# Algorithm 743: WAPR: A Fortran Routine for Calculating Real Values of the *W*-Function

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We implement the $W$-function approximation scheme described by Barry et al. [1995]. A range of tests of the approximations is included so that the code can be assessed on any given machine. Users can calculate $W(x)$ by specifying $x$ itself or by specifying an offset from $-\exp(-1)$, the latter option necessitated by rounding errors that can arise for $x$ close to $-\exp(-1)$. Results of running the code on a SUN workstation are included.

Categories and Subject Descriptors: G.1.2 [**Numerical Analysis**]: Approximation—*nonlinear approximation*; G.1.5 [**Numerical Analysis**]. Roots of Nonlinear Equations—*iterative methods*

General Terms: Algorithms

Additional Key Words and Phrases: $W$-function

## 1. INTRODUCTION

This package complements Barry et al. [1995], where the details of the $W$-function approximations and implementation are given. Note that this package supersedes ACM Algorithm 443 [Fritsch et al. 1973] and makes use of its iterative improvement scheme.

The $W$-function is defined by solutions of:

$$W \exp(W) = x, \; x \geq -\exp(-1). \tag{1}$$

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There are two solution branches to (1), $W_p$ and $W_m$ as discussed by Barry et al. [1995]. The package consists of three main components: a driver program, a bisection routine (BISECT), and the function WAPR, which implements the formulas in Barry et al. As explained by Barry et al., a single iteration of the algorithm produces approximations to $W(x)$ accurate to at least 16 significant digits, which corresponds to the 15-digit precision available in the mantissa of the floating-point number representation (53–56 bits) in most computers. The appropriate bit length is calculated automatically by WAPR. This calculation need only be carried out once on any given machine.

## 2. TESTING WAPR

The purpose of the driver program is solely to test the precision of WAPR. The behavior of the $W$-function presents computational difficulties for $x$ near $-\exp(-1)$ (both $W_p$ and $W_m$) or 0 ($W_p$) [Barry et al. 1995]. In these regions, rounding error affects the iterative improvement scheme of Fritsch et al. [1973], as well as any other root-finding scheme that might be applied to (1), such as the bisection method used here. Near $x = 0$, $W_p$ can be estimated directly using a truncated form of the expansion (2) given by Barry et al. This expansion can be compared with a continued fraction approximation (Eq. (14) in Barry et al.). For $x$ close to $-\exp(-1)$ another rounding-error difficulty arises. Users of WAPR will usually want to calculate $W(x)$ for specified values of $x$. However, if $W(x)$ is to be evaluated near $x = -\exp(-1)$, it is not possible to specify $x$ due to roundoff. For example, if WAPR is being used in single precision (usually a 7-digit mantissa), and $W(-\exp(-1) + 10^{-10})$ is to be evaluated, then the required $x$ cannot be represented directly as a floating-point number. However, $W_p(x)$, for example, will be affected because

$$W_p \approx -1 + \sqrt{2 \times 10^{-10} \exp(1)}\,.$$

Thus, the $10^{-10}$ increment to $x$ at $x = -\exp(-1)$ increments $W$ from $-1$ by about $2 \times 10^{-5}$. For this reason WAPR is used in either of two ways. First, $x$ is specified directly, and $W(x)$ is returned. Alternatively, $x$ is specified by its offset $(\Delta x)$ from $-\exp(-1)$, i.e., $\Delta x = x + \exp(-1)$. In this case $W(-\exp(-1) + \Delta x)$ is returned. Note that it is not necessary for $\Delta x$ to be small.

Specifying $x$ by its offset, $\Delta x$, from $-\exp(-1)$ resolves the problem of obtaining approximations from WAPR, but it does not deal with the rounding error in the bisection routine (BISECT). Higher-precision arithmetic is the most convenient way to deal with this rounding error, and thereby test the calculations of WAPR.

So that WAPR can be easily tested on various platforms, a suite of test values of the $W$-function, precise to 40 digits, was calculated using the arbitrary-precision arithmetic of MAPLE V [Char et al. 1991]. These test values for both $W_p$ and $W_m$ were computed for $x$ near $-\exp(-1)$ and 0 (i.e., where the bisection method results are not wholly reliable), as well as a few other values. Note that the bisection method results were found to be reliable elsewhere.

