Sampling signals with finite rate of innovation

Martin Vetterli[1,2] Pina Marziliano[1] Thierry Blu[3]

Abstract

Consider the problem of sampling signals which are not bandlimited, but still have a finite number of degrees of freedom per unit of time, such as, for example, piecewise polynomials. Call the number of degrees of freedom per unit of time the rate of innovation. We demonstrate that by using an adequate sampling kernel and a sampling rate greater or equal to the rate of innovation, one can uniquely reconstruct such signals.

We thus prove theorems for classes of signals and sampling kernels that generalize the classic "bandlimited and sinc kernel" case. In particular, we show sampling theorems for periodic as well as finite length piecewise polynomials, using a bandlimited derivative kernel, as well as a Gaussian kernel. For infinite length piecewise polynomials with a finite local rate of innovation, we show exact local reconstruction using sampling with spline kernels.

All the results presented lead to computational procedures that are readily implementable, which is shown through experimental results. Applications of these new sampling results can be found in signal processing, communications systems and biological systems.

Index Terms

Sampling, generalized sampling, poisson processes, piecewise polynomials, non-bandlimited signals, analog-to-digital conversion, annihilating filters.

[1] LCAV, DSC, Ecole Polytechnique Fédérale de Lausanne, Switzerland,
[2] EECS Dept., University of California at Berkeley, USA
[3] BIG, DMT, Ecole Polytechnique Fédérale de Lausanne, Switzerland .

I. Introduction

Most continuous-time phenomenas can only be seen through sampling the continuous-time waveform, and typically, the sampling is uniform. Very often, instead of the waveform itself, one has only access to a smoothed or filtered version of it. This may be due to the physical set up of the measurement, or may be by design.

Calling $x(t)$ the original waveform, its filtered version is $x(t)*\tilde{\varphi}(t)$, where $\tilde{\varphi}(t)=\varphi(-t)$ is the convolution kernel. Then, uniform sampling with a sampling interval T leads to samples $x[n]$ given by

$$x[n] = <\varphi(t-nT), x(t)> = \int_{-\infty}^{\infty} \varphi(t-nT)\, x(t)\, dt. \tag{1}$$

This setup is shown in Fig. 1.

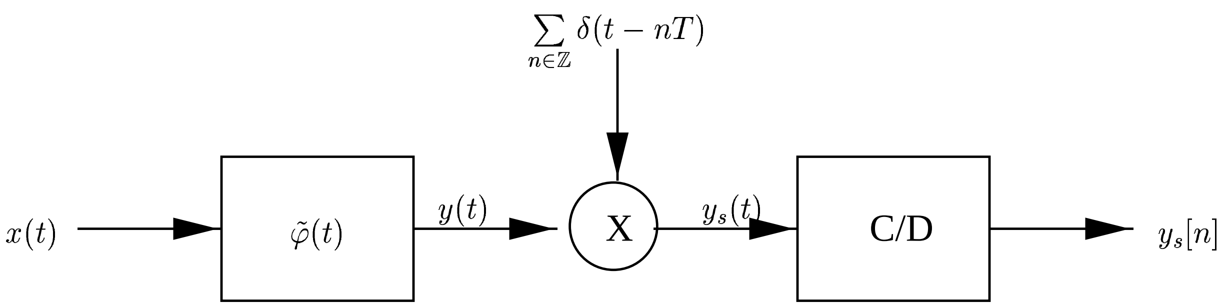


Fig. 1. Sampling set up: $x(t)$ is the continuous-time signal; $\tilde{\varphi}(t)=\varphi(-t)$ is the smoothing kernel; $y(t)$ is the filtered signal; T is the sampling interval; $y_s[n]=x[n], n\in\mathbb{Z}$ are the sample values.

When no smoothing kernel is used, we simply have $x[n]=x(nT)$, which is equivalent to (1) with $\varphi(t)=\delta(t)$. This simple model for having access to the continuous-time world is typical for acquisition devices in many areas of science and technology, including scientific measurements, medical and biological signal processing and analog-to-digital converters.

The key question is of course if the samples $x[n]$ are a faithful representation of the original signal $x(t)$. If so, how can we reconstruct $x(t)$ from $x[n]$, and if not, what approximation $\hat{x}(t)$ do we get based on the samples $x[n]$? This question is at heart of signal processing, and the dominant result is the well-known sampling theorem of Whittaker, Kotelnikov and Shannon which states that if $x(t)$ is bandlimited, or $X(\omega)=0, |\omega|>\omega_m$, then samples $x[n]=x(nT)$ with $T\leq\pi/\omega_m$ are sufficient to reconstruct $x(t)$ [4], [6], [9]. The reconstruction formula is given by

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT)\, sinc(t/T - n), \qquad (2)$$

with

$$sinc(t) = \frac{sin(\pi t)}{\pi t}. \qquad (3)$$

If $x(t)$ is not bandlimited, convolution with $\tilde{\varphi}(t) = sinc(t/T)$ (an ideal lowpass filter with support $[-\pi/T, \pi/T]$) allows to apply sampling and reconstruction of $\hat{x}(t)$, the lowpass approximation of $x(t)$, or the restriction of $X(\omega)$ to the interval $[-\pi/T, \pi/T]$.

A possible interpretation of the interpolation formula (2) is the following. Any real bandlimited signal can be seen as having $1/T$ degrees of freedom per unit of time, which is the number of samples per unit of time that specify it. In the present paper, this **number of degrees of freedom per unit of time** is called the **rate of innovation** of a signal [1] , and is denoted by ρ. In the bandlimited case above, the rate of innovation is $\rho = 1/T = \omega_m/\pi$. In the sequel, we are interested in signals that have a **finite rate of innovation**, either on intervals, or on average. Take a Poisson process, which generates Diracs with independent and identically distributed (i.i.d.) interarrival times, the distribution being exponential with probability density function $\mu e^{-\mu t}$. The expected interarrival time is given by $1/\mu$. Thus, the rate of innovation is μ, since on average, μ real numbers per unit of time fully describe the process.

Given a signal with finite rate of innovation, it seems attractive to sample it with a rate of ρ samples per unit of time. We know it will work with bandlimited signals, but will it work with a larger class of signals? Thus, the natural questions to pursue are the following:

1. What classes of signals of finite rate of innovation can be sampled uniquely, especially using uniform sampling ?
2. What kernels $\varphi(t)$ allow for such sampling schemes?
3. What algorithms allow the reconstruction of the signal based on the samples?

In the present paper, we concentrate on stream of Diracs and on piecewise polynomials, which are classes for which we are able to derive sampling theorems under certain condi-

[1] This is different from the rate used in rate-distortion theory [1]. Here, rate corresponds to a degree of freedom that are specified by real numbers. In rate-distortion, the rate corresponds to bits.

tions. The kernels involved are related to the sinc (the bandlimited derivative), the Gaussian, and the spline kernels. The algorithms, while more complex than the standard sinc sampling of bandlimited signals, are still reasonable (structured linear systems) but also often involve root finding.

The outline of the paper is as follows. Section II formally defines signals with finite rate of innovation. Section III and Section IV consider periodic signals in discrete and continuous time respectively, and derive sampling theorems for streams of Diracs and piecewise polynomials. Both of these type of signals have a finite number of degrees of freedom, and a sampling rate that is sufficiently high to capture these degrees of freedom, together with appropriate sampling kernels, allows perfect reconstruction. Section V addresses the sampling of finite length signals having a finite number of degrees of freedom, using infinitely supported kernels like the sinc kernel and the Gaussian kernel. Again, if the critical number of samples is taken, we can derive a sampling theorem. Section VI concentrates on local reconstruction schemes. Given that the local rate of innovation is bounded, local reconstruction is possible, using for example spline kernels. Finally, Section VII derives applications of the above results, in particular to piecewise bandlimited signals, and to filtered streams of Diracs.

In the Appendix, we introduce the "annihilating filter" method borrowed from spectral analysis. This method will be referred to in all of the proofs in Section III, Section IV as well as one in Section V-B.

II. Signals with finite rate of innovation

In the introduction, we have informally discussed the intuitive notion of signals with **finite rate of innovation**. More formally, consider functions or signals having a parametric representation.[2] Then:

Definition 1: The rate of innovation ρ is the average number of degrees of freedom per unit of time, or, with $C_x(t_0, t_1)$ giving the number of degrees of freedom of $x(t)$ over the interval $[t_0, t_1]$,

$$\rho = \lim_{\tau \to \infty} \frac{1}{\tau} C_x(-\frac{\tau}{2}, \frac{\tau}{2}). \tag{4}$$

[2] In the sequel, we consider real signals unless specified otherwise.

