Stereographic Frames of Wavelets on the Sphere

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Abstract

In this technical report we build Discrete Wavelet Frames on the sphere S^2, discretizing the existing Spherical Continuous Wavelet Transform (CWT). We first explore the spherical half-continuous frames, i.e. where the position remains a continuous variable; and then we proceed to the fully discrete frames. We introduce the notion of controlled frames, which reflects the particular nature of the underlying theory, namely, the apparent conflict between dilation and the compacity of the spherical manifold. We conclude with our perspectives for future work.

1 Introduction

Many examples in (astro-)physics, geodesics and medicine require existing of suitable tools for analysing data on spherical manifolds. As an analysing tool, the main advantage of CWT is operating by dilation and translation of the wavelet on the analyzed data. Existing of CWT on the sphere, is a challenge for verifying existing of its discetized form, namely spherical discrete frames.

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1.1 Continuous Wavelet Transform on the Sphere

The CWT on the sphere is based on the affine transformations on the sphere, namely: rotations, defined by the element ρ of the group $SO(3)$; and dilations, parametrized by the scale $a \in \mathbb{R}_+^*$ [2]. In other words, if $f \in L^2(S^2) \equiv L^2(S^2, d\mu)$, with the rotation inavarient measure on the sphere $d\mu(\theta, \varphi) = \sin\theta d\theta d\varphi$, we have

- rotation $R_\rho(\rho \in SO(3))$:

$$(R_\rho f)(\omega) = f(\rho^{-1}\omega), \quad \omega \equiv (\theta, \varphi). \tag{1}$$

- dilation $D_a(a \in \mathbb{R}_+^*)$:

$$(D_a f)(\omega) = \lambda(a, \theta)^{\frac{1}{2}} f(\omega_{\frac{1}{a}}), \tag{2}$$

  where $\omega_a \equiv (\theta_a, \varphi)$ with $\tan\frac{\theta_a}{2} = a\tan\frac{\theta_a}{2}$; $a > 0, \theta \in [0, \pi], \varphi \in [0, 2\pi)$; and λ is a normalization factor associated to the cocycle and needed for making the dilation $d\mu$ dilation-invariant. This cocycle is given by

$$\lambda(a, \theta) = \frac{4a^2}{[(a^2-1)\cos\theta + (a^2+1)]^2}. \tag{3}$$

Intuitively, the action of dilation D_a on a function $f \in L^2(S^2)$ corresponds to an Euclidean dilation of the projected function in the tangent to the North Pole plane, by a stereographic projection trough the South Pole, and lifting it back to the sphere by inverse stereographic projection. In the language of group theory, these two affine transformations, which do not generate a group neither they commute, are found in the conformal of the sphere S^2 group - the Lorentz group $SO(3,1)$, where each subgroup is isolated using the Iwasawa decomposition. The convenience of this approach is existing of unitary irreducible representation of $SO(3,1)$ in $L^2(S^2)$ from where the square-integrable representation on $|R_+^* \times SO(3)$ is found. Using so defined schema for construction of wavelets on the sphere, an wavelet $\psi \in L^2(S^2)$ is *admissible* if there is a constant $c \in \mathbb{R}_+^*$, such that for all $l \in \mathbb{N}$

$$G_\psi(l) = \frac{8\pi^2}{2l+1} \sum_{|m| \leq l} \int_{\mathbb{R}_+^*} \frac{da}{a^3} |\hat{\psi}_a(l,m)|^2 < c, \tag{4}$$

where $\hat{\psi}_a(l,m) = \langle Y_l^m | \psi_a \rangle$ is the Fourier transform of $\psi_a = D_a\psi$. In particular, for $\phi = \exp(-\tan^2(\frac{\theta}{2}))$, which is the inverse stereographic projection of

The Gaussian on the sphere, We obtaine the *Difference of Gaussian (DOG)* spherical wavelet

$$\psi(\theta,\varphi) = \exp(-\tan^2(\frac{\theta}{2})) - \frac{1}{\alpha}\lambda(\alpha,\theta)^{\frac{1}{2}}\exp(-\frac{1}{\alpha^2}\tan^2(\frac{\theta}{2})). \quad (5)$$

Thus, with given rotation, dilation and an admissible wavelet $\psi \in L^2(S^2)$, the CWT of a function $f \in L^2(S^2)$ is:

$$W_f(\rho,a) = \langle \psi_{\rho,a} | f \rangle = \int_{S^2} d\mu(\omega) f(\omega) [R_\rho D_a \psi]^*(\omega). \quad (6)$$

Since the stereographic dilation is radial around the North Pole, an axisymmetric wavelet on the sphere ψ is such that

$$W_f(\rho,a) = (f * \psi_a^*)(\rho) = (f \star \psi_a^*)(\omega) \equiv W_f(\omega,a) \quad (7)$$

with $a \in \mathbb{R}_+^*, \rho \in SO(3)$ and $\omega \in S^2$.