Typical double-precision results for the suite of test values of the $W$-function are included in Table I. All the results presented here were obtained using a SUN SPARC II machine. Note that the corresponding bisection method results (not shown) are less accurate than those calculated using WAPR for $x$ close to $-\exp(-1)$, e.g., $-\exp(-1) < x < -\exp(-1) + 10^{-4}$. The figures in the final column in the table are the number of significant digits (rounded to the nearest integer) of agreement between the results $W$-function values in the second and third columns. The significant digits were calculated using the formula:

$$\text{Significant Digits} = \text{Int}\left[-\log_{10}\left|\frac{W_{\text{approx}} - W}{W}\right| + \frac{1}{2}\right],$$

where Int( ) refers to the integer part of the argument. Typical results for $W_p$ and $W_m$ are presented in Tables II and III, respectively. In all cases satisfactory results are obtained.

## 3. CALLING SEQUENCE

The WAPR function is accessed using the call:

```
WAPR(X,NB,NERROR,L,M)
```

where the arguments are defined as

| | |
|---|---|
| X | argument of $W(x)$; |
| NB | branch of the $W$-function needed (0 $\leftrightarrow$ upper branch; otherwise lower branch); |
| NERROR | output error flag (0 $\leftrightarrow$ no error; 1 $\leftrightarrow$ X is out of range); |
| L | determines how WAPR is to treat the argument (L = 1 $\leftrightarrow$ X is the offset from $-\exp(-1)$; otherwise X is the actual X, so compute $W(x)$; |
| M | warning message flag (M = 1 $\leftrightarrow$ print warnings; otherwise no warnings). |

## 4. EFFICIENCY

WAPR requires at most two passes of the iterative improvement scheme to achieve results precise to the mantissa length of the host machine. On most platforms only a single pass is needed. The rate of convergence of a standard root-finding scheme, such as Newton's method, applied to (1) is much slower than the convergence rate of Fritsch et al.'s [1973] scheme. Additionally, Newton's method will not always converge, in which case other direct approximations (such as those contained in WAPR) must be employed. The bisection method, although slowly convergent, will always find the correct solution (subject to roundoff). A comparison of the CPU time used to calculate $W$-function values calculated from the BISECT routine and to use WAPR showed that the latter is—not surprisingly—faster by an order of magnitude.

Table I. Test Run Comparing WAPR Calculations with the Exact Results

| $W_p$ Results for $x$ Near -exp (-1) | | | |
|---|---|---|---|
| Offset $x$ | $W(x)$ (WAPR) | $W(x)$ (EXACT) | Digits Correct |
| 1.00000000D-40 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 2.00000000D-40 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 3.00000000D-40 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 4.00000000D-40 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 5.00000000D-40 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 6.00000000D-40 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 7.00000000D-40 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 8.00000000D-40 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 9.00000000D-40 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 1.00000000D-39 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 1.00000000D-30 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 2.00000000D-30 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 3.00000000D-30 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 4.00000000D-30 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 5.00000000D-30 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 6.00000000D-30 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 7.00000000D-30 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 8.00000000D-30 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 9.00000000D-30 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 1.00000000D-29 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 1.00000000D-20 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 2.00000000D-20 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 3.00000000D-20 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 4.00000000D-20 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 5.00000000D-20 | -9.99999999D - 01 | -9.99999999D - 01 | 16 |
| 6.00000000D-20 | -9.99999999D - 01 | -9.99999999D - 01 | 16 |
| 7.00000000D-20 | -9.99999999D - 01 | -9.99999999D - 01 | 16 |
| 8.00000000D-20 | -9.99999999D - 01 | -9.99999999D - 01 | 16 |
| 9.00000000D-20 | -9.99999999D - 01 | -9.99999999D - 01 | 16 |
| 1.00000000D-19 | -9.99999999D - 01 | -9.99999999D - 01 | 16 |
| 1.00000000D-10 | -9.99976684D - 01 | -9.99976684D - 01 | 16 |
| 2.00000000D-10 | -9.99967026D - 01 | -9.99967026D - 01 | 16 |
| 3.00000000D-10 | -9.99959615D - 01 | -9.99959615D - 01 | 16 |
| 4.00000000D-10 | -9.99953368D - 01 | -9.99953368D - 01 | 16 |
| 5.00000000D-10 | -9.99947864D - 01 | -9.99947864D - 01 | 16 |
| 6.00000000D-10 | -9.99942888D - 01 | -9.99942888D - 01 | 16 |
| 7.00000000D-10 | -9.99938312D - 01 | -9.99938312D - 01 | 16 |
| 8.00000000D-10 | -9.99934053D - 01 | -9.99934053D - 01 | 16 |
| 9.00000000D-10 | -9.99930052D - 01 | -9.99930052D - 01 | 16 |
| 1.00000000D-09 | -9.99926269D - 01 | -9.99926269D - 01 | 16 |
| 1.00000000D-05 | -9.92644755D - 01 | -9.92644755D - 01 | 16 |
| 2.00000000D-05 | -9.89608643D - 01 | -9.89608643D - 01 | 16 |
| 3.00000000D-05 | -9.87283109D - 01 | -9.87283109D - 01 | 16 |
| 4.00000000D-05 | -9.85325390D - 01 | -9.85325390D - 01 | 16 |
| 5.00000000D-05 | -9.83602718D - 01 | -9.83602718D - 01 | 16 |
| 6.00000000D-05 | -9.82047003D - 01 | -9.82047003D - 01 | 16 |
| 7.00000000D-05 | -9.80617797D - 01 | -9.80617797D - 01 | 16 |
| 8.00000000D-05 | -9.79288746D - 01 | -9.79288746D - 01 | 16 |
| 9.00000000D-05 | -9.78041545D - 01 | -9.78041545D - 01 | 16 |
| 1.00000000D-04 | -9.76862866D - 01 | -9.76862866D - 01 | 16 |
| 2.00000000D-04 | -9.67382627D - 01 | -9.67382627D - 01 | 15 |

