

CORRESPONDENCE PRINCIPLE BETWEEN SPHERICAL AND EUCLIDEAN WAVELETS

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ABSTRACT

Wavelets on the sphere are reintroduced and further developed independently of the original group theoretic formalism, in an equivalent, but more straightforward approach. These developments are motivated by the interest of the scale-space analysis of the cosmic microwave background (CMB) anisotropies on the sky. A new, self-consistent, and practical approach to the wavelet filtering on the sphere is developed. It is also established that the inverse stereographic projection of a wavelet on the plane (*i.e.* Euclidean wavelet) leads to a wavelet on the sphere (*i.e.* spherical wavelet). This new correspondence principle simplifies the construction of wavelets on the sphere and allows to transfer onto the sphere properties of wavelets on the plane. In that regard, we define and develop the notions of directionality and steerability of filters on the sphere. In the context of the CMB analysis, these notions are important tools for the identification of local directional features in the wavelet coefficients of the signal, and for their interpretation as possible signatures of non-gaussianity, statistical anisotropy, or foreground emission. But the generic results exposed may find numerous applications beyond cosmology and astrophysics.

Subject headings: cosmology: cosmic microwave background — methods: data analysis

1. INTRODUCTION

The last decade has recognized the cosmic microwave background (CMB) as a unique laboratory for achieving precision cosmology. The cosmological parameters defining the structure, the energy content, and the evolution of the universe are now determined with an impressive precision (Page et al. 2003; Spergel et al. 2003; Bouchet 2004). However, the theoretical hypotheses on which the corresponding concordance cosmological model relies still must be fully investigated. They notably extend from the cosmological principle of homogeneity and isotropy, to the models of inflation for the physics of the early universe (Wandelt 2004; Bartolo et al. 2004), or the theory of gravitation itself, namely general relativity (Boucher et al. 2004a; Boucher et al. 2004b).

In the context of the concordance cosmological model, the cosmic background radiation is understood as a unique realization of a gaussian and stationary random signal on the sphere, arising from quantum energy density perturbations developed in a primordial inflationary era of the universe. In this respect, the analysis of the CMB anisotropies on the sky is essentially confined to the study of its temperature (and polarization) angular power spectrum.

But questioning the basic hypotheses of inflation, or of the cosmological principle notably amounts to raise the questions of the gaussianity and statistical isotropy (stationarity) of the CMB. New methods of analysis of the CMB must therefore be considered. Notably, in addition to the information on scales given through a pure spherical harmonics decomposition, the scale-space analysis is essential to allow the localization of

features on the sky. Such local features might for example be associated with non-gaussianity or non-stationarity of the statistical distribution from which the CMB arises, or with foreground emission such as point sources. First analyses on the one-year data of the ongoing Wilkinson Microwave Anisotropy Probe (WMAP) satellite mission suggest a departure from both statistical isotropy and gaussianity of the signal. The various methods applied extend from the analysis of phase correlations (Coles et al. 2004), N-point correlation functions (Eriksen et al. 2004; Eriksen et al. 2005), bipolar power spectra (Hajian & Souradeep 2003; Hajian et al. 2005; Hajian & Souradeep 2005), or local power spectra (Hansen et al. 2002; Hansen et al. 2004a; Hansen et al. 2004b), to multipole vectors (Copi et al. 2004; Katz & Weeks 2004; Lachièze-Rey 2004). In addition, the efficiency of the wavelet signal processing for detecting non-gaussianities in the CMB signal was also recently established (Hobson et al. 1999; Barreiro & Hobson 2001; Martínez-González et al. 2002; Starck et al. 2004). While no evidence for non-gaussianity was found through wavelet analyses in the former COsmic Background Explorer (COBE) satellite mission data (Barreiro et al. 2000; Cayón et al. 2001), various spherical wavelet analyses of the one-year WMAP data also suggest a departure from non-gaussianity or statistical isotropy (Vielva et al. 2004; Mukherjee & Wang 2004; McEwen et al. 2005; Cruz et al. 2005). Other works also explicitly established the efficiency of the scale-space wavelet processing for point sources detection in the foreground, or for technical purposes such as denoising and deconvolution (Sanz et al. 1999; Tenorio et al. 1999; Vielva et al. 2001; Vielva et al. 2003; Maisinger et al. 2004). But a huge amount

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of work is still needed in this context, notably for the identification of not only the position, but also the precise direction, and possibly the morphology of the observed features, and for their interpretation (see (McEwen et al. 2005) for a first approach to the directional wavelet analysis).

In this context, a new approach to the formalism of spherical wavelets is developed here. In § 2, we briefly review the formalism for the construction of Euclidean wavelets. In § 3, we reintroduce spherical wavelets independently of the original group theoretic formalism, in a new, equivalent, and self-consistent approach. We adopt a practical philosophy, considering wavelets as localized filters which enable scale-space analysis, and offer an explicit reconstruction formula for the signal considered, from its wavelet coefficients. We also prove that the fundamental operation of dilation on the sphere is uniquely determined from the requirement of basic natural properties. This establishes the uniqueness of the wavelet formalism considered. Related technical proofs are postponed to appendix A. In § 4, we prove that the inverse stereographic projection of a wavelet on the plane gives a wavelet on the sphere. It is also established that the stereographic projection is the unique projection through which this correspondence principle holds. This new principle simplifies the construction of wavelets on the sphere and allows to transfer wavelet properties from the plane onto the sphere. Notice that it has been misleadingly suggested in the original approach (Antoine et al. 2002), while not established. The related technical proofs are detailed in appendix B. In § 5, the concepts of filter directionality and steerability on the sphere are defined from the corresponding notions on the plane. In the context of the wavelet formalism, the correspondence principle enables to transfer these properties from wavelets on the plane to wavelets on the sphere. The property of filter steerability allows the computation of the rotation of a filter on itself in any direction from a simple finite linear combination of basis filters. We also study the angular band limitation of steerable filters and explicitly treat the examples of steerable wavelets on the sphere defined as inverse stereographic projection of the derivatives of radial functions on the plane. In § 6, we discuss the interest of directional and steerable filters for the identification and interpretation of local directional features in the context of the CMB analysis. A numerical example illustrates our discussion. In § 7, we briefly conclude.

2. WAVELETS ON THE PLANE

In this section we briefly sketch the well-known formalism of wavelets on the plane.

On the plane as well as on the line, the notion of wavelet transform of a signal is a powerful method of signal decomposition (Torrésani 1995; Mallat 1998; Antoine et al. 2004). We consider here a practical approach for the definition of wavelets on the plane, which will easily be translated on the sphere. A “mother wavelet” $\psi(\vec{x})$ is first defined as a localized function on the plane, on which affine transformations may be applied: translations, rotations, and dilations. Second, the wavelet transform of a signal on the plane is defined as the correlation of the signal with the dilated and rotated versions of the mother wavelet, leading to wavelet coefficients. This explicitly defines the scale-space nature of the decomposition. In this context, an admissibility condition is finally imposed on the mother wavelet by explicitly requiring the exact reconstruction formula of the signal from its wavelet coefficients. Notice for completeness that, in terms of the original group theoretic approach, the wavelet decomposition is defined by

the construction of the coherent states of the group of affine transformations (translations, rotations and dilations) on the plane. In this context, a wavelet must satisfy an admissibility condition which ensures the square-integrability of the unitary and irreducible representations of that group on the Hilbert space of square-integrable functions in which the signals are defined. This square-integrability implies that the family of wavelets obtained by affine transformations from a mother wavelet constitutes an over-complete frame in the considered Hilbert space, and that an exact reconstruction formula of a signal in terms of its wavelet coefficients may be obtained. Our more practical considerations lead identically to the same wavelet formalism.

First, the affine transformations are defined as follows. Let us consider the Hilbert space of square-integrable functions on the plane: $g(\vec{x})$ in $L^2(\mathbb{R}^2, d^2\vec{x})$. In the coordinate system $(o, o\hat{x}, o\hat{y})$, the point \vec{x} on the plane is given in Cartesian coordinates as $\vec{x} = (x, y)$, and in polar coordinates as $\vec{x} = (r, \varphi)$. The invariant measure on the plane, relative to the canonical Euclidean metric in \mathbb{R}^2 is simply $d^2\vec{x} = dx dy$. Generically, the action of an operator on a function in $L^2(\mathbb{R}^2, d^2\vec{x})$ is defined by the action of the inverse of the corresponding operator on \mathbb{R}^2 , applied to the function’s argument. The operator $t(\vec{x}_0)$ in $L^2(\mathbb{R}^2, d^2\vec{x})$ for a translation by an amplitude $\vec{x}_0 = (x_0, y_0)$ is defined in terms of the inverse of the translation $t_{\vec{x}_0}$ on points in \mathbb{R}^2 . Its action reads:

$$[t(\vec{x}_0)g](\vec{x}) = g(t_{\vec{x}_0}^{-1}\vec{x}), \quad (1)$$

with $t_{\vec{x}_0}(x, y) = (x + x_0, y + y_0)$. The rotation operator around the origin of coordinates $r(\chi)$ in $L^2(\mathbb{R}^2, d^2\vec{x})$, rotation of the wavelet around itself by an angle $\chi \in [0, 2\pi[$, is also given in terms of the inverse of the rotation r_χ on points in \mathbb{R}^2 . It reads

$$[r(\chi)g](\vec{x}) = g(r_\chi^{-1}\vec{x}), \quad (2)$$

where $r_\chi(x, y)$ follows from the action of the two-dimensional rotation matrix r_χ on the Cartesian coordinates (x, y) , or equivalently in polar coordinates $r_\chi(r, \varphi) = (r, \varphi + \chi)$. The dilation operator $d(a)$ on functions in $L^2(\mathbb{R}^2, d^2\vec{x})$ with a dilation factor $a \in \mathbb{R}_+^*$ is again defined, in terms of the inverse of the corresponding dilation d_a on points in \mathbb{R}^2 . It reads

$$[d(a)g](\vec{x}) = a^{-1}g(d_a^{-1}\vec{x}), \quad (3)$$

with $d_a(x, y) = (ax, ay)$ in Cartesian coordinates, or equivalently $d_a(r, \varphi) = (r_a(r), \varphi)$ for $r_a(r) = ar$ in polar coordinates. The dilation operator on the plane is uniquely defined (see appendix A) by the requirement of the following properties. The operator d_a on the points on \mathbb{R}^2 must be a radial (*i.e.* only affecting the radial variable r independently of φ , and leaving φ invariant) and conformal (*i.e.* preserving the measure of angles in the tangent plane at each point of \mathbb{R}^2) diffeomorphism (*i.e.* a continuously differentiable bijection on \mathbb{R}^2). The normalization factor a^{-1} in (3) is also uniquely determined by the requirement that the dilation $d(a)$ of functions in $L^2(\mathbb{R}^2, d^2\vec{x})$ be a unitary operator (*i.e.* preserving the scalar product in $L^2(\mathbb{R}^2, d^2\vec{x})$, and specifically the norm of functions).