The reconstruction of a spherical function from its coefficients W_f is specific since the wavelet ψ is such that $\int_{S_1} d\varphi\psi(\theta,\varphi) \neq 0$, then the familly $\{\psi_{\rho,a} : \rho \in SO(3), a > 0\}$ constitute a continuous frame in $L^2(S^2)$. Consequently, we give the following proposition

Proposition 1.1.1 *Let $f \in L^2(S^2)$. If ψ is an admissible wavelet such that $\int_{S^2} d\varphi\psi(\theta,\varphi) \neq 0$, then*

$$f(\omega) = \int_{\mathbb{R}_+^*} \int_{SO(3)} \frac{da d\nu(\rho)}{a^3} W_f(\rho,a) [R_\rho L_\psi^{-1} D_a \psi](\omega), \quad (8)$$

where the coefficients are given by (6), L is the ***frame operator*** *defined by*

$$\widehat{[L_\psi h]}(l,m) = G_\psi(l)\hat{h}(l,m), \quad \forall h \in L^2(S^2), \quad (9)$$

and $G_\psi(l)$ defined by (4)

Corollary 1.1.2 *Under the condition of the previous proposition, the following Plancharel relation is satisfied*

$$\|f\|_2 = \int_{\mathbb{R}_+^*} \int_{SO(3)} \frac{da d\nu(\rho)}{a^3} W_f(\rho,a) \tilde{W}_f^*(\rho,a) \quad (10)$$

with

$$\tilde{W}_f(\rho,a) = \langle \tilde{\psi}_(\rho,a) | f \rangle = \langle R_\rho L_\psi^{-1} D_a \psi | f \rangle. \quad (11)$$

The proof of this proposition and corollary and more details on CWT on the sphere and its implementation can be found in [3]

2 Discrete Wavelet Frames on the Sphere

In this section we describe under which conditions the parameters of the continuous wavelet transform can be descretized. We will study only the case of axisymmetric wavelets.

2.1 Half-continuous Spherical Frame

2.1.1 A Frame

For a frame, with a given function $f : S^2 \mapsto \mathbb{R}$ and an axisymmetric wavelet ψ satisfying the admissibility condition, the spherical CWT of f is defined by

$$W_f(\omega, a) = \int_{S^2} d\mu(\omega')f(\omega')[R_{[\omega]}D_a\psi]^*(\omega'), \tag{12}$$

with $\omega = (\theta, \varphi) \in S^2, [\omega] = \rho(\varphi, \theta, 0) \in SO(3)$ and $a \in \mathbb{R}_+^*$. The R_ρ and dilation D_a perators are defined in (1) and (2), respectively.

For an axisymmetric spherical wavelet, the reconstruction is given by

$$f(\omega) = \int_{\mathbb{R}_+^*} \int_{S^2} \frac{da d\mu(\omega')}{a^3} W_f(\omega', a)\tilde{\psi}_{\omega,a}(\omega'), \tag{13}$$

with $\tilde{\psi}_a = R_{[\omega]}L_\psi^{-1}D_a\psi$, and L_ψ is the frame operator such that

$$\begin{aligned} & \widehat{[L_\psi^{-1}\psi_a]}(l,m) && (14)\\ = \; & G_\psi(l)^{-1}\hat{\psi}_a(l,0)\delta_{0,m} && (15)\\ = \; & \left[\frac{4\pi}{2l+1}\int_{\mathbb{R}_+^*}\frac{da}{a^3}|\hat{\psi}_a(l,0)|^2\right]^{-1}\hat{\psi}_a(l,0)\delta_{0,m} && (16)\end{aligned}$$

2.1.2 First Approach

We propose now to discretize the scale of the CWT on the sphere as we leave the position varying continualy. In other words, we choose

$$\begin{aligned} \omega &\in S^2 && (17)\\ a &\in A \equiv \{a_j \in \mathbb{R}_+^* : j \in \mathbb{Z}, a_j > a_{j+1}\} && (18)\end{aligned}$$

which build the half-continuous grid

$$\Lambda(A) = \{(\omega, a_j) : \omega \in S^2, j \in \mathbb{Z}\}. \tag{19}$$

In order to have a reconstruction of all the functions $f \in L^2(S^2)$, one first approach would be to impose

$$A\|f\|_2^2 \leq \sum_{j\in\mathbb{Z}} \nu_j \int_{S^2} d\mu(\omega)|W_f(\omega, a_j)|^2 \leq B\|f\|_2^2, \tag{20}$$

with $A, B \in \mathbb{R}_+^*$ indipendant of f, and for some weights $\nu_j > 0$ taking into account the discretizatrion of the continuous measure $\frac{da}{a^3}$. In this case, the familly

$$\{\psi_{\omega,a_j} = R_{[\omega]} D_{a_j}\psi : (\omega, a_j) \in \Lambda(A)\}, \tag{21}$$

constitutes a half-continuous frame in $L^2(S^2)$. The following proposition translates this last condition into the Fourier space.

Proposition 2.1.1 *If there are two constants $A, B \in \mathbb{R}_+^*$ such that*

$$A \leq \frac{4\pi}{2l+1} \sum_{j\in\mathbb{Z}} \nu_j |\hat{\psi}_{a_j}(l, 0)|^2 \leq B \tag{22}$$

for all $l \in \mathbb{N}$, then (20) is fulfilled.