Table I—*Continued*

| | | | |
|---|---|---|---|
| 3.00000000D-04 | -9.60148542D - 01 | -9.60148542D - 01 | 15 |
| 4.00000000D-04 | -9.54076869D - 01 | -9.54076869D - 01 | 15 |
| 5.00000000D-04 | -9.48747869D - 01 | -9.48747869D - 01 | 15 |
| 6.00000000D-04 | -9.43946291D - 01 | -9.43946291D - 01 | 15 |
| 7.00000000D-04 | -9.39544278D - 01 | -9.39544278D - 01 | 15 |
| 8.00000000D-04 | -9.35458531D - 01 | -9.35458531D - 01 | 15 |
| 9.00000000D-04 | -9.31631195D - 01 | -9.31631195D - 01 | 15 |
| 1.00000000D-03 | -9.28020150D - 01 | -9.28020150D - 01 | 15 |
| 2.00000000D-03 | -8.99185766D - 01 | -8.99185766D - 01 | 15 |
| 3.00000000D-03 | -8.77428717D - 01 | -8.77428717D - 01 | 15 |
| 4.00000000D-03 | -8.59327504D - 01 | -8.59327504D - 01 | 16 |
| 5.00000000D-03 | -8.43558002D - 01 | -8.43558002D - 01 | 15 |
| 6.00000000D-03 | -8.29441686D - 01 | -8.29441686D - 01 | 16 |
| 7.00000000D-03 | -8.16575805D - 01 | -8.16575805D - 01 | 16 |
| 8.00000000D-03 | -8.04698156D - 01 | -8.04698156D - 01 | 15 |
| 9.00000000D-03 | -7.93626754D - 01 | -7.93626754D - 01 | 16 |
| 1.00000000D-02 | -7.83229199D - 01 | -7.83229199D - 01 | 16 |