Second, the analysis of signals goes as follows. The wavelet transform of a signal $f(\vec{x})$ with the wavelet $\psi(\vec{x})$, localized analysis function in $L^2(\mathbb{R}^2, d^2\vec{x})$, is defined as the correlation between the signal $f(\vec{x})$ and the dilated and rotated wavelet $\psi_{\chi, a} = r(\chi)d(a)\psi$, that is as the following scalar product:

$$W_\psi^f(\vec{x}_0, \chi, a) = \int_{\mathbb{R}^2} d^2\vec{x} \psi_{\vec{x}_0, \chi, a}^*(\vec{x}) f(\vec{x}) \\ = \langle \psi_{\vec{x}_0, \chi, a} | f \rangle, \quad (4)$$

for $\psi_{\vec{x}_0, \chi, a} = t(\vec{x}_0)\psi_{\chi, a}$. The wavelet coefficients $W_{\psi}^f(\vec{x}_0, \chi, a)$ represent the characteristics of the signal for each analysis scale a , direction χ , and position \vec{x}_0 .

Finally, the synthesis of a signal $f(\vec{x})$ from its wavelet coefficients reads:

$$f(\vec{x}) = \int_0^{2\pi} d\chi \int_0^{+\infty} \frac{da}{a^3} \int_{\mathbb{R}^2} d^2\vec{x}_0 W_{\psi}^f(\vec{x}_0, \chi, a) [t(\vec{x}_0)r(\chi)L_{\psi}\psi_a](\vec{x}). \quad (5)$$

In this relation, the operator L_{ψ} in $L^2(\mathbb{R}^2, d^2\vec{x})$ is defined by the simple division by a constant: $[L_{\psi}g](\vec{x}) = g(\vec{x})/C_{\psi}$. This exact reconstruction formula holds if and only if the wavelet $\psi(\vec{x})$ satisfies the following admissibility condition:

$$0 < C_{\psi} = \int_{\mathbb{R}^2} d^2\vec{k} \frac{|\widehat{\psi}(\vec{k})|^2}{|\vec{k}|^2} < \infty, \quad (6)$$

with the normalization convention $\widehat{\psi}(\vec{k}) = \int d\vec{x} e^{-i\vec{k}\cdot\vec{x}}\psi(\vec{x})$ for the Fourier transform of functions in the plane. The zero-mean is therefore a necessary condition for the wavelet admissibility in $L^2(\mathbb{R}^2, d^2\vec{x})$:

$$\int_{\mathbb{R}^2} d^2\vec{x} \psi(\vec{x}) = 0. \quad (7)$$

It is also well-known that under the additional requirement that $\psi(\vec{x})$ be in $L^1 \cap L^2(\mathbb{R}^2, d^2\vec{x})$, the zero-mean condition (7) implies the exact admissibility condition (6). Wavelets on the plane may therefore be easily built.

3. WAVELETS ON THE SPHERE

The formalism of wavelets on the sphere was originally established in a group theoretic framework (Antoine & Vandergheynst 1999; Antoine & Vandergheynst 1998; Antoine et al. 2002; Demanet & Vandergheynst 2003; Bogdanova et al. 2005). In this section we reintroduce the notion of wavelets on the sphere through a new and completely self-consistent approach. The resulting formalism is equivalent to the original one, but more practical and straightforward. The structure of our approach follows, in perfect analogy with the formalism of wavelets on the plane introduced in the former section.

For the clarity of the expressions, we denote functions and operators on the sphere in uppercase letters, by opposition to the lowercase letters denoting functions and operators on the plane. Identically to our approach in the plane, a ‘‘mother wavelet’’ $\Psi(\omega)$ is first defined as a localized function on the unit sphere, on which affine transformations may be applied: translations, rotations, and dilations. Second, the wavelet transform of a signal on the sphere is defined as the correlation of the signal with the dilated and rotated versions of the mother wavelet, leading to wavelet coefficients, defining the scale-space nature of the decomposition on the sphere. Third, an admissibility condition is imposed on the mother wavelet by explicitly requiring the exact reconstruction formula of the signal from its wavelet coefficients. We again notice that, similarly to the formalism in the plane, in a group theoretic approach, the wavelet decomposition is defined by the construction of generalized coherent states for the affine transformations on the sphere (translations, rotations and dilations) contained in the conformal group of the sphere, $SO(1, 3)$. As

already stated, our approach leads to the same final wavelet formalism.

First, the affine transformations are defined as follows on square-integrable functions $G(\omega)$ in $L^2(S^2, d\Omega)$ on the unit sphere. The point ω on the sphere is given in spherical coordinates as $\omega = (\theta, \varphi)$. In an orthonormal Cartesian coordinate system $(o, o\hat{x}, o\hat{y}, o\hat{z})$ centered on the unit sphere, the polar angle, or co-latitude, $\theta \in [0, \pi]$ represents the angle between the vector identifying ω and the axis $o\hat{z}$. The azimuthal, or longitudinal, angle $\varphi \in [0, 2\pi[$, not defined though for $\theta \in \{0, \pi\}$, represents the angle between the projection of this vector in the plane $(o, o\hat{x}, o\hat{y})$ and the axis $o\hat{x}$. The invariant measure on the sphere, relative to the canonical metric on S^2 induced from the Euclidean metric in three dimensions is $d\Omega = d\cos\theta d\varphi$. Once again, generically, the action of an operator on a function in $L^2(S^2, d\Omega)$ is defined by the action of the inverse of the corresponding operator on S^2 , applied to the function’s argument.

The action of a rotation $\rho \in SO(3)$ in three dimensions on a function G on the sphere, is defined by the operator $R(\rho) : G(\omega) \rightarrow [R(\rho)G](\omega) = G(R_{\rho}^{-1}\omega)$ in $L^2(S^2, d\Omega)$. The operator R_{ρ} may be decomposed in three consecutive rotations, respectively around the axes of coordinates $o\hat{z}$, $o\hat{y}$, and $o\hat{x}$, and defined by the Euler angles $(\varphi_0, \theta_0, \chi)$, with $\theta_0 \in [0, \pi]$ and $\varphi_0, \chi \in [0, 2\pi[$: $R_{\rho} = R_{\varphi_0, \theta_0, \chi} = R_{\varphi_0}^z R_{\theta_0}^y R_{\chi}^x$. The inverse rotation R_{ρ}^{-1} is characterized by opposite Euler angles in the reverse order: $R_{\varphi_0, \theta_0, \chi}^{-1} = R_{-\chi, -\theta_0, -\varphi_0} = R_{\pi-\chi, \theta_0, \pi-\varphi_0}$. In the particular context of the analysis of functions on the sphere, the variable $(\varphi_0, \theta_0, \chi)$ in the parameter space of the group $SO(3)$ may be decomposed as (ω_0, χ) in $S^2 \otimes [0, 2\pi[$, where ω_0 defines a position on the sphere S^2 , and χ a direction in $[0, 2\pi[$ at each point. In that context, the rotation operator on a function G on the sphere is decomposed as $R(\rho) = R(\omega_0)R^z(\chi)$, where $R(\omega_0) = R^z(\varphi_0)R^y(\theta_0)$ defines the motion or translation of the function by ω_0 , and $R^z(\chi)$ defines its rotation by χ on itself. The operator $R(\omega_0)$ on the sphere for a motion of amplitude ω_0 is defined in terms of the inverse of the motion operator R_{ω_0} on points in S^2 . It reads

$$[R(\omega_0)G](\omega) = G(R_{\omega_0}^{-1}\omega), \quad (8)$$

where $R_{\omega_0}(\theta, \varphi)$ readily follows from the action of the three-dimensional rotation matrices $R_{\varphi_0}^y$ and $R_{\varphi_0}^z$ on the Cartesian coordinates in three dimensions associated with the point ω on the sphere. The rotation operator $R^z(\chi)$ of a function around itself, by an angle $\chi \in [0, 2\pi[$, follows again from the inverse of the rotation operator R_{χ}^z on points in S^2 . It is given as

$$[R^z(\chi)G](\omega) = G(R_{\chi}^z^{-1}\omega), \quad (9)$$

where $R_{\chi}^z(\theta, \varphi) = (\theta, \varphi + \chi)$ also follows from the action of the three-dimensional rotation matrix R_{χ}^z on the Cartesian coordinates in three dimensions associated with the point ω .

The dilation operator $D(a)$ on functions in $L^2(S^2, d\Omega)$, for a dilation factor $a \in \mathbb{R}_+^*$, is defined in terms of the inverse of the corresponding dilation D_a on points in S^2 as

$$[D(a)G](\omega) = \lambda^{1/2}(a, \theta)G(D_a^{-1}\omega). \quad (10)$$

In spherical coordinates, the dilated point is given by $D_a(\theta, \varphi) = (\theta_a(\theta), \varphi)$ with $\theta_a(\theta) = 2 \arctan(a \tan(\theta/2))$. This corresponds to a linear dilation of $\tan(\theta/2)$: $\tan(\theta_a(\theta)/2) = a \tan(\theta/2)$. The dilation operator therefore maps the sphere without its South pole on itself: $\theta_a(\theta) : \theta \in [0, \pi[\rightarrow \theta_a \in [0, \pi[$.

It is established in appendix A that this dilation operator on the sphere is uniquely defined by the requirement of the same basic and natural properties as on the plane. The dilation D_a of points on S^2 must be a radial (*i.e.* only affecting the radial variable θ independently of φ , and leaving φ invariant) and conformal (*i.e.* preserving the measure of angles in the tangent plane at each point of S^2) diffeomorphism (*i.e.* a continuously differentiable bijection on S^2). The conformal factor $\lambda(a, \theta)$ reads $\lambda^{1/2}(a, \theta) = a^{-1}[1 + \tan^2(\theta/2)]/[1 + a^{-2}\tan^2(\theta/2)]$, also equal to $\lambda^{1/2}(a, \theta) = 2a/[(a^2 - 1)\cos\theta + (a^2 + 1)]$ as defined in the original formalism (Antoine & Vandergheynst 1999). The normalization by $\lambda^{1/2}(a, \theta)$ in (10) is uniquely determined by the requirement that the dilation $D(a)$ of functions in $L^2(S^2, d\Omega)$ be a unitary operator (*i.e.* preserving the scalar product in $L^2(S^2, d\Omega)$, and specifically the norm of functions). Notice that in the limit $\theta \rightarrow 0$, this dilation factor on the sphere naturally reduces to first order in θ to the normalization constant on the plane: $\lambda^{1/2}(a, \theta) \rightarrow a^{-1}$.

Second, the analysis of signals goes as follows. The wavelet transform $W_{\Psi}^F(\omega_0, \chi, a)$ of a signal $F(\omega)$ with the wavelet $\Psi(\omega)$, localized analysis function in $L^2(S^2, d\Omega)$ on the sphere, is defined as the correlation between $F(\omega)$ and the dilated and rotated wavelet $\Psi_{\chi, a} = R^{\xi}(\chi)D(a)\Psi$, that is again as the scalar product:

$$W_{\Psi}^F(\omega_0, \chi, a) = \int_{S^2} d\Omega \Psi_{\omega_0, \chi, a}^*(\omega) F(\omega) = \langle \Psi_{\omega_0, \chi, a} | F \rangle, \quad (11)$$

with $\Psi_{\omega_0, \chi, a} = R(\omega_0)\Psi_{\chi, a}$. The wavelet coefficients $W_{\Psi}^F(\omega_0, \chi, a)$ represent the characteristics of the signal for each analysis scale a , direction χ , and position ω_0 .

