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On an Exact Analytical Solution of the Boussinesq Equation

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Abstract. A useful exact analytical solution of the Boussinesq equation is discussed and is the most general solution presently available, and in particular yields a solution for a finite aquifer. It provides insight into the physical processes arising during the exchange of water between an aquifer and a free body of water of varying height as an application and extension of Barenblatt's solution. We also illustrate the value of such a solution to check numerical and approximate schemes.

Key words: Boussinesq equation, exact solution, saturated flow, unsteady profiles, finite aquifers.

1. Introduction

The movement of water in an aquifer is represented by Boussinesq's equation,

$$\frac{\partial h}{\partial t} = \frac{K}{S}\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right), \tag{1}$$

where $h(x, t)$ represents the depth of the water table with respect to a horizontal impervious layer. The boundary conditions are given by $h(x, 0)$ and

$$h(0, t) = H(t), \tag{2}$$

$$\frac{\partial h}{\partial t}(\infty, t) = 0, \tag{3}$$

for a semi-infinite aquifer, the case of a finite aquifer will be discussed later.

This problem can be interpreted as describing the interaction between a free body of water of height H in a channel located at $x < 0$ and an unconfined aquifer located at $x > 0$, S being the specific yield and K the hydraulic conductivity.

We obtain a unique exact solution as the most general polynomial solution of the form

$$\frac{K}{S}(H-h)=\sum_{i=1}^{n}A_i(t)x^i. \tag{4}$$

Substituting this solution in Equation (1), $\partial h/\partial t$ is a polynomial of degree n, whereas the RHS is a polynomial of degree $2n-2$. Thus, the highest value of n must be such that $n=2n-2$ or $n=2$. Writing then

$$\frac{K}{S}(H-h)=A_1x+A_2x^2, \tag{5}$$

Equation (1) gives

$$-\frac{K}{S}\dot{H}+\dot{A}_1x+\dot{A}_2x^2=\frac{\partial}{\partial x}\left[\left(\frac{K}{S}H-A_1x-A_2x^2\right)(A_1+2A_2x)\right], \tag{6}$$

and balancing terms in x^0, x and x^2 give

$$\frac{K}{S}\dot{H}=-A_1^2+2A_2\frac{K}{S}H, \tag{7}$$

$$\dot{A}_1=-6A_1A_2, \tag{8}$$

$$\dot{A}_2=-6A_2^2, \tag{9}$$

or, by integration, we finally obtain the solution

$$\frac{K}{S}(H-h)=\sqrt{\frac{K}{S}}\frac{\beta}{t+\alpha}x+\frac{x^2}{6(t+\alpha)}, \tag{10}$$

which includes known limiting cases as discussed below. Here α and β are two arbitrary constants and $H(t)$ is given by

$$H=\frac{1.5\beta^2}{t+\alpha}[C(t+\alpha)^{2/3}-1]. \tag{11}$$

By changing the origin of time, we could always take $\alpha = 0$. However, this would impose a spatial singularity for the initial profile so that we keep α explicitly in the formulae.

In particular, the flux, q, at $x=0$ is given by

$$q=H\sqrt{\frac{K}{S}}\frac{\beta}{t+\alpha}. \tag{12}$$

Hence, if $\beta>0$, water flows from the channel into the bank and the opposite, for $\beta<0$, and if $\beta=0$, the interface at $x=0$ is impervious to water movement. The initial water table in the aquifer is then given by

$$\frac{K}{S}[H(0)-h(x,0)]=\sqrt{\frac{K}{S}}\frac{\beta}{\alpha}x+\frac{x^2}{6\alpha}, \tag{13}$$

with

$$H(0) = \frac{3\beta^2}{2\alpha}(C\alpha^{2/3} - 1). \tag{14}$$

Two particular limiting cases are already known. (See Barenblatt *et al.* (1990) and the discussion in Chen *et al.* (1995).) One solution is obtained for $\beta = 0$ and is discussed below. The other for $\beta \to \infty$, $\beta/\alpha = 1$, has also been discussed in Hogarth *et al.* (1997) and gives h linearly dependent on x.