Proof : With a given admissible spherical wavelet, the Fourier coefficinets of a function $f \in L^2(S^2)$ are given by

$$W_f(\omega, a) \;=\; \sum_{(l,m)\in\mathcal{N}} \sqrt{\tfrac{4\pi}{2l+1}}\, \hat{f}(l, m)\, \hat{\psi}_a^*(l, 0)\, Y_l^m(\omega).$$

Developping (15) using these coefficients, we have

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}} \nu_j \int_{S^2} d\mu(\omega)|W_f(\omega, a_j)|^2 \\
&= \sum_{j\in\mathbb{Z}} \nu_j \sum_{(l,k)\in\mathcal{N}} \sum_{(l',k')\in\mathcal{N}} \tfrac{4\pi}{\sqrt{(2l+1)(2l'+1)}}\, \hat{f}(l, k)\, \hat{f}^*(l', k') \\
&\qquad \hat{\psi}_{a_j}^*(l, 0)\, \hat{\psi}_{a_j}(l', 0) \int_{S^2} d\mu(\omega)\; Y_l^k(\omega)\, Y_{l'}^{k'*}(\omega) \\
&= \sum_{j\in\mathbb{Z}} \nu_j \sum_{(l,k)\in\mathcal{N}} \tfrac{4\pi}{2l+1}\, |\hat{f}(l, k)|^2\, |\hat{\psi}_{a_j}(l, 0)|^2 \\
&= \sum_{(l,k)\in\mathcal{N}} |\hat{f}(l, k)|^2 \sum_{j\in\mathbb{Z}} \tfrac{4\pi}{2l+1}\, \nu_j\, |\hat{\psi}_{a_j}(l, 0)|^2,
\end{aligned}$$

where we have used the orthonormality of the spherical harmonics, namely:

$$\langle Y_l^k Y_{l'}^{k'} \rangle = \delta_{ll'} \delta_{kk'}.$$

The inferior and superior bounds of (15), are well defined if there are two constants $A, B \in \mathbb{R}_+^*$ such that

$$A \;\leq\; \tfrac{4\pi}{2l+1} \sum_{j \in \mathbb{Z}} \nu_j \,|\hat{\psi}_{a_j}(l,0)|^2 \;\leq\; B,$$

for all $l \in \mathbb{N}$. ■

Let us choose a DOG wavelet ($\alpha = 1.25$) with a discretized dyadic scale with $K \in \mathbb{N}^0$, namely

$$a_j = a_0 2^{-\frac{j}{K}}, \quad j \in \mathbb{Z}. \tag{23}$$

For simplifying the notation, we replace the indeces a_j by j, so, for instance ψ_{a_j} becomes ψ_j. As well, we take the weights ν_j which take into account the discretization of the continuous measure $\frac{da}{a^3}$, which means

$$\nu_j = \frac{a_j - a_{j+1}}{a_j^3} = \frac{2^{\frac{1}{K}} - 1}{2^{\frac{1}{K}} a_j^2} \tag{24}$$

We have estimated the bounds A and B, based on the respectively, minimum and maximum of the quantity

$$S(l) = \frac{4\pi}{2l+1} \sum_{j \in \mathbb{Z}} \nu_j |\hat{\psi}_j(l,0)|^2, \tag{25}$$

over $l \in [0, 31]$ and for $K = [1, 4]$. The results are shown in the table 1.

K	A	B	B/A
1	0.5281	0.9658	1.8288
2	0.6817	1.1203	1.8107
3	0.6537	1.1836	1.8107
4	0.6722	1.2171	1.8107

Table 1: Estimation of the bounds A et B as a function of extremum of $S(l)$ for some values of K.

We can see that for $K > 2$, the relation B/A converges toward the value 1.8107. So, it is not a convergence toward a strict frame, for which $A = B$.

2.1.3 Second Approach

For this second approach, we start from the Plancharel relation as defined in corollary (1.1.2). In other words, we will observe under which exclusion of the following condition for *controlled* frame, it is satisfied. For $A, B \in \mathbb{R}_+^*$, we want

$$A\|f\|_2^2 \leq \sum_{j\in\mathbb{Z}} \nu_j \int_{S^2} d\mu(\omega) W_f(\omega, a_j) \tilde{W}_f^*(\omega, a_j) \leq B\|f\|_2^2, \tag{26}$$

for any $f \in L^2(S^2)$ and $\tilde{W}_f(\omega, a_j) = \langle R_{[\omega]} L_\psi^{-1} D_a \psi | f\rangle$. The operator L_ψ controls the frame and it is limited if the wavelet ψ is admissible.