Third, the synthesis of a signal $F(\omega)$ from its wavelet coefficients reads as:

$$F(\omega) = \int_0^{2\pi} d\chi \int_0^{+\infty} \frac{da}{a^3} \int_{S^2} d\Omega_0 W_{\Psi}^F(\omega_0, \chi, a) [R(\omega_0, \chi)L_{\Psi}\Psi_a](\omega). \quad (12)$$

In this relation, the operator L_{Ψ} in $L^2(S^2, d\Omega)$ is defined by the following action on the spherical harmonics coefficients of functions: $\widehat{L_{\Psi}}G_{lm} = \widehat{G}_{lm}/C_{\Psi}^l$, with $l \in \mathbb{N}$, $m \in \mathbb{Z}$, and $|m| \leq l$. On the contrary to L_{ψ} on the plane, the operator L_{Ψ} does not summarize to the division by a constant. Notice that the *a priori* arbitrary choice of scale integration measure da/a^3 is uniquely fixed by requiring the correspondence principle discussed in the next section. In appendix A, a detailed proof shows that this exact reconstruction formula holds if and only if the spherical harmonics transform $\widehat{\Psi}_{lm}$ of the wavelet $\Psi(\omega)$ satisfies the following admissibility condition:

$$0 < C_{\Psi}^l = \frac{8\pi^2}{2l+1} \sum_{|m| \leq l} \int_0^{+\infty} \frac{da}{a^3} |\widehat{(\Psi_a)}_{lm}|^2 < \infty, \quad (13)$$

for all $l \in \mathbb{N}$. This condition is slightly less restrictive than the admissibility condition required in the original group theoretic formalism for the square-integrability of the group representation considered, and given in (B4). It may also be shown that the following relation defines a necessary condition for the wavelet admissibility in $L^2(S^2, d\Omega)$ (Antoine & Vandergheynst 1999):

$$\int_{S^2} d\Omega \frac{\Psi(\theta, \varphi)}{1 + \cos\theta} = 0. \quad (14)$$

Finally, in the limit $\theta \rightarrow 0$, the formalism of spherical wavelets simply identifies with the formalism of Euclidean wavelets. This Euclidean limit (Antoine & Vandergheynst 1999) is naturally established by direct identification of the relations (1) to (7) on the plane, one by one with the relations (8) to (14) on the sphere, to first order in θ . In particular, the identification of the admissibility conditions (6) and (13) is made obvious in terms of their reformulation, as expressed in appendix B.

Also notice that the wavelet formalism may be similarly defined in the space of integrable functions, on the plane, $L^1(\mathbb{R}^2, d^2\vec{x})$, as on the sphere, $L^1(S^2, d\Omega)$. The dilation operator on functions must be redefined consistently in such a way that it still preserves the norm of the functions on which it is applied. The measure of integration on scales changes and the admissibility condition on the sphere is modified consistently. In that formalism, the reconstruction formula contains an additional low frequency term, and the corresponding necessary admissibility condition on the sphere reduces to an exact zero-mean condition: $\int_{S^2} d\Omega \Psi(\theta, \varphi) = 0$ (Antoine et al. 2002). But we do not develop this formalism here.

4. CORRESPONDENCE PRINCIPLE

In the previous sections, we established the wavelet formalism, independently on the plane and on the sphere. Wavelets on the plane are well-known, and may be easily defined in terms of the zero-mean condition (7) for a function both integrable and square-integrable, implying the admissibility condition (6). On the contrary, the admissibility condition (13) for the definition of wavelets on the sphere is difficult to check in practice. In this section, we prove that the inverse stereographic projection of a wavelet on the plane leads to a wavelet on the sphere. For clarity, the related technical proofs are postponed to appendix B. Beyond its pure theoretical interest, this new correspondence principle between the wavelet formalisms on the plane and on the sphere is of great practical use. Indeed, it enables the construction of wavelets on the sphere by simple projection of wavelets on the plane. In that respect, it also naturally allows to transfer wavelet properties from the plane onto the sphere, as illustrated in the next section.

First, appendix B establishes that the projection of a wavelet on the plane leads to a wavelet on the sphere under specific requirements on the corresponding projection operator. That is to say, if the function $\psi(r, \varphi)$ in $L^2(\mathbb{R}^2, d^2\vec{x})$ satisfies the wavelet admissibility condition (6) on the plane, then the function

$$\Psi(\theta, \varphi) = [\Pi^{-1}\psi](\theta, \varphi), \quad (15)$$

in $L^2(S^2, d\Omega)$ satisfies the wavelet admissibility condition (13) on the sphere. The projection operator Π between functions in $L^2(S^2, d\Omega)$ and in $L^2(\mathbb{R}^2, d^2\vec{x})$ on the two manifolds is defined in terms of the inverse of the corresponding projection operator π between points on the sphere S^2 and on the plane \mathbb{R}^2 , applied to the argument of the function considered. The proof of this correspondence principle relies on the requirement that the projection operator be a unitary, radial, and conformal diffeomorphism. Let us recall that the unit sphere on which spherical wavelets are defined is centered at the origin of the orthonormal Cartesian coordinate system $(o, o\hat{x}, o\hat{y}, o\hat{z})$ in three dimensions, with spherical coordinates (θ, φ) . For simplicity, and without loss of generality, we consider a geometrical setting where the plane on which the Euclidean wavelets are defined is parallel to the plane $(o\hat{x}, o\hat{y})$, at an arbitrary

height z_0 . The polar coordinates (r, φ) on the plane are defined relatively to the coordinate system $(o, \alpha\hat{x}, \alpha\hat{y})$. The polar angle φ on the plane and the azimuthal angle φ on the sphere are therefore identified with one another. We consider specifically, still without loss of generality, the plane tangent to the sphere at the North pole, $z_0 = 1$ (see Fig. 1 below). With this choice of coordinates, the operator π of projection of points from S^2 onto \mathbb{R}^2 must naturally be a radial diffeomorphism. That is, it must be a continuously differentiable bijection between S^2 and \mathbb{R}^2 , which only relates the radial variables r on the plane and θ on the sphere independently of φ , and which leaves φ invariant. It must also define a conformal mapping between the sphere and the plane (*i.e.* the metric induced on the plane by the transformation $r(\theta)$ is conformally equivalent to the Euclidean metric on the plane). This property is essential to ensure that the measure of angles and directions (orientations) is preserved by projection. The projection operator Π between functions in $L^2(S^2, d\Omega)$ and $L^2(\mathbb{R}^2, d^2\vec{x})$ on the two manifolds must also be unitary (*i.e.* preserve the scalar product between $L^2(S^2, d\Omega)$ and $L^2(\mathbb{R}^2, d^2\vec{x})$, and specifically the norm of functions).

Second, the correspondence principle may also be extended in the following way. The rotation $R^z(\chi)$, acting on functions on the sphere through the azimuthal angular variable φ , is conjugate to the rotation $r(\chi)$, acting on functions on the tangent plane at the North pole through the polar angular variable φ , by any radial projection:

$$R^z(\chi) = \Pi^{-1} r(\chi) \Pi. \quad (16)$$

As shown in appendix B, for a unitary, radial and conformal projection operator Π between $L^2(S^2, d\Omega)$ and $L^2(\mathbb{R}^2, d^2\vec{x})$, a conjugation relation holds between the dilation operators (10) and (3), through the projection Π :

$$D(a) = \Pi^{-1} d(a) \Pi. \quad (17)$$

It appears from these two conjugation relations that the operations of dilation by $a \in \mathbb{R}_+^*$ and rotation by $\chi \in [0, 2\pi[$ of a spherical wavelet defined as projection of a Euclidean wavelet may be simply performed on the plane before inverse projection on the sphere:

$$\Psi_{\chi,a}(\theta, \varphi) = [\Pi^{-1} \psi_{\chi,a}](\theta, \varphi). \quad (18)$$

The dilated and rotated wavelets on the sphere may therefore be built in an extremely straightforward way with the intuitive operations of dilation (3) and rotation (2) on the plane, forgetting the corresponding operators (10) and (9) on the sphere. Also notice that, as all considered operators are unitary, if the wavelet on the plane is normalized, the corresponding wavelet on the sphere remains naturally normalized. Only motions (translations) by ω_0 have to be explicitly performed on the sphere as the corresponding operator (8) is not the conjugate of the operator (1) of translation by any \vec{x}_0 on the plane.

Third, appendix B also establishes that the stereographic projection is the unique radial conformal diffeomorphism mapping the sphere S^2 onto the plane \mathbb{R}^2 . The unitary stereographic projection operator between functions G in $L^2(S^2, d\Omega)$ and g in $L^2(\mathbb{R}^2, d^2\vec{x})$ and its inverse respectively read

$$[\Pi G](\vec{x}) = \mu^{1/2}(r) G(\pi^{-1}\vec{x}), \quad (19)$$

and

$$[\Pi^{-1} g](\omega) = \mu^{-1/2}(r(\theta)) g(\pi\omega). \quad (20)$$

The radial conformal diffeomorphism between points is given as $\pi(\theta, \varphi) = (r(\theta), \varphi)$ and $\pi^{-1}(r, \varphi) = (\theta(r), \varphi)$, and for $r(\theta) =$

$2 \tan(\theta/2)$ and $\theta(r) = 2 \arctan(r/2)$. The diffeomorphism $r(\theta)$ and its inverse $\theta(r)$ explicitly define the stereographic projection of points on the sphere onto points on the plane, and its inverse. This stereographic projection maps the sphere, without its South pole, on the entire plane: $r(\theta) : \theta \in [0, \pi[\rightarrow [0, \infty[$. Geometrically, it projects a point $\omega = (\theta, \varphi)$ on the unit sphere onto a point $\vec{x} = (r, \varphi)$ on the tangent plane at the North pole, co-linear with ω and the South pole (see Fig. 1). The conformal factor $\mu(r)$ is given as $\mu^{1/2}(r) = (1 + (r/2)^2)^{-1}$. The normalization $\mu^{1/2}(r)$ is required to ensure the unitarity of the projection operator Π between $L^2(S^2, d\Omega)$ and $L^2(\mathbb{R}^2, d^2\vec{x})$. Conversely, $\mu^{-1/2}(r(\theta)) = 1 + \tan^2(\theta/2)$ is the conformal factor associated with the inverse radial conformal mapping π^{-1} , which ensures the unitarity of Π^{-1} .

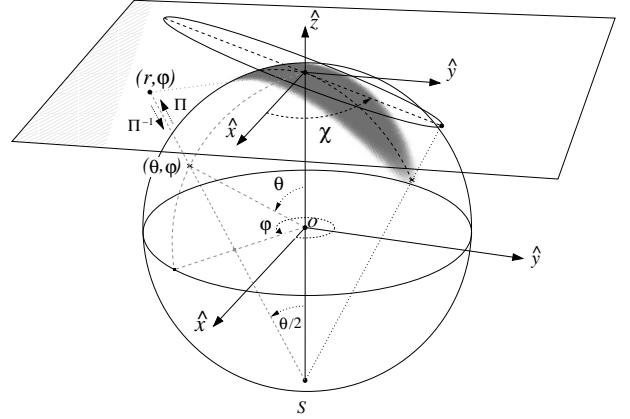


FIG. 1.— Relation established between points on the sphere (θ, φ) and on its tangent plane at the North pole (r, φ) through the stereographic projection π and its inverse π^{-1} . The same relation holds between functions living on each of the two manifolds, as illustrated by the shadow on the sphere and the localized region on the plane. The corresponding operators Π and Π^{-1} enable the establishment of a correspondence principle between the wavelet formalisms on the plane and on the sphere.

