A note on the error analysis of time compression approximations

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Abstract. The accuracy of the time compression analysis (TCA) is analyzed by comparison with a numerical solution. Both the standard TCA and a new modified TCA are considered for a power law diffusivity and constant surface flux. As expected, the error of the approximations decreases with increasing power, and the error of the modified TCA is about half the error of the standard TCA. In a second part, the errors of the two TCAs are measured using a simple analytical solution instead of a numerical solution. It is shown that the conclusions remain the same for the analytical and numerical solutions. The advantage of using the analytical solution is to obtain simple analytical expressions, showing the influence of parameters. This is done to estimate the maximum error of both TCAs. A practical estimate of the errors can be obtained from equations which only require knowledge of the soil water diffusivity. It appears that for real soils the errors of the TCA are always <1% and thus are a very reliable tool for practical problems. Although not studied systematically, it also appears that gravity effects reduce the errors of the TCA so that the error obtained in the absence of gravity provides a conservative estimate when gravity is present.

1. Introduction

An earlier paper [*Liu et al*., 1998] showed that the time compression approximation (TCA) predicts the cumulative infiltration with remarkable accuracy for linear soils. As pointed out by *Salvucci and Entekkabi* [1994], knowledge of the cumulative infiltration is most important in practice, whereas others, like infiltration rates after ponding, are less important.

In the following we first extend the analysis of *Liu et al*. [1998], which applies to unrealistic soils, to one with a realistic soil water diffusivity obeying a power law. For the linear soil, exact analytical solutions are available. For a power law diffusivity it is necessary to use numerical tools and/or analytical approximations.

As usually done, when TCA is used, the possibly complex dependence of the rainfall rate on time is replaced, until ponding, by its average. Hence, if t_p and I_p are the ponding time and the cumulative infiltration at ponding, then the average rainfall rate $\bar{q}$ is

$$\bar{q} = I_p/t_p. \tag{1}$$

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This points out that the actual rainfall rate could be very different from $\bar{q}$ at any time before ponding, so that infiltration rates are theoretically, as well as in practice, less meaningful than their cumulative values. However, the additional errors introduced by the time variation of the rainfall rate and associated problems, like redistribution and hysteresis, are ignored, even though they can be crucial in practice [*Ibrahim and Brutsaert*, 1968; *Reeves and Miller*, 1975; *Smith et al*., 1993; *Corradini et al*., 1994].

The modified TCA then assumes that I_p is known rather than the flux at and before ponding, but the cumulative infiltration is measured up to ponding time (the flux, which could then be obtained by differentiation, is known with less accuracy).

We may note that in the field the cumulative rainfall is typically measured. Thus, to obtain I_p, one must have an estimate of the ponding time, which may be somewhat crude given the large spatial variability of soil properties normally encountered in the field. However, even for the standard TCA, $\bar{q}$ requires some knowledge of t_p (see (4)). It is quite interesting that in his application of TCA to the study of evaporation, *Salvucci* [1997] used the equivalent of the modified TCA for evaporation, i.e., an average evaporation rate till the actual start of stage 2.

2. Time Compression Analysis

We consider the case of infiltration into a soil initially at uniform water content with constant (average) rainfall $\bar{q}$ until ponding. (Again, the additional problems associated with the time dependence of rainfall are ignored here.) We take a realistic representation of the soil water diffusivity

$$D = D_s\theta^\alpha, \tag{2}$$

where θ is the reduced water content (water content measured relative to its initial value divided by its saturated value). The

Table 1a. Estimates of $I(t)$ for $\alpha = 0$

t	I_{exact}	$I[9]$	$I[12]$	$I[26]$
1.000000	9.547269E-01*	9.316409E-01	9.434457E-01	9.489191E-01
2.000000	1.484097	1.463282	1.470826	1.488126
4.000000	2.180562	2.165104	2.170209	2.204621
6.000000	2.702410	2.689638	2.693750	2.741088
10.000000	3.520974	3.510999	3.514149	3.581651
15.000000	4.331751	4.323576	4.326135	4.413441
20.000000	5.013040	5.005947	5.008158	5.111995
25.000000	5.612214	5.605864	5.607838	5.726146
30.000000	6.153313	6.147512	6.149312	6.280643
40.000000	7.113073	7.108045	7.109602	7.263945
50.000000	7.957908	7.953408	7.954800	8.129324
60.000000	8.721281	8.717173	8.718442	8.911154
70.000000	9.423014	9.419209	9.420383	9.629782
80.000000	10.075990	10.072430	10.073530	10.298430
90.000000	10.689150	10.685800	10.686830	10.926270
100.000000	11.269000	11.265820	11.266800	11.519970