Definition 2: A signal with finite rate of innovation is such that $\rho < \infty$.

If we consider finite length or periodic signals of length τ, then the number of degrees of freedom is finite, and the rate of innovation is $1/\tau \, C_x[0, \tau]$.

Bandlimited signals with frequency support $[-\pi/T, \pi/T]$ have a rate of innovation $\rho = 1/T$ since they are uniquely specified by samples taken every T seconds. Bandpass signals having support $(-\omega_0 - \Delta, -\omega_0] \cup [\omega_0, \omega_0 + \Delta]$ have a rate of innovation $\rho = \Delta/\pi$, since appropriate demodulation allows sampling with a period $T = 2\pi/2\Delta$.

If we consider discrete-time sequences, then general sequences have a (normalized) rate of innovation of 1 (one degree of freedom per sample). If the underlying sampling rate is taken into account, then we have again a rate of innovation $\rho = 1/T$.

A stream of Diracs in discrete-time with K locations over an interval of size N has a rate of innovation of the order K/N,[3] but these are degrees of freedom over the integers. A piecewise polynomial in discrete-time has thus a combination of integer and real degrees of freedom.

One can also define a **local** rate of innovation with respect to a moving window of size τ.

Definition 3: Given a window of size τ, the local rate of innovation at time t is

$$\rho_\tau(t) = \frac{1}{\tau} C_x(t - \tau/2, t + \tau/2).$$

In this case, one is often interested in the maximal local rate, or $\rho_m(\tau)$

$$\rho_m(\tau) = \max_{t \in \mathbb{R}} \rho_\tau(t).$$

As $\tau \to \infty$, $\rho_m(\tau)$ tends to ρ. To illustrate the differences between ρ and ρ_m, consider again the Poisson process with expected interarrival time $1/\mu$. The rate of innovation ρ is given by μ. However, for any finite τ, there is no bound on $\rho_m(\tau)$, even though its expected value is μ.

While one can define many parametric signals which have a finite rate of innovation, in the sequel we will concentrate on streams of Diracs and piecewise polynomials which are classes for which we are able to give sampling theorems and reconstruction formulae.

Combinations of bandlimited signals and piecewise polynomials are also of interest, as are

[3] Actually, slightly less because there can only be one Dirac at any one location.

filtered versions of stream of Diracs.

III. Discrete-time periodic signals with finite rate of innovation

The discrete-time periodic signals we consider are streams of weighted Diracs and piecewise polynomials. Through appropriate differentiation, piecewise polynomials can be reduced to streams of Diracs, so we begin with these.

A. Stream of Diracs

Consider a discrete-time periodic signal, with one period given by

$$\mathbf{x} = (x[0], x[1], \ldots, x[N-1])^T \tag{5}$$

and containing K weighted Diracs at locations $\{n_0, n_1, \ldots, n_{K-1}\}$, $n_k \in [0, N-1]$ and $K < \lfloor N/2 \rfloor$,

$$x[n] = \sum_{k=0}^{K-1} c_k \, \delta[n - n_k], \tag{6}$$

where $\delta[n]$ is the Kronecker delta and equal to 1 if $n = 0$ and 0 if $n \neq 0$.

Denote by $\mathbf{X} = (X[0], X[1], \ldots, X[N-1])^T$ the discrete-time Fourier series (DTFS) coefficients of $\mathbf{x}$ where

$$X[m] = \sum_{k=0}^{K-1} c_k \, W_N^{n_k m}, \quad m = 0, \ldots, N-1 \tag{7}$$

and $W_N = e^{-i2\pi/N}$.

Consider filtering the signal $x[n]$ with a lowpass filter $\tilde{\varphi}[n] = \varphi[-n]$ with bandwidth $[-K, K]$ then the sample values $y_s[l]$ are simply a subsampled version (by M) of the filtered signal $y[n] = x[n] * \tilde{\varphi}[n]$. The DTFS coefficients of $y[n]$ are given by

$$Y[m] = \begin{cases} X[m] & \text{if } m \in [-K, K] \\ 0 & \text{else} \end{cases} \tag{8}$$

and those of the subsampled signal $y_s[l] = y[lM]$ are given by the usual subsampling formula

$$Y_s[m] = \frac{1}{M} \sum_{l=0}^{M-1} Y[(m + lN)/M]. \tag{9}$$

With appropriate re-indexing it follows that

$$Y_s[m] = \frac{1}{M} X[m], \quad m \in [-K, K]. \tag{10}$$

Figure 2 illustrates that we can recover $2K$ spectral values $X[m]$ of the original signal from the subsampled spectra of the lowpass approximation $Y_s[m]$ as long as there is no overlapping in the spectra of the lowpass approximation $Y[m]$ and this occurs only if $N/M \geq 2K$.

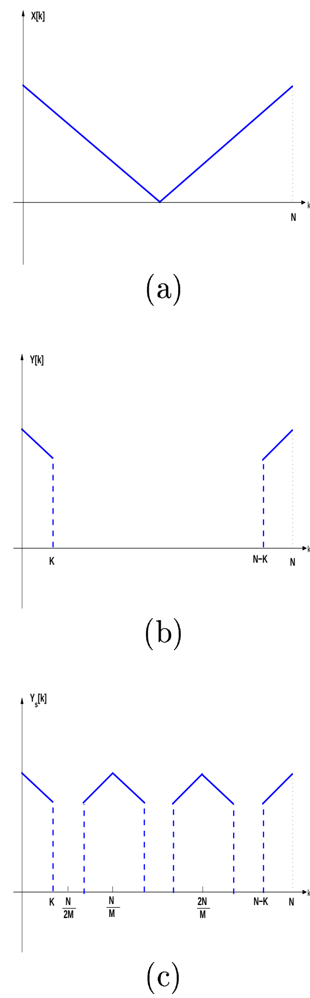


Fig. 2. (a)DTFS of stream of Diracs, $X[k], k \in [0, N]$; (b) DTFS of lowpass approximation $Y[k] = X[k], k \in [-K, K], 0$ otherwise; (c) DTFS of lowpass approximation subsampled by $M = 3$, $Y_s[k] = 1/MX[k], k \in [-K, K]$.

This leads us to

Proposition 1: Consider a discrete-time periodic signal $x[n]$ of period N containing K weighted Diracs. Let M be an integer divisor of N satisfying $N/M \geq 2K+1$. Consider the discrete-time periodized sinc sampling kernel $\varphi[n] = \frac{1}{N} \sum_{m=-K}^{K} W_N^{-mn}$, that is, the inverse DTFS of the $Rect_{[-K,K]}$. Then the $N/M \in \mathbb{N}$ samples defined by

$$y_s[l] \quad = \quad < x[n], \varphi[n - lM] >_{circ}, \quad l = 0, \ldots, N/M - 1 \tag{11}$$

are a sufficient representation of the signal.

Proof: We start by showing that the DTFS coefficients $X[m], m \in [-K, K]$ are sufficient to determine the stream of K weighted Diracs . Then we show that the N/M samples $y_s[l]$ are a sufficient representation of $X[m], m \in [-K, K]$.

1. Since $X[m]$ is a linear combination of K complex exponentials, u_k^m, with $u_k = W_N^{n_k}$, the locations n_k of the Diracs can be found using the annihilating filter method described in Appendix A. It suffices to determine the annihilating filter $H(z)$ whose coefficients are $(1, H[1], \ldots, H[K])$ or

$$H(z) = 1 + H[1]\, z^{-1} + H[2]\, z^{-2} + \cdots + H[K]\, z^{-K} \tag{12}$$

which factors as

$$H(z) = \prod_{k=0}^{K-1} (1 - z^{-1} W_N^{n_k}) \tag{13}$$

and satisfies

$$\sum_{k=0}^{K} H[k]\, X[m-k] = 0, \quad m = 0, \ldots, N-1 \tag{14}$$

Since $H[0] = 1$, K equations (14) will be sufficient to determine the K unknown filter coefficients $H[k], k = 1, \ldots, K$. Let $m = 1, \ldots, K$ then the system in (14) is equivalent to

$$\sum_{k=1}^{K} H[k]\, X[m-k] = -X[m], \quad m = 1, \ldots, K. \tag{15}$$

For example take $N = 8$, $K = 3$ and let $m = 1, 2, 3$ then in matrix/vector form the system is

$$\begin{bmatrix} X[0] & X[-1] & X[-2] \\ X[1] & X[0] & X[-1] \\ X[2] & X[1] & X[0] \end{bmatrix} \cdot \begin{pmatrix} H[1] \\ H[2] \\ H[3] \end{pmatrix} \quad = \quad - \begin{pmatrix} X[1] \\ X[2] \\ X[3] \end{pmatrix}. \tag{16}$$

Given that these are K sinusoids the matrix in (16) is full rank ($= K$) and thus there is a unique solution $H[1], \dots, H[K]$. The set of locations $\{n_0, n_1, \dots, n_{K-1}\}$ are given by the the zeros of $H(z)$.