Proposition 2.1.2 *If there exist two constants $A, B \in \mathbb{R}_+^*$ such that*

$$A \leq \frac{4\pi}{2l+1} G_\psi(l)^{-1} \sum_{j\in\mathbb{Z}} \nu_j |\hat{\psi}_j(l,0)|^2 \leq B, \tag{27}$$

with $G_\psi(l)$ given by (16) and for all $l \in \mathbb{N}$, then

$$A\|f\|2^2 \leq \sum_{j\in\mathbb{Z}} \sum_{p,q=0}^{2B_j-1} \nu_j \omega_{jp} W_f(\omega_{jpq}, a_j) \tilde{W}_f^*(\omega_{jpq}, a_j) \leq B\|f\|_2^2 \tag{28}$$

is verified.

Proof : As in the previous proposition, we start from the Fourier coefficients

$$W_f(\omega, a) \;=\; \sum_{(l,m)\in\mathcal{N}} \sqrt{\tfrac{4\pi}{2l+1}}\, \hat{f}(l,m)\, \hat{\psi}_a^*(l,0)\, Y_l^m(\omega).$$

Then $\tilde{W}_f(\omega, a) = \langle R_{[\omega]} L_\psi^{-1} D_a \psi | f\rangle$ reads

$$\tilde{W}_f(\omega, a) \;=\; \sum_{(l,m)\in\mathcal{N}} \sqrt{\tfrac{4\pi}{2l+1}}\, G_\psi(l)^{-1}\, \hat{f}(l,m)\, \hat{\psi}_a^*(l,0)\, Y_l^m(\omega),$$

since the frame operator depends only on l and commutes with the rotations.

Developping (15) using these coefficinets, we have

$$\begin{aligned} &\sum_{j\in\mathbb{Z}} \nu_j \int_{S^2} d\mu(\omega)\; W_f(\omega, a_j)\, \tilde{W}_f(\omega, a) \\ &= \sum_{(l,k)\in\mathcal{N}} |\hat{f}(l,k)|^2 \sum_{j\in\mathbb{Z}} \tfrac{4\pi}{2l+1}\, G_\psi(l)^{-1}\, \nu_j\, |\hat{\psi}_{a_j}(l,0)|^2, \end{aligned}$$

using the orthonormality of the spherical harmonics. Then the equation (15) is fulfilled if there exist two constants $A, B \in \mathbb{R}_+^*$, such that

$$A \;\leq\; \tfrac{4\pi}{2l+1}\, G_\psi(l)^{-1} \sum_{j\in\mathbb{Z}} \nu_j\, |\hat{\psi}_{a_j}(l,0)|^2 \;\leq\; B,$$

for all $l \in \mathbb{N}$. ■

In this case we evaluate the quantity

$$S(l) \;=\; \frac{4\pi}{2l+1}\, G_\psi(l)^{-1} \sum_{j\in\mathbb{Z}} \nu_j\, |\hat{\psi}_j(l,0)|^2. \tag{29}$$

Taking into account

$$G_\psi(l) \;=\; \lim_{K\to\infty} \frac{4\pi}{2l+1} \sum_{j\in\mathbb{Z}} \nu_j\, |\hat{\psi}_j(l,0)|^2, \tag{30}$$

where the weights ν_j discretize the continuous measure $\frac{da}{a^3}$.

The behaviour of $S(l)$ is presented in the Table 2 for $K \in [1, 4]$.

K	A	B	B/A
1	0.7313	0.7628	1.0431
2	0.8747	0.8766	1.0021
3	0.9242	0.9254	1.0014
4	0.9503	0.9512	1.0009

Table 2: Estimation of the bounds A et B in function of extremum of $S(l)$ for some values of K.

It shows that the relation B/A converges towards 1, so, the controlled half-continuous spherical frame is better then the classical frame from the first approach.

2.1.4 Construction of a strict half-continuous frame

It is possible to create a strict half-continuous frame on the sphere using the preveous considerations.

Proposition 2.1.3 *Let $\{a_j : j \in \mathbb{Z}, a_j > a_{j+1}\}$ be a sequance of scales. If ψ is a axisymmetric wavelet such that*

$$g_\psi(l) = \frac{4\pi}{2l+1} \sum_{j\in\mathbb{Z}} \nu_j |\hat{\psi}_j(l,0)|^2 \neq 0, \quad \forall l \in \mathbb{N}, \tag{31}$$

with $\psi_{\omega,a_j} = R_{[\omega]} l_\psi^{-1} D_{a_j}\psi$ *and* l_ψ *is a operator of a discretized frame defined in Fourier domain by*

$$\widehat{l_\psi^{-1} h}(l,m) = g_\psi^{-1}(l) h(l,m). \tag{32}$$

Then, the reconstruction is:

$$f(\omega) = [S_f \star \zeta^\sharp](\omega) + \sum_{j\in\mathbb{N}} \nu_j [W_f(\cdot, a_j) \star \psi_j^\sharp](\omega), \tag{33}$$

with $S_f(\omega) = \langle R_{[\omega]}\zeta | f\rangle$ and $\zeta^\sharp = R_\omega l_\psi^{-1}\zeta$.