I is dimensionless cumulative infiltration, and t is a dimensionless time.
*Read 9.547269E-01 as 9.547269×10^{-1}.

constant α varies between 4 (sand) and 8 (clay) [*Brooks and Corey*, 1964]. We shall consider the representative cases, $\alpha = 1$, 5, 10. The results for the case $\alpha = 0$ are by *Liu et al*. [1998]. The approximate analytical results will be based on a method [*Parlange et al*., 1992, 1998] which is more accurate for larger α, i.e., values normally encountered in the field. Thus the cases $\alpha = 0$ and 1, although not realistic, are important to check the analytical results when their accuracy should be the worst.

We solve the equation

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left[\theta^{\alpha}\frac{\partial \theta}{\partial x}\right] \qquad (3)$$

with the initial condition

$$\theta = 0, \qquad t < 0, \qquad x \geq 0 \qquad (4)$$

and the boundary conditions

$$-\theta^{\alpha}\frac{\partial \theta}{\partial x} = 1, \qquad 0 \leq t \leq t_p, \qquad x = 0, \qquad (5)$$

or

$$\theta = 1, \qquad t \geq t_p, \qquad x = 0. \qquad (6)$$

Here x is a dimensionless distance (distance multiplied by $\bar{q}/D_s$) and t is a dimensionless time (time multiplied by $\bar{q}^2/D_s$), which is equivalent to taking $\bar{q}$ and D_s equal to one. By first ignoring gravity, α is the only parameter remaining in the problem. We are looking at the simplest application possible to evaluate TCA (both standard and modified), i.e., as done by *Sivapalan and Milly* [1989] for the standard method but with $\alpha \neq 0$.

The problem is first solved numerically to obtain $I(t)$ by linearizing the right-hand side of (3) using the transform $u = \int D\, d\theta$ and then applying the method of lines [*Schiesser*, 1991] with a finite volume discretization on a fine variable spatial grid and a stiff ordinary differential equation integrator. In particular, for $\alpha = 0$, numerical and exact analytical results [*Liu et al*., 1998] are identical for the number of significant figures given in Table 1a. Tables 1b, 1c, and 1d correspond to $\alpha = 1$, 5, and 10, respectively.

The standard TCA gives, for the dimensionless cumulative infiltration,

$$I = \int_0^{\infty} \theta\, dx, \qquad (7)$$

Table 1b. Estimates of $I(t)$ for $\alpha = 1$

t	I_{exact}	$I[9]$	$I[12]$	$I[26]$
5.000000E-01*	4.954358E-01	4.885782E-01	4.929449E-01	4.935538E-01
6.000000E-01	5.712031E-01	5.634329E-01	5.672235E-01	5.684866E-01
7.000000E-01	6.370115E-01	6.294478E-01	6.328431E-01	6.346942E-01
1.000000	8.017018E-01	7.952661E-01	7.979562E-01	8.012950E-01
1.200000	8.946338E-01	8.887879E-01	8.911958E-01	8.953707E-01
1.500000	1.018201	1.013009	1.015122	1.020400
2.000000	1.196076	1.191607	1.193404	1.200250
2.500000	1.350716	1.346724	1.348315	1.356508
2.999999	1.489384	1.485734	1.487175	1.496565
3.999998	1.733759	1.730573	1.731811	1.743287
4.999998	1.947709	1.944829	1.945931	1.959215
5.999997	2.140379	2.137718	2.138721	2.153620
6.999996	2.317083	2.314588	2.315514	2.331887
7.999995	2.481234	2.478870	2.479734	2.497471
8.999998	2.635180	2.632922	2.633735	2.652746
10.000000	2.780616	2.778445	2.779217	2.799426

*Read 5.000000E-01 as 5.000000×10^{-1}.

