Extension of the Heaslet-Alksne Technique to Arbitrary Soil Water Diffusivities

M. B. Parlange,[1] S. N. Prasad,[2] J.-Y. Parlange,[3] and M. J. M. Römkens[4]

The Heaslet-Alksne technique solved the nonlinear diffusion equation by expansion around the wetting front for power law diffusivities. Essentially the same technique has been applied when a well-defined wetting front exists at a finite distance. In this paper, the method is extended for an arbitrary diffusivity and to the case when there is no well-defined wetting front at a finite distance. Two illustrations for exponential and power law diffusivities show the excellent accuracy of the method.

Introduction

The nonlinear diffusion equation reduces to the *Bruce and Klute* [1956] equation when the similarity variable $\phi = xt^{-1/2}$ exists (x is the distance from the surface where water is applied, and t is the time). This equation is now so well understood that any new technique must first pass a test of simplicity and accuracy for this particular problem before applying it to more complex situations. In addition, the Bruce and Klute equation provides a severe test of any theory since the fundamental difficulty in solving Richards' equation for general boundary and initial conditions is still mainly caused by the rapidly increasing soil water diffusivity.

There are at present two main approaches to obtain analytical descriptions of liquid movement in porous media. The double integral technique, initiated by *Parlange* [1971], which has been used extensively in recent years, is based on the observation that even though the soil water diffusivity is a difficult function, its integral is far easier to handle. Hence this approach can be seen as similar to the *Green and Ampt* [1911] method, and indeed it leads to accurate infiltration predictions [*Haverkamp et al.*, 1990] which are extensions of the well-known Green and Ampt equation.

The second approach can be traced back to *Heaslet and Alksne* [1961] who described the wetting profile by an expansion around the wetting front for a power law diffusivity. A diffusivity of this form has remarkable similarity properties [e.g., *Shampine*, 1973; *Parlange et al.*, 1980a, b] which can be studied systematically using Lie's method [*Bluman and Kumei*, 1980; *Lisle and Parlange*, 1992]. There have been attempts to generalize the method of Heaslet and Alksne when the diffusivity does not obey a power law [*Prasad and Römkens*, 1982] for all water content, θ, except when $\theta \to 0$ so that the Heaslet and Alksne expression remains formally valid near the wetting front. This rather strong requirement was later relaxed [*Parlange et al.*, 1984] as long as a well-defined wetting front still exists. This

remaining condition is still very limiting: For instance, it prevents us from using the method for those standard diffusivities which have an exponential dependence on θ [*Reichardt et al.*, 1972]. In this note, this last constraint of a well-defined wetting front will be removed.

Theory

We are solving the *Bruce and Klute* [1956] equation

$$D = -\frac{1}{2}\frac{d\phi}{d\theta}\int_0^\theta \phi \, d\bar{\theta} \tag{1}$$

where D is the soil water diffusivity, and $\phi = xt^{-1/2}$, where x is the distance from the soil surface where water is applied, and t is the time, θ is a reduced water content so that $\theta = 0$ is the initial water content, and $\theta = 1$ at the surface. If there is a well-defined wetting front ϕ_f, equation (12) of *Parlange et al.* [1984] gives

$$2\int_0^\theta D/\bar{\theta} \, d\bar{\theta} = \phi_f(\phi_f - \phi) - \frac{A}{2}(\phi_f - \phi)^2$$
$$+ B(\phi_f - \phi)^3 + \cdots \tag{2}$$

where A is an unknown constant, and ϕ_f is given by applying (2) at $\theta = 1$. Ignoring terms of order $B(\phi_f - \phi)^3$, which are shown in the appendix to introduce small corrections only,

$$2\int_0^1 D/\theta \, d\theta = \phi_f^2\left(1 - \frac{A}{2}\right) \tag{3}$$

and, by subtraction,

$$2\int_\theta^1 \frac{D}{\bar{\theta}} \, d\bar{\theta} = \phi_f(1 - A)\phi + \frac{A}{2}\phi^2 \tag{4}$$

The advantage of (4) over (2) is that (4) remains formally meaningful even if $\phi \to \infty$ as $\theta \to 0$. That is, (2) can be seen as an expansion around ϕ_f, but ϕ_f is the wetting front only if it is well defined, and the expansion is used for $\phi \le \phi_f$. If not, ϕ_f is the value of ϕ for some small (unknown) value of θ which is not zero.