The weights of the Diracs are obtained by solving K equations in (7), let $m = 0, \dots, K-1$, this leads to the following Vandermonde system

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ W_N^{n_0} & W_N^{n_1} & \dots & W_N^{n_{K-1}} \\ \vdots & \vdots & \dots & \vdots \\ W_N^{n_0(K-1)} & W_N^{n_1(K-1)} & \dots & W_N^{n_{K-1}(K-1)} \end{bmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{K-1} \end{pmatrix} = \begin{pmatrix} X[0] \\ X[1] \\ \vdots \\ X[K-1] \end{pmatrix} \tag{17}$$

and has a unique solution since the $n_k \neq n_l, \forall k \neq l$.

Therefore, given $2K$ contiguous DTFS coefficients

$$\{X[-K+1], X[-K+2], \dots, X[0], \dots, X[K]\}$$

we have found a unique set of locations $\{n_k\}_{k=0}^{K-1}$ and a unique set of weights $\{c_k\}_{k=0}^{K-1}$.

2. We need to show that $2K$ spectral values $X[m], m \in [-K, K]$ can be obtained from the N/M sample values $y_s[l]$ defined in (11).

We substitute the discrete-time periodized sinc kernel in the expression of the sample values and we obtain the following:

$$\begin{aligned} y_s[l] &= < x[n], \varphi[n - lM] >_{circ} \quad l = 0, \dots, N/M - 1 && (18) \\ &= \sum_{n=0}^{N-1} x[n]\, \varphi[n - lM] && (19) \\ &= \frac{1}{N} \sum_{n=0}^{N-1} x[n] \sum_{m=-K}^{K} W_N^{-m(n-lM)} && (20) \\ &= \frac{1}{N} \sum_{n=0}^{N-1} x[n] \sum_{m=-K}^{K} W_N^{-mn}\, W_N^{mlM} && (21) \\ &= \frac{1}{N} \sum_{m=-K}^{K} W_{N/M}^{ml} \underbrace{\sum_{n=0}^{N-1} x[n]\, W_N^{-nm}}_{X[-m]} && (22) \\ &= \frac{1}{N} \sum_{m=-K}^{K} X[-m]\, W_{N/M}^{ml}. && (23) \end{aligned}$$

If we calculate the DTFS coefficients of the sample values $y_s[l]$ we obtain an expression in terms of the DTFS of the signal,

$$\begin{aligned} Y_s[k] &= \sum_{l=0}^{N/M-1} y_s[l]\, W_{N/M}^{lk}, \quad k = 0, \dots, N/M-1 && (24) \\ &= \frac{1}{N} \sum_{l=0}^{N/M-1} \sum_{m=-K}^{K} X[-m]\, W_{N/M}^{ml} W_{N/M}^{lk} && (25) \\ &= \frac{1}{N} \sum_{m=-K}^{K} X[-m] \underbrace{\sum_{l=0}^{N/M-1} W_{N/M}^{l(k+m)}}_{=\begin{cases} N/M & \text{if } k+m=0 \\ 0 & \text{otherwise} \end{cases}} && (26) \\ &= \frac{1}{M} X[k], \quad k = 0, \dots, \min\{K, N/M-1\} && (27) \\ \Rightarrow X[k] &= M Y_s[k], \quad k = 0, \dots, K && (28) \end{aligned}$$

by hypothesis, $N/M \geq 2K+1 > K$. Since we are dealing with real signals the DTFS is Hermitian, that is, $X[-k] = X^*[k]$, $k = 0, \dots, K$, so we have the $2K+1$ spectral values $X[k]$, $k \in [-K, K]$ obtained from the N/M DTFS coefficients of the sample values $y_s[l]$. Therefore we have a sufficient number of spectral values which uniquely define the stream of weighted Diracs.

■

Figure 3 illustrates in time and frequency domain the sampling of a discrete-time periodic stream of Diracs with period $N = 256$ and $K = 15$ weighted Diracs. The signal is perfectly reconstructed within machine precision, $MSE = 10^{-11}$.

Note that in the proof of Proposition 1 the locations of the Diracs are determined by finding the roots of the annihilating filter $H(z)$. If the locations are bunched up or there are a large number of Diracs then finding the roots of the polynomial is numerically unstable. An alternative method that is commonly used in error correction coding involves extrapolating the $N-K$ spectral values of the signal using K first spectral $X[k], k = 1, \dots, K$ components and the error locating polynomial which in our case corresponds to the annihilating filter

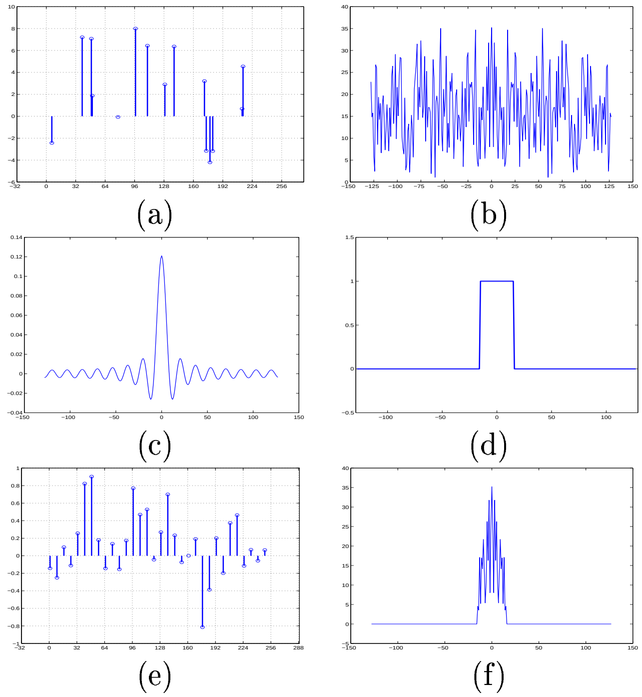


Fig. 3. (a) Periodic discrete-time signal with $K = 15$ weighted Diracs with period $N = 256$;(b) DTFS $X[m]$; (c) Discrete-time periodized sinc sampling kernel, $\varphi[n]$; (d) DTFS $Rect_{[-K,K]}$, $K = 15$(e) Sample values $y_s[l] =< x[n], \varphi[n - lM] >, l = 0, \ldots, 31$ with $M = 8$; (f) DTFS $\mathbf{Y}_s$.

$H[k], k = 1, \ldots, K,$

$$X[k] = -\sum_{l=1}^{K} H[l]\, X[k-l], \quad k = K+1, \ldots, N-K. \tag{29}$$

Consider a signal of length $N = 64$ where there are $K = 16$ Diracs in an interval of size $2K$, see Figure 4.

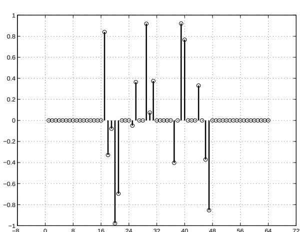


Fig. 4. Stream of $K = 16$ bunched Diracs with period $N = 64$.

Figure 5 compares the relative reconstruction error between the root finding method and the spectral extrapolation method for different values of K.

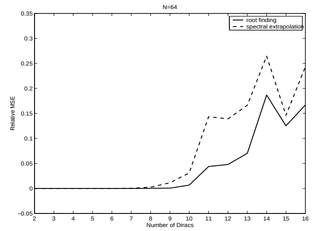


Fig. 5. Comparison between the root finding method and the spectral extrapolation method on a signal of length $N = 64$, K varying between 2 and 16 on interval $2K$, 100 simulations.