**$W_p$ Results for $x$ Near 0**

| $x$ | $W(x)$ (WAPR) | $W(x)$ (EXACT) | Digits Correct |
|---|---|---|---|
| 1.00000000D-09 | 1.00000000D-09 | 1.00000000D-09 | 16 |
| 2.00000000D-09 | 2.00000000D-09 | 2.00000000D-09 | 16 |
| 3.00000000D-09 | 2.99999999D-09 | 2.99999999D-09 | 16 |
| 4.00000000D-09 | 3.99999998D-09 | 3.99999998D-09 | 16 |
| 5.00000000D-09 | 4.99999998D-09 | 4.99999998D-09 | 16 |
| 6.00000000D-09 | 5.99999996D-09 | 5.99999996D-09 | 16 |
| 7.00000000D-09 | 6.99999995D-09 | 6.99999995D-09 | 16 |
| 8.00000000D.09 | 7.99999994D-09 | 7.99999994D-09 | 16 |
| 9.00000000D-09 | 8.99999992D-09 | 8.99999992D-09 | 16 |
| 1.00000000D-08 | 9.99999990D-09 | 9.99999990D-09 | 16 |
| 1.00000000D-02 | 9.90147384D-03 | 9.90147384D-03 | 16 |
| 2.00000000D-02 | 1.96115893D-02 | 1.96115893D-02 | 16 |
| 3.00000000D-02 | 2.91384592D-02 | 2.91384592D-02 | 16 |
| 4.00000000D-02 | 3.84896659D-02 | 3.84896659D-02 | 16 |
| 5.00000000D-02 | 4.76723086D-02 | 4.76723086D-02 | 16 |
| 6.00000000D-02 | 5.66930438D-02 | 5.66930438D-02 | 16 |
| 7.00000000D-02 | 6.55581227D-02 | 6.55581227D-02 | 16 |
| 8.00000000D-02 | 7.42734246D-02 | 7.42734246D-02 | 16 |
| 9.00000000D-02 | 8.28444857D-02 | 8.28444857D-02 | 16 |
| 1.00000000D-01 | 9.12765272D-02 | 9.12765272D-02 | 16 |
| -1.00000000D-09 | -1.00000000D-09 | -1.00000000D-09 | 16 |
| -2.00000000D-09 | -2.00000000D-09 | -2.00000000D-09 | 16 |
| -3.00000000D-09 | -3.00000001D-09 | -3.00000001D-09 | 16 |
| -4.00000000D-09 | -4.00000002D-09 | -4.00000002D-09 | 16 |
| -5.00000000D-09 | -5.00000002D-09 | -5.00000003D-09 | 16 |
| -6.00000000D-09 | -6.00000004D-09 | -6.00000004D-09 | 16 |
| -7.00000000D-09 | -7.00000005D-09 | -7.00000005D-09 | 16 |
| -8.00000000D-09 | -8.00000006D-09 | -8.00000006D-09 | 16 |
| -9.00000000D-09 | -9.00000008D-09 | -9.00000008D-09 | 16 |
| -1.00000000D-08 | -1.00000001D-08 | -1.00000001D-08 | 16 |
| -1.00000000D-02 | -1.01015272D-02 | -1.01015272D-02 | 16 |
| -2.00000000D-02 | -2.04124441D-02 | -2.04124441D-02 | 16 |
| -3.00000000D-02 | -3.09427950D-02 | -3.09427950D-02 | 16 |
| -4.00000000D-02 | -4.17034084D-02 | -4.17034084D-02 | 16 |
| -5.00000000D-02 | -5.27059836D-02 | -5.27059836D-02 | 16 |

Table I—*Continued*

| | | | |
|---|---|---|---|
| -6.00000000D-02 | -6.39631894D-02 | -6.39631894D-02 | 16 |
| -7.00000000D-02 | -7.54887789D-02 | -7.54887789D-02 | 16 |
| -8.00000000D-02 | -8.72977209D-02 | -8.72977209D-02 | 16 |
| -9.00000000D-02 | -9.94063528D-02 | -9.94063528D-02 | 16 |
| -1.00000000D-01 | -1.11832559D-01 | -1.11832559D-01 | 16 |

**Other $W_p$ Results**

| $x$ | $W(x)$ (WAPR) | $W(x)$ (EXACT) | Digits Correct |
|---|---|---|---|
| 1.00000000D+01 | 1.74552800D+00 | 1.74552800D+00 | 16 |
| 1.00000000D+02 | 3.38563014D+00 | 3.38563014D+00 | 16 |
| 1.00000000D+03 | 5.24960285D+00 | 5.24960285D+00 | 16 |
| 1.00000000D+04 | 7.23184604D+00 | 7.23184604D+00 | 16 |
| 1.00000000D+05 | 9.28457143D+00 | 9.28457143D+00 | 16 |
| 1.00000000D+06 | 1.13833581D+01 | 1.13833581D+01 | 16 |
| 1.00000000D+07 | 1.35143440D+01 | 1.35143440D+01 | 16 |
| 1.00000000D+08 | 1.56689967D+01 | 1.56689967D+01 | 16 |
| 1.00000000D+09 | 1.78417260D+01 | 1.78417260D+01 | 16 |
| 1.00000000D+10 | 2.00286854D+01 | 2.00286854D+01 | 16 |