Let us consider one example, as illustration of the correspondence principle through the stereographic projection. On the plane, the derivatives of Gaussians in a specific direction, say \hat{x} , are well-known examples of wavelets. The normalized (negative) first derivative of Gaussian in direction \hat{x} reads, after dilation by a and rotation by χ : $\psi_{\chi,a}(r, \varphi) = \sqrt{2/\pi} r e^{-r^2/2a^2} \cos(\varphi - \chi)/a^2$. The corresponding wavelet on the sphere, normalized, dilated by a and rotated by χ , but still at the North pole, is simply obtained by the action of the inverse stereographic projection (20): $\Psi_{\chi,a}(\theta, \varphi) = (1 + \tan^2(\theta/2)) \sqrt{2/\pi} 2 \tan(\theta/2) e^{-2 \tan^2(\theta/2)/a^2} \cos(\varphi - \chi)/a^2$. Notice that, depending on the application, the exact zero mean of a filter is often an appreciated property, as the corresponding filtering completely erases constant signals. In the L^2 formalism, this property may be considered as an additional practical requirement for the choice of the mother wavelet. As illustrated by the present example, the stereographic projection of any odd-numbered derivative of a radial function already satisfies this zero mean condition, in addition to the admissibility condition (13).

Finally, these developments necessitate the following remarks. Notice on the one hand, that the conformal relation between the Euclidean invariant measure on the plane and the measure induced from the stereographic projection, $r(\theta) dr(\theta) d\varphi = \mu^{-1}(r(\theta)) \sin \theta d\theta d\varphi$ with $r(\theta) = 2 \tan(\theta/2)$,

readily implies the correspondence of the necessary wavelet admissibility conditions on the plane and on the sphere. If a function satisfies the necessary wavelet admissibility condition (7) on the plane, then its inverse stereographic projection satisfies the necessary wavelet admissibility condition (14) on the sphere. This result gives us intuition on the otherwise non-trivial complete demonstration of the correspondence principle (15) established in appendix B, which links the necessary and sufficient wavelet admissibility conditions. On the other hand, we also emphasize that, in the limit $\theta \rightarrow 0$, we get the first order relation $r(\theta) \rightarrow \theta$, which corresponds to the identification between the considered portion of the sphere around the North pole, with the tangent plane at that same point. The factors of unitarity also tend to unity. The stereographic projection summarizes to the identity operator. The wavelets on the plane are therefore identified with their projection on the sphere, in complete coherence with the Euclidean limit discussed in § 3.

5. FILTER DIRECTIONALITY AND STEERABILITY

In this section, we discuss the notions of directionality and steerability of filters on the plane and on the sphere. On the sphere, these developments constitute a new advance for the scale-space analysis of signals. Their interest will be illustrated in the next section in the context of the CMB analysis. In a first subsection, we recall the well-known notions of directionality and steerability on the plane. We introduce the corresponding definitions on the sphere. We also show that these notions are transferred from the plane onto the sphere through the stereographic projection. In the context of the wavelet formalism, the correspondence principle established in the former section therefore enables to transfer the directionality and steerability of wavelets, from the plane onto the sphere. In a second subsection, we characterize steerable filters in terms of their band limitation in the Fourier index conjugate to the angular variable φ . In the last subsection, we finally develop the examples of steerable wavelets on the sphere defined as inverse stereographic projection of the derivatives of radial functions on the plane.

5.1. Definitions and correspondence

First, we recall the notions of directionality and steerability on the plane. Locally, at each point of coordinates $\vec{x} = (r, \varphi)$ on the plane \mathbb{R}^2 , directions are defined in the tangent plane, in terms of the rotation angle $\chi \in [0, 2\pi[$. The origin of angles ($\chi = 0$) in the tangent plane is defined by the direction tangent to the line passing through the point considered, from the origin o of coordinates, and making angle φ with the axis $o\hat{x}$ (direction of increasing r). The directionality of a filter $g(r, \varphi)$ in $L^2(\mathbb{R}^2, d^2\vec{x})$ on the plane may be measured through its auto-correlation function, defined as the scalar product of the rotations of the filter in two different directions χ and χ' , and depending on the difference $\Delta\chi = \chi - \chi'$:

$$C^g(\Delta\chi) = \langle r(\chi)g | r(\chi')g \rangle. \quad (21)$$

The steerability of a directional filter $g(r, \varphi)$ is defined in $L^2(\mathbb{R}^2, d^2\vec{x})$ on the plane (Freeman & Adelson 1991; Simoncelli et al. 1992) by the requirement that any rotated version $r(\chi)g$ of the filter be expressed as a linear combination of a finite number of rotations of the filter in specific directions χ_m . This is mathematically defined by the relation $[r(\chi)g](r, \varphi) = \sum_{m=1}^M k_m(\chi)[r(\chi_m)g](r, \varphi)$, where the weights $k_m(\chi)$, with $1 \leq m \leq M$, and $M \in \mathbb{N}$, are called interpolation functions. More generally, one requires that $r(\chi)g$ be expressed in terms of a finite number of independent basis filters

$g_m(r, \varphi)$ in $L^2(\mathbb{R}^2, d^2\vec{x})$, which are not necessarily specific rotations of $g(r, \varphi)$:

$$[r(\chi)g](r, \varphi) = \sum_{m=1}^M k_m(\chi)g_m(r, \varphi). \quad (22)$$

Second, we can define the notions of directionality and steerability on the sphere in perfect analogy with their definition on the plane. Locally, at each point of coordinates $\omega = (\theta, \varphi)$ on the sphere S^2 , directions are defined in terms of the third Euler angle $\chi \in [0, 2\pi[$, which identifies the directions in the tangent plane. The origin of angles ($\chi = 0$) in the tangent plane is defined by the direction tangent to the meridian passing through the point considered (direction of increasing θ). Let us recall that the non-existence of differentiable vector fields on S^2 rules out the definition of directions globally on the sphere. The directionality of a filter $G(\theta, \varphi)$ in $L^2(S^2, d\Omega)$ on the sphere is measured through its auto-correlation function, also defined as the scalar product of the rotations of the filter in two different directions χ and χ' , and depending on difference $\Delta\chi = \chi - \chi'$:

$$C^G(\Delta\chi) = \langle R^{\vec{z}}(\chi)G | R^{\vec{z}}(\chi')G \rangle. \quad (23)$$

The steerability of a directional filter $G(\theta, \varphi)$ is defined in $L^2(S^2, d\Omega)$ on the sphere by the relation

$$[R^{\vec{z}}(\chi)G](\theta, \varphi) = \sum_{m=1}^M k_m(\chi)G_m(\theta, \varphi). \quad (24)$$

Again, the weights $k_m(\chi)$, with $1 \leq m \leq M$, and $M \in \mathbb{N}$ are the interpolation functions. The basis filters $G_m(\theta, \varphi)$ are not necessarily specific rotations of $G(\theta, \varphi)$.

Third, we show that these properties are transferred from the plane onto the sphere through the stereographic projection. Considering the stereographic projection $G = \Pi^{-1}g$, the unitarity of the operator Π ensures that the auto-correlation function of the projected filter is identical to the auto-correlation function of the original filter on the plane: $C^{(\Pi^{-1}g)}(\Delta\chi) = C^g(\Delta\chi)$. Moreover, if the relation of steerability (22) holds for $g(r, \varphi)$ on the plane, then the relation of steerability (24) obviously holds for $G = \Pi^{-1}g$ on the sphere, with the same interpolation functions $k_m(\chi)$, for $1 \leq m \leq M$, and $M \in \mathbb{N}$, and for the basis filters $G_m(\theta, \varphi)$ defined as $G_m = \Pi^{-1}g_m$. This property explicitly relies on the conjugation relation (16) induced by any radial projection operator. If wavelet filters are considered, the correspondence principle established in § 4 therefore enables to transfer the directionality and steerability of wavelets, from the plane onto the sphere.

We finally discuss the interest of the concepts introduced, on the plane as on the sphere. Let us consider on the one hand the notion of filter directionality. Any non-axisymmetric filter will be defined as directional in the sense that its auto-correlation is not a constant function of $\Delta\chi$. We consider as a good directional filter, a filter for which the auto-correlation, (21) or (23), is a rapidly decreasing function of $\Delta\chi$. In this regard, the ideal directional filter would have the expression of a δ distribution in the angle φ , in such a way that its auto-correlation be a $\delta(\Delta\chi)$ distribution. Notice that other definitions of directionality on the plane and on the sphere may be found in the literature (Antoine et al. 2002; Demanet & Vandergheynst 2003; Antoine et al. 2004). Our definitions have the non-negligible advantage of corresponding to one another through the stereographic projection between the sphere and the plane. In these terms, the peakedness

of the auto-correlation function is a measure of the sensitivity of the filter to directions. From a practical point of view, it also measures how the identification of directions, in terms of the maximization of the filtering coefficients of a considered signal, is sensitive to any kind of noise inevitably affecting the estimation of these coefficients. The directionality is consequently a key criterion for the choice of the filter for the identification of local directions of a signal. Consider on the other hand the notion of filter steerability. Through a linear filtering, if a relation of steerability holds for a filter, (22) or (24), then the same relation holds for the filtering coefficients of the signal considered. This is specifically true in the analysis of a signal with wavelet filters. The steerability therefore allows the computation of filtering coefficients of a signal in all directions at the cost of the computation of M filtering coefficients. This property may therefore reduce the computation cost of filtering by a non-negligible factor. Its direct interest in the perspective of the CMB analysis is suggested in § 6. In the following subsection, we discuss the band limitation in the Fourier index conjugate to the variable φ for directional and steerable filters.

5.2. Angular band limitation

We show here that the notions of ideal directionality and steerability represent competing concepts on the plane as on the sphere, in terms of the band limitation of the considered filters in the Fourier index conjugate to the angular variable φ . The following results are discussed on the plane, for functions $g(r, \varphi)$ in $L^2(\mathbb{R}^2, d^2\vec{x})$. They may be identically read on the sphere for $G(\theta, \varphi)$ in $L^2(S^2, d\Omega)$, through the substitution of g by G . Let us thus consider the Fourier decomposition of the filter $g(r, \varphi)$ in the variable φ : $g(r, \varphi) = \sum_{n=-N}^{+N} g_n(r) e^{in\varphi} / 2\pi$. The function $g_n(r)$ stands for the n^{th} Fourier coefficient, and $N+1$ represents the band limitation in the Fourier index n : $n < N+1$. On the one hand, an ideal directional filter has an angular dependence associated with a $\delta(\varphi)$ distribution, which corresponds to a null angular width, that is no band limit, $N \rightarrow \infty$, and constant Fourier coefficients. On the other hand, conceptually, the basis filters of a steerable filter must have a non-zero angular width (hence also the filter itself by linearity of the steerability relation). This ensures that they are sensitive to a whole range of directions. In that case only, one may imagine to steer the filter in all directions from a finite number, M in relation (22), of these basis filters. The non-zero angular width is naturally associated with a band limitation N of the steerable filter and its basis filters. It may be proved rigorously that, if T is the finite number of non-zero coefficients $g_n(r)$ for a filter $g(r, \varphi)$, then this filter is steerable and T is the minimum number of basis filters $g_m(r, \varphi)$ required in relation (22) to steer $g(r, \varphi)$. In other words, the following inequality holds: $M \geq T$ (Freeman & Adelson 1991). Consequently, if the filter $g(r, \varphi)$ is steerable with a number M of basis filters, then it has a finite number T of non-zero coefficients and, as suggested here above, it is inevitably limited in band at some band limit N . The notion of ideal steerable filter (M small, hence N finite) is therefore clearly in opposition with the notion of ideal directional filter ($N \rightarrow \infty$, hence $M \rightarrow \infty$).

