rho(0) + o(h) + \int_0^t \big(G_h(t-s)P_hF(\rho(s)) + o(h)\big)\mathrm{d}s \\ &= G_h(t)P_h\rho(0) + \int_0^t G_h(t-s)F(P_h\rho(s))\mathrm{d}s + o(h)\end{aligned} \tag{2.25}$$

for $t \in [0, T]$. Therefore,

$$P_h\rho(t) - \rho_h(t) = G_h(t)\big(P_h\rho(0) - \rho_h(0)\big) + o(h) \tag{2.26}$$

$$+ \int_0^t G_h(t-s)\big(F(P_h\rho(s)) - F(\rho_h(s))\big)\mathrm{d}s. \tag{2.27}$$

Set

$$\phi_h(t) := \|P_h\rho(t) - \rho_h(t)\|_h \tag{2.28}$$

and note that by assumption (1.6) we have:

$$\phi_h(0) = o(h). \tag{2.29}$$

Applying the local Lipschitz property established in eq.(2.14) and using the fact that $\|G_h(t)f\|_h \leq \|f\|_h$ one concludes on the existence of a constant K depending on T (and $\tilde{M}$) such that:

$$\phi_h(t) \leq Kh + K\int_0^t \phi_h(s)\mathrm{d}s. \tag{2.30}$$

Using Gronwall's inequality, the above directly implies that $\phi_h(t) \leq Kh\exp(Kt)$ for $t \in [0, T]$ and therefore:

$$\lim_{h\searrow 0}\phi_h(t) = \lim_{h\searrow 0}\|P_h\rho(t) - \rho_h(t)\|_h = 0, \quad \text{for } \ t \in [0, T]. \tag{2.31}$$

which proves eqs.(1.10) and (1.11).

3. Application. The RW-model eq.(1.8) was originally motivated by practical considerations in connection with controlled thermonuclear fusion. The interesting feature for applications in general is that the eq.(1.8) can be solved explicitly for a large class of initial conditions in terms of modified Bessel functions. Consequently, shock waves and approach to equilibrium can be investigated analytically. Here we extend the fields of applications to a non-linear transport phenomena encountered in vehicular traffic flow (see e.g., [4, 16] for comprehensive reviews). At the proposed level of description, the main ingredients for non-linearity comes from a certain anisotropic collision behaviour (a fast driver behind a slow one has to slow down or to overtake if he can). This is taken into account by the RW-model. Indeed, it is seen from section 1 that there is one binary collision of the form $(+,-) \to (-,-)$. The presence of this collision mechanism together with the absence of the inverse collision $(-,-) \to (+,-)$ means the violation of the detailed balance of momentum which is the mentioned desired anisotropic collision feature encountered in vehicular traffic flow.

Traffic flow modelling. A traffic system, comprised of drivers, vehicles and roadways, exhibits extremely complex behaviour including congestion formation, stop-and-go traffic and hysteresis due to the heterogeneous drivers behaviour, the highly nonlinear group dynamics and large system dimensions. Traffic theory proposes mathematical descriptions of the processes in order to understand the dynamics of traffic flow. Two complementary approaches have dominated traffic flow modelling:

i) a purely *microscopic* approach in which the individual vehicular interactions are taken into account (see [16, 17] and the references therein), and

ii) a *macroscopic* approach which is based on fluid dynamical equations describing the behaviour of a compressible fluid (see [4, 18, 19] and the references therein).