**$W_m$ Results for $x$ Near -exp (-1)**

| Offset $x$ | $W(x)$ (WAPR) | $W(x)$ (EXACT) | Digits Correct |
|---|---|---|---|
| 1.00000000D-40 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 2.00000000D-40 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 3.00000000D-40 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 4.00000000D-40 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 5.00000000D-40 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 6.00000000D-40 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 7.00000000D-40 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 8.00000000D-40 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 9.00000000D-40 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 1.00000000D-39 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 1.00000000D-30 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 2.00000000D-30 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 3.00000000D-30 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 4.00000000D-30 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 5.00000000D-30 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 6.00000000D-30 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 7.00000000D-30 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 8.00000000D-30 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 9.00000000D-30 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 1.00000000D-29 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 1.00000000D-20 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 2.00000000D-20 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 3.00000000D-20 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 4.00000000D-20 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 5.00000000D-20 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 6.00000000D-20 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 7.00000000D-20 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 8.00000000D-20 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 9.00000000D-20 | -1.00000000D+00 | -1.00000000D+00 | 16 |

Table I—*Continued*

| | | | |
|---|---|---|---|
| 1.00000000D-19 | -1.00000000D+00 | -1.00000000D+00 | 16 |
| 1.00000000D-10 | -1.00002332D+00 | -1.00002332D+00 | 16 |
| 2.00000000D-10 | -1.00003297D+00 | -1.00003297D+00 | 16 |
| 3.00000000D-10 | -1.00004039D+00 | -1.00004039D+00 | 16 |
| 4.00000000D-10 | -1.00004663D+00 | -1.00004663D+00 | 16 |
| 5.00000000D-10 | -1.00005214D+00 | -1.00005214D+00 | 16 |
| 6.00000000D-10 | -1.00005711D+00 | -1.00005711D+00 | 16 |
| 7.00000000D-10 | -1.00006169D+00 | -1.00006169D+00 | 16 |
| 8.00000000D-10 | -1.00006595D+00 | -1.00006595D+00 | 16 |
| 9.00000000D-10 | -1.00006995D+00 | -1.00006995D+00 | 16 |
| 1.00000000D-09 | -1.00007373D+00 | -1.00007373D+00 | 16 |
| 1.00000000D-05 | -1.00739149D+00 | -1.00739149D+00 | 16 |
| 2.00000000D-05 | -1.01046385D+00 | -1.01046385D+00 | 16 |
| 3.00000000D-05 | -1.01282563D+00 | -1.01282563D+00 | 16 |
| 4.00000000D-05 | -1.01481959D+00 | -1.01481959D+00 | 16 |
| 5.00000000D-05 | -1.01657851D+00 | -1.01657851D+00 | 16 |
| 6.00000000D-05 | -1.01817048D+00 | -1.01817048D+00 | 16 |
| 7.00000000D-05 | -1.01963593D+00 | -1.01963593D+00 | 16 |
| 8.00000000D-05 | -1.02100123D+00 | -1.02100123D+00 | 16 |
| 9.00000000D-05 | -1.02228469D+00 | -1.02228469D+00 | 16 |
| 1.00000000D-04 | -1.02349962D+00 | -1.02349962D+00 | 16 |
| 2.00000000D-04 | -1.03334244D+00 | -1.03334244D+00 | 15 |
| 3.00000000D-04 | -1.04093919D+00 | -1.04093919D+00 | 15 |
| 4.00000000D-04 | -1.04737363D+00 | -1.04737363D+00 | 15 |
| 5.00000000D-04 | -1.05306550D+00 | -1.05306550D+00 | 15 |
| 6.00000000D-04 | -1.05823003D+00 | -1.05823003D+00 | 15 |
| 7.00000000D-04 | -1.06299509D+00 | -1.06299509D+00 | 15 |
| 8.00000000D-04 | -1.06744399D+00 | -1.06744399D+00 | 16 |
| 9.00000000D-04 | -1.07163456D+00 | -1.07163456D+00 | 15 |
| 1.00000000D-03 | -1.07560894D+00 | -1.07560894D+00 | 15 |
| 2.00000000D-03 | -1.10808188D+00 | -1.10808188D+00 | 15 |
| 3.00000000D-03 | -1.13348700D+00 | -1.13348700D+00 | 15 |
| 4.00000000D-03 | -1.15524585D+00 | -1.15524585D+00 | 16 |
| 5.00000000D-03 | -1.17468261D+00 | -1.17468261D+00 | 15 |
| 6.00000000D-03 | -1.19247585D+00 | -1.19247585D+00 | 15 |
| 7.00000000D-03 | -1.20902838D+00 | -1.20902838D+00 | 16 |
| 8.00000000D-03 | -1.22460245D+00 | -1.22460245D+00 | 16 |
| 9.00000000D-03 | -1.23938010D+00 | -1.23938010D+00 | 15 |
| 1.00000000D-02 | -1.25349379D+00 | -1.25349379D+00 | 15 |