5.3. Examples of steerable wavelets

The present subsection introduces the inverse stereographic projection of derivatives of radial functions on the plane as examples of steerable wavelets on the sphere. On the plane, if a sufficiently regular radial function $\phi(r)$ is considered, its

N^{th} derivative in direction \hat{x} ,

$$\psi_{\hat{x}}^{\partial_x^N}(r, \varphi) = \partial_{\hat{x}}^N [\phi(r)], \quad (25)$$

satisfies the wavelet admissibility condition for any $N \geq 1$ and may be called a wavelet. In terms of the Fourier index n conjugate to the angular variable φ , this function is limited in band at the level $N+1$. It contains $T = N+1$ non-zero coefficients of indices $n = -N, -(N-2), \dots, 0, \dots, N-2, N$ for N even, and $n = -N, -(N-2), \dots, 1, -1, \dots, N-2, N$ for N odd. The rotation by an angle χ of the N^{th} derivative in direction \hat{x} of a radial function is given as the N^{th} derivative in direction $\hat{u}(\chi) = r_\chi \hat{x}$: $\psi_{\chi}^{\partial_x^N}(r, \varphi) = \partial_{\hat{u}(\chi)}^N [\phi(r)]$. It may thus be expanded as:

$$\psi_{\chi}^{\partial_x^N}(r, \varphi) = \sum_{i_1 \dots i_N=1}^2 u_{i_1} \dots u_{i_N} \psi^{\partial_{s_{i_1}} \dots \partial_{s_{i_N}}}(r, \varphi), \quad (26)$$

with $\hat{x}_1 = \hat{x}$ and $\hat{x}_2 = \hat{y}$, and where the coordinates of $\hat{u}(\chi)$ read $(u_1, u_2) = (\cos \chi, \sin \chi)$. It is therefore a steerable filter, expressed in terms of $M = \gamma_2^N = N+1 = T$ combinations with repetitions of the basis filters $\psi^{\partial_x^{N-k} \partial_y^k} = \partial_{\hat{x}}^{N-k} \partial_{\hat{y}}^k [\phi(r)]$, for $0 \leq k \leq N$. In the following paragraphs, we explicitly consider the examples of the first and second derivatives of a gaussian.

The first derivative of a radial function in direction \hat{x} on the plane reads:

$$\psi^{\partial_x}(r, \varphi) = \partial_r \phi(r) \cos \varphi, \quad (27)$$

where ∂_r stands for the radial derivative. It therefore has an angular band limit at $N+1 = 2$. It is far from being an ideal directional filter as its auto-correlation function reads for a normalized filter: $C^{\partial_x}(\Delta\chi) = \cos(\Delta\chi)$. In terms of steerability however, a first derivative is an ideal filter, as it only requires $N+1 = 2$ weights. The steerability relation reads, in terms of the specific rotations ψ^{∂_x} and ψ^{∂_y} at $\chi = 0$ and $\chi = \pi/2$ respectively:

$$\psi_{\chi}^{\partial_x}(r, \varphi) = \psi^{\partial_x}(r, \varphi) \cos \chi + \psi^{\partial_y}(r, \varphi) \sin \chi. \quad (28)$$

As a concrete example, let us consider once more the normalized (negative) first derivative of a gaussian already discussed at the end of § 4. It is given here through the general relation (27) for $\phi(r) = -\sqrt{2/\pi} e^{-r^2/2}$. The section of this wavelet at $y = 0$ is illustrated by Fig. 2, while its section at constant x has a gaussian shape:

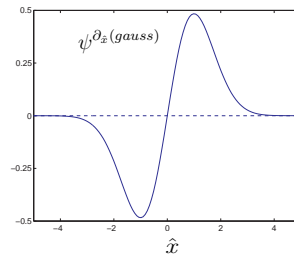


FIG. 2.— Section at $y = 0$ of the first derivative of gaussian in direction \hat{x} on the plane.

The inverse stereographic projection on the sphere of the wavelet rotated by χ and dilated by a (see relation (18)), is given, as already discussed, by:

$$\Psi_{\chi, a}^{\partial_x(\text{gauss})}(\theta, \varphi) = \sqrt{\frac{2}{\pi}} \frac{2}{a^2} \left(1 + \tan^2 \frac{\theta}{2} \right) e^{-2 \tan^2(\theta/2)/a^2} \left[\tan \frac{\theta}{2} \cos(\varphi - \chi) \right]. \quad (29)$$

Fig. 3 illustrates the directionality of first derivatives of radial functions and their steerability relation (28), understood on the sphere, through the specific example of the first gaussian derivative:

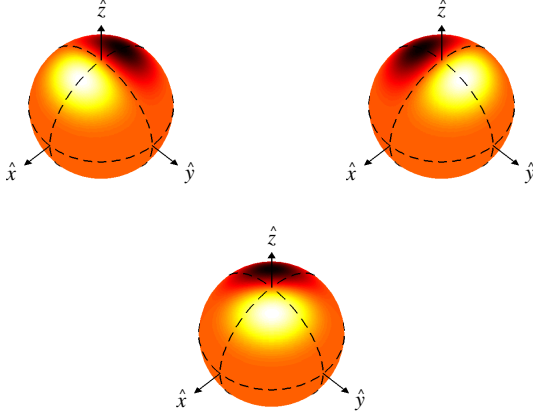


FIG. 3.— First derivative of gaussian on the sphere for a dilation factor $a = 0.4$: (top left) $\partial_{\hat{x}}$, (top right) $\partial_{\hat{y}}$, and (bottom) $\partial_{\hat{r}(\chi)}$ with $\chi = \pi/4$. Dark and light regions respectively identify negative and positive values of the functions.

The second derivative of a radial function in direction \hat{x} on the plane reads:

$$\psi^{\partial_{\hat{x}}^2}(r, \varphi) = \frac{1}{r} \partial_r \phi(r) \sin^2 \varphi + \partial_r^2 \phi(r) \cos^2 \varphi. \quad (30)$$

It is limited in band at $N + 1 = 3$. Its auto-correlation function reads $C^{\partial_{\hat{x}}^2}(\Delta\chi) = A + B \cos 2\Delta\chi$, with the values A and B functions of $\phi(r)$, and $A + B = 1$ for a normalized filter. A second order derivative is therefore not necessarily a better directional filter than a first order derivative as its auto-correlation function is not generically better peaked. The general steerability relation (22) holds, the rotated filter being expressed in terms of basis filters, which are not specific rotations of the second derivative itself. It requires $N + 1 = 3$ weights:

$$\begin{aligned} \psi_{\chi}^{\partial_{\hat{x}}^2}(r, \varphi) = & \psi^{\partial_{\hat{x}}^2}(r, \varphi) \cos^2 \chi + \psi^{\partial_{\hat{y}}^2}(r, \varphi) \sin^2 \chi \\ & + \psi^{\partial_{\hat{x}\hat{y}}}(r, \varphi) \sin 2\chi. \end{aligned} \quad (31)$$

Again, we consider the example of the normalized (negative) second derivative of a gaussian, that is for $\phi(r) = -\sqrt{4/3\pi}e^{-r^2/2}$ in relation (30). The section of this wavelet at $y = 0$ is illustrated by Fig. 4, while its section at constant x has a gaussian shape:

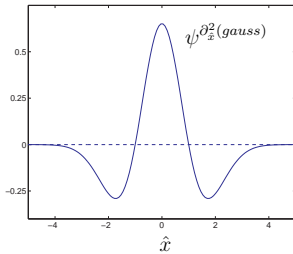


FIG. 4.— Section at $y = 0$ of the second derivative of gaussian in direction \hat{x} on the plane.

The inverse stereographic projection on the sphere of the

wavelet rotated by χ and dilated by a reads:

$$\begin{aligned} \Psi_{\chi, a}^{\partial_{\hat{x}}^2(\text{gauss})}(\theta, \varphi) = & \frac{1}{a} \sqrt{\frac{4}{3\pi}} \left(1 + \tan^2 \frac{\theta}{2} \right) e^{-2 \tan^2(\theta/2)/a^2} \\ & \left[1 - \frac{4}{a^2} \tan^2 \frac{\theta}{2} \cos^2(\varphi - \chi) \right]. \end{aligned} \quad (32)$$

Fig. 5 illustrates the directionality of second derivatives of radial functions and their steerability relation (31), understood on the sphere, through the specific example of the second gaussian derivative:

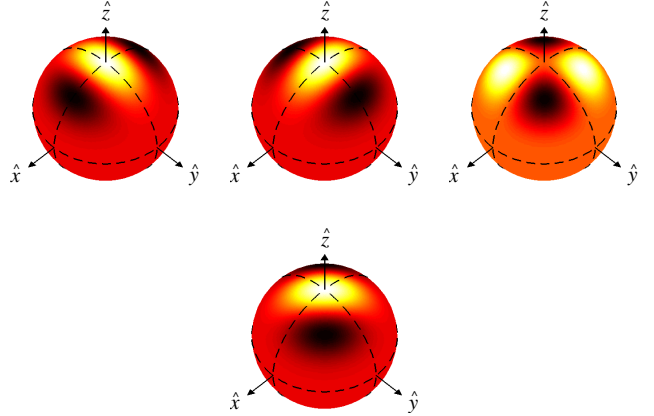


FIG. 5.— Second derivative of gaussian on the sphere for a dilation factor $a = 0.4$: (top left) $\partial_{\hat{x}}^2$, (top centre) $\partial_{\hat{y}}^2$, (top right) $\partial_{\hat{x}}\partial_{\hat{y}}$, and (bottom) $\partial_{\hat{r}(\chi)}^2$ with $\chi = \pi/4$. Dark and light regions again respectively identify negative and positive values of the functions.

6. CMB LOCAL DIRECTIONAL FEATURES

In this last section, we discuss the interest of directional and steerable filters for the identification and interpretation of local directional features on the sphere, with direct application to the CMB analysis. First, we show that steerable filters may efficiently detect local directional features on the sphere with the same angular precision as ideal directional filters. Second, the discussion is illustrated by a simple numerical example. Third, we emphasize on how the property of filter steerability is essential to reduce the computation cost in the search for directional features through the wavelet analysis of the CMB.

The theoretical angular resolution power of any directional filter remains infinite, independently of its possible steerability. This infinite resolution assumes however that the morphology of the signal considered is known *a priori*, in particular in the variable φ . In the context of the CMB analysis, this might be the case when looking for the imprint of cosmic strings or other pre-defined features in the background radiation. In that case indeed, the wavelet coefficient is known analytically as a pure function of the difference $\Delta\chi = \chi - \chi^*$ between the wavelet rotated by χ , and the direction χ^* of the considered feature. This direction χ^* may therefore be identified exactly in the analysis of the wavelet coefficients of the signal. Notice however that for a given experimental precision, the resolution power of filters is directly related to their directionality, through the peakedness of their auto-correlation function, and is therefore a function of the angular band limitation and the number of weights M in the case of steerable filters.

The following simple example illustrates the theoretical infinite angular resolution power of steerable filters.