We will now consider the solution of Equation (10) for the three cases $\beta = 0$, $\beta > 0$ and $\beta < 0$, separately.

Case $\beta = 0$

There is no water flux at $x = 0$. Since $H(t)$ cannot be zero for all times, $C\beta^2$ must be finite, e.g.,

$$\tfrac{3}{2}\beta^2 C \equiv D > 0. \tag{15}$$

Then from Equation (11)

$$H = \frac{D}{(t+\alpha)^{1/3}} \tag{16}$$

and from Equation (10)

$$\frac{K}{S}(H - h) = \frac{x^2}{6(t+\alpha)}. \tag{17}$$

In particular the position x_0, such that $h = 0$ for $x \geqslant x_0$, is given by

$$x_0^2 = \frac{K}{S} 6D(t+\alpha)^{2/3}. \tag{18}$$

The solution for $\alpha = 0$ corresponds to Barenblatt's solution for no water in the aquifer at $t = 0$, except at $x = 0$, where

$$h(x,0) = \frac{2}{3} D\sqrt{\frac{6DK}{S}}\,\delta(x), \tag{19}$$

where $\delta(x)$ is a delta function with $\int_0^\infty \delta(x)\,\mathrm{d}x = 1$. Of course, for $t > 0$ or $\alpha > 0$, the solution is finite everywhere and for all times (but there is no real difference between the solution for $\alpha = 0$ and $\alpha > 0$).

Figure 1 illustrates the results taking $\alpha = 0$; $\bar{h} = h/D$ for normalized height and $\bar{x} = x\sqrt{S/KD}$ for normalized distance. Then Equations (10) and (11) become, without loss of generality

$$\bar{H} = t^{-1/3}, \qquad \bar{H} - \bar{h} = \frac{\bar{x}^2}{6t}. \tag{20}$$