Even within the microscopic approach there are different types of mathematical descriptions. The so-called car-following theory for example provides a deterministic, Newton-like description of the motion of individual vehicles. In contrast, particle hopping modelling (also stochastic microscopic modelling) describe traffic in terms of a stochastic dynamics of individual vehicles and will be the point of view adopted in this application.
The macroscopic description, is always based on a continuity equation,

$$\partial_t \rho(x,t) + \partial_x J(x,t) = 0 \tag{3.1}$$

and completed by a relation between the current $J(x,t)$ and the vehicle density $\rho(x,t)$, which is known in traffic engineering as the *fundamental diagram*. This relation contains all the dynamic information specific to a particular macroscopic model. Among the various fundamental diagrams which have been explored a very simple and popular one is the Lighthill-Whitham (LW) equation [18] assuming that there exists an equilibrium flow-density relationship of the form:

$$J(x,t) = j(\rho(x,t)). \tag{3.2}$$

Moreover, on the basis of experimental observations, B.D. Greenshields [20] proposed the choice:

$$j(\rho(x,t)) = V_{\max}\rho(x,t)\big(1-\rho(x,t)\big) \tag{3.3}$$

where the phenomenological parameter $V_{\max}$ is the maximum average speed for $\rho \to 0$. The nonlinear model can explain the formation of shock waves which corresponds to congestion formation in traffic flow [18]. Despite this success in describing congestion formation in traffic flow the LW theory fails in describing more complicated traffic flow phenomena such as stop-and-go traffic or hysteresis (see e.g., [21]). This is due to the unrealistic assumption that the traffic flow is always in equilibrium. In reality, the dynamics is a result of the retarded response of drivers to various (mostly) frontal stimuli [22]. Among different non-equilibrium models [19, 22, 23] we proposed in [10] the exactly solvable two velocity RW-model eq.(1.8) which takes into account acceleration behaviour and anisotropic interactions of vehicles with different speed in the most simple manner. Despite its simplicity, it is shown in [10] that the RW-model relaxes in a diffusive limit to the viscous LW model specified by the Greenshields flux relation eq.(3.3). This meso-macro link explains the importance of the empirical density-flux relation eq.(3.3) and reciprocally, corroborates the relevance of the RW-model in traffic theoretical contexts. Moreover, the RW-model shows the signature of hysteretic behaviour for a specific range of parameters. Indeed, for $\alpha > \beta$ there exists a class of spatially inhomogeneous equilibrium distributions of the RW-model indexed by two continuous real parameters $x_0 \in \mathbb{R}$ and C, $0 \le C \le (\sqrt{\alpha} - \sqrt{\beta})^2$ given by (see [9] Section 5):

$$\rho_x^+(x) = \frac{\mu C + \alpha - \beta}{2\mu} + \frac{r}{2\mu}\tanh((x - x_0)r/2)$$
$$\rho_x^-(x) = \frac{-\mu C + \alpha - \beta}{2\mu} + \frac{r}{2\mu}\tanh((x - x_0)r/2)$$

with r depending on the parameters α, β, μ and C only. The parameter C corresponds to the equilibrium flow $C = \rho^+(x) - \rho^-(x)$ and is determined by the initial conditions $g^\pm$ of eq.(1.8). The equilibrium density ρ at x_0, is simply $\rho(x_0) = \rho^+(x_0) + \rho^+(x_0) = \frac{\alpha-\beta}{\mu}$ and hence independent of the flow C. Therefore, without changing the equilibrium density $\rho(x_0)$, different initial conditions lead to different equilibrium flow states at x_0 which is a typical signature for hysteresis (see e.g., [16] p. 264).

The two-velocity RW-model however is too simple to explain stop and go waves. In this sense the model can not compete with more sophisticated models such as the non-local gas-kinetic based traffic models presented in [23].

We complete now the meso-macro picture by a micro-meso link based on the limit result of section 2. The stochastic microscopic description (i.e. the particle hopping model) is specified by the following considerations:

We observe cars on a long highway without on/off ramps. We further suppose that at any instant of time t, the heterogeneous drivers behaviour can be classified into "slow" resp. "fast" drivers corresponding to cars with speed v_1 (slow) resp. cars with speed v_2 (fast). Informally, the basic modelling assumptions are:

- A1) The fairly diverse driving habits of the people is modelled by spontaneous (Markovian) transitions from one behaviour to the other.
- A2) The interactions are typically short ranged in the sense that only consecutive cars (with different speed) can interact. The rate of interactions within a short region is proportional to the number of drivers of each type in this region.
- A3) The anisotropic character of traffic flow is taken into account by saying that vehicles from behind should not influence the actions of their leading vehicles.