$W_m$ Results for $x$ Near 0

| $x$ | $W(x)$ (WAPR) | $W(x)$ (EXACT) | Digits Correct |
|---|---|---|---|
| -1.00000000D-40 | -9.66747560D+01 | -9.66747560D+01 | 16 |
| -2.00000000D-40 | -9.59743374D+01 | -9.59743374D+01 | 16 |
| -3.00000000D-40 | -9.55645938D+01 | -9.55645938D+01 | 16 |
| -4.00000000D-40 | -9.52738649D+01 | -9.52738649D+01 | 16 |
| -5.00000000D-40 | -9.50483515D+01 | -9.50483515D+01 | 16 |
| -6.00000000D-40 | -9.48640895D+01 | -9.48640895D+01 | 16 |
| -7.00000000D-40 | -9.47082952D+01 | -9.47082952D+01 | 16 |
| -8.00000000D-40 | -9.45733378D+01 | -9.45733378D+01 | 16 |
| -9.00000000D-40 | -9.44542952D+01 | -9.44542952D+01 | 16 |
| -1.00000000D-39 | -9.43478067D+01 | -9.43478067D+01 | 16 |
| -1.00000000D-30 | -7.33731103D+01 | -7.33731103D+01 | 16 |
| -2.00000000D-30 | -7.26703389D+01 | -7.26703389D+01 | 16 |
| -3.00000000D-30 | -7.22592002D+01 | -7.22592002D+01 | 16 |