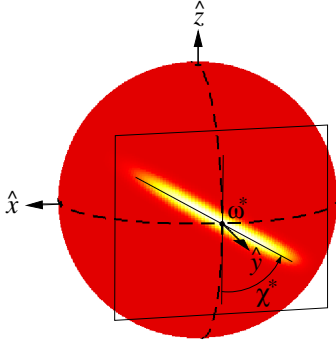


FIG. 6.— Test-signal: elongated feature centered at the point $\omega^* = (\theta^*, \varphi^*) = (\pi/2, \pi/2)$ of the sphere, and making an angle $\chi^* = \pi/3$ with the meridian at that point.

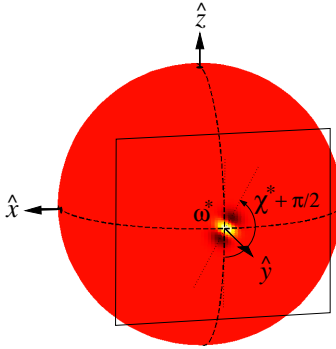


FIG. 7.— Analysis-wavelet: second gaussian derivative in direction \hat{x} , projected on the sphere. It is shown centered at the point $\omega^* = (\theta^*, \varphi^*) = (\pi/2, \pi/2)$ of the sphere, at a scale $a = 0.05$, and making an angle $\chi_0 = \chi^* + \pi/2 = 5\pi/6$ with the meridian at that point, that is in the configuration which maximizes the corresponding wavelet coefficient of the test-signal.

First, the test-signal is defined as an elongated feature centered at the point $\omega^* = (\theta^*, \varphi^*) = (\pi/2, \pi/2)$ of the sphere, and making an angle $\chi^* = \pi/3$ with the meridian at that point. The signal is analytically defined by the function $F(\theta, \varphi) = \exp[-(\cos \chi^* x - \sin \chi^* z)^2 / 2\sigma^2] \exp[-16(y-1)^2]$, with the Cartesian coordinates of points of the sphere related to their spherical coordinates by $(x, y, z) = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$, and for a half width $\sigma = 0.05$ (see Fig. 6). Second, the signal is analyzed by its correlation with a second derivative of a gaussian $\Psi_{\partial_x^2}$, that is in terms of its wavelet coefficients $W_{\partial_x^2}^F(\omega_0, \chi, a)$ at each point ω_0 of the sphere, and for any scale a and direction χ of the wavelet. We might have equivalently chosen the first gaussian derivative or any other steerable wavelet. Notice that the zeros of the second gaussian derivative at the North pole are located at $\theta_0 = 2\arctan(a/2)$ in the direction of the wavelet $\varphi = \chi$, and in the opposite direction $\varphi = \chi + \pi$ (see equation (32), and Fig. (5)). This angular opening θ_0 may be understood as a qualitative measure of the half width of the wavelet. For a dilation factor $a = 1$, we get $\theta_0 \simeq 0.9$ rad. For a small dilation factor a , the half width is simply given as $\theta_0 \simeq arad$. We consider a highly localized analysis-wavelet for a dilation factor $a = 0.05$, which corresponds to a typical half width $\theta_0 \simeq 0.05$ rad. It is therefore of the same size as the width of the feature in the direction $\chi^* + \pi/2$ as $a = \sigma = 0.05$, but much smaller than its elongation in the direction χ^* . At the North

pole, the non-rotated second gaussian derivative with $a = 0.05$ has essentially zero mean in direction \hat{x} (this is only exact in the Euclidean limit, that is $a \rightarrow 0$), and a maximum mean in direction \hat{y} . At any point ω_0 on the signal, the wavelet transform is therefore minimum if the wavelet is directed along the signal, which is essentially constant on an interval of the size of the wavelet. On the contrary, it is maximum if it is directed at an angle $\pi/2$ relative to the feature, direction in which it has the same scale as the signal itself (see Fig. 7).

Through the correspondence principle and the linearity of the wavelet filtering, the steerability relation (31) for the second derivative of a gaussian may be equivalently written for the spherical wavelet coefficients as:

$$W_{\partial_x^2}^F(\chi) = W_{\partial_x^2}^F \cos^2 \chi + W_{\partial_y^2}^F \sin^2 \chi + W_{\partial_x \partial_y}^F \sin 2\chi, \quad (33)$$

where we dropped the dependence of each coefficient in ω_0 and a . Each basis coefficient is evaluated at $\chi = 0$ for the corresponding basis wavelet. This relation may also be written in a form similar to the expression of the auto-correlation function of the second gaussian derivative. This is natural as the auto-correlation function is nothing else but the particular wavelet coefficient of the rotated wavelet analyzed by itself. We get indeed $W_{\partial_x^2}^F(\chi) = A' + B' \cos 2(\chi - \chi_0)$, for $A' = (W_{\partial_x^2}^F + W_{\partial_y^2}^F)/2$ and $B' = (W_{\partial_x^2}^F - W_{\partial_y^2}^F)/2 \cos \chi_0$, with a maximum at χ_0 defined by

$$\tan 2\chi_0 = \frac{2W_{\partial_x \partial_y}^F}{W_{\partial_x^2}^F - W_{\partial_y^2}^F}. \quad (34)$$

At each point ω_0 considered, the cost of the analysis is therefore reduced to the computation of the three wavelet coefficients for the basis wavelets at the chosen scale $a = 0.05$, and in their original direction $\chi = 0$. The direction χ^* of the feature at that point is given as $\chi^* = \chi_0 - \pi/2$. Fig. 8 represents the wavelet coefficient $W_{\partial_x^2}^F(\chi)$ as a function of χ , resulting from this directional analysis at the point $\omega_0 = \omega^* = (\pi/2, \pi/2)$, through the relation (33).

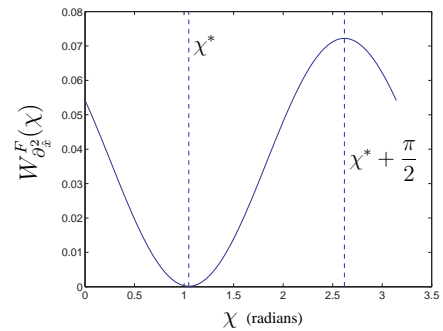


FIG. 8.— Wavelet coefficient as a function of χ for the considered elongated feature, analyzed with a second gaussian derivative at the point $\omega_0 = (\pi/2, \pi/2)$ and at the chosen scale $a = 0.05$.

The direction of the feature is precisely recovered at $\chi^* = \pi/3$, up to negligible numerical errors. This clearly illustrates the fact that steerable wavelets have an infinite angular resolution power for an ideal experiment where no error affects the signal.

In conclusion, the steerability of wavelets is a property of fundamental interest in the analysis of local directional features on the sphere. First, the filter steerability allows a

drastic reduction in computation cost for analyzing all possible directions of features at each point on the sphere, as only a small number M of basis directions must be considered at each point. Second, as the above example illustrates, this reduction is achieved theoretically without any precision loss in the identification of the directions. In the context of the CMB study, let us recall that local directional features may be associated with non-gaussianity, statistical anisotropy, or foreground emission. Their identification requires for each analysis scale a , the correlation of a filter in all directions χ and at all points ω_0 of the sphere with numerous theoretical simulations of the signal. Such an analysis is currently hardly affordable in terms of computation time at the already high resolution level of the CMB maps, when one wants the same precision in the identification of the direction ($> 10^3$ sampling points) as of the localization (10^6 sampling points on the sphere) of singular features (McEwen et al. 2004; McEwen et al. 2005). The property of filter steerability may therefore be used to reduce the complexity of such calculations and eventually render them accessible. However, Fig. 8 also illustrates the non optimal peakedness of the wavelet coefficients of the considered signal as a function of directions ($W_{\partial_{\chi}^2}^F(\chi) = A' + B' \cos 2(\chi - \chi_0)$), equivalently to the peakedness of the wavelet auto-correlation. As already discussed, in a practical situation, a better directionality of the filter would enhance the stability of the measurement of directions relatively to noise. The specific choice of filter, optimizing the compromise between ideal directionality and steerability, will depend very much on the application considered, in terms of noise, precision requirements, and computational resources.

7. CONCLUSION

The recent developments in the analysis of the cosmic microwave background (CMB) radiation ask for new methods

of scale-space signal analysis on the sphere, notably for the identification of foreground emission, or for testing the hypotheses of gaussianity and statistical isotropy of the CMB, on which the concordance cosmological model relies today.

In the work presented here, we reintroduced the formalism of wavelets on the sphere in a practical and self-consistent approach, simply understanding wavelets as localized filters which enable a scale-space analysis and provide an explicit reconstruction of the signal considered from its wavelet coefficients. We also proved a correspondence principle which states that the inverse stereographic projection of a wavelet on the plane gives a wavelet on the sphere, and allows to transfer wavelet properties from the plane onto the sphere. In that context, we finally defined and discussed the notions of directionality and steerability of filters on the sphere for the analysis of local directional features in the signal considered.

The practical formalism introduced provides a method for the detection of local features in the CMB, and the identification of their precise direction, through the analysis of the wavelet coefficients of the signal. Notice however that these generic developments of signal processing on the sphere may find numerous applications beyond cosmology and astrophysics, as soon as the data to be analyzed are distributed on the sphere.

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APPENDIX

TECHNICAL PROOFS FOR WAVELETS ON THE SPHERE

In this first appendix, we establish technical proofs for the wavelet formalism on the sphere defined in § 3. First, we prove the uniqueness of the unitary, radial, and conformal operator $D(a)$ of dilation on functions in $L^2(S^2, d\Omega)$ on the sphere, and give its expression. Second, we explicitly prove the admissibility condition for wavelets on the sphere (13). This condition is required to ensure the exact reconstruction formula (12) for any signal $F(\omega)$ from its wavelet coefficients (11), in a decomposition with the wavelet $\Psi(\omega)$ in $L^2(S^2, d\Omega)$.

Uniqueness of the dilation operator

The dilation operator $D(a)$ on functions in $L^2(S^2, d\Omega)$ is defined in terms of the inverse of the corresponding dilation D_a on points in S^2 , applied to the argument of the function considered. The dilation operator D_a must be a radial and conformal diffeomorphism. A radial operator only affects the radial variable θ independently of φ , and leaves φ invariant. This property is natural to define a dilation of points on the sphere relative to the North pole. In this context, the operator $D(a)$ on functions in $L^2(S^2, d\Omega)$ takes a generic form given by relation (10), in terms of a radial diffeomorphism $D_a(\theta, \varphi) = (\theta_a(\theta), \varphi)$ on S^2 . The diffeomorphism $\theta_a(\theta)$ and the function $\lambda^{1/2}(a, \theta)$ have to be determined from the additional properties required.