If we differentiate (4) and compare with (1) at $\theta = 1$, it is clear that

$$\phi_f(1 - A) = S \tag{5}$$

where S is the sorptivity,

[1] Hydrologic Science, Land, Air and Water Resources, Department of Agricultural Engineering, University of California, Davis.
[2] Civil Engineering Department, University of Mississippi, Oxford.
[3] Department of Agricultural and Biological Engineering, Cornell University, Ithaca, New York.
[4] USDA Sedimentation Laboratory, Oxford, Mississippi.

$$S = \int_0^1 \phi \, d\theta. \qquad (6)$$

Since ϕ_f is unknown, we rewrite (4) as

$$2\int_\theta^1 \frac{D}{\bar{\theta}} \, d\bar{\theta} = S\phi + \frac{A}{2} \phi^2 \qquad (7)$$

As long as we keep only two terms in the expression above, we require two conditions to estimate the two parameters S and A. One fundamental condition is provided by integrating (1) or [Parlange, 1975a]

$$2\int_0^1 D \, d\theta = \int_0^1 \phi^2 \, d\theta, \qquad (8)$$

which applied to (7) yields at once

$$2\int_0^1 D \, d\theta \left(1 - \frac{A}{2}\right) = S^2. \qquad (9)$$

Thus, if S^2 were known, (9) would provide A. To calculate S^2 and complete the determination of the profile, a variety of choices are available. For instance, Parlange [1975b] suggested the use of

$$S^2 = \int_0^1 (1 + \theta) \, D \, d\theta. \qquad (10)$$

Other expressions are also available and could be used equally well [Parlange, 1975b; Brutsaert, 1976; Parlange et al., 1980a, b, 1987; Lockington et al., 1988; Parslow et al., 1988]. In practice, (10) is quite accurate enough [Elrick and Robin, 1981].

Let us return briefly to the situation which arises when ϕ_f is the finite wetting front and (2), (4) or (7) are identical. Combining (3), (5) and (9) gives A from

$$\frac{A}{2} = \left\{ \left[\int_0^1 \frac{D}{\theta} \, d\theta \bigg/ \int_0^1 D \, d\theta \right]^{1/2} - 1 \right\}$$

$$\cdot \left\{ 2 \left[\int_0^1 \frac{D}{\theta} \, d\theta \bigg/ \int_0^1 D \, d\theta \right]^{1/2} - 1 \right\}^{-1} \qquad (11)$$

with S^2 obtained from (9). Thus when $\int_0^1 D/\theta \, d\theta$ exists (i.e., is finite), (9) and (11) provide yet another estimate of S which can be used conveniently instead of (10). Obviously, for this particular case the result reduces almost exactly to the earlier one of Parlange et al. [1984]. The analytical form of the results is slightly changed, as the knowledge that A is small was used to simplify the results further, but this is not necessary in general. In fact, the present result of (11) is slightly more accurate than the earlier one of Parlange et al. [1984]. For instance, if we apply the present theory when [Brutsaert, 1968]

$$D = \frac{1}{2} D_0 n \theta^n [1 - \theta^n/(n + 1)] \qquad (12)$$

then the exact solution is

$$\phi = D_0^{1/2}(1 - \theta^n). \qquad (13)$$

TABLE 1. Soil Water Profile, ϕ, for an Exponential Diffusivity, $D = 9 \times 10^{-4} \exp [8.36\theta]$

θ	Numerical	Analytical
1.0	0.0	0.0
0.9	0.56658	0.56569
0.8	0.82683	0.82622
0.7	0.95120	0.95127
0.6	1.01244	1.01308
0.5	1.04351	1.04455
0.4	1.05987	1.06114
0.3	1.06893	1.07036
0.2	1.07441	1.07595
0.1	1.07839	1.08000
0.01	1.08382	1.08555
0.001	1.08776	1.08959
0.0001	1.09156	1.09351
0.00001	1.09534	1.09742
0.000001	1.09902	1.10133

The numerical result is from Braddock and Parlange [1980]. The analytical results follow (15), (16), and (17).