2.2 Discrete Spherical Frame

In this section, we will complitely discretize the CWT on the sphere. The scales are discretized as previously, namely

$$a \in A = \{a_j \in \mathbb{R}_+^* : a_j > a_{j+1}, j \in \mathbb{Z}\}, \tag{34}$$

and the positions are taken at equi-angular grid, related to the scale in way such that $\omega \in \mathcal{G}_j$ with

$$\mathcal{G}_j = \{(\theta_{jp}, \varphi_{jq}) \in S^2 : \theta_{jp} = \frac{(2p+1)\pi}{4B_j}, \varphi_{jq} = \frac{q\pi}{B_j}\} \tag{35}$$

with $p, q \in \mathbb{N}$, $0 \leq p, q < 2B_j$; for some range of bandwidth $B = \{B_j \in 2\mathbb{N}, j \in \mathbb{Z}\}$. Actually, θ_{jp} form a *pseudo-spectral* grid and are localized on the knots of a Chebishev polynomial of order $2B$ [4, 5]. In general, the space of discretization is

$$\Lambda(A, B) = \{(a_j, \omega_{jpq}) : j \in \mathbb{Z}, p, q \in \mathbb{N}, 0 \leq p, q < 2B_j\}, \tag{36}$$

with $\omega_{jpq} = (\theta_{jp}, \varphi_{jq})$.

In this case, for an axisymmetric and admissible mothet-wavelet $\psi \in S^2$, the familly of wavelets

$$\{\psi_{jpq} = R_{[\omega]jpq} D_{a_j}\psi : j \in \mathbb{Z}, p, q \in \mathbb{N}, 0 \leq p, q < 2B_j\} \tag{37}$$

constitutes in an weighted frame and controlled by the operator L_ψ, if there are two constants $A, B \in \mathbb{R}_+^*$ such that for all functions $f \in L^2(S^2)$ we have

$$A\|f\|_2^2 \leq \sum_{j\in\mathbb{Z}} \sum_{p,q=0}^{2B_j-1} \nu_j \omega_{jp} W_f(\omega_{jpq}, a_j) \tilde{W}_f^*(\omega_{jpq}, a_j) \leq B\|f\|_2^2, \tag{38}$$

with $\omega_{jp} = \omega_p^{B_j}$ and weights as defined in (). Here, $\nu_j\omega_{jp}$ replaces the measure $\frac{da}{a^3} d\mu(\theta, \varphi)$.

Theorem 2.2.1 *Let the discretized grid $\Lambda(A,B)$ be given as in (36), ψ is an axisymmetric and admissible wavelet on S^2, and*

$$K_0 = \inf_{l\in\mathbb{N}} \sum_{j\in\mathbb{Z}} \frac{4\pi\nu_j}{2l+1} \mathbb{1}_{[0,B_j[}(l) G_\psi^{-1} |\hat{\psi}_{a_j}(l,0)|^2, \tag{39}$$

$$K_1 = \sup_{l\in\mathbb{N}} \sum_{j\in\mathbb{Z}} \frac{4\pi\nu_j}{2l+1} \mathbb{1}_{[0,B_j[}(l) G_\psi^{-1} |\hat{\psi}_{a_j}(l,0)|^2, \tag{40}$$

$$\delta = \|\mathcal{X}\| = \sup_{(H_l)_{l\in\mathbb{N}}} \frac{\|\mathcal{X}H\|}{\|H\|}, \tag{41}$$

with the infinite matrix

$$\mathcal{X} = \left(\sum_{j\in\mathbb{N}} \frac{2\pi\nu_j c_j(l,l')}{B_j} \mathbb{1}_{[2B_j,+\infty[}(l+l') G_\psi^{-1}(l) |\hat{\psi}_{a_j}(l,0)| |\hat{\psi}_{a_j}(l',0)| \right)_{l,l'\in\mathbb{N}} \tag{42}$$

and $c_j(l,l') = \big(2(l+B_j)+1\big)^{\frac{1}{2}} \big(2(l'+B_j)+1\big)^{\frac{1}{2}}$*. If*

$$0 \le \delta < K_0 \le K_1 < \infty, \tag{43}$$

then the defined wavelet family in (37) is an weighted spherical wavelet controlled by the operator L_ψ and the bounds $K_0-\delta$, $K_0+\delta$.

Proof : Let us define the sum

$$S = \sum_{j\in\mathbb{Z}} \sum_{p,q=0}^{2B_j-1} \nu_j w_{jp} \, W_f(\omega_{jpq}, a_j) \, \tilde{W}_f^*(\omega_{jpq}, a_j).$$

Using

$$W_f(\omega,a) = \sum_{(l,m)\in\mathcal{N}} \sqrt{\tfrac{4\pi}{2l+1}} \, \hat{f}(l,m) \, \hat{\psi}_a^*(l,0) \, Y_l^m(\omega)$$

$$\tilde{W}_f(\omega,a) = \sum_{(l,m)\in\mathcal{N}} \sqrt{\tfrac{4\pi}{2l+1}} \, G_\psi(l)^{-1} \, \hat{f}(l,m) \, \hat{\psi}_a^*(l,0) \, Y_l^m(\omega),$$