Table 1c. Estimates of $I(t)$ for $\alpha = 5$

t	I_{exact}	$I[9]$	$I[12]$	$I[26]$
1.700000E-01*	1.697305E-01	1.691987E-01	1.695651E-01	1.695769E-01
1.800000E-01	1.786473E-01	1.780406E-01	1.783889E-01	1.784154E-01
1.900000E-01	1.870754E-01	1.864638E-01	1.867963E-01	1.868366E-01
2.000000E-01	1.951207E-01	1.945225E-01	1.948413E-01	1.948945E-01
2.200000E-01	2.102751E-01	2.097130E-01	2.100088E-01	2.100855E-01
2.500000E-01	2.311441E-01	2.306303E-01	2.308993E-01	2.310074E-01
3.000000E-01	2.622595E-01	2.618046E-01	2.620416E-01	2.621945E-01
3.499999E-01	2.900558E-01	2.896429E-01	2.898571E-01	2.900484E-01
3.999999E-01	3.154120E-01	3.150307E-01	3.152276E-01	3.154530E-01
4.499998E-01	3.388762E-01	3.385198E-01	3.387031E-01	3.389592E-01
4.999998E-01	3.608176E-01	3.604816E-01	3.606537E-01	3.609381E-01
5.999997E-01	4.011160E-01	4.008112E-01	4.009660E-01	4.013014E-01
6.999996E-01	4.377199E-01	4.374383E-01	4.375802E-01	4.379609E-01
7.999995E-01	4.714906E-01	4.712270E-01	4.713587E-01	4.717806E-01
8.999994E-01	5.029991E-01	5.027500E-01	5.028735E-01	5.033333E-01
9.999993E-01	5.326469E-01	5.324098E-01	5.325264E-01	5.330217E-01

*Read 1.700000E-01 as 1.700000×10^{-1}.

$$I = t \qquad t \le t^*, \tag{8}$$

$$I = S(t - t^*/2)^{1/2} \qquad t \ge t^*, \tag{9}$$

$$2t^* = S^2, \tag{10}$$

with the dimensionless sorptivity S^2 being known exactly. Note that with this method, t^* is different from the exact value of t_p, see Table 2.

For the modified TCA, we take

$$I = t \qquad t \le t_p, \tag{11}$$

$$I = S(t - t_p + t_c)^{1/2}, \tag{12}$$

where I_p and S have their exact values and t_c is given by

$$S^2 t_c = t_p^2. \tag{13}$$

Note that the modified TCA gives a discontinuity for the flux at t_p as encountered by *Salvucci* [1997, p. 116] at the onset of stage 2 of evaporation and for exactly the same reason: "The discontinuity is an artifact of using the time compression approximation after t_D while using the flux boundary condition solution to predict t_D," where t_p in our case is replaced by the time t_D when stage 2 of evaporation begins.

We now apply the analytical method of *Parlange et al*. [1998, 1992, 1997] as an alternative to the numerical method.

3. Analytical Approximation

Parlange et al. [1998] solved explicitly the problem of constant flux until ponding and the derivation will not be repeated here. The two necessary results (using the dimensionless variables previously defined) are

$$t_p = 2(\alpha + 2)/(2\alpha^2 + 6\alpha + 5) \tag{14}$$

$$\int_0^1 x_p^2 \, d\theta = 2(\alpha + 2)t_p/[(\alpha + 1)(2\alpha + 3)], \tag{15}$$

where $x_p(\theta, t_p) = x(\theta, t)$ at $t = t_p$.