Remarkably, (11), (9) and (7) yield the exact solution, whereas the earlier solution of Parlange et al. [1984] did not. (It can easily be shown that (11), (9) and (7) with two terms in the expansion yield the exact solution of (1) only when D obeys (12)).

It remains to show that the B term in (2) introduces negligible corrections compared to the A term. This is done in the appendix.

ILLUSTRATION

We shall now illustrate the accuracy of (7) with two terms only, when A is obtained from (9) and S is known independently, e.g., from (10). Such an approximation is obviously very simple to apply, and we are now going to check its accuracy for a standard exponential diffusivity when no wetting front exists at a finite distance [Reichardt et al., 1972]. Then

$$D = D_i \exp n\theta \qquad (14)$$

Equation (7) becomes

$$2[E_i(n) - E_i(n\theta)] = (S/D_i^{1/2})(\phi/D_i^{1/2}) + \frac{A}{2} \phi^2/D_i \qquad (15)$$

with, from (10),

$$S^2/D_i = (\exp n)(2n^{-1} - n^{-2}) - n^{-1} + n^{-2} \qquad (16)$$

and A, from (9),

$$A = [(\exp n)n^{-1} - 1 - n^{-1}]/[(\exp n) - 1] \qquad (17)$$

Note that for an exponential diffusivity the result is fully analytical, i.e., expressible in terms of tabulated functions, as suggested by Koussis [1982]. For most soils the value of n is close to 8.36 [Reichardt et al., 1972]. The value of D_i is clearly irrelevant, but to facilitate comparison with the exact numerical results of Braddock and Parlange [1980] we shall take $D_i = 9 \times 10^{-4}$. Then the exact value of S is 0.929178, and the approximate value from (16) is 0.929969, a 1% error, with $A = 0.11938$. Table 1 gives $\phi(\theta)$ obtained from (15) and the exact results. We note that, as expected, the error in the profile position is of the order of the error on S. We note

TABLE 2. Soil Water Profiles for a Power Law Diffusivity

θ	n = 1			n = 2			n = 3			n = 4			n = 5		
	Numerical	Analytical (21)	Analytical (22)	Numerical	Analytical (21)	Analytical (22)	Numerical	Analytical (21)	Analytical (22)	Numerical	Analytical (21)	Analytical (22)	Numerical	Analytical (21)	Analytical (22)
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0.9	0.21495	0.21096	0.21545	0.24119	0.23938	0.24114	0.26019	0.25929	0.26004	0.27411	0.27365	0.27395	0.28428	0.28406	0.28413
0.8	0.41200	0.40781	0.41125	0.44057	0.43970	0.43971	0.45400	0.45422	0.45337	0.45879	0.45856	0.45753	0.45595	0.45661	0.45561
0.7	0.59455	0.59305	0.59196	0.60574	0.60737	0.60412	0.59730	0.59937	0.59634	0.57913	0.58100	0.57853	0.55657	0.55814	0.55618
0.6	0.76506	0.76852	0.76061	0.74184	0.74670	0.73969	0.70106	0.70504	0.69995	0.65597	0.65895	0.65534	0.61238	0.61457	0.61199
0.5	0.92540	0.93562	0.91934	0.85246	0.86072	0.85002	0.77350	0.77909	0.77234	0.70186	0.70559	0.70123	0.64082	0.64336	0.64043
0.4	1.07702	1.09544	1.06971	0.94014	0.95161	0.93759	0.82121	0.82800	0.82006	0.72685	0.73103	0.72624	0.65359	0.65630	0.65321
0.3	1.22106	1.24886	1.21291	1.00671	1.02089	1.00416	0.84983	0.85739	0.84871	0.73865	0.74305	0.73804	0.65833	0.66111	0.65795
0.2	1.35844	1.39660	1.34990	1.05343	1.06967	1.05092	0.86443	0.87240	0.86333	0.74302	0.74750	0.74241	0.65961	0.66240	0.65923
0.1	1.48992	1.53923	1.48140	1.08114	1.09866	1.07868	0.86980	0.87791	0.86870	0.74403	0.74853	0.74342	0.65980	0.66259	0.65942
0	1.61612	1.67726	1.60803	1.09032	1.10828	1.08788	0.87057	0.87870	0.86947	0.74410	0.74860	0.74349	0.65981	0.66260	0.65943
$SD_i^{-1/2}$	0.88749	0.91287	0.87946	0.75305	0.76376	0.75053	0.66516	0.67082	0.66402	0.60213	0.60553	0.60150	0.55412	0.55634	0.55373