B. Piecewise polynomials of degree R

The previous result on the stream of Diracs is extended to piecewise polynomials. Consider a discrete-time periodic piecewise polynomial defined by [4]

$$x[n] = \frac{1}{R!} \sum_{k=0} c_k \, (n - n_k)_+^R \tag{30}$$

of period N with K pieces each with maximum degree R. Suppose a discrete-time difference operator $d[n] = \delta[n] - \delta[n-1]$ is applied $R+1$ times to the piecewise polynomial signal. The differentiated signal $x^{R+1}[n]$ in frequency domain is

$$X^{(R+1)}[m] = (D[m])^{R+1} \, X[m], \quad m = 0, \ldots, N-1 \tag{31}$$

where $D[m] = 1 - W_N^m$ is the DTFS of the discrete-time difference operator. This results in putting to zero all the polynomial pieces. Assume there are discontinuities between pieces (but no Diracs), then K transitions can lead to at most $K(R+1)$ weighted Diracs and thus the rate of innovation is $\rho = 2K(R+1)/N$. From Proposition 1 we can uniquely recover the $K(R+1)$ Diracs from $2K(R+1)$ DTFS coefficients of the differentiated signal $X^{(R+1)}[k]$. The piecewise polynomial signal is reconstructed by applying the inverse discrete-time difference operator $R+1$ times on the stream of weighted Diracs. The discrete-time difference operator $d[n]$ is a singular operator (since $D[0] = 0$) and so we define the inverse discrete-time difference operator as $D^{-1}[m] = 0$ for $m = 0$ and $D^{-1}[m] = (1 - W_N^m)^{-1}$ for $m = 1, \ldots, N-1$. Hence instead of using the sinc sampling kernel $\varphi[n]$ we will use the derivative sinc sampling kernel defined by $\psi[n] = (\underbrace{d * d * \cdots * d}_{R+1} * \varphi)[n]$ which has at least $R+1$ zeros at the origin $z = 1$. Then the DTFS of $\psi[n]$ is

$$\Psi[m] \quad = \quad (1 - W_N^m)^{R+1} \, \Phi[m], \quad m = 0, \ldots, N-1 \tag{32}$$

[4] $n_+ = n$, if $n \geq 0$, and 0 else.

where $\Phi[m]$ is the $Rect_{[-K(R+1),K(R+1)]}$ function. This brings us to the following theorem.

Theorem 1: Consider a discrete-time periodic piecewise polynomial signal of period N with K pieces of degree R and with zero mean. [5] Let M be an integer and a divisor of N such that $N/M \geq (2K(R+1)+1)$. Take a sampling kernel $\psi[n]$ with DTFS coefficients defined in (32). Then we can recover the signal from the $N/M \in \mathbb{N}$ samples

$$y_s[l] \quad = \quad < x[n], \psi[n - lM] > \qquad l = 0, \ldots, N/M - 1. \tag{33}$$

Proof: First we show that $2K(R+1)$ DTFS coefficients of the signal, $X[m], m \in [-K(R+1), K(R+1)]$ are sufficient to determine the piecewise polynomial signal, $x[n]$. Then we show that the N/M samples $y_s[l]$ are sufficient to determine the $(2K(R+1)+1)$ values $X[m]$.

1. If we have the DTFS coefficients $X[m], m \in [-K(R+1), K(R+1)]$ then from (31) we have the DTFS coefficients of the $(R+1)$th discrete-time differentiated signal, $X^{(R+1)}[m]$. From Proposition 1 these are sufficient to reconstruct the stream of $K(R+1)$ Diracs. Thus, the signal is recovered by applying $R+1$ times the inverse discrete-time difference operator, $d^{-1}[n]$, on the stream of Diracs, that is,

$$x[n] = \big(\underbrace{d^{-1} * d^{-1} * \cdots * d^{-1}}_{R+1} * x^{(R+1)}\big)[n].$$

2. Similar to the second part in the proof of Proposition 1 we expand the inner product between the piecewise polynomial signal and the differentiated sinc sampling

[5] We consider zero mean signals since $D \circ D^{-1}$ is a projector on the space of signals having zero mean.

kernel:

$$\begin{aligned}
y_s[l] &= < x[n], \psi[n - lM] >, \quad l = 0, \ldots, N/M - 1 && (34)\\
&= \sum_{n=0}^{N-1} x[n]\, \psi[n - lM] && (35)\\
&= \frac{1}{N} \sum_{n=0}^{N-1} x[n] \sum_{m=-K(R+1)}^{K(R+1)} (1 - W_N^m)^{R+1}\, W_N^{-m(n-lM)} && (36)\\
&= \frac{1}{N} \sum_{n=0}^{N-1} x[n] \sum_{m=-K(R+1)}^{K(R+1)} (1 - W_N^m)^{R+1}\, W_N^{-mn}\, W_N^{mlM} && (37)\\
&= \frac{1}{N} \sum_{m=-K(R+1)}^{K(R+1)} (1 - W_N^m)^{R+1} W_{N/M}^{ml} \underbrace{\sum_{n=0}^{N-1} x[n]\, W_N^{-nm}}_{X[-m]} && (38)\\
&= \frac{1}{N} \sum_{m=-K(R+1)}^{K(R+1)} (1 - W_N^m)^{R+1}\, X[-m]\, W_{N/M}^{ml}. && (39)
\end{aligned}$$

Taking the DTFS of the sample values $y_s[l]$ we obtain

$$\begin{aligned}
Y_s[k] &= \sum_{l=0}^{N/M-1} y_s[l]\, W_{N/M}^{lk}, \quad k = 0, \ldots, N/M - 1 && (40)\\
&= \frac{1}{N} \sum_{l=0}^{N/M-1} \sum_{m=-K(R+1)}^{K(R+1)} (1 - W_N^m)^{R+1}\, X[-m]\, W_{N/M}^{ml} W_{N/M}^{lk} && (41)\\
&= \frac{1}{N} \sum_{m=-K(R+1)}^{K(R+1)} (1 - W_N^m)^{R+1}\, X[-m] \underbrace{\sum_{l=0}^{N/M-1} W_{N/M}^{l(k+m)}}_{= \begin{cases} N/M & \text{if } k+m=0 \\ 0 & \text{otherwise} \end{cases}} && (42)\\
&= \frac{1}{M} (1 - W_N^{-k})^{R+1}\, X[k], \quad k = 0, \ldots, \min\{N/M, K(R+1)\} && (43)\\
\Rightarrow X[k] &= \begin{cases} M\left[(1 - W_N^{-k})^{R+1}\right]^{-1} Y_s[k] & \text{for } k = 1, \ldots, K(R+1) \\ 0 & \text{for } k = 0 \end{cases}. && (44)
\end{aligned}$$

Since $N/M \geq (2K(R+1) + 1)$ we have a sufficient representation for the spectral values of the signal. This completes the proof.

■

Figure 6 illustrates the reconstruction of a discrete-time periodic piecewise linear signal of period $N = 1024$ with $K = 6$ pieces.

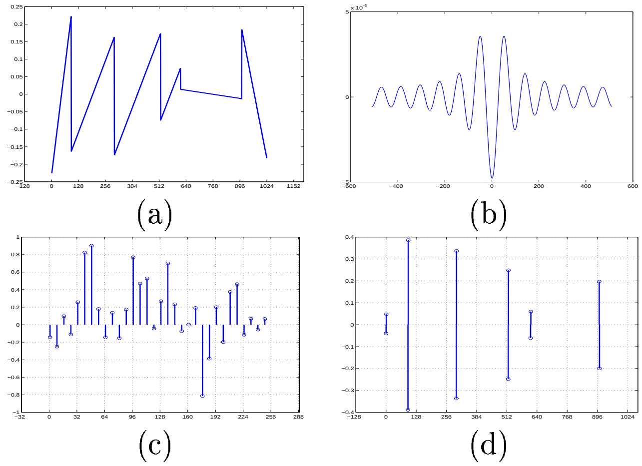


Fig. 6. (a) Discrete-time periodic piecewise linear ($R = 1$) signal of period $N = 1024$ with $K = 6$ pieces; (b) Differentiated sinc sampling kernel,$\psi[n] = d[n] * d[n] * \varphi[n]$ with DTFS $D[m] \cdot Rect_{[-K(R+1),K(R+1)]}$ (c) Sample values $y_s[l] =< x[n], \psi[n - lM] >, l = 0, \ldots, 31$ with $M = 32$; (d) Stream of $K(R + 1) = 12$ Diracs obtained from $X[m], m \in [-K(R + 1), K(R + 1)]$.

IV. Continuous-time periodic signals with finite rate of innovation

We derive now the equivalent results but for continuous-time periodic signals, again building up from a stream of Diracs to piecewise polynomials. We will put in evidence the common points.