It is of some interest to mention that other estimates of the ponding time could also be used. For instance, *Smith et al*. [1993] and *Corradini et al*. [1994] were careful to use an optimal estimate of t_p valid for α large [*Smith and Parlange*, 1977],

$$t_p = S^2(1 + 2\alpha)/4\alpha, \tag{16}$$

which is equivalent to the first two orders to (14) if S^2 is estimated to the same order. For instance, we can take

Table 1d. Estimates of $I(t)$ for $\alpha = 10$

t	I_{exact}	$I[9]$	$I[12]$	$I[26]$
1.000000E-01*	9.925776E-02	9.913861E-02	9.920613E-01	9.921078E-02
1.100000E-01	1.076586E-01	1.075478E-01	1.076100E-01	1.076188E-01
1.200000E-01	1.154486E-01	1.153455E-01	1.154035E-01	1.154159E-01
1.500000E-01	1.361703E-01	1.360836E-01	1.361328E-01	1.361544E-01
1.700000E-01	1.483858E-01	1.483066E-01	1.483518E-01	1.483784E-01
2.000000E-01	1.650222E-01	1.649516E-01	1.649922E-01	1.650255E-01
2.500000E-01	1.895318E-01	1.894712E-01	1.895065E-01	1.895491E-01
3.000000E-01	2.112162E-01	2.111625E-01	2.111942E-01	2.112448E-01
3.500000E-01	2.308728E-01	2.308243E-01	2.308533E-01	2.309110E-01
3.999999E-01	2.489824E-01	2.489380E-01	2.489649E-01	2.490290E-01
4.999998E-01	2.817307E-01	2.816925E-01	2.817163E-01	2.817917E-01
5.999997E-01	3.110501E-01	3.110164E-01	3.110380E-01	3.111233E-01
6.999996E-01	3.378344E-01	3.378043E-01	3.378242E-01	3.379184E-01
7.999995E-01	3.626459E-01	3.626187E-01	3.626372E-01	3.627396E-01
8.999994E-01	3.858652E-01	3.858404E-01	3.858578E-01	3.859679E-01
3.999993E-01	4.077646E-01	4.077418E-01	4.077583E-01	4.078755E-01

*Read 1.000000E-01 as 1.000000×10^{-1}.

Table 2. Sorptivity, Ponding Times, and Maximum Relative Errors of the TCA Approximations for Cumulative Infiltration $I(t)$

	α			
	0	1	5	10
S_{exact}	$\sqrt{4/\pi}$	0.8874	0.5541	0.4169
$S[17]$	$\sqrt{4/3}$	0.8944	0.5547	0.4170
$t_{p\ \text{exact}}$	0.7854	0.4592	0.16465	0.09056
$t_p[14]$	0.8	0.4615	0.1647	0.09057
$t^*[10]$	0.6366	0.3937	0.1535	0.08690
Maximum relative error [9]	0.024	0.014	0.0035	0.0012
Maximum relative error [12]	0.012	0.007	0.0015	0.00055
ε_1 [31]	0.0257	0.00971	0.00147	0.000472
ε_2 [33]	0.007	0.0045	0.0011	0.0004

$$S^2 \simeq 4/(2\alpha + 3), \tag{17}$$

as first suggested by *Brutsaert* [1976], as a particular case of the estimate

$$S^2 \simeq 2(\theta_s - \theta_i)^{1/2} \int_{\theta_i}^{\theta_s} (\theta - \theta_i)^{1/2} D \, d\theta \tag{18}$$

in agreement with the optimization approach [*Parlange*, 1975]. Table 2 shows the accuracy of (17). Then the two estimates $t_p[14]$ and $t_p[16]$ given by (14) and (16), respectively, yield

$$t_p[16]/t_p[14] = (1 + 2\alpha)(2\alpha^2 + 6\alpha + 5)/[2\alpha(2\alpha + 3) \cdot (\alpha + 2)] \tag{19}$$

and for α large,

$$t_p[16]/t_p[14] \simeq 1 + \alpha^{-2} + \cdots, \tag{20}$$

showing the agreement of the two expressions to the first two orders (α^0 and α^{-1}) as expected. One advantage of (14) over (16) is that the former can be applied even for $\alpha = 0$ (even if we do not expect the result to be accurate).