θ	n = 6			n = 7			n = 8			n = 9			n = 10		
	Numerical	Analytical (21)	Analytical (22)	Numerical	Analytical (21)	Analytical (22)	Numerical	Analytical (21)	Analytical (22)	Numerical	Analytical (21)	Analytical (22)	Numerical	Analytical (21)	Analytical (22)
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0.9	0.29158	0.29151	0.29146	0.29668	0.29669	0.29658	0.30003	0.30008	0.29994	0.30198	0.30206	0.30190	0.30281	0.30290	0.30274
0.8	0.45004	0.45069	0.44980	0.44171	0.44231	0.44153	0.43193	0.43246	0.43180	0.42135	0.42182	0.42125	0.41042	0.41803	0.41034
0.7	0.53260	0.53386	0.53234	0.50883	0.50985	0.50864	0.48611	0.48693	0.48598	0.46486	0.46552	0.46476	0.44521	0.44576	0.44513
0.6	0.57257	0.57419	0.57231	0.53716	0.53839	0.53698	0.50605	0.50699	0.50592	0.47881	0.47954	0.47871	0.45493	0.45552	0.45485
0.5	0.58992	0.59171	0.58966	0.54764	0.54894	0.54746	0.51233	0.51330	0.51219	0.48255	0.48330	0.48245	0.45715	0.45774	0.45707
0.4	0.59634	0.59820	0.59609	0.55084	0.55217	0.55066	0.51391	0.51489	0.51378	0.48332	0.48408	0.48322	0.45753	0.45812	0.45745
0.3	0.59822	0.60010	0.59797	0.55157	0.55291	0.55140	0.51419	0.51518	0.51406	0.48343	0.48419	0.48333	0.45757	0.45817	0.45749
0.2	0.59859	0.60047	0.59834	0.55168	0.55302	0.55150	0.51422	0.51521	0.51409	0.48344	0.48420	0.48334	0.45757	0.45817	0.45750
0.1	0.59862	0.60051	0.59838	0.55169	0.55302	0.55151	0.51423	0.51522	0.51409	0.48344	0.48420	0.48334	0.45757	0.45817	0.45750
0	0.59863	0.60051	0.59838	0.55169	0.55302	0.55151	0.51423	0.51522	0.51409	0.48344	0.48420	0.48334	0.45757	0.45817	0.45750
$SD_i^{-1/2}$	0.51599	0.51754	0.51573	0.48477	0.48591	0.48459	0.45861	0.45946	0.45847	0.43626	0.43693	0.43616	0.41689	0.41742	0.41681

The scaled profile $\phi D_i^{-1/2}$ is given as a function of the water content θ. The first analytical result is based on the use of (21) and the second result on (22).

further that for $\theta < 0.1$, i.e., in the "tail" [*Parlange and Braddock*, 1980], (15) remains quite accurate.

To complete this illustration we shall revisit the case of a power law diffusivity, in part because of its historical significance, since the Heaslet-Alksne method was developed for that case, and also because it will allow us to compare the numerical accuracy of using (11) with that of using (10).

In both cases, (9) yields

$$S^2/D_i = 2(n + 1)^{-1}(1 - A/2) \qquad (18)$$

with

$$D = D_i\theta^n \qquad (19)$$

and, from (7),

$$2n^{-1}(1 - \theta^n) = S\phi/D_i + A\phi^2/2D_i \qquad (20)$$

If (10) is used, then the value of

$$S^2/D_i = (2n + 3)(n + 1)^{-1}(n + 2)^{-1} \qquad (21)$$

completes the determination of $\phi/D_i^{1/2}$ (A being obtained from (18)).