A. Stream of Diracs

Consider a continuous-time periodic signal $x(t)$ of period τ containing K weighted Diracs at locations $\{t_k\}_{k=0}^{K-1}$ with $t_k \in [0, \tau)$, or

$$\begin{aligned} x(t) &= \sum_{n\in\mathbb{N}} c_n\, \delta(t - t_n) \\ &= \sum_{n\in\mathbb{N}} \sum_{k=0}^{K-1} c_k\, \delta(t - (t_k + n\tau)) \end{aligned} \tag{45}$$

since $t_{n+K} = t_n + \tau$ and $c_{n+K} = c_n$ for all $n \in \mathbb{N}$.

The continuous-time Fourier series (CTFS) coefficients of $x(t)$ are defined by

$$\begin{aligned} X[m] &= \frac{1}{\tau} \int_0^{\tau} x(t)\, e^{-i2\pi tm/\tau}\, dt, \quad m \in \mathbb{Z} \\ &= \frac{1}{\tau} \sum_{k=0}^{K-1} c_k\, e^{-i2\pi t_k m/\tau}. \end{aligned} \tag{46}$$

If the signal $x(t)$ is convolved with a sinc filter of bandwidth $[-K, K]$ then we have a lowpass approximation $y(t)$ given by

$$y(t) = \sum_{m=-K}^{K} X[m]\, e^{i2\pi mt/\tau}. \tag{47}$$

Suppose the lowpass approximation $y(t)$ is sampled at multiples of T, we obtain $\tau/T \in \mathbb{N}$ samples defined by

$$y_s[l] = y(lT) = \sum_{m=-K}^{K} X[m]\, e^{i2\pi mlT/\tau}, \quad l = 0, \ldots, \tau/T - 1. \tag{48}$$

Similar to the discrete-time case as long as the number of samples is larger than the number of values in the spectral support of the lowpass signal, that is, $\frac{\tau}{T} \geq 2K+1$, (48) can be used to recover $2K+1$ values of $X[m]$. Thus we can state:

Proposition 2: Consider a continuous-time periodic stream of K weighted Diracs with period τ and a continuous-time periodic sinc sampling kernel $\varphi(t)$ with bandwidth $[-K, K]$. Taking a sampling period T such that $\tau/T \in \mathbb{N}$ and $\tau/T \geq 2K+1$. Then the samples defined by

$$y_s[l] = < x(t), \varphi(t - lT) >, \quad l = 0, \ldots, \tau/T - 1 \tag{49}$$

are a sufficient representation of $x(t)$.

Proof: Similar to the discrete-time case first we show that $2K+1$ CTFS coefficients $X[m]$ are sufficient to find the locations and the weights of the Diracs. From (46) we have that the CTFS coefficients $X[m]$ are linear combinations of complex exponentials. Thus to find the locations t_k we need to find the annihilating filter $\mathbf{H} = (1, H[1], H[2], \ldots, H[K])$ such that

$$\mathbf{H} *_c \mathbf{X} = 0. \tag{50}$$

This is the same Toeplitz system as in (14) considered in Sec. III-A and therefore a solution exists. Factoring the $z-$transform of $\mathbf{H}$, or $H(z) = \sum_{k=0}^{K} H[k]\, z^{-k}$, into

$$H(z) = \prod_{k=0}^{K-1} (1 - z^{-1} u_k), \tag{51}$$

we then find the K locations $\{t_0, t_1, \ldots, t_{K-1}\}$ from the zeros of $H(z)$, that is, from

$$u_k = e^{-i2\pi t_k/\tau}. \tag{52}$$

Given the locations $\{t_k\}_{k=0}^{K-1}$ and K values $X[m], m = 0, \ldots, K-1$, we find the weights $\{c_k\}_{k=0}^{K-1}$ of the Diracs by solving the Vandermonde system in (46). Since the locations t_k are distinct, $t_k \neq t_l, \forall k \neq l$, the Vandermonde system admits a solution.

The second part of the proof consists in showing that the τ/T samples $y_s[l]$ are sufficient to determine the CTFS coefficients $X[m], m \in [-K, K]$. We substitute the continuous-time periodic sinc function $\varphi(t)$ with bandwidth $[-K, K]$ defined by

$$\varphi(t) = \sum_{m=-K}^{K} e^{i2\pi mt}. \tag{53}$$

in (49) and we obtain

$$\begin{aligned} y_s[l] &= < x(t), \varphi(t - lT) >, \quad l = 0, \ldots, \tau/T - 1 & (54)\\ &= \int_0^\tau x(t) \sum_{m=-K}^{K} e^{i2\pi m(t-lT)/\tau}\, dt & (55)\\ &= \sum_{m=-K}^{K} e^{-i2\pi ml/(\tau/T)} \underbrace{\int_0^\tau x(t) e^{i2\pi mt/\tau}\, dt}_{\tau X[-m]} & (56)\\ &= \tau \sum_{m=-K}^{K} X[-m]\, e^{-i2\pi mlT/\tau}. & (57) \end{aligned}$$

Note that $y_s[l]$ is periodic with period τ/T, thus the DTFS coefficients are $Y_s[k] = TX[k], k = 0, \ldots, \tau/T - 1$. Since $\tau/T \geq 2K+1$, we have a sufficient number of samples that determine the CTFS $X[m], m \in [-K, K]$. ■

B. Piecewise polynomials of degree R

Here we consider continuous-time periodic piecewise polynomial signal of period τ, containing K pieces of maximum degree R and $R-1$ continuous derivatives, $\mathcal{C}^{R-1}$,

$$x(t) = \frac{1}{R!} \sum_{k=0}^{K-1} c_k\, (t - t_k)_+^R, \quad t \in [0, \tau]. \tag{58}$$

We differentiate the signal $R+1$ times and we obtain a continuous-time stream of K weighted Diracs, $x^{(R+1)}(t)$. The CTFS of the derivative operator is defined by $D[m] = i2\pi m, m \in \mathbb{Z}$ and therefore the CTFS coefficients of the differentiated signal $x^{(R+1)}(t)$ are equal to

$$X^{(R+1)}[m] = (i2\pi m)^{R+1} X[m], \quad m \in \mathbb{Z}. \tag{59}$$

From Proposition 2 we can recover the continuous-time periodic stream of K Diracs from the CTFS coefficients, $X^{(R+1)}[m], m \in [-K, K]$. Therefore we can sample the signal with the differentiated sinc sampling kernel whose CTFS coefficients are defined by

$$\Psi[m] = (i2\pi m)^{R+1} \Phi[m], \quad m \in \mathbb{Z} \tag{60}$$

where $\Phi[m] = Rect_{[-K,K]}$ is the CTFS of the continuous-time periodized sinc sampling kernel.

Theorem 2: Consider a continuous-time periodic piecewise polynomial signal $x(t)$ with period τ, containing K pieces of maximum degree R, belonging to $\mathcal{C}^{R-1}$ and having zero mean. Consider a sampling kernel $\psi(t)$ with its CTFS coefficients defined in (60). Let $\tau/T \in \mathbb{N}$ and $\tau/T \geq 2K+1$. Then $x(t)$ can be uniquely recovered from the τ/T samples

$$y_s[l] = < x(t), \psi(t - lT) >, \quad l = 0, \ldots, \tau/T - 1. \tag{61}$$

Proof: Similar to the proof of Theorem. 1, we first show that CTFS coefficients $X[m], m \in [-K, K]$ are sufficient to determine the piecewise polynomial signal, $x(t)$. Then we show that the τ/T samples $y_s[l]$ are sufficient to determine the values $X[m], m \in [-K, K]$.