On the other hand, the estimate t^* of t_p given by (10) from the standard TCA is only correct to order α^0. Even for the unrealistic case of constant D, taking the exact value $S^2 = 4/\pi$ in (10), the latter yields $t^*[10] = 2/\pi$, a 19% error compared to the exact result $t_p = \pi/4$, whereas (14) gives $t_p[14] = 0.8$, only a 2% error. Hence, since (14) is more accurate as α increases, it is a most reliable estimate; see Table 2.

After ponding, we approximate the profile by (using (8) of *Parlange et al.* [1998])

$$1 - \theta^\alpha = \alpha[qx + q^2x^2/2], \tag{21}$$

where $q(t)$ is the surface flux. To estimate q after ponding, (21) is integrated to yield

$$(\alpha + 1)^{-1} = qI + q^2 \int_0^1 x^2 \, d\theta/2 \tag{22}$$

and (see (5) of *Parlange et al.* [1998])

$$\int_0^1 x^2 \, d\theta = 2(\alpha + 1)^{-1}(t - t_p) + \int_0^1 x_p^2 \, d\theta, \tag{23}$$

the last integral in (23) being given by (15). Thus

$$1 = (\alpha + 1)qI + q^2[t - (\alpha + 1)t_p/(2\alpha + 3)]. \tag{24}$$

For convenience, let us call

$$\tilde{t} = t - (\alpha + 1)t_p/(2\alpha + 3) \tag{25}$$

and define a new variable y by

$$(\alpha + 1)I\tilde{t}^{-1/2} = y^{1/2} - y^{-1/2}, \tag{26}$$

then from (24),

$$\tilde{t}/\tilde{t}_p = \{(y/y_p)[(2\alpha + 3 - y_p)/(2\alpha + 3 - y)]^{2(\alpha+2)}\}^{1/(2\alpha+3)}, \tag{27}$$

where y_p and $\tilde{t}_p$ are the value of y and $\tilde{t}$ at ponding. Equations (25)–(27) yield $I(t)$.

4. Results and Discussion

Table 1 gives detailed $I(t)$. Obviously, the two TCA approximations and the analytical approximations of (26) are excellent, with the new modified TCA having about half the error of the standard one; see Table 2. That is, the modified TCA gives a significant improvement with no additional work, as long as I_p is known.

By definition, as $t \to \infty$, $I \to S(t)^{1/2}$, in agreement with the TCA approximations, as long as S is known exactly. The analytical approximation, on the other hand, gives $I \to 2[t(2\alpha + 3)]^{1/2}$, in agreement with the estimate of S in (17). Because this estimate is approximate (see Table 2), it results in the error apparent as $t \to \infty$ (see Tables 1a–1d). We could, as suggested by *Parlange et al.* [1997], modify the analytical approximation building in the knowledge of S. However, this is not necessary to understand, from the analysis, the reason for the accuracy of the TCA approximation and to quantify their errors, as done in the following.

The fundamental property of any TCA method is that $I \sim (t - A)^{1/2}$, where A is a constant. This is in agreement with (26) as long as y is near constant after ponding. In general,

$$y_p < y < y_\infty \tag{28}$$

with, from (27),

$$y_\infty = 2\alpha + 3 \tag{29}$$

and, from (25) and (26),

$$y_p^{1/2} - y_p^{-1/2} = (\alpha + 1)\sqrt{2(2\alpha + 3)/(2\alpha^2 + 6\alpha + 5)}. \tag{30}$$

Table 3. Errors in the Prediction of I_p or t_p From (38) and (36) for the Standard TCA for Various Values of $K_{sat}I_p/(1-\nu)$ and ν

	$K_{sat}I_p/(1-\nu)$						
	0.01	0.10	0.20	0.50	1.00	2.00	5.00
$\nu = 0.95$	0.27	0.26	0.24	0.20	0.16	0.11	0.00
$\nu = 0.90$	1.09	1.04	0.99	0.84	0.66	0.44	0.24
$\nu = 0.80$	3.66	3.53	3.39	3.00	2.47	1.72	0.84
$\nu = 0.70$	6.66	6.46	6.25	5.67	4.84	3.62	1.72

Errors are in percent.