A1) implies transitions of the form $v_1 \to v_2$ and $v_2 \to v_1$. A2) together with A3) states that (only) consecutive cars can interact in the following way $(v_2, v_1) \to (v_1, v_1)$ or $(v_1, v_2) \to (v_2, v_2)$ where (x, y) means an ordered couple of consecutive cars with velocities x and y respectively. The former interaction is quite natural for highway traffic, saying that a fast car behind a slow one has to slow down (or to overtake, if he can). The latter interaction is somewhat special (nevertheless not completely lacking in real traffic) and we will neglect them. Hence we extend the assumptions by:

- A4) slow cars behind fast ones do not interact.

With the assumptions A1)-A4), which is our stochastic microscopic traffic model, we are able to apply the convergence result of section 1. Indeed, describing the cars in a moving framework which links the coordinates to the center of inertia:

$$\big(x, t\big) \mapsto \big(y, \tau\big) := \Big(x - \frac{v_1 + v_2}{2}t, t\Big), \tag{3.4}$$

the velocities of the vehicles are transformed as

$$v_1 \mapsto -v_0, \quad v_2 \mapsto v_0, \tag{3.5}$$

where $v_0 := (v_2 - v_1)/2$. Partitioning the y-axes into equally spaced cells $C(j) = [jh, (j+1)h[$ of length h we see that in the moving coordinates cars migrate from $C(j)$ to $C(j-1)$ or $C(j+1)$ at the rate v_0/h.
A1) implies that within a time interval $[t, t+\Delta t[$, the number of particles $N_h^+(jh,t)$ (resp. $N_h^-(jh,t)$) in $C(j)$ with speed v_0 (resp. $-v_0$) at time t increase by an amount proportional to $N_h^-(jh,t)\Delta t + o(\Delta t)$ (resp. $N_h^+(jh,t)\Delta t + o(\Delta t)$) and decrease by an amount proportional to $N_h^+(jh,t)\Delta t + o(\Delta t)$ (resp. $N_h^-(jh,t)\Delta t + o(\Delta t)$).
From A2)-A4) we infer the local collision rule $(v_0, -v_0) \to (-v_0, -v_0)$. By the second part of assumption A2), this will increase (decrease) the number of slow (fast) cars within the time interval $[t, t+\Delta t[$ by the amount:

$$\frac{\mu}{h} N_h^+(jh,t) N_h^-(jh,t)\Delta t + o(\Delta t).$$

Hence using the convergence result of section 1, we see that under the stochastic microscopic assumptions A1)-A4), the resulting mesoscopic description of the cars in traffic is given by the RW-model:

$$\begin{cases} \partial_t \rho^-(x,t) - v_0 \rho^-(x,t) &= -\alpha\rho^-(x,t) + \beta\rho^+(x,t) + \mu\rho^-(x,t)\rho^+(x,t) \\ \partial_t \rho^+(x,t) + v_0 \rho^+(x,t) &= +\alpha\rho^-(x,t) - \beta\rho^+(x,t) - \mu\rho^-(x,t)\rho^+(x,t). \end{cases} \tag{3.6}$$

It describes in a moving coordinate system the evolution of the distribution functions of fast cars ρ^+ and slow cars ρ^-. An explicit solution of the traffic density $\rho = \rho^- + \rho^+$ is sketched in figure 3. The initial conditions reflect a situation where a platoon of fast cars is behind a platoon of slow cars. When the fast cars ketch up with the slow ones the collision mechanism $(+,-) \longrightarrow (-,-)$ increase (decrease) the concentration of slow (fast) cars and only a view fast cars passe the train of slow cars without undergoing collisions.