we have

$$\begin{aligned} S &= \sum_{j\in\mathbb{N}} \sum_{p,q=0}^{2B_j-1} \sum_{(l,m)\in\mathcal{N}} \sum_{(l',m')\in\mathcal{N}} \frac{4\pi}{\sqrt{(2l+1)(2l'+1)}} \hat{f}(l,m)\, \hat{f}^*(l',m') \\ &\qquad \nu_j w_{jp}\, G_\psi^{-1}(l)\, \hat{\psi}^*_{a_j}(l,0)\, \hat{\psi}_{a_j}(l',0)\, Y_l^m(\omega_{jpq})\, Y_{l'}^{m'*}(\omega_{jpq}) \\ &= \sum_{j\in\mathbb{N}} 4\pi\nu_j \sum_{(l,m)\in\mathcal{N}} \sum_{(l',m')\in\mathcal{N}} \frac{\hat{f}(l,m)\, \hat{f}^*(l',m')}{\sqrt{(2l+1)(2l'+1)}} G_\psi^{-1}(l)\, \hat{\psi}^*_{a_j}(l,0)\, \hat{\psi}_{a_j}(l',0) \\ &\qquad \sum_{p,q=0}^{2B_j-1} w_{jp}\, Y_l^k(\omega_{jpq})\, Y_{l'}^{k'*}(\omega_{jpq}). \end{aligned}$$

If $l + l' < B_j$, the order of the product $Y_l^m Y_{l'}^{m'}$ is equal to $l + l'$ and the weights ω_{jp} constitute the quadrature [4, 5]

$$\sum_{p,q=0}^{2B_j-1} w_{jp}\, Y_l^m(\omega_{jpq})\, Y_{l'}^{m'*}(\omega_{jpq}) = \int_{S^2} d\mu(\omega)\; Y_l^m(\omega)\, Y_{l'}^{*m'}(\omega) = \delta_{ll'}\delta_{mm'}, \tag{44}$$

for all $|m| \leq l$ and $|m' \leq l'|$.

The sum S is separated in two parts:

$$\begin{aligned} S &= \sum_{j\in\mathbb{N}} \sum_{p,q=0}^{2B_j-1} \sum_{\substack{(l,m)\in\mathcal{N} \\ (l',m')\in\mathcal{N} \\ l+l'<2B_j}} \cdots \quad + \quad \sum_{j\in\mathbb{N}} \sum_{p,q=0}^{2B_j-1} \sum_{\substack{(l,k)\in\mathcal{N}(l',m')\in\mathcal{N} \\ l+l'\geq 2B_j}} \cdots \\ &= C + D. \end{aligned}$$

The first part C, where (44) is reduced to

$$\begin{aligned} C &= \sum_{j\in\mathbb{N}} 4\pi\nu_j \sum_{\substack{(l,m)\in\mathcal{N} \\ l<B_j}} \frac{1}{(2l+1)} |\hat{f}(l,m)|^2\, G_\psi^{-1}(l)\, |\hat{\psi}_{a_j}(l,0)|^2 \\ &= \sum_{(l,m)\in\mathcal{N}} |\hat{f}(l,m)|^2 \sum_{j\in\mathbb{N}} \frac{4\pi\nu_j}{(2l+1)} \mathbb{1}_{[0,B_j[}(l)\, G_\psi^{-1}(l)\, |\hat{\psi}_{a_j}(l,0)|^2. \end{aligned}$$

If the equation (43) is satisfied, then

$$K_0\|f\|^2 \quad \leqslant \quad C \quad \leqslant \quad K_1\|f\|^2. \tag{45}$$

Now, let us develope the part D. Since $Y_l^m(\omega_{jpq}) = Y_l^m(\theta_{jp},0)\, e^{im\varphi_{jq}}$, we have

$$\begin{aligned}
\sum_{q=0}^{2B_j-1} Y_l^m(\omega_{jpq})\, Y_{l'}^{*m'}(\omega_{jpq}) &= Y_l^m(\theta_{jp},0)\, Y_{l'}^{*m'}(\theta_{jp},0) \sum_{q=0}^{2B_j-1} e^{i(m-m')\varphi_{jq}} \\
&= 2B_j\, Y_l^m(\theta_{jp},0)\, Y_{l'}^{*m'}(\theta_{jp},0) \sum_{\substack{t\in\mathbb{Z} \\ |m+2tB_j|\leqslant l'}} \delta_{m',m+2tB_j} \\
&= 2B_j \sum_{\substack{t\in\mathbb{Z} \\ |m+2tB_j|\leqslant l'}} Y_l^m(\theta_{jp},0)\, Y_{l'}^{*m+2tB_j}(\theta_{jp},0)\, \delta_{m',m+2tB_j}.
\end{aligned}$$