Thus the difference between the two values of I for $y = y_p$ and $y = 2\alpha + 3$ in (26) gives an estimate of the TCA's maximum error, ε_1, or

$$\varepsilon_1 \simeq \frac{1}{2}\left[1 - \frac{2\alpha + 3}{\sqrt{2(2\alpha^2 + 6\alpha + 5)}}\right] \simeq \frac{1}{4(2\alpha^2 + 6\alpha + 5)}, \tag{31}$$

which as expected, is maximum for $\alpha = 0$ and goes to zero as $\alpha \to \infty$. Table 2 also gives ε_1 and shows its essential agreement, especially as α increases with the numerical estimates of the standard TCA errors. (The error of the modified TCA being $\varepsilon_1/2$.) The error is small, not only for $\alpha = 0$, decreasing very rapidly with α, i.e., for real soils.

A cruder way to estimate the maximum error, ε_2, of the TCA is to exploit the observation that the standard method error is twice the error of the modified method. Thus, comparing I in (9) and the exact $I = t_p$ at $t = t_p$ gives 1/2 of the error of the standard method, or

$$2\varepsilon_2 = 1 - S^2(t_p - S^2/4)/t_p^2. \tag{32}$$

In particular, in the present case and estimating ε_2 from the analytical results, (14) and (17) yield

$$2\varepsilon_2 = (\alpha + 1)^2/\lfloor(\alpha + 2)^2(2\alpha + 3)^2\rfloor, \tag{33}$$

which provides a rough estimate of the maximum errors of the standard (ε_2) and the modified ($\varepsilon_2/2$) methods, as shown in Table 2.

5. Gravity Effects

We are now going to discuss an example showing that gravity effects reduce the error of the TCA even further. To keep the discussion analytic and to see the source of errors clearly, we need to consider a case with gravity, where two analytic solutions are available. The first exact solution applies for any soil properties when the flux at the surface is proportional to the surface water content [*Fleming et al.*, 1984]. The second exact result applies for constant surface flux $\bar{q}$ [*Rogers et al.*, 1983; *Sander et al.*, 1988] but for special soil properties

$$D(\theta) = (1 - \nu)/(1 - \nu\theta)^2 \tag{34}$$

and the soil water conductivity

$$K(\theta) = K_{sat}(1 - \nu)\theta^2/(1 - \nu\theta), \tag{35}$$

where K_{sat} is the saturated conductivity. As in section 4, we effectively take $\bar{q} = 1$ as well as $\theta = D(\theta) = 1$ at saturation.

At ponding, $t_p = I_p$ and K_{sat} are related by [*Rogers et al.*, 1983; *Sander et al.*, 1988, 1999]

$$\frac{2K_{sat}}{\nu} = 1 - e^{-I_pK_{sat}}/I_pK_{sat}/(1-\nu)\ \mathrm{erfc}\left[-\frac{\nu}{2(1-\nu)}I_p^{1/2}\right] + [1 + 4(1-\nu)K_{sat}/\nu^2]^{1/2} \cdot \mathrm{erf}\left[I_p\frac{K_{sat} + \nu^2/4(1-\nu)}{1-\nu}\right]^{1/2}. \tag{36}$$

The general solution for I when $q/\theta_s = A$ is constant (q is the surface flux, and θ_s is the surface water content) is at ponding [*Fleming et al.*, 1984],

$$I_p = \int_0^1 \frac{\theta D(\theta)\, d\theta}{A\theta - K(\theta)}. \tag{37}$$

In particular, when (34) and (35) hold (37) gives

$$K_{sat}I_p = (1 - \nu)\ln\frac{f}{f-1}, \tag{38}$$

$$f = A/K_{sat}. \tag{39}$$

The two exact solutions in (36) and (38) (for $\bar{q} = 1$ and $q/\theta_s = A$) have not been extended beyond ponding. Thus, at the difference of the case without gravity we cannot determine the error of the cumulative infiltration after ponding. This was the situation encountered by *Sivapalan and Milly* [1989], and like them, we shall look at the error in the cumulative infiltration prediction of the standard TCA when $q = \bar{q}$. Although not as complete as the discussion of section 4, we can still deduce from it whether gravity effects are helpful or not. We shall operate in the same manner for the modified TCA looking at the error of the predicted flux at ponding time.