Table I—*Continued*

| | | | |
|---|---|---|---|
| **-4.00000000D-30** | -7.19674727D+01 | -7.19674727D+01 | **16** |
| **-5.00000000D-30** | -7.17411798D+01 | -7.17411798D+01 | **16** |
| **-6.00000000D-30** | -7.15562776D+01 | -7.15562776D+01 | **16** |
| **-7.00000000D-30** | -7.13999397D+01 | -7.13999397D+01 | **16** |
| **-8.00000000D-30** | -7.12645097D+01 | -7.12645097D+01 | **16** |
| **-9.00000000D-30** | -7.11450489D+01 | -7.11450489D+01 | **16** |
| **-1.00000000D-29** | -7.10381852D+01 | -7.10381852D+01 | **16** |
| **-1.00000000D-20** | -4.99629843D+01 | -4.99629843D+01 | **16** |
| **-2.00000000D-20** | -4.92555773D+01 | -4.92555773D+01 | **16** |
| **-3.00000000D-20** | -4.88416735D+01 | -4.88416735D+01 | **16** |
| **-4.00000000D-20** | -4.85479597D+01 | -4.85479597D+01 | **16** |
| **-5.00000000D-20** | -4.83201118D+01 | -4.83201118D+01 | **16** |
| **-6.00000000D-20** | -4.81339297D+01 | -4.81339297D+01 | **16** |
| **-7.00000000D-20** | -4.79765031D+01 | -4.79765031D+01 | **16** |
| **-8.00000000D-20** | -4.78401250D+01 | -4.78401250D+01 | **16** |
| **-9.00000000D-20** | -4.77198242D+01 | -4.77198242D+01 | **16** |
| **-1.00000000D-19** | -4.76122059D+01 | -4.76122059D+01 | **16** |
| **-1.00000000D-10** | -2.62952388D+01 | -2.62952388D+01 | **16** |
| **-2.00000000D-10** | -2.55742914D+01 | -2.55742914D+01 | **16** |
| **-3.00000000D-10** | -2.51521833D+01 | -2.51521833D+01 | **16** |
| **-4.00000000D-10** | -2.48525155D+01 | -2.48525155D+01 | **16** |
| **-5.00000000D-10** | -2.46199710D+01 | -2.46199710D+01 | **16** |
| **-6.00000000D-10** | -2.44298992D+01 | -2.44298992D+01 | **16** |
| **-7.00000000D-10** | -2.42691466D+01 | -2.42691466D+01 | **16** |
| **-8.00000000D-10** | -2.41298594D+01 | -2.41298594D+01 | **16** |
| **-9.00000000D-10** | -2.40069706D+01 | -2.40069706D+01 | **16** |
| **-1.00000000D-09** | -2.38970196D+01 | -2.38970196D+01 | **16** |
| **-1.00000000D-05** | -1.14163600D+01 | -1.14163600D+01 | **16** |
| **-2.00000000D-05** | -1.34162445D+01 | -1.34162445D+01 | **16** |
| **-3.00000000D-05** | -1.29775328D+01 | -1.29775328D+01 | **16** |
| **-4.00000000D-05** | -1.26655140D+01 | -1.26655140D+01 | **16** |
| **-5.00000000D-05** | -1.24230404D+01 | -1.24230404D+01 | **16** |
| **-6.00000000D-05** | -1.22246178D+01 | -1.22246178D+01 | **16** |
| **-7.00000000D-05** | -1.20566300D+01 | -1.20566300D+01 | **16** |
| **-8.00000000D-05** | -1.19109413D+01 | -1.19109413D+01 | **16** |
| **-9.00000000D-05** | -1.17822992D+01 | -1.17822992D+01 | **16** |
| **-1.00000000D-04** | -1.16671145D+01 | -1.16671145D+01 | **16** |
| **-2.00000000D-04** | -1.09065574D+01 | -1.09065574D+01 | **16** |
| **-3.00000000D-04** | -1.04592111D+01 | -1.04592111D+01 | **16** |
| **-4.00000000D-04** | -1.01405924D+01 | -1.01405924D+01 | **16** |
| **-5.00000000D-04** | -9.89269952D+00 | -9.89269952D+00 | **16** |
| **-6.00000000D-04** | -9.68963797D+00 | -9.68963797D+00 | **16** |
| **-7.00000000D-04** | -9.51756976D+00 | -9.51756976D+00 | **16** |
| **-8.00000000D-04** | -9.36822217D+00 | -9.36822217D+00 | **16** |
| **-9.00000000D-04** | -9.23625197D+00 | -9.23625197D+00 | **16** |
| **-1.00000000D-03** | -9.11800647D+00 | -9.11800647D+00 | **16** |
| **-2.00000000D-03** | -8.33508138D+00 | -8.33508138D+00 | **16** |
| **-3.00000000D-03** | -7.87252138D+00 | -7.87252138D+00 | **16** |
| **-4.00000000D-03** | -7.54194042D+00 | -7.54194042D+00 | **16** |
| **-5.00000000D-03** | -7.28399714D+00 | -7.28399714D+00 | **16** |
| **-6.00000000D-03** | -7.07216205D+00 | -7.07216205D+00 | **16** |
| **-7.00000000D-03** | -6.89224149D+00 | -6.89224149D+00 | **16** |
| **-8.00000000D-03** | -6.73574166D+00 | -6.73574166D+00 | **16** |
| **-9.00000000D-03** | -6.59717173D+00 | -6.59717173D+00 | **16** |
| **-1.00000000D-02** | -6.47277512D+00 | -6.47277512D+00 | **16** |