First, the diffeomorphism D_a may be understood as a coordinate transformation on the sphere, $D_a : \omega = (x^1, x^2) = (\theta, \varphi) \rightarrow \omega' = (x'^1, x'^2) = (\theta_a, \varphi) \in S^2$. It must be conformal in order to preserve the measure of angles and directions, which are defined locally in the tangent plane at each point of S^2 . This explicitly requires that the transformed metric $g'_{ij}(\theta_a, \varphi) = (\partial x^k / \partial x'^i)(\partial x^l / \partial x'^j) g_{kl}(\theta, \varphi)$ is conformally equivalent to the original metric $g_{ij}(\theta, \varphi)$ on the sphere, with $i, j, k, l \in \{1, 2\}$. We consider the metric induced on the sphere from the Euclidean metric in three dimensions: $g_{ij}(\theta, \varphi) = \text{diag}(1, \sin^2 \theta)$. The conformal equivalence reads by definition: $g'_{ij}(\theta_a, \varphi) = e^{\phi(a, \theta)} g_{ij}(\theta, \varphi)$, for some strictly positive conformal factor $e^{\phi(a, \theta)}$, that is for a real function $\phi(a, \theta)$. This condition implies that the operator D_a is linear in $\tan \theta/2$: $\tan \theta_a(\theta)/2 = \alpha(a) \tan \theta/2$. The function $\alpha(a)$ must be strictly increasing in the dilation factor $a \in \mathbb{R}_+^*$, with the limits $\alpha(0) = 0$ and $\alpha(\infty) = \infty$. The group structure for the composition of dilations must also be fulfilled. This constrains $\alpha(a)$ to $\alpha(a) = a^{\alpha_0}$ for a strictly positive exponent $\alpha_0 \in \mathbb{R}_+^*$. We

take, without loss of generality, a function linear in the dilation factor a , corresponding to $\alpha_0 = 1$:

$$\tan \frac{\theta_a(\theta)}{2} = a \tan \frac{\theta}{2}. \quad (\text{A1})$$

Any other choice for α_0 would simply correspond to a rescaling of the dilation factor. By dilation, the sphere without its South pole is thus mapped on itself: $\theta_a(\theta) : \theta \in [0, \pi[\rightarrow \theta_a \in [0, \pi[$. The conformal equivalence also determines the conformal factor: $e^{-\phi(a,\theta)/2} = a[1 + \tan^2(\theta/2)] / [(1 + a^2 \tan^2(\theta/2))]$.

Second, the unitarity of the dilation operator $D(a)$ on functions in $L^2(S^2, d\Omega)$ identifies $\lambda(a, \theta)$ with the conformal factor: $\lambda(a, \theta) = e^{\phi(a, \theta_{1/a}(\theta))} = e^{-\phi(a^{-1}, \theta)}$, or equivalently $\lambda^{1/2}(a, \theta) = a^{-1} [1 + \tan^2(\theta/2)] / [(1 + a^{-2} \tan^2(\theta/2))]$. This identity simply relies on the relation between the invariant measure on the sphere and the measure obtained after the conformal transformation D_a : $\sin \theta_a(\theta) d\theta_a(\theta) d\varphi = e^{-\phi(a,\theta)} \sin \theta d\theta d\varphi$.

On the plane, the construction of the dilation $d(a)$ on functions, unique unitary, radial, and conformal operator in $L^2(\mathbb{R}^2, d^2\vec{x})$, relies on an identical reasoning. The canonical Euclidean metric on the plane in polar coordinates simply reads $g_{ij}(\theta, \varphi) = \text{diag}(1, r^2)$, for $i, j \in \{r, \varphi\}$. The expression (3) trivially follows.

Establishment of the admissibility condition

First, we recall the expressions for the Fourier decomposition of a function $G(\omega)$ in $L^2(S^2, d\Omega)$ on the sphere. The spherical harmonics $Y_{lm}(\omega)$, with $l \in \mathbb{N}$, $m \in \mathbb{Z}$, and $|m| \leq l$, form an orthonormal basis in $L^2(S^2, d\Omega)$:

$$\int_{S^2} d\Omega Y_{lm}^*(\omega) Y_{l'm'}(\omega) = \delta_{ll'} \delta_{mm'}. \quad (\text{A2})$$

The direct and inverse spherical harmonics transforms of a function $G(\omega)$ are respectively defined as:

$$\widehat{G}_{lm} = \int_{S^2} d\Omega Y_{lm}^*(\omega) G(\omega) \quad (\text{A3})$$

$$G(\omega) = \sum_{l \in \mathbb{N}} \sum_{|m| \leq l} \widehat{G}_{lm} Y_{lm}(\omega). \quad (\text{A4})$$

Second, we introduce the transformation law of functions on the sphere under rotation, in terms of the Wigner D -functions (Brink & Satchler 1993). Let ρ be an element of $SO(3)$, with $\rho = (\varphi, \theta, \chi)$, in a decomposition in the Euler angles φ , θ , and χ . The Wigner D -functions $D_{mn}^l(\rho)$, with $l \in \mathbb{N}$, $m, n \in \mathbb{Z}$, and $|m|, |n| \leq l$, are the matrix elements of the irreducible unitary representations of weight l of the rotation group in the space of square-integrable functions $L^2(SO(3), d\rho)$ on $SO(3)$, with the invariant measure $d\rho = d\varphi d\cos\theta d\chi$. By the Peter-Weyl theorem on compact groups, the matrix elements D_{mn}^l also form an orthogonal basis in $L^2(SO(3), d\rho)$, with the following orthogonality relation:

$$\int_{SO(3)} d\rho D_{mn}^l(\rho) D_{m'n'}^{l'*}(\rho) = \frac{8\pi^2}{2l+1} \delta_{ll'} \delta_{mm'} \delta_{nn'}. \quad (\text{A5})$$

Let us consider the decomposition $\rho = (\omega_0, \chi)$, with $\omega_0 = (\theta_0, \varphi_0)$ identifying a point on the sphere S^2 , and $\chi \in [0, 2\pi[$ identifying a direction at each point. The action of the corresponding operator $R(\omega_0, \chi)$ on a function $G(\omega)$ in $L^2(S^2, d\Omega)$ on the sphere reads in terms of its spherical harmonics coefficients and the Wigner D -functions:

$$[R(\widehat{\omega_0, \chi}) G]_{lm} = \sum_{|n| \leq l} D_{mn}^l(\omega_0, \chi) \widehat{G}_{ln}. \quad (\text{A6})$$

Finally, we prove the admissibility condition for a wavelet on the sphere. Let $F(\omega)$ be a function in $L^2(S^2, d\Omega)$ on the sphere, and $\Psi(\omega)$ in $L^2(S^2, d\Omega)$, the wavelet considered for the decomposition into the coefficients $W_{\Psi}^F(\omega_0, \chi, a)$ given in (11). We want to establish the condition under which the explicit reconstruction formula (12) holds. From the expressions (A4) and (A6) and the definition of the operator L_{Ψ} , the function $[R(\omega_0, \chi) L_{\Psi} \Psi_a](\omega) = [L_{\Psi} \Psi_a](R_{\omega_0, \chi}^{-1} \omega)$ takes the form

$$[R(\omega_0, \chi) L_{\Psi} \Psi_a](\omega) = \sum_{l \in \mathbb{N}} \sum_{|m|, |n| \leq l} \frac{1}{C_{\Psi}^l} D_{mn}^l(\omega_0, \chi) (\widehat{\Psi_a})_{ln} Y_{lm}(\omega). \quad (\text{A7})$$

The wavelet coefficient $W_{\Psi}^F(\omega_0, \chi, a)$ defined in (11) may be written as:

$$W_{\Psi}^F(\omega_0, \chi, a) = \sum_{l \in \mathbb{N}} \sum_{|m|, |n| \leq l} D_{mn}^{l*}(\omega_0, \chi) (\widehat{\Psi_a})_{ln}^* \widehat{F}_{lm}. \quad (\text{A8})$$

Inserting these last two expressions in (12), and using the orthogonality relation (A5) for the Wigner D -functions, we obtain:

$$F(\omega) = \sum_{l \in \mathbb{N}} \sum_{|m| \leq l} \widehat{F}_{lm} Y_{lm}(\omega) \frac{1}{C_{\Psi}^l} \left[\frac{8\pi^2}{2l+1} \sum_{|n| \leq l} \int_0^{+\infty} \frac{da}{a^3} |(\widehat{\Psi_a})_{ln}|^2 \right]. \quad (\text{A9})$$

From this last expression it is obvious that the reconstruction formula (12) holds if and only if the coefficients C_{Ψ}^l defined in (13) are finite and non-zero for any $l \in \mathbb{N}$. This explicitly establishes the wavelet admissibility condition (13) on the sphere.

TECHNICAL PROOFS FOR THE CORRESPONDENCE PRINCIPLE

In this second appendix, we prove the correspondence principle defined in § 4, which states that the inverse stereographic projection of a wavelet $\psi(\vec{x})$ in $L^2(\mathbb{R}^2, d^2\vec{x})$ on the plane, thus satisfying the admissibility condition (6), gives a wavelet $\Psi(\omega)$ in $L^2(S^2, d\Omega)$ on the sphere, satisfying the admissibility condition (13). First, we reformulate the admissibility conditions on the plane and on the sphere. Second, we establish explicitly that the projection of a wavelet on the plane gives a wavelet on the sphere, in terms of these reformulated admissibility conditions. In that respect, we only require that the projection Π between functions in $L^2(S^2, d\Omega)$ and in $L^2(\mathbb{R}^2, d^2\vec{x})$ be a unitary, radial, and conformal diffeomorphism. Finally, we prove that the stereographic projection is the unique projection operator satisfying these properties.

Reformulation of the admissibility conditions

First, let $\psi(\vec{x})$ be a function in $L^2(\mathbb{R}^2, d^2\vec{x})$. We show that the wavelet admissibility condition (6) on the plane applied to $\psi(\vec{x})$ is equivalent to the condition:

$$0 < I_\psi^f = \int_0^{2\pi} d\chi \int_0^{+\infty} \frac{da}{a^3} \int_{\mathbb{R}^2} d^2\vec{x}_0 |\langle \psi_{\vec{x}_0, \chi, a} | f \rangle|^2 < \infty, \quad (\text{B1})$$

for any $f(\vec{x}) \neq 0$ in $L^2(\mathbb{R}^2, d^2\vec{x})$. To prove this statement, we simply notice that the scalar product is preserved up to a factor 2π by the Fourier transform. This implies that $2\pi \langle \psi_{\vec{x}_0, \chi, a} | f \rangle = \langle \widehat{(\psi_{\vec{x}_0, \chi, a})} | \widehat{f} \rangle$, with $\widehat{(\psi_{\vec{x}_0, \chi, a})}(\vec{k}) = a\widehat{\psi}(a\vec{\chi}^{-1}\vec{k})e^{-i\vec{k}\cdot\vec{x}_0}$. Using the orthogonality relation of the imaginary exponentials, the integral I_ψ^f therefore takes the form

$$I_\psi^f = \left[\int_0^{2\pi} d\chi \int_0^{+\infty} \frac{da}{a} |\widehat{\Psi}(a\vec{\chi}^{-1}\vec{k})|^2 \right] \|f\|^2, \quad (\text{B2})$$

where $\|f\|$ stands for the norm of $f(\vec{x})$ in $L^2(\mathbb{R}^2, d^2\vec{x})$. A simple change of variable in the integrals leads to the equality

$$I_\psi^f = C_\psi \|f\|^2, \quad (\text{B3})$$

which proves the equivalence between the condition (B1) and the wavelet admissibility condition (6) on the plane.