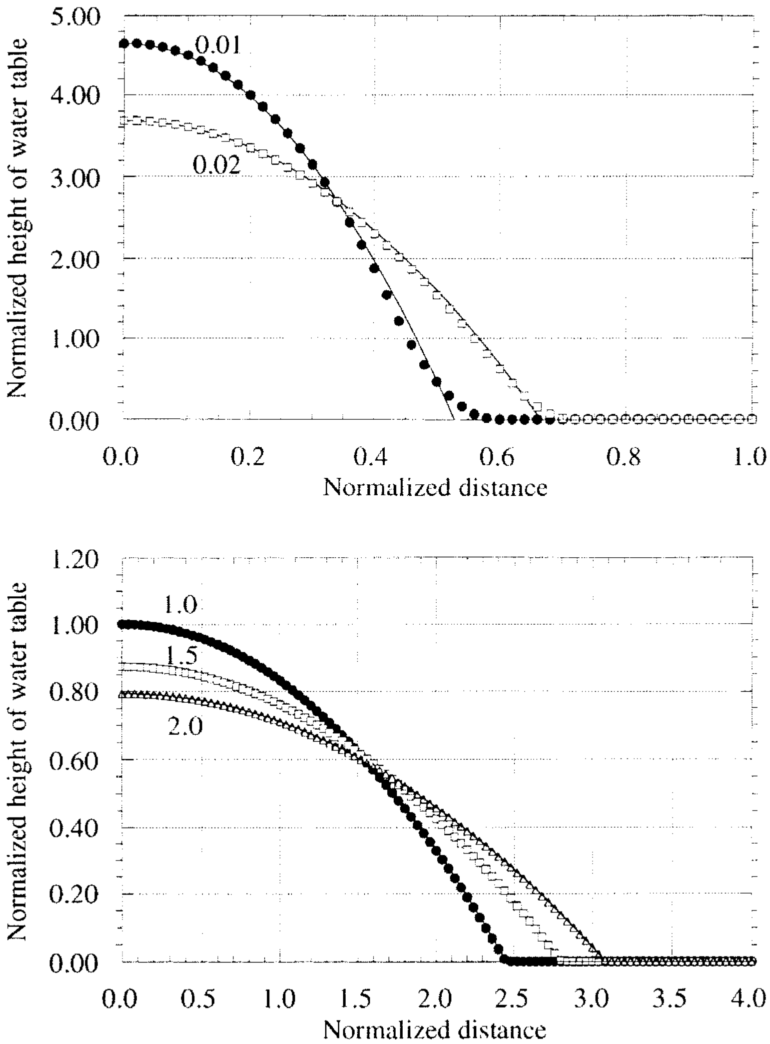

Figure 1. Illustration of Barenblatt's solution ($\alpha = \beta = 0$) for early times (Figure 1a) and longer times (Figure 1(b)). The dots represent the numerical results for the profiles and show some numerical dispersion for early times (the solid line is the analytical result). In that case, water redistributes in the aquifer and the wall at $x = 0$ is impervious. Times are indicated on the profiles.

Also as an application of the method we plot profiles obtained by a standard numerical method (Govindaraju and Koelliker, 1994) which shows excellent accuracy except at very short time where some numerical dispersion is apparent at the wetting front. As in Hogarth *et al.* (1997), who use the linear solution, we illustrate here the importance of exact analytical results to validate numerical scheme to solve the partial differential equation (1). Chen *et al.* (1995) also used Barenblatt's solution to check their numerical scheme of the ODE associated with Equation (1), that is, when there is a similarity solution.

We now discuss the cases for β finite and nonzero. There are two different physical situations depending on the sign of β.

Case $\beta > 0$
Taking

$$\bar{h} = \frac{h\alpha}{\beta^2}, \qquad \bar{x} = \frac{x(S/K)^{1/2}}{|\beta|}, \qquad \tau = \frac{t}{\alpha}, \tag{21}$$

as normalized variables, Equations (10) and (11) become, without loss of generality

$$\bar{H} - \bar{h} = \frac{\bar{x}}{\tau + 1} + \frac{\bar{x}^2}{6(\tau + 1)} \tag{22}$$

and

$$\bar{H} = \frac{3}{2(\tau + 1)}[C\alpha^{2/3}(\tau + 1)^{2/3} - 1]. \tag{23}$$

The only remaining parameter being $C\alpha^{2/3} \geqslant 1$. The case, $C\alpha^{2/3} = 1$, is especially interesting as it leads to $h(x, 0) = 0$ for all x. In that case $H(t)$ increases in time from zero, reaches a maximum value and then decreases to zero but slowly enough so that q remains positive for all times. Figure 2 illustrates that case at time $t^* = 3^{3/2} - 1$, such that $H(t^*)$ is maximum. The numerical scheme shows little numerical dispersion, but the linearized solution of Govindaraju and Koelliker (1994) shows a significant discrepancy (this illustrates the value of an exact solution to check numerical results).

Case $\beta < 0$
For this case water flows from the aquifer into the channel, so that for this case we always have water in the aquifer, with $C \geqslant 1$. Using the same normalized variables of the previous case Equation (10) becomes, $\bar{H}$ being as above (Equation (23))

$$\bar{H} - \bar{h} = -\frac{\bar{x}}{\tau + 1} + \frac{\bar{x}^2}{6(\tau + 1)}, \tag{24}$$

so that the solution here is formally the same as above if the former was applied for $\bar{x} < 0$. The solution has an interesting property that for all times there is a fixed position at $\bar{x} = 3$, where $\partial\bar{h}/\partial\bar{x} = 0$, so that for $\bar{x} < 3$ water drains out from the aquifer into the channel. Hence, the solution obtained here is also applicable to a finite aquifer such that at $\bar{x} = 3$ there is an impervious surface. Mathematically one could formally reconstruct the solutions for $\beta \neq 0$ by an appropriate translation of the spatial coordinates from the case $\beta = 0$, that is, Barenblatt's case. However, the more direct approach used here makes the properties of the general solution far more transparent, for example the solution for a finite aquifer.

Figure 3 illustrates the results for $C\alpha^{2/3} = 1$. The numerical solution is in excellent agreement with the exact one for all times.