**Compare WAPR with the given exact *W*-function results (Y or N)? Yes.**

Table II. Example Output of Algorithm for $W_p$

**Results for $W_p(x)$**

| $x$ | $W(x)$ (WAPR) | $W(x)$ (BISECT) | Digits Correct |
|---|---|---|---|
| 0.00000000D+00 | 0.00000000D+00 | 0.00000000D+00 | 16 |
| 5.00000000D+18 | 3.93826444D+01 | 3.93826444D+01 | 16 |
| 1.00000000D+19 | 4.00587692D+01 | 4.00587692D+01 | 16 |
| 1.50000000D+19 | 4.04544063D+01 | 4.04544063D+01 | 16 |
| 2.00000000D+19 | 4.07351721D+01 | 4.07351721D+01 | 16 |
| 2.50000000D+19 | 4.09529829D+01 | 4.09529828D+01 | 16 |
| 3.00000000D+19 | 4.11309677D+01 | 4.11309677D+01 | 16 |
| 3.50000000D+19 | 4.12814661D+01 | 4.12814661D+01 | 16 |
| 4.00000000D+19 | 4.14118442D+01 | 4.14118442D+01 | 16 |
| 4.50000000D+19 | 4.15268539D+01 | 4.15268539D+01 | 16 |
| 5.00000000D+19 | 4.16297399D+01 | 4.16297399D+01 | 16 |
| 5.50000000D+19 | 4.17228167D+01 | 4.17228167D+01 | 16 |
| 6.00000000D+19 | 4.18077935D+01 | 4.18077935D+01 | 16 |
| 6.50000000D+19 | 4.18859681D+01 | 4.18859681D+01 | 16 |
| 7.00000000D+19 | 4.19583495D+01 | 4.19583495D+01 | 16 |
| 7.50000000D+19 | 4.20257376D+01 | 4.20257376D+01 | 16 |
| 8.00000000D+19 | 4.20887772D+01 | 4.20887772D+01 | 16 |
| 8.50000000D+19 | 4.21479958D+01 | 4.21479958D+01 | 16 |
| 9.00000000D+19 | 4.22038304D+01 | 4.22038304D+01 | 16 |
| 9.50000000D+19 | 4.22566469D+01 | 4.22566469D+01 | 16 |
| 1.00000000D+20 | 4.23067551D+01 | 4.23067551D+01 | 16 |

**Compare WAPR with the given exact *W*-function results (Y or N)? No.**
**Is $x$ to be specified as an offest from -exp (-1) (Y or N)? No.**
**Input the minimum and maximum $x$: 0.D0, 1.D20.**
**How many values to be calculated in this range? 20.**
**$W_p$ has been selected (maximum $x$ is greater than 0).**

Table III. Example Output of Algorithm for Both $W_p$ and $W_m$

**Results for $W_p(x)$**

| $x$ | $W(x)$ (WAPR) | $W(x)$ (BISECT) | Digits Correct |
|---|---|---|---|
| -3.50000000D-01 | -7.16638816D-01 | -7.16638816D-01 | 16 |
| -3.15001000D-01 | -5.41184568D-01 | -5.41184568D-01 | 16 |
| -2.80002000D-01 | -4.30764280D-01 | -4.30764280D-01 | 16 |
| -2.45003000D-01 | -3.46440430D-01 | -3.46440430D-01 | 16 |
| -2.10004000D-01 | -2.77041793D-01 | -2.77041793D-01 | 16 |
| -1.75005000D-01 | -2.17532161D-01 | -2.17532161D-01 | 16 |
| -1.40006000D-01 | -1.65146267D-01 | -1.65146267D-01 | 16 |
| -1.05007000D-01 | -1.18179749D-01 | -1.18179749D-01 | 16 |
| -7.00080000D-02 | -7.54981107D-02 | -7.54981107D-02 | 16 |
| -3.50090000D-02 | -3.63032934D-02 | -3.63032934D-02 | 16 |
| -1.00000000D-05 | -1.00001000D-05 | -1.00001000D-05 | 16 |

**Results for $W_m(x)$**

| $x$ | $W(x)$ (WAPR) | $W(x)$ (BISECT) | Digits Correct |
|---|---|---|---|
| -3.50000000D-01 | -1.34971725D+00 | -1.34971725D+00 | 16 |
| -3.15001000D-01 | -1.66501149D+00 | -1.66501149D+00 | 16 |
| -2.80002000D-01 | -1.93099282D+00 | -1.93099282D+00 | 16 |
| -2.45003000D-01 | -2.19071095D+00 | -2.19071095D+00 | 16 |
| -2.10004000D-01 | -2.46133093D+00 | -2.46133093D+00 | 16 |
| -1.75005000D-01 | -2.75713160D+00 | -2.75713160D+00 | 16 |
| -1.40006000D-01 | -3.09626728D+00 | -3.09626728D+00 | 16 |
| -1.05007000D-01 | -3.50908302D+00 | -3.50908302D+00 | 16 |
| -7.00080000D-02 | -4.06043615D+00 | -4.06043615D+00 | 16 |
| -3.50090000D-02 | -4.95192688D+00 | -4.95192688D+00 | 16 |
| -1.00000000D-05 | -1.41636008D+01 | -1.41636008D+01 | 16 |

**Compare WAPR with the given exact *W*-function results (Y or N)? No.**
**Is *x* to be specified as an offset from -exp (-1) (Y or N)? No.**
**Input the minimum and maximum *x*: -0.35D0, -1.D-5.**
**How many values to be calculated in this range? 10.**
**Both branches of the *W*-function will be checked.**

## REFERENCES


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