1. If we have the CTFS coefficients $X[m], m \in [-K, K]$ then from (59) we have the CTFS coefficients of the $(R+1)$th differentiated signal, $X^{(R+1)}[m]$. From Proposition 2 these are sufficient to reconstruct the stream of K Diracs. Thus, the signal is recovered by integrating $R+1$ times the stream of Diracs, that is,

$$x(t) = \underbrace{\int \int \cdots \int}_{R+1} x^{(R+1)}(t)\, dt\, dt \cdots dt$$

or in frequency domain from (59)

$$\begin{aligned} X[m] &= (D^{-1}[m])^{R+1} X^{(R+1)}[m], \quad m \in \mathbb{Z}/\{0\} & (62) \\ &= (i2\pi m)^{-(R+1)} X^{(R+1)}[m], \quad m \in \mathbb{Z}/\{0\} & (63) \end{aligned}$$

with $D^{-1}[m] = 0$ for $m = 0$ and thus

$$x(t) = \sum_{m\in\mathbb{Z}} X[m]\, e^{i2\pi mt/\tau}.$$

2. Similar to the second part in the proof of Proposition 2 we expand the inner product between the piecewise polynomial signal and the differentiated sinc sampling kernel defined by $\psi(t) = \sum_{m=-K}^{K} (i2\pi m)^{R+1} e^{i2\pi mt/\tau}$. That is, the sample values are

$$\begin{aligned}
y_s[l] &= < x(t), \psi(t-lT) >, \quad l = 0, \dots, \tau/T - 1 && (64)\\
&= \int_0^\tau x(t) \sum_{m=-K}^{K} (i2\pi m)^{R+1} e^{i2\pi m(t-lT)/\tau}\, dt && (65)\\
&= \sum_{m=-K}^{K} (i2\pi m)^{R+1}\, e^{-i2\pi mlT/\tau} \underbrace{\int_0^\tau x(t)\, e^{i2\pi mt/\tau}\, dt}_{\tau X[-m]} && (66)\\
&= \tau \sum_{m=-K}^{K} X[-m]\, (i2\pi m)^{R+1}\, e^{-i2\pi ml/(\tau/T)}. && (67)
\end{aligned}$$

Since $y_s[l]$ is periodic with period τ/T, the DTFS coefficients of $y_s[l]$ are given by

$$\begin{aligned}
Y_s[k] &= \sum_{l=0}^{\tau/T-1} y_s[l]\, e^{-i2\pi kl/(\tau/T)} && (68)\\
&= \tau \sum_{l=0}^{\tau/T-1} \sum_{m=-K}^{K} X[-m]\, (i2\pi m)^{R+1}\, e^{-i2\pi ml/(\tau/T)}\, e^{-i2\pi kl/(\tau/T)} && (69)\\
&= \tau \sum_{m=-K}^{K} X[-m]\, (i2\pi m)^{R+1} \underbrace{\sum_{l=0}^{\tau/T-1} e^{-i2\pi(k+m)l/(\tau/T)}}_{=\begin{cases} \tau/T & \text{if } k+m=0 \\ 0 & \text{otherwise} \end{cases}} && (70)\\
&= \frac{\tau^2}{T}\, (-i2\pi m)^{R+1}\, X[k] && (71)
\end{aligned}$$

Therefore the CTFS coefficients of the signal are obtained by the DTFS coefficients of the samples values $Y_s[m], m = 0, \dots, \tau/T - 1$ and are defined by

$$X[m] = \begin{cases} T\, Y_s[m]/(\tau^2\, (-i2\pi m)^{R+1}) & \text{for } m = 1, \dots, \tau/T - 1 \\ 0 & \text{for } m = 0 \end{cases}. \qquad (72)$$

Since $\tau/T \geq 2K + 1$ the sample values are a sufficient representation of the spectral values of the signal. This completes the proof.

■

Note that removing the restriction $x(t) \in \mathcal{C}^{R-1}$ leads to the same result as in Theorem. 1.

V. Finite length signals with finite rate of innovation

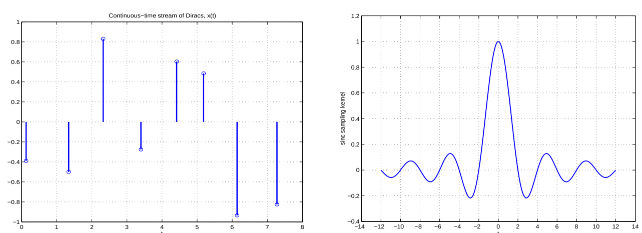
A finite length signal with finite rate of innovation ρ clearly has a finite number of degrees of freedom. The question of interest is: Given a sampling kernel with *infinite support*, is there a *finite set of samples* that uniquely specifies the signal? In the following sections we will sample signals with finite number of weighted Diracs with infinite support sampling kernels such as the sinc and Gaussian.

A. Sinc sampling kernel

Consider a continuous-time signal with a finite number of weighted Diracs

$$x(t) = \sum_{k=0}^{K-1} c_k \, \delta(t - t_k) \tag{73}$$

and an infinite length sinc sampling kernel, see Figure 7.



(a) (b)

Fig. 7. (a) Example of a finite length continuous-time stream of $K = 8$ Diracs randomly spread on an interval $[0, \tau]$ with $\tau = 8$; (b) Sinc sampling kernel, $\operatorname{sinc}(t/T), T = 2$.

The sample values are obtained by filtering the signal with a $\operatorname{sinc}(t/T), t \in \mathbb{R}$, sampling kernel. This is equivalent to taking the inner product between the signal and a shifted version of the sinc

$$y_n = < x(t), \operatorname{sinc}(t/T - n) > \tag{74}$$

where $\operatorname{sinc}(t) = \sin(\pi t)/\pi t$. The question that arises is: How many of these samples do we need to recover the signal? The signal has $2K$ degrees of freedom, K from the weights and K from the locations of the Diracs and thus N samples, $N \geq 2K$, will be sufficient to recover the signal. Similar to the previous cases, the reconstruction method will require solving two systems of linear equations: one for the locations of the Diracs and the second for the weights of the Diracs. These systems admit solutions if the following conditions are satisfied:

[C1] Rank($\mathbf{V}$) $< K+1$ where $v_{nk} = \Delta^K\left((-1)^n n^k y_n\right)$ and $\mathbf{V} \in \mathbb{R}^{(N-K)\times(K+1)}$;

[C2] Rank($\mathbf{A}$) $= K$ where $a_{nk} = \frac{\sin(\pi t_k/T)}{\pi\,(t_k/T-n)}$ and $\mathbf{A} \in \mathbb{R}^{K\times K}$.

Theorem 3: Given a finite stream of K weighted Diracs and a sinc sampling kernel sinc(t/T). If conditions [C1] and [C2] are satisfied then N samples with $N \geq 2K$

$$y_n \quad = \quad < x(t), \mathrm{sinc}(t/T-n) > \tag{75}$$

are a sufficient representation of the signal.

Proof: Taking the inner products between the signal and shifted versions of the sinc sampling kernel yields a set of N samples

$$\begin{aligned}
y_n &= < x(t), \mathrm{sinc}(t/T-n) >, \quad n = 0, \ldots, N-1 && (76)\\
&= \int_{-\infty}^{\infty} \sum_{k=0}^{K-1} c_k\, \delta(t-t_k)\, sinc(t/T-n)\, dt && (77)\\
&= \sum_{k=0}^{K-1} c_k\, \mathrm{sinc}\big(t_k/T-n\big) && (78)\\
&= \sum_{k=0}^{K-1} \frac{c_k \sin\left(\pi t_k/T - \pi n\right)}{\pi\big(t_k/T-n\big)} && (79)\\
&= (-1)^n \sum_{k=0}^{K-1} \frac{c_k \sin\left(\pi t_k/T\right)}{\pi\big(t_k/T-n\big)} && (80)\\
\Leftrightarrow (-1)^n\, y_n &= \frac{1}{\pi} \sum_{k=0}^{K-1} c_k \sin\left(\pi t_k/T\right) \cdot \frac{1}{\big(t_k/T-n\big)} && (81)
\end{aligned}$$

The denominator of the previous expression (81) can be rewritten as follows:

$$\frac{1}{(t_k/T-n)} \quad = \quad \frac{\prod_{l\neq k}\left(t_l/T-n\right)}{\prod_{l=0}^{K-1}\left(t_l/T-n\right)} = \frac{P_k(n)}{P(n)} \tag{82}$$

where $P(u)$ is a polynomial of degree K with zeros at all values of t_k/T,

$$P(u) \quad = \quad \prod_{l=0}^{K-1}\left(t_l/T-u\right) = \sum_{k=0}^{K} p_k\, u^k \tag{83}$$

and the $P_k(u)$ is a polynomial of degree $K-1$ and has zeros at all locations except at location t_k

$$P_k(u) \;=\; \prod_{l\neq k} \left(t_l/T - u\right). \tag{84}$$

Therefore if the coefficients of the polynomial $P(u)$ are determined then the locations of the Diracs are simply the K roots of $P(u)$. We can now find an equivalent expression to (81) in terms of the interpolating polynomials:

$$(-1)^n\, P(n)\, y_n \;=\; \frac{1}{\pi}\sum_{k=0}^{K-1} c_k \sin\left(\pi t_k/T\right) P_k(n). \tag{85}$$