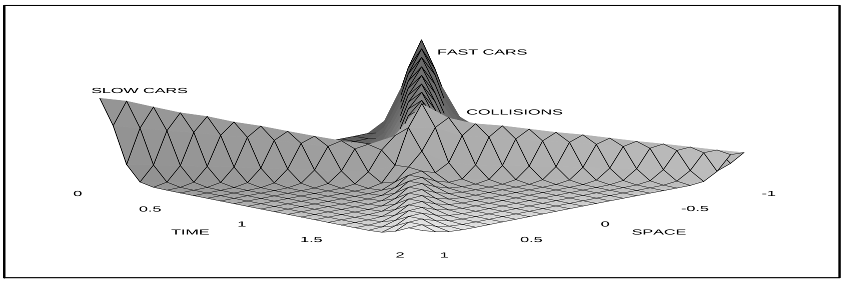

Fig. 3.1. *Sketch of the density profiles in a moving coordinate system for Gaussian initial conditions. When the platoons meet (fast cars catch up with the slow ones), the collision term will become important and decrease the amount of fast cars and increase the amount of slow cars. Parameters:* $\alpha = 50$, $\beta = 10$, $\mu = 80$ *and* $v_0 = 1$.

Remark. It is worthwhile noting that part two of assumption A2) (i.e., collisions are of mass action type) can be justified on the basis of simple kinetic considerations and a uniformity assumption. Indeed, suppose that a cell of length h contains $N = N^+ + N^-$ particles (N^+ of type + and N^- of type −). Divide the cell in N boxes of length

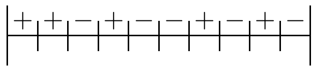

FIG. 3.2. *Uniform distribution of $N = N^+ + N^-$ particles (N^+ of type + and N^- of type −) in a cell with N boxes. A collision-configuration is of the form* $(+,-)$.

h and suppose that the particles are uniformly distributed over the boxes such that every box contains exactly one particle (see figure 3). The configurations contributing eventually to collisions are of the form $(+,-)$ and it is an easy combinatorial exercise to show that the probability of having exactly r-configurations of the form $(+,-)$ in the cell is:

$$P(\text{r-config.}) = \binom{N^+}{r}\binom{N^- - 1}{r-1}\Big/\binom{N}{N^+}$$

and that the expected number of collision-configurations is $(N^+)^2N^-/((N-1)N)$. A collision-configuration $(+,-)$ turns into a real collision when it remains stable during the mean free time $h/(N\Delta v)$ where $\Delta v = v - (-v) = 2v$ is the velocity difference between + and − particles. The stability however is inversely proportional to the "total temperature" $\alpha + \beta$ and therefore the frequency of collision $2vN/h$ multiplied by the expected collision configurations which remain stable yields the mass type hypotheses:

$$\frac{2vN}{h}\frac{1}{\alpha+\beta}\frac{(N^+)^2N^-}{(N-1)N} = \frac{2v/(\alpha+\beta)}{h}\frac{N^+}{(N-1)}N^+N^-.$$

Supposing further that the number of + particles and the number of − particles in the cell are proportional (i.e. $N^+/(N-1) = \text{const.}$) we have:

$$(+,-) \to (-,-), \quad \text{with rate } \frac{\mu}{h}N^+N^-$$

for some constant μ.

4. Conclusion. We established the convergence of a space discrete approximation for the nonlinear two-velocity Boltzmann model of Ruijgrok and Wu. The derivation shows the main kinetic features of the equations which are besides the migration term a reaction and a collision mechanism of mass action type.
An application within traffic theory is considered which joins on a minimal level of detailed knowledge the above kinetic features. The convergence scheme completes a micro-meso-macro link for the popular macroscopic traffic model of Lighthill, Whitham and Greenshields.

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