Lexico-smallest representations, duality and matching polyhedra

Komei Fukuda

Institute For Operations Research, ETH, Switzerland

Stefano Picozzi

Institute of Mathematics, EPFL, Switzerland

Abstract

Every convex polyhedron in $\mathbb{R}^d$ admits both H- and V-representations. In both types, a representation is *canonical* if it is minimal and unique up to some elementary operations. In this paper, we show the duality of canonical representations and that the canonical V-representations coincide with certain canonical H-representations. Also, we show how the lexico-smallest representation, a computationally convenient alternative to the usual orthogonal representation, can be computed efficiently. Finally, we illustrate our results by considering H-representations of the perfect matching polytope. In particular, we show that using the properties of the underlying graph results in sensible improvements in the running time of the computation of canonical representation.

1 Introduction

A *(convex) polyhedron* in $\mathbb{R}^d$ is the solution set to a finite system of inequalities with real coefficients in d real variables. For a matrix $A \in \mathbb{R}^{m\times d}$, a vector $b \in \mathbb{R}^m$ and a partition (I, L) of $[m] := \{1, 2, \ldots, m\}$, a quadruple (b, A, I, L) is said to be an *H-representation* of a convex polyhedron P if $P = \{x \in \mathbb{R}^d \mid b_I + A_I x \geq 0,\ b_L + A_L = \mathbf{0}\}$. For matrices $V \in \mathbb{R}^{p\times d}$, $R \in \mathbb{R}^{q\times d}$ and $M \in \mathbb{R}^{r\times d}$, a triple (V, R, M) is said to be a *V-representation* of a polyhedron P if $P = conv(V) + cone(R) + lin(M)$, where $conv(A)$, $cone(A)$ and

Email addresses: fukuda@ifor.math.ethz.ch (Komei Fukuda), stefano.picozzi@epfl.ch (Stefano Picozzi).

Preprint submitted to Elsevier Science *10 August 2004*

$lin(A)$, denote respectively the convex hull, the nonnegative hull and the linear hull of the row vectors of the matrix A. Motzkin's decomposition theorem (see, e.g. (2; 3)) states that every polyhedron has both H- and V-representations.

Clearly, neither H- nor V-representation is unique. In (1), we described a family of polynomialy computable H- and V-representations that were unique up to some elementary operations. In particular we defined the *lexico-smallest representation* which guarantees certain sparsity properties. In the present paper, we propose a slightly different definition which makes the computations easier. Also, we show the duality of canonical representations and that the canonical V-representations coincide with certain canonical H-representations. Finally, we illustrate our results by considering the case of the perfect matching polytope. In particular, we determine its lexico-smallest H-representation in terms of the underlying graph, which results in sensible improvements in the running times of the computation. Also, we show that in the case of complete bipartite graphs, the lexico-smallest representation is simpler than the *orthogonal representation*, its usual alternative described in (2; 3). Finally, it is worth mentionning that a C-package computing the canonical representations of a polyhedron P from any other representation of P will be released in the future.

2 Representations of convex polyhedra

We define a quadruple (b, A, I, L) to be an H-representation of the polyhedron $P_H = \{x \in \mathbb{R}^d \mid b_I + A_I x \geq \mathbf{0},\ b_L + A_L x = \mathbf{0}\}$ and a V-representation of the polyhedron $P_V = \{x \in \mathbb{R}^d \mid x = A^T y,\ y_I \geq \mathbf{0},\ b^T y = 1\}$. We will also refer to H- and V-representations as $*$*-representation*, intended that $*$ refers to one of H or V. A V-representation is called *standard* if $b_i \in \{0, 1\}$ and $b_L = \mathbf{0}$. Note that the quadruple (b, A, I, L) is a standard V-representation of $P_V = conv(A_{I^+}) + cone(A_{I^0}) + lin(A_L)$, where $I^0 := \{i \in I \mid b_i = 0\}$ and $I^+ := \{i \in I \mid b_i = 1\}$. As any V-representation can be transformed to a standard V-representation of the same polyhedron in quadratic time, we assume for the sequel of the paper that every V-representation is standard.

2.1 Canonical representations

We say two representations (b, A, I, L) and (b', A', I', L') of the same type *equivalent* if the represented polyhedra are equal. They are said to be *equal* if $b_L + A_L x = 0 \Leftrightarrow b_{L'} + A_{L'} x = 0$ and if there is a permutation π of I such that $\pi(I) = I'$ and each (b'_i, A'_i) is a positive multiple of $(b_{\pi(i)}, A_{\pi(i)})$ for any $i \in I$. Note that for V-representations, $b_L = \mathbf{0}$ and then the first equivalence coincides with the statement $\{x \in \mathbb{R}^d \mid x = (A_L)^T y\} = \{x \in \mathbb{R}^d \mid x = (A_{L'})^T y\}$.