Second, let $\Psi(\omega)$ be a function in $L^2(S^2, d\Omega)$. We show in a similar way that the wavelet admissibility condition (13) on the sphere applied to $\Psi(\omega)$ is induced by the condition:

$$0 < I_\Psi^F = \int_0^{2\pi} d\chi \int_0^{+\infty} \frac{da}{a^3} \int_{S^2} d\Omega_0 |\langle \Psi_{\omega_0, \chi, a} | F \rangle|^2 < \infty, \quad (\text{B4})$$

for any $F(\omega) \neq 0$ in $L^2(S^2, d\Omega)$. Decomposing $\Psi(\omega)$ and $F(\omega)$ in spherical harmonics through (A4), we get the following relation: $\langle \Psi_{\omega_0, \chi, a} | F \rangle = \sum_{l \in \mathbb{N}} \sum_{|m| \leq l} (\widehat{\Psi}_{\omega_0, \chi, a})_{lm}^* \widehat{F}_{lm}$, with $(\widehat{\Psi}_{\omega_0, \chi, a})_{lm} = \sum_{|n| \leq l} D_{mn}^l(\omega_0, \chi) (\widehat{\Psi}_a)_{ln}$ (see relation (A6)). From the orthogonality relation (A5) for the Wigner D -functions, the integral I_Ψ^F finally reads:

$$I_\Psi^F = \sum_{l \in \mathbb{N}} C_\Psi^l \sum_{|m| \leq l} |\widehat{F}_{lm}|^2. \quad (\text{B5})$$

This proves that the condition (B4) implies the wavelet admissibility condition (13) on the sphere. The condition (B4) indeed requires that C_Ψ^l be strictly positive and bounded for all l by a strictly positive constant c , $0 < C_\Psi^l < c$, while the wavelet admissibility condition (13) reads $0 < C_\Psi^l < \infty$. In the original group theoretic approach, the condition (B4) defines the admissibility, through the square-integrability of the considered group representation (Antoine & Vandergheynst 1999). Our admissibility condition for the reconstruction of the signal is therefore slightly less restrictive.

Establishment of the correspondence principle

We can now prove the correspondence principle by showing that, if $\psi(\vec{x})$ in $L^2(\mathbb{R}^2, d^2\vec{x})$ satisfies the reformulated wavelet admissibility condition (B1), then the inverse stereographic projection $\Psi(\omega) = [\Pi^{-1}\psi](\omega)$ in $L^2(S^2, d\Omega)$ satisfies the condition (B4) (Bogdanova et al. 2005), and therefore the admissibility condition (13).

First, we establish the upper bound $I_{(\Pi^{-1}\psi)}^F < \infty$ for any $F(\omega) \neq 0$ in $L^2(S^2, d\Omega)$. The scalar product of functions in $L^2(S^2, d\Omega)$ is invariant under translation by ω_0 , rotation by χ . Considering a unitary operator Π between $L^2(S^2, d\Omega)$ and $L^2(\mathbb{R}^2, d^2\vec{x})$ implies by definition that the scalar product of functions is preserved by the projection. Moreover, for a unitary, radial, and conformal projection operator Π , the conjugation relation (17) holds between the dilation operator $D(a)$ in $L^2(S^2, d\Omega)$ on the sphere and the dilation operator $d(a)$ in $L^2(\mathbb{R}^2, d^2\vec{x})$ on the plane. Indeed, the dilation operator $d(a)$ of functions in $L^2(\mathbb{R}^2, d^2\vec{x})$ on the plane defined by (3) is also unitary, radial and conformal. The conjugate operator $\Pi^{-1}d(a)\Pi$ on functions in $L^2(S^2, d\Omega)$ on the sphere therefore also satisfies the same properties. Consequently, the uniqueness of the unitary, radial and conformal dilation operator in $L^2(S^2, d\Omega)$ on the sphere (see appendix A) implies that the operator $\Pi^{-1}d(a)\Pi$ is identified with the dilation $D(a)$ defined in (10). We therefore get:

$$I_{(\Pi^{-1}\psi)}^F = \int_{SO(3)} d\rho \left[i_{(\Pi^{-1}\psi)}^F(\rho) \right], \quad (\text{B6})$$

with

$$I_{(\Pi^{-1}\psi)}^F(\rho) = \int_0^{+\infty} \frac{da}{a^3} |\langle d(a)\psi | \Pi R^{-1}(\rho)F \rangle|^2. \quad (\text{B7})$$

Similarly, from the invariance of the scalar product of functions in $L^2(\mathbb{R}^2, d^2\vec{x})$ under translation by \vec{x}_0 and rotation by χ we may write the upper bound of the wavelet admissibility condition (B1) on $\psi(\vec{x})$ on the plane as

$$I_{\psi}^f = \int_0^{2\pi} d\chi \int_0^{+\infty} \frac{da}{a^3} \int_{\mathbb{R}^2} d^2\vec{x}_0 |\langle d(a)\psi | r^{-1}(\chi)t^{-1}(\vec{x}_0)f \rangle|^2 < \infty, \quad (\text{B8})$$

for any $f(\vec{x}) \neq 0$ in $L^2(\mathbb{R}^2, d^2\vec{x})$. By continuity of the integrand in the variables \vec{x}_0 and χ , this finally implies that

$$\int_0^{+\infty} \frac{da}{a^3} |\langle d(a)\psi | f \rangle|^2 < \infty, \quad (\text{B9})$$

for any $f(\vec{x}) \neq 0$ in $L^2(\mathbb{R}^2, d^2\vec{x})$. Consequently, for any $F(\omega) \neq 0$ in $L^2(S^2, d\Omega)$, the function $f(\vec{x}) = [\Pi R^{-1}(\rho)F](\vec{x})$ is in $L^2(\mathbb{R}^2, d^2\vec{x})$ and differs from zero, and we readily obtain that $I_{(\Pi^{-1}\psi)}^F(\rho) < \infty$, for any $\rho \in SO(3)$. The compactness of the group $SO(3)$ finally ensures that $I_{(\Pi^{-1}\psi)}^F < \infty$ for any $F(\omega) \neq 0$ in $L^2(S^2, d\Omega)$.

Notice that the choice of the measure of integration on scales da/a^3 is natural on the plane (see (5)), while *a priori* arbitrary on the sphere (see (12)). It is however required to establish the correspondence principle between the wavelet formalisms on the plane and on the sphere, as it clearly appears from the former proof.

Second, the lower bound $0 < I_{(\Pi^{-1}\psi)}^F$ for any $F(\omega) \neq 0$ in $L^2(S^2, d\Omega)$ remains to be established. In that regard, we simply notice that the set of functions obtained by translation by ω_0 , rotation by χ , and dilation by a of any non-identically null function is dense in $L^2(S^2, d\Omega)$. The lower bound of the wavelet admissibility condition on the plane (B1), $0 < I_{\psi}^f$ for any $f(\vec{x}) \neq 0$ in $L^2(\mathbb{R}^2, d^2\vec{x})$, ensures that a wavelet on the plane cannot be identically null. The unitarity of the stereographic projection therefore implies that the inverse stereographic projection $\Psi(\omega) = [\Pi^{-1}\psi](\omega)$ is also different from zero in $L^2(S^2, d\Omega)$. Consequently, the set of functions $\Psi_{\omega_0, \chi, a}$ is dense in $L^2(S^2, d\Omega)$, and considering any $F(\omega) \neq 0$ in $L^2(S^2, d\Omega)$, the scalar product $\langle \Psi_{\omega_0, \chi, a} | F \rangle$ cannot be identically null in ω_0 , χ , and a . Again, the continuity of this function in all its arguments ω_0 , χ , and a ensures that it is non-zero on a set of non-zero measure in the corresponding Hilbert space. This strict positivity of the integrand in $I_{(\Pi^{-1}\psi)}^F$ finally guarantees that $0 < I_{(\Pi^{-1}\psi)}^F$ for any $F(\omega) \neq 0$ in $L^2(S^2, d\Omega)$. This completes the proof of the correspondence principle.

Uniqueness of the stereographic projection operator

Let us prove that the stereographic projection defined in (19) is the unique unitary, radial, and conformal diffeomorphism between the sphere and the plane. The projection operator Π between functions G in $L^2(S^2, d\Omega)$ on the sphere and g in $L^2(\mathbb{R}^2, d^2\vec{x})$ on the plane is generically expressed in terms the inverse of the corresponding projection operator π between points on the sphere S^2 and on the plane \mathbb{R}^2 , applied to the argument of the function considered. The projection π is a radial diffeomorphism, that is a continuously differentiable bijection between S^2 and \mathbb{R}^2 , which only relates the radial variables r on the plane and θ on the sphere independently of φ , and which leaves φ invariant. In these terms, the operator Π and its inverse Π^{-1} are given by the relations (19) and (20), with $\pi(\theta, \varphi) = (r(\theta), \varphi)$, and its inverse $\pi^{-1}(r, \varphi) = (\theta(r), \varphi)$. The functions $r(\theta)$ and its inverse $\theta(r)$, together with the function $\mu(r)$, have to be fixed by the additional properties required.

First, the projection π is a mapping between the sphere and the plane, $\pi : \omega = (x^1, x^2) = (\theta, \varphi) \in S^2 \rightarrow \vec{x}^T = (x'^1, x'^2) = (r, \varphi) \in \mathbb{R}^2$. It must be conformal in order to preserve the measure of angles and directions, which are defined locally in tangent plane at each point of S^2 and \mathbb{R}^2 . This explicitly means that the metric $g'_{ij}(r, \varphi) = (\partial x^k / \partial x'^i)(\partial x^l / \partial x'^j) g_{kl}(\theta(r), \varphi)$ induced on the plane by the projection from the sphere is conformally equivalent to the Euclidean metric $g_{ij}(r, \varphi) = \text{diag}(1, r^2)$ on the plane, with $i, j, k, l \in \{1, 2\}$. The original metric on the sphere is the canonical metric $g_{ij}(\theta, \varphi) = \text{diag}(1, \sin^2 \theta)$. The conformal equivalence reads by definition: $g'_{ij}(r, \varphi) = e^{\phi(r)} g_{ij}(r, \varphi)$, for some strictly positive conformal factor $e^{\phi(r)}$, that is for a real function $\phi(r)$. This condition requires that $r(\theta)$ is a linear function of $\tan \theta/2$. Let us recall that we consider the plane tangent to the sphere at the North pole ($z_0 = 1$). For the choice of coordinates (r, φ) and (θ, φ) defined in § 4, this gives uniquely:

$$r(\theta) = 2 \tan \frac{\theta}{2}. \quad (\text{B10})$$

This radial diffeomorphism $r(\theta)$ and its inverse $\theta(r)$ explicitly define the stereographic projection π of points on the sphere S^2 onto points on the plane \mathbb{R}^2 , and its inverse. By stereographic projection, the sphere without its South pole is mapped onto the entire plane: $r(\theta) : \theta \in [0, \pi[\rightarrow r \in [0, \infty[$. A point $\omega = (\theta, \varphi)$ on the unit sphere is projected onto a point $\vec{x} = (r, \varphi)$ on the tangent plane at the North pole, co-linear with ω and the South pole (see Fig. 1). The conformal factor is also explicitly given by the condition of conformal mapping: $e^{\phi(r)/2} = (1 + (r/2)^2)^{-1}$, or equivalently $e^{-\phi(r(\theta))/2} = 1 + \tan^2(\theta/2)$.

Second, the unitarity of the operator Π on functions between $L^2(S^2, d\Omega)$ and $L^2(\mathbb{R}^2, d^2\vec{x})$ identifies $\mu(r)$ with the conformal factor: $\mu(r) = e^{\phi(r)}$. This identity simply relies on the relation between the invariant Euclidean measure on the plane and the measure induced from the projection: $r(\theta)dr(\theta)d\varphi = e^{-\phi(r(\theta))} \sin \theta d\theta d\varphi$.

In conclusion, the unique projection operator satisfying the properties required to ensure a correspondence principle between the formalisms of wavelets on the plane and on the sphere is therefore the stereographic projection operator (19).

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