Table 3 indicates the errors in $I_pK_{sat}/(1 - \nu)$ from (36) and (38) for various ν. The column heads give the exact values of $I_pK_{sat}/(1 - \nu)$; from the tabulated errors, one can reconstruct the values of $f = K_s^{-1}$ in each case. Clearly, as $I_pK_{sat}/(1 - \nu)$ increases, $K_{sat}/\bar{q}$ increases (from $\sim 10^{-2}$ to ~ 0.99). Table 3 then shows that the error decreases by a factor of 4, for all ν, as gravity increases. Table 3 also shows that the error decreases as ν approaches 1 (the error is exactly zero for $\nu = 1$, i.e., when D is a delta function, as the TCA is well known to be exact in that limit). It is of some interest in that respect to look at the relation between the values of ν in Table 3 and the values of α in section 4. This can be done easily from the calculation of $\int \theta D\, d\theta / \int D\, d\theta$, which approaches 1 for a delta function; we have

$$\int_0^1 \theta D\, d\theta \Bigg/ \int_0^1 D\, d\theta = \frac{\alpha + 1}{\alpha + 2} = \nu^{-1} - (1 - \nu)\nu^{-2}\ln(1 - \nu)^{-1}, \tag{40}$$

Table 4. Errors on the Flux at Ponding for the Modified TCA from (38) and (36) for Various Values of $K_{sat}I_p/(1-\nu)$ and ν

	$K_{sat}I_p/(1-\nu)$						
	0.01	0.1	0.2	0.5	1.0	2.0	5.0
$\nu = 0.95$	0.27	0.24	0.22	0.16	0.09	0.03	0.00
$\nu = 0.90$	1.09	0.99	0.89	0.65	0.38	0.13	0.01
$\nu = 0.80$	3.64	3.35	3.05	2.29	1.42	0.53	0.02
$\nu = 0.70$	6.62	6.12	5.61	4.31	2.75	1.09	0.05

Errors are in percent.

so that for $\nu = 0.7$, 0.8, 0.9, and 0.95 the corresponding values of α are 1.24, 1.95, 3.78, and 6.82, covering the relevant range of values.

Table 4 makes a similar calculation for the modified TCA; now taking the same $I_pK_{sat}/(1-\nu)$ in (36) and (38), we give the error in f, from (38), and the exact value of K_s^{-1}, from (36). The same conclusions obtained from Table 3 can be made here. Note, however, that the effect of gravity in reducing the error is much more pronounced (most likely because when gravity effects increase, then, by definition, the flux approaches K_{sat}).

6. Conclusion

A numerical solution confirms that TCA provides an excellent prediction of cumulative flux after ponding, assuming that the rainfall rate can be replaced by its average before ponding. A modified procedure is discussed which halves the error of the standard method. An approximate analytical solution provides the means to analyze the reason for the TCA accuracy and, in particular, yields an analytical estimate of the errors. As already pointed out by *Sivapalan and Milly* [1989], the errors are smaller as the soil water diffusivity increases more rapidly with water content. The result was quantified for a power law diffusivity, $D \sim \theta^{\alpha}$. However, if for a general diffusivity the first moments, $\int D\, d\theta$ and $\int \theta D\, d\theta$, are calculated as we have done in (40), then α can be defined by the following relation, which is exact for a power law:

$$(2+\alpha)^{-1} \simeq \int_0^{\theta_{sat}} (\theta_{sat}-\theta) D\, d\theta / \theta_{sat} \int_0^{\theta_{sat}} D\, d\theta, \tag{41}$$

where θ_{sat} is the saturated water content. Then the error of the TCAs can be estimated from (31) or (33) for a general diffusivity.

Gravity effects seem to reduce the error of the TCA even further. Even though both standard and modified TCAs have similar, and small, errors, the latter appears to be slightly more accurate. More importantly, when a reasonable estimate of I_p is available, it is more in line with the basic premise of TCA of [*Smith et al.*, 1993, p. 137] "using I as a surrogate for time."

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