For an index set J we let $J + i := J \cup \{i\}$ and (provided that $i \in J$) $J - i := J \setminus \{i\}$. A row index $i \in [m]$ is called *redundant in a representation* (b, A, I, L) *of* P if $(b, A, I - i, L)$ is a representation of P. We say that $i \in I$ is in the *implicit linearity* of (b, A, I, L) if $(b, A, I - i, L + i)$ is a representation of P. It is *minimal* if it has no redundant row index and has empty implicit linearity. For every polyhedron P, we let $L_V(P) := \mathit{lin.space}(P)$ and $L_H(P) := \mathit{aff}(P)^{\perp}$, where $\mathit{lin.space}(P) = \{z \in \mathbb{R}^d \mid x + \lambda z \in P, \forall x \in P, \lambda \in \mathbb{R}\}$ and $\mathit{aff}(P)^{\perp}$ is the orthogonal complement of the affine hull $\mathit{aff}(P)$ of P. Finally, we say two linear subspaces S_1 and S_2 of $\mathbb{R}^d$ are *complementary* if every basis of S_1 and every basis of S_2 form, together, a basis of $\mathbb{R}^d$.

Theorem 1 *A minimal $*$-representation (b, A, I, L) of a nonempty polyhedron P such that all row vectors A_i, $i \in I$, belong to a fixed linear subspace S complementary with $L_*(P)$ exists and is unique.*

Selecting S as the orthogonal complement $L_*(P)^{\perp}$ of $L_*(P)$ results in the *orthogonal representation*. Another choice is to let S be a *coordinate subspace*, which is any vector subspace of $\mathbb{R}^d$ generated by some unit vectors e^j, $j = 1, 2, \ldots, d$. It is easy to show that $S := \mathit{lin}(\{e^j\})_{j \notin J}$ and $L_*(P)$ are complementary if and only if the columns of A_{LJ} form a basis of the space spanned by the columns of A_L. Requiring that J is lexicographically largest results in the *lexico-smallest representation*. Clearly, the matrix A_I of this representation has at least $|L| = \mathit{rank}(A_L)$ zero columns. This definition of the lexico-smallest representation, which differs from the one proposed in (1), has the advantage that the computation of S amounts to apply a single gaussian elimination on the matrix A_L, instead of the $O(d)$ eliminations required with the previous definition.

2.2 Canonical V-representations via H-representations

Here, we show the duality of canonical representations and that the canonical V-representations coincide with certain canonical H-representations. Firstly,

Theorem 2 *A representation $(\mathbf{0}, A, I, L)$ is a canonical H-representation of a cone C if and only if it is a canonical V-representation of the polar C^* of C.*

Now, for each V-representation (H-representation, respectively) (b, A, I, L) of a nonempty polyhedron P, we define $[b^0\ A^0] := [b\ A]$ ($[b^0\ A^0] := [b\ A]$ if P is bounded and $[b^0\ A^0] := \begin{bmatrix} 1 & \mathbf{0}^T \\ b & A \end{bmatrix}$ otherwise.)

Theorem 3 *Let (b, A, I, L) be a $*$-representation of a nonempty polyhedron*

P. Then, (b, A, I, L) is a canonical $*$-representation $\Leftrightarrow$ $[b^0 A^0]$ is a canonical $*$-representation.

Corollary 4 *Let (b, A, I, L) be a V-representation of $P_V \ni \mathbf{0}$. Then, (b, A, I, L) is a canonical V-representation if and only if it is a canonical H-representation.*

2.3 From minimal to canonical representations

Let (b, A, I, L) be any minimal $*$-representation of a nonempty polyhedron P.

Lemma 5 *Let $S := lin(\{e^j\}_{j\in J})$ be complementary $L_*(P)$, and let $\bar{A}_L$ the matrix arising from A_L by setting to zero its columns $j \notin J$. Then,*

(1) the rows A'_i of the matrix A'_I of the orthogonal $*$-representation are the rows $A'_i := A_i - \lambda A_L,\ i \in I$, where $\lambda(A_L A_L^T) = A_i A_L^T$;
(2) the rows of the matrix $A'_{I'}$ of the lexico-smallest $*$-representation are the rows $A'_i := A_i - \lambda A'_{L'},\ i \in I$, where $\lambda(\bar{A}_L \bar{A}_L^T) = A_i \bar{A}_L^T$.