If, instead, (11) is used, then

$$A/2 = \{[(n + 1)/n]^{1/2} - 1\}/\{2[(n + 1)/n]^{1/2} - 1\} \qquad (22)$$

with S being obtained from (18).

Table 2 gives results for n varying between 1 and 10, although most soils would be well represented with n varying only between 4 and 8 [*Brooks and Corey*, 1964]. The exact numerical results, using *Shampine*'s [1973] technique are given, as well as the two approximations based on (21) or (22). We observe that the errors in both cases decrease as n increases. This is as expected, since the A term, and all higher-order terms, tend to zero as n increases. However, even for $n = 1$, the errors are remarkably and quite surprisingly small, with a slight advantage (the error is halved) to the results using (22) over (21). In both cases, as in the previous example, the error in the position of the profile and the sorptivity are of the same order.

CONCLUSION

We have extended the Heaslet-Alksne approach to arbitrary diffusivities even when there is no finite wetting front. If there is a finite wetting front, the earlier result of *Parlange et al.* [1984] is slightly improved. More importantly, when there is no finite wetting front and the diffusivity is arbitrary, a very simple description of the water profile is obtained. The accuracy of the result is also checked for two standard soil water diffusivities. In all cases, the error in profile prediction seems to be of the order of sorptivity prediction, using optimal techniques.

APPENDIX

The fundamental reason for being able to neglect higher-order terms in (2) is that the estimates of S^2 from (10) or any other similar expression are consistent with those terms' being an order of magnitude ε smaller than the A term, where [*Parslow et al.*, 1988],

$$\varepsilon = \int_0^1 (1 - \theta) D \, d\theta \Big/ \int_0^1 D \, d\theta \qquad (A1)$$

For ε to be small, D must increase rapidly with θ, as usual for normal soils. Then the sorptivity is,

$$S^2 = 2\int_0^1 D \, d\theta \, [1 - \tfrac{1}{2}\varepsilon + O(\varepsilon^2)]. \qquad (A2)$$

Note that the approximation given in (10) is consistent with (A2) if the terms of order ε^2 are strictly zero.

We are now going to show that if we are not interested in terms of that order, then the B terms are negligible, i.e., $B\phi_f = O(\varepsilon^2)$ whereas $A = O(\varepsilon)$.

It is clear from (9) that,

$$A = \varepsilon + O(\varepsilon^2). \qquad (A3)$$

Then let us rewrite (5) keeping the B terms in (2), or

$$\phi_f(1 - A + 3B\phi_f) \approx S \qquad (A4)$$

which redefines ϕ_f. If (2) is integrated,

$$S^2 - 2\int_0^1 D \, d\theta \, (1 - A/2)$$

which was zero in (9), is now proportional to B,

$$S^2 - 2\int_0^1 D \, d\theta \, (1 - A/2) \approx 2B\phi_f^3. \qquad (A5)$$

The left-hand side of (A5) is clearly of order $\phi_f^2 O(\varepsilon^2)$, using (A2) and (A3). Thus $B\phi_f$ is only of order ε^2 as desired to neglect the B term compared to the A term.

We may note here that the derivation of (10) was based on a similar argument and also required the rapid increase of D with θ. However, it is well known that (10) is remarkably accurate even when D is not a rapidly increasing function of θ, i.e., the result is more general than the formal proof suggests. This holds here as well; e.g., Table 2 shows the remarkable accuracy of the results down to $n = 1$ when ε is only 1/3.

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J.-Y. Parlange, Department of Agricultural and Biological Engineering, Cornell University, Ithaca, NY 14853.

M. B. Parlange, Hydrologic Science, Land, Air and Water Resources, Department of Agricultural Engineering, University of California, Davis, CA 95616.

S. N. Prasad, Civil Engineering Department, University of Mississippi, Oxford, MS 38677.

M. J. M. Römkens, USDA Sedimentation Laboratory, P. O. Box 1157, Oxford, MS 38655.