It tends out that

$$\begin{aligned}
D &= \sum_{j\in\mathbb{N}} 8\pi\nu_j\, B_j \sum_{(l,m)\in\mathcal{N}} \sum_{l'\in\mathbb{N}} \sum_{t\in\mathbb{Z}} \frac{\mathbb{1}_{[2B_j,+\infty[}(l+l')\, \mathbb{1}_{[-l',l']}(m+2tB_j)}{\sqrt{(2l+1)(2l'+1)}} \\
&\quad \times \hat{f}(l,m)\, \hat{f}^*(l',m+2tB_j) G_\psi^{-1}(l) \\
&\quad \times \hat{\psi}^*_{a_j}(l,0)\, \hat{\psi}_{a_j}(l',0) \sum_{p=0}^{2B_j-1} w_{jp}\, Y_l^m(\theta_{jp},0)\, Y_{l'}^{*m+2tB_j}(\theta_{jp},0).
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
|D| &\leqslant \sum_{j\in\mathbb{N}} 8\pi\nu_j\, B_j \sum_{(l,m)\in\mathcal{N}} \sum_{l'\in\mathbb{N}} \sum_{t\in\mathbb{Z}} \frac{\mathbb{1}_{[2B_j,+\infty[}(l+l')\, \mathbb{1}_{[-l',l']}(m+2tB_j)}{\sqrt{(2l+1)(2l'+1)}} \\
&\quad \times |\hat{f}(l,m)|\, |\hat{f}(l',m+2tB_j)|\, G_\psi(l)^{-1} \\
&\quad \times |\hat{\psi}_{a_j}(l,0)|\, |\hat{\psi}_{a_j}(l',0)| \sum_{p=0}^{2B_j-1} w_{jp}\, |Y_l^m(\theta_{jp},0)|\, |Y_{l'}^{m+2tB_j}(\theta_{jp},0)| \\
&\leqslant \sum_{j\in\mathbb{N}} 4\pi\nu_j \sum_{(l,m)\in\mathcal{N}} \sum_{l'\in\mathbb{N}} \sum_{t\in\mathbb{Z}} |\hat{f}(l,m)|\, |\hat{f}(l',m+2tB_j)|\, \mathbb{1}_{[-l',l']}(m+2tB_j) \\
&\quad \mathbb{1}_{[B_j,+\infty[}(l+l')\, G_\psi^{-1}(l)\, |\hat{\psi}_{a_j}(l,0)|\, |\hat{\psi}_{a_j}(l',0)|
\end{aligned}$$

where we have used the fact, that $|Y_l^m| \leqslant \sqrt{\frac{2l+1}{4\pi}}$ pour tout $(l,m)\in\mathcal{N}$, and that $\sum_{p=0}^{2B_j-1} w_{jp} = \frac{4\pi}{2B_j}$.

The sums over m and t can be bounded because

$$\begin{aligned}
&\sum_{t\in\mathbb{Z}}\sum_{|m|\leqslant l} |\hat{f}(l,m)|\,|\hat{f}(l',m+2tB_j)|\,\mathbb{1}_{[-l',l']}(m+2tB_j)\\
&\leqslant \sum_{t\in\mathbb{Z}} \big[\sum_{|m|\leqslant l} |\hat{f}(l,m)|^2\,\mathbb{1}_{[-l',l']}(m+2tB_j)\big]^{\frac{1}{2}} \big[\sum_{|m|\leqslant l} |\hat{f}(l',m+2tB_j)|^2\,\mathbb{1}_{[-l',l']}(m+2tB_j)\big]^{\frac{1}{2}}\\
&\leqslant \big[\sum_{t\in\mathbb{Z}}\sum_{|m|\leqslant l} |\hat{f}(l,m)|^2\,\mathbb{1}_{[-l',l']}(m+2tB_j)\big]^{\frac{1}{2}} \big[\sum_{t\in\mathbb{Z}}\sum_{|m|\leqslant l} |\hat{f}(l',m+2tB_j)|^2\,\mathbb{1}_{[-l',l']}(m+2tB_j)\big]^{\frac{1}{2}}\\
&\leqslant \big[\sum_{|m|\leqslant l} |\hat{f}(l,m)|^2\,\big[\frac{2l'+1}{2B_j}+1\big]\big]^{\frac{1}{2}} \big[\sum_{t\in\mathbb{Z}}\sum_{m'=-l+2tB_j}^{l+2tB_j} |\hat{f}(l',m')|^2\,\mathbb{1}_{[-l',l']}(m')\big]^{\frac{1}{2}}\\
&\leqslant \big[\sum_{|m|\leqslant l} |\hat{f}(l,m)|^2\,\big[\frac{2l'+1}{2B_j}+1\big]\big]^{\frac{1}{2}} \big[\sum_{t\in\mathbb{Z}}\sum_{m'=-l'}^{l'} |\hat{f}(l',m')|^2\,\mathbb{1}_{[-l,l]}(m'-2tB_j)\big]^{\frac{1}{2}}\\
&\leqslant \big[\sum_{|m|\leqslant l} |\hat{f}(l,m)|^2\,\big[\frac{2l'+1}{2B_j}+1\big]\big]^{\frac{1}{2}} \big[\sum_{|m'|\leqslant l'} |\hat{f}(l',m')|^2\,\big[\frac{2l+1}{2B_j}+1\big]\big]^{\frac{1}{2}}\\
&\leqslant (2B_j)^{-1}\big(2(l+B_j)+1\big)^{\frac{1}{2}}\big(2(l'+B_j)+1\big)^{\frac{1}{2}}\,\big[\sum_{|m|\leqslant l}|\hat{f}(l,m)|^2\big]^{\frac{1}{2}}\,\big[\sum_{|m'|\leqslant l'}|\hat{f}(l',m')|^2\big]^{\frac{1}{2}},
\end{aligned}$$