Note that the right-hand side of (85) is a polynomial of degree $K-1$ in the variable n, applying K finite differences makes the left-hand side vanish,[6] that is,

$$\Delta^K\left((-1)^n\, P(n)\, y_n\right) = 0, \quad n = K, \ldots, N-1 \tag{86}$$

$$\Leftrightarrow \quad \sum_{k=0}^{K} p_k \underbrace{\Delta^K\left((-1)^n\, n^k\, y_n\right)}_{v_{n\,k}} = 0 \tag{87}$$

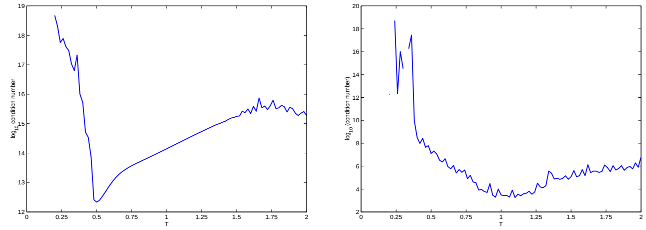
$$\Leftrightarrow \quad \mathbf{V}\cdot\mathbf{p} = 0 \tag{88}$$

where the matrix $\mathbf{V}$ is an $(N-K)\times(K+1)$ matrix and admits a solution when $N-K \geq K$ and the rank($\mathbf{V}$) is less than $K+1$, that is, condition [C1]. Therefore (87) can be used to find, up to a normalization, the $K+1$ unknowns p_k which lead to the K locations t_k. Once the K locations t_k are determined the weights of the Diracs c_k are found by solving the system in (81) for $n = 0, \ldots, K-1$. Since $t_k \neq t_l, \forall k \neq l$, the system admits a solution from condition [C2]. ■

Note that the result does not depend on T. In practice if T is not chosen appropriately then the matrices $\mathbf{V}$ may be ill-conditioned. Figure 8(a) illustrates the conditioning of the matrix $\mathbf{V}$ is the least for T close to 0.5 and that the matrix $\mathbf{A}$ is well-conditioned on average.

By choosing more adequately the interpolating polynomials, for example by taking the Lagrange polynomials, we may reduce the conditioning of the matrix $\mathbf{V}$, but this remains to be investigated. The algorithm is as follows:

[6] Note that the K finite difference operator plays the same role as the annihilating filter in the previous chapter.



(a) (b)

Fig. 8. (a) Average condition number of the matrix that leads to the locations of the Diracs, $\mathbf{V}$, versus the sampling interval T, optimal $T \approx 0.5$; (b) Average condition number of the matrix that leads to the weights of the Diracs, $\mathbf{A}$, versus the sampling interval T, optimal $T \approx 1$. Average is taken on 100 signals with 8 Diracs uniformly spread in the interval $[0, 8]$.

Algorithm 1: Finite length stream of Diracs sampled with a sinc sampling kernel

Given $y_n = < x(t), \mathrm{sinc}(t/T - n) >, \quad n = 0, 1, \ldots, N-1;$

Calculate $v_{nk} = \Delta^K \left((-1)^n\, n^k\, y_n\right), \quad n = K, \ldots, N-1, \quad k = 0, \ldots, K;$

Solve the linear system $\mathbf{V} \cdot \mathbf{p} = 0 \longrightarrow \{p_0, p_1, \ldots, p_K\};$

Find the K roots of $P(u) = \sum_{k=0}^{K} p_k\, u^k \longrightarrow \{t_0/T, t_1/T, \ldots, t_{K-1}/T\};$

Calculate $a_{nk} = \frac{\sin\left(\pi t_k/T\right)}{\pi\left(t_k/T - n\right)}, \quad n = 0, 1, \ldots, N-1\ ;$

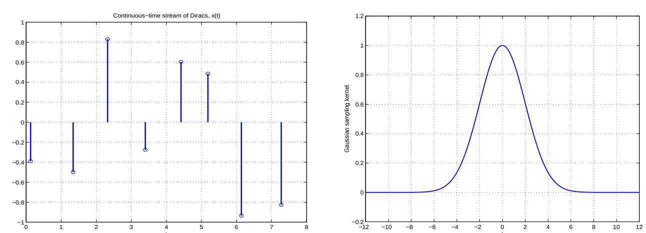
Calculate $Y_n = (-1)^n\, P(n)\, y_n, \quad n = 0, 1, \ldots, N-1;$

Solve the linear system $\mathbf{A} \cdot \mathbf{c} = \mathbf{Y} \longrightarrow \{c_0, c_1, \ldots, c_{K-1}\}.$

This method can be extended to piecewise polynomials, similarly to Theorem. 2. Also, there is an obvious equivalent for discrete-time signals in $\ell^2(\mathbb{Z})$ and discrete-time sinc kernels.

B. Gaussian sampling kernel

Consider sampling the same signal as in (73) but this time with a Gaussian sampling kernel, $\varphi_\sigma(t) = e^{-t^2/2\sigma^2}$, see Figure 9.



(a) (b)

Fig. 9. (a) Example of a finite length continuous-time stream of $K = 8$ Diracs randomly spread on an interval $[0, \tau]$ with $\tau = 8$; (b) Gaussian sampling kernel, $\varphi_\sigma(t) = e^{-t^2/2\sigma^2}, \sigma = 2$.

Similar to the sinc sampling kernel, the samples are obtained by filtering the signal with a Gaussian kernel. Since there are $2K$ unknown variables we show next that N samples with $N \geq 2K$ are sufficient to represent the signal.

Theorem 4: Given a finite stream of K weighted Diracs and a Gaussian sampling kernel $\varphi_\sigma(t) = e^{-t^2/2\sigma^2}$. If $N \geq 2K$ then the N sample values

$$y_n = < x(t), \varphi_\sigma(t/T - n) > \tag{89}$$

are sufficient to reconstruct the signal.

Proof: The sample values are given by

$$\begin{aligned} y_n &= < x(t), e^{-(t/T-n)^2/2\sigma^2} >, \quad n = 0, \ldots, N-1 & (90)\\ &= \int_{-\infty}^{\infty} \sum_{k=0}^{K-1} c_k\, \delta(t - t_k)\, e^{-(t/T-n)^2/2\sigma^2}\, dt & (91)\\ &= \sum_{k=0}^{K-1} c_k\, e^{-(t_k/T-n)^2/2\sigma^2}. & (92)\end{aligned}$$

We expand (92) and regroup the terms so as to have variables that depend solely on n and solely on k. We obtain

$$y_n = \sum_{k=0}^{K-1} \left(c_k\, e^{-t_k^2/2\sigma^2 T^2}\right) \cdot e^{n t_k/\sigma^2 T} \cdot e^{-n^2/2\sigma^2} \tag{93}$$

which is equivalent to

$$Y_n = \sum_{k=0}^{K-1} a_k\, u_k^n \tag{94}$$

where we let $Y_n = e^{n^2/2\sigma^2}\, y_n$, $a_k = c_k\, e^{-t_k^2/2\sigma^2 T^2}$ and $u_k = e^{t_k/\sigma^2 T}$. Note that we reduced the expression Y_n to a linear combination of real exponentials. This hints that the annihilating filter method described in the Section III-A seems appropriate to find the K values u_k. Let $H(z) = h_0 + h_1\, z^{-1} + \cdots + h_K\, z^{-K}$ be an annihilating filter, that is, $\mathbf{h}$ is such that

$$\begin{aligned} \mathbf{h} * \mathbf{Y} &= 0 & (95)\\ \Leftrightarrow \sum_{k=0}^{K} h_k\, Y_{n-k} &= 0, \quad n = K, \ldots, N-1. & (96)\end{aligned}$$

Note that this is a Toeplitz system with real exponential components $Y_n = e^{n^2/2\sigma^2}\, y_n$ and therefore a solution exists when the number of equations is greater than the number of unknowns, that is, $N-K \geq K$ and the rank of the system is less than $K+1$ which is the case by hypothesis. Furthermore σ must be carefully chosen otherwise the system is ill-conditioned. If we factor $H(z) = \prod_{k=0}^{K-1}(1-z^{-1}u_k)$ then we obtain the locations of the Diracs t_k from the roots of the polynomial $H(z)$, that is,

$$t_k = \sigma^2 T \ln u_k. \tag{97}$$

Once the values of the Diracs t_k are obtained then we solve for a_k the Vandermonde system in (94) for which a solution exists since $u_k \neq u_l, \forall k \neq l$. The weights of the Diracs are simply given by

$$c_k = a_k \, e^{t_k^2/2\sigma^2 T^2}. \tag{98}$$

■

The reconstruction scheme is given in the following

Algorithm 2: Finite length stream of Diracs sampled with a Gaussian sampling kernel