3 Perfect matching polyhedra

We apply the results of the last section to the perfect matching polytope. Given a bipartite graph $G = (V, E)$ the perfect matching polytope $P_{MA}(G)$ is

$$P_{MA}(G) = \left\{ x \in \mathbb{R}^{|E|} \;\middle|\; \begin{array}{ll} \sum\limits_{e\in\delta(v)} x_e = 1 & \forall v \in V \\ x_e \geq 0 & \forall e \in E, \end{array} \right\} \tag{3.1}$$

where $\delta(V)$ denotes the set of edges which have exactly one endnode in V. In the following, G denotes a bipartite graph which contains a perfect matching, and we assume that the implicit linearity of the corresponding representation is empty.

Lemma 6 *A subspace $S; = lin(\{e^j\}_{j\in F})$ is complementary with aff$(P_{MA}(G))^\perp$ if and only if (V, F) is a spanning forest in $G = (V, E)$. In turn, if G has c connected components, the affine hull of $P_{MA}(G)$ has dimension $|E| - |V| + c$.*

From now on, we assume that G is connected. We let $T := (V, E_T)$ be any fixed spanning tree in G, and for every $W \subset V$ we let $T[W]$ be the *subtree of T induced by W*. For each $\bar{e} \in E_T$, we denote by $FC_G(T, \bar{e})$ the *fundamental cycle of T with $\bar{e}$*, by $FC^*_G(T, \bar{e})$ the *fundamental cut of T with $\bar{e}$*, and by V_e any of the nodesets such that $FC^*_G(T, \bar{e}) = \delta(V_{\bar{e}})$. We define $\delta^+(V_{\bar{e}})$ ($\delta^-(V_{\bar{e}})$, repsectively) as the set of edges $e \neq \bar{e} \in \delta(V_{\bar{e}}))$ such that $FC^*_G(T, e) \cap T[V_{\bar{e}}]$ has an odd (even) number of vertices. Finally, $E^+(V_{\bar{e}})$ ($E^+(V_{\bar{e}})$, respectively) denotes the set of edges $e \neq \bar{e}$ in $T[V_{\bar{e}}]$ such that the smallest path in T containing both e and $\bar{e}$ has an odd (even) number of edges. If T is the lexicographically largest spanning tree in G and $\Delta_{E(V_{\bar{e}})} := |E^+(V_{\bar{e}})| - |E^-(V_{\bar{e}})|$ we have,

Theorem 7 *The affine hull of $P_{MA}(G)$ is the solution set to the following minimal system of equations: For all $\bar{e} \in T$,*

$$x_{\bar{e}} + \sum_{e \in \delta^+(V_{\bar{e}})} x_e - \sum_{e \in \delta^-(V_{\bar{e}})} x_e = 1 + \Delta_{E(V_{\bar{e}})}.$$

Theorem 8 *Let $x_{\bar{e}} \geq 0$ be any nonredundant inequality in (3.1), and assume that it is not an implicit equation. Then, the corresponding inequality in the lexico-smallest representation is $x_{\bar{e}} \geq 0$ if $\bar{e} \in E \setminus E_T$ and $\sum_{e \in \delta^+(V_{\bar{e}})} x_e \leq 1 + \Delta_{E(V_{\bar{e}})}$ otherwise.*

As a consequence, A_I is a $(\{-1, 0, 1\})$-matrix, and its $2n - 1$ columns corresponding to edge in $T = (V, E_T)$, $|V| = 2n$, are completely zero. Furthermore $|b_i| \leq n$ for all $i \in [m]$. In the case when $G = K_{n,n}$, we have

Property 9 *Let $x_{\bar{e}} \geq 0$, $\bar{e} = (u, v)$, be any inequality in (3.1). Then, the corresponding inequality in the orthogonal representation is*

$$(n-1)^2 x_{\bar{e}} - (n-1) \sum_{e \in \delta(\{u,v\})} x_e + \sum_{e \notin \delta(\{u,v\})} x_e \geq -n,$$

Clearly, A_I is completely dense. Also, the numbers in the lexico-smallest representation are smaller than the one in the orthogonal representation. To conclude, note that using properties of the graph results in sensible improvements of the running time of the computation: removing linearly independent rows of A_L amounts to computing a spanning tree, while transforming the inequalities amounts to computing the length of certain paths in a tree. In the full paper, we will discuss how similar improvements can be obtained for the computation of minimal representations.

References

[1] D. Avis, K. Fukuda, S. Picozzi. On Convex Representation of Convex Polyhedra. *Proc. of the First International Congress of Mathematical Software*, 350–360, 2002.

[2] A. Schrijver. *Theory of Linear and Integer Programming.* John Wiley & Sons, New York, 1986.

[3] G.M. Ziegler. *Lectures on polytopes.* Graduate Texts in Mathematics 152. Springer-Verlag, 1994.