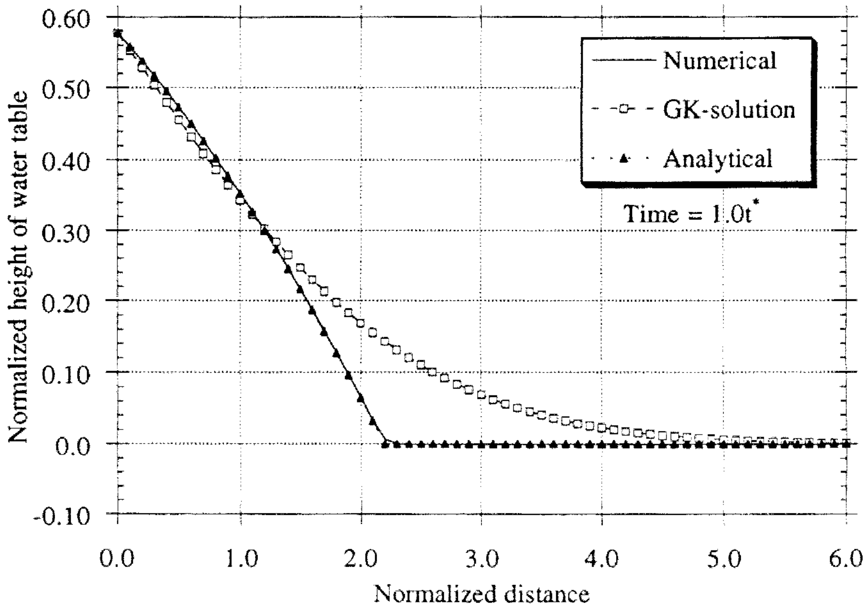

Figure 2. Case $\beta > 0$, $C\alpha^{2/3} = 1$, the time is chosen for the maximum height at $x = 0$. There is minimal numerical dispersion but the linear approximation of Govindaraju and Koelliker (1994) is significantly incorrect. In that case, water flows from the channel into the aquifer.

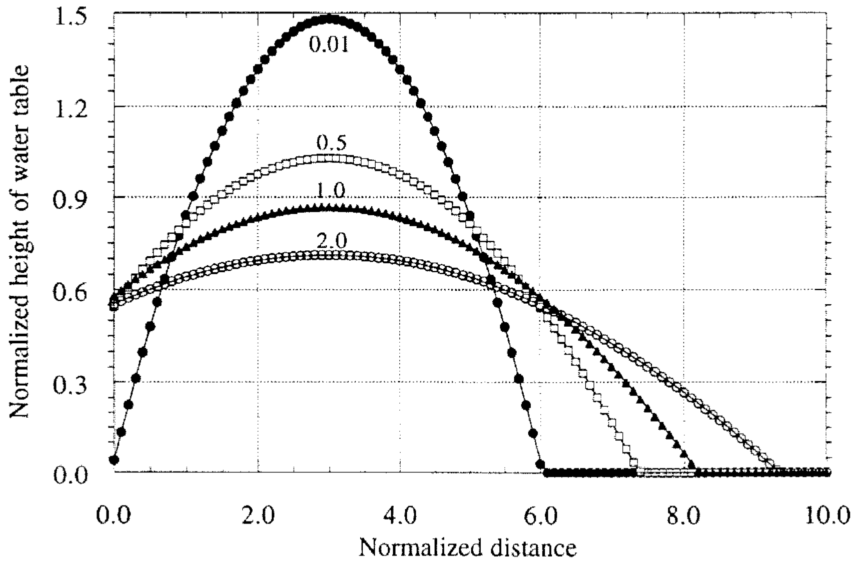

Figure 3. Case $\beta < 0$, $C\alpha^{2/3} = 1$, time is normalized so that at unit time the water level in the channel is maximum. Profiles at different times are shown. In that case, the solutions for $\bar{x} < 3$ and $\bar{x} > 3$ are independent. For $\bar{x} < 3$, water flows out of the (finite aquifer) and dries after an infinite time. For $\bar{x} > 3$ the water redistributes itself in the aquifer, and is basically Barenblatt's solution.

2. Conclusion

We have obtained an exact solution to the Boussinesq equation which extends and includes, as particular cases, two earlier solutions. The merit of this solution is that, it can describe water movement both in and out of an aquifer and includes the case of a finite aquifer as well. The solutions will be useful to check approximate (analytical) and numerical schemes.

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