Given $y_n = < x(t), e^{-(t/T-n)^2/2\sigma^2} >, \quad n = 0, \ldots, N-1$;
Calculate $Y_n = e^{n^2/2\sigma^2}, \quad n = 0, \ldots, N-1$;
Solve the linear system $\mathbf{h} * \mathbf{Y} = 0 \longrightarrow \{h_0, h_1, \ldots, h_K\}$;
Find the K roots of $H(z) = \sum_{k=0}^{K} h_k \, z^k \longrightarrow \{u_0, u_1, \ldots, u_{K-1}\} \longrightarrow t_k = \sigma^2 T \ln u_k$;
Solve the linear system $Y_n = \sum_{k=0}^{K-1} a_k \, u_k^n, \quad n = 0, \ldots, K-1 \longrightarrow \{a_0, a_1, \ldots, a_{K-1}\} \longrightarrow$ $c_k = a_k \, e^{t_k^2/2\sigma^2 T^2}$.

Here unlike in the sinc case, we have an almost local reconstruction because of the exponential decay of the Gaussian sampling kernel which brings us to the next topic.

VI. Infinite length signals with finite local rate of innovation

In this section we consider the dual problem of Sec. V, that is, *infinite* length *signals* $x(t), t \in \mathbb{R}^+$ with a finite *local* rate of innovation and sampling kernels with *compact support*. In particular, the $\beta-$splines of different degree d are considered [8]

$$\varphi_d(t) = (\varphi_{d-1} * \varphi_0)(t), \quad d \in \mathbb{N}^+ \tag{99}$$

where $\varphi_0(t)$ is the box spline defined by

$$\varphi_0(t) = \begin{cases} 1 & \text{if } 0 \leq t < 1 \\ 0 & \text{else} \end{cases} . \quad (100)$$

We develop local reconstruction algorithms which depend on moving intervals equal to the size of the support of the sampling kernel.[7] The advantage of local reconstruction algorithms is that their complexity does not depend on the length of the signal. We begin by considering bilevel signals, followed by piecewise polynomial signals.

A. Bilevel signals

Consider an infinite length continuous-time signal $x(t), t \in \mathbb{R}^+$ which takes on two values, 0 and 1, with initial condition $x(t)\big|_{t=0} = 1$ with a finite local rate of innovation, ρ. These are called bilevel signals and are completely represented by their transition values t_k. For example, binary signals such as amplitude or position modulated pulses or PAM, PPM signals [3], see Figure 10.

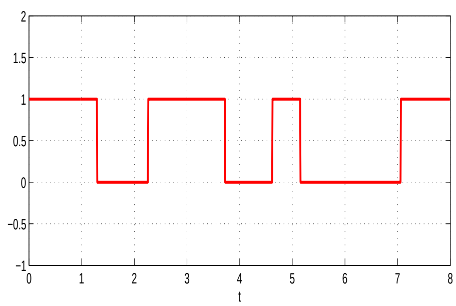


Fig. 10. Bilevel signal.

Suppose a bilevel signal is sampled with a box spline $\varphi_0(t/T)$. Then the sample values are given by the inner products between the bilevel signal and the box function,

$$y_n =< x(t), \varphi_0(t/T - n) >= \int_{-\infty}^{\infty} x(t)\, \varphi_0(t/T - n)\, dt. \quad (101)$$

It can bee seen in Figure 11 that the sample value y_n corresponds to the area occupied by the signal in the interval $[nT, (n+1)T]$. Thus if there is at most one transition per box then we can recover the transition from the sample. This leads us to

[7] The size of the support of $\varphi_d(t/T)$ is equal to $(d+1)T$.

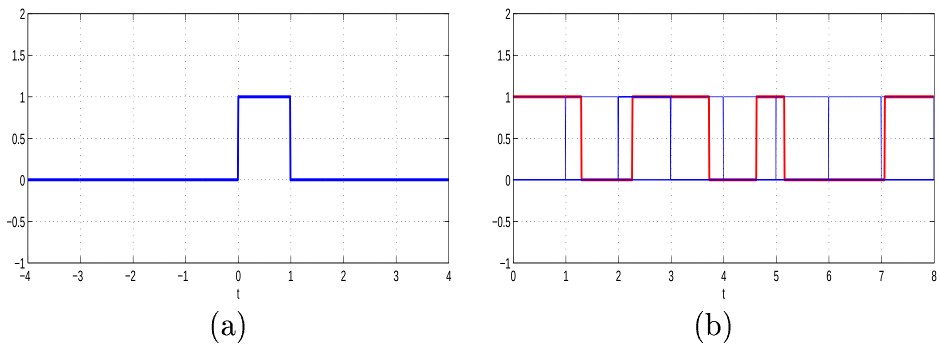


Fig. 11. (a) Box spline sampling kernel, $\varphi_0(t)$, $T = 1$. (b) Bilevel signal sampled with the box sampling kernel.

Proposition 3: A bilevel signal $x(t), t > 0$, with initial condition $x(t)\big|_{t=0} = 1$, is uniquely determined from the samples $y_n =< x(t), \varphi_0(t/T - n) >$ where $\varphi_0(t)$ is the box spline defined in (100) if and only if there is at most one transition t_k in each interval $[nT, (n+1)T]$.

Proof: For simplicity let $T = 1$. Consider an interval $[n, n+1]$ and suppose $x(n) = 1$. First we show sufficiency followed by necessity .

$\Leftarrow$: If there are 0 transitions in the interval $[n, n+1]$ then the area under the bilevel signal, or the sample value, is $y_n = 1$ since we supposed that $x(t)\big|_{t=n} = 1$. If there is one transition in $[n, n+1]$ then the sample value is equal to

$$y_n = < x(t), \varphi_0(t-n) >= \int_{-\infty}^{\infty} x(t)\,\varphi_0(t-n)\,dt \tag{102}$$

$$= \int_{n}^{n+1} x(t)\,dt = \int_{n}^{t_k} 1\,dt = t_k - n \tag{103}$$

This implies that $t_k = y_n - n$. Similarly if $x(n) = 0$ then we have $t_k = n + 1 - y_n$. Therefore we can uniquely determine the signal in the interval $[n, n+1]$.

$\Rightarrow$: Necessity is shown by counterexample.

Suppose $x(n) = 1$ and there are two transitions t_k, t_{k+1} in the interval $[n, n+1]$ then the sample value is equal to

$$y_n = \int_{n}^{n+1} x(t)\,dt = \int_{n}^{t_k} 1\,dt + \int_{t_{k+1}}^{n+1} 1\,dt \tag{104}$$

$$= t_k - n + n + 1 - t_{k+1} = t_k - t_{k+1} + 1. \tag{105}$$

That is, there is one equation with two unknowns and therefore insufficient samples to determine both transitions. Thus there must be at most one transition in an interval $[n, n+1]$ to uniquely define the signal.

■

Now consider shifting the bilevel signal by an unknown shift ϵ, see Figure 12, then there will be two transitions in an interval of length T and one box function will not be sufficient to recover the transitions.

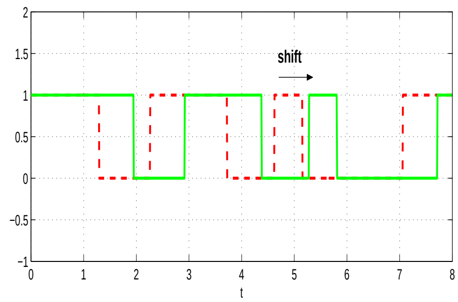


Fig. 12. Shifted bilevel signal with two transitions in the interval $[5, 6]$.

Suppose we double the sampling rate, then the support of the box sampling kernel is doubled and we have two sample values y_n, y_{n+1} covering the interval $[nT, (n+1)T]$ but these values are identical (see their areas). Therefore increasing the sampling rate is still insufficient. This brings us to consider a sampling kernel not only with a larger support but with added information. For example, the hat spline function $\varphi_1(t/T)$ defined by

$$\varphi_1(t) = \begin{cases} 1 - |t| & \text{if } |t| < 1 \\ 0 & \text{else} \end{cases} \tag{106}$$