applying the Cauchy-Schwarz inequality on the sum over m, and after this on the sum over t. From here it follows:

$$|D| \leqslant \sum_{l,l'\in\mathbb{N}} \left[\sum_{|m|\leqslant l}|\hat{f}(l,m)|^2\right]^{\frac{1}{2}} \left[\sum_{|m'|\leqslant l'}|\hat{f}(l',m')|^2\right]^{\frac{1}{2}} \chi(l,l')$$

with

$$\chi(l,l') = \sum_{j\in\mathbb{N}} \frac{2\pi\nu_j\, c_j(l,l')}{B_j}\,\mathbb{1}_{[2B_j,+\infty[}(l+l')\,G_\psi^{-1}(l)\,|\hat{\psi}_{a_j}(l,0)|\,|\hat{\psi}_{a_j}(l',0)|.$$

and $c_j(l,l') = \big(2(l+B_j)+1\big)^{\frac{1}{2}}\big(2(l'+B_j)+1\big)^{\frac{1}{2}}$.

Putting $F_l^2 = \sum_{|m|\leqslant l}|\hat{f}(l,m)|^2$, by the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}
|D| &\leq \sum_{l\in\mathbb{N}} F_l \sum_{l'\in\mathbb{N}} \chi(l,l')\,F_{l'}\\
&\leq \|F\|\|\mathcal{X}F\|\\
&= \|f\|\|\mathcal{X}F\|,
\end{aligned}$$

avec $F=(F_l)_{l\in\mathbb{N}}$, $\|F\|^2=\sum_{l\in\mathbb{N}}|F_l|^2=\|f\|^2$, $\mathcal{X}=\big(\chi(l,l')\big)_{l,l'\in\mathbb{N}}$ et $(\mathcal{X}F)_l = \sum_{l'\in\mathbb{N}}\chi(l,l')\,F_{l'}$.

If (43) is verified, we have

$$|D| \le \|f\| \, \|\mathcal{X}\| \, \|f\| = \delta \, \|f\|^2,$$

with the norm

$$\|\mathcal{X}\| = \sup_{(G_l)_{l\in\mathbb{N}}} \frac{\|\mathcal{X}G\|}{\|G\|}.$$

The proof of the theorem is done with the fact that

$$0 < (K_0 - \delta)\|f\|^2 < C - |D| \leqslant S \leqslant C + |D| < (K_1 + \delta)\|f\|^2 < \infty.$$

■

The evaluation of $\|\mathcal{X}\|$ could be complex in case when the character of $\mathcal{X}$ is infinite. However, in the practice we work on functions $f \in L^2(S^2)$ at limited band, namely, $\hat{f}(l,m) = 0$, for all $l \geq B$, where $B \in \mathbb{N}^*$ is the bandwidth of f. Consequently, $\|\mathcal{X}\|$ could be changed with the norm of the finite matrix $(\mathcal{X}_{l,l'})_{0\le l,l'<B}$.

We have estimated the bounds of a spherical DOG wavelet frame choosing a scale, dyadicaly discretized with

$$a_j = \frac{a_0}{2^j}, \quad a_0 = 1, \quad j \in \mathbb{Z}, \tag{46}$$

and the bandwidth, associated to the grid size supporting each resolution j, was fixed at

$$B_j = B_0 2^{|j|}, \quad B_0 \in \mathbb{N}, \tag{47}$$

where B_0 is the mimnimal bandwidth associated to ψ_1.

The Table 3 presents the results of the evaluation of K_0, K_1 and δ as well as the bounds of the associated frames. One can see that for $B_0 \geq 4$, the

	K_0	K_1	δ	$A = K_0 - \delta$	$B = K_1 + \delta$	B/A
$B_0 = 2$	0.6807	0.7700	84.1502	–	–	–
$B_0 = 4$	0.7402	0.7790	0.0594	0.6808	0.8384	1.2314
$B_0 = 8$	0.7402	0.7790	0.0014	0.7388	0.7804	1.0564

Table 3: Evaluation of K_0, K_1 and δ on the fonctions $f \in L^2(S^2)$ at bandwidth 64.

condition (43) is reached. A strict frame cannot be reached while we increase B_0. Actually, if B_0 tends to infinity, the spherical grids at each resolution gets finer and finer and we approch to the half-continuous frames, but (it is shown) in the previous section, the discretization of the scale only, is not sufficient.

3 Conclusions and Future Work

Conditions on the existence of half-continuous and discrete spherical frames have been established from the (stereographical) spherical CWT. An example of a discrete frame using the results of Theorem 2.2.1 has still to be designed. These techniques could serve for instance to discover the Gaussian anisotropies in the astronomical *Cosmic Microwave Background* [7], or to track the orientations in $\mathbb{R}^3$ of fibre in the human brain connectivity [8]. Some works in that sense are currently undertaken by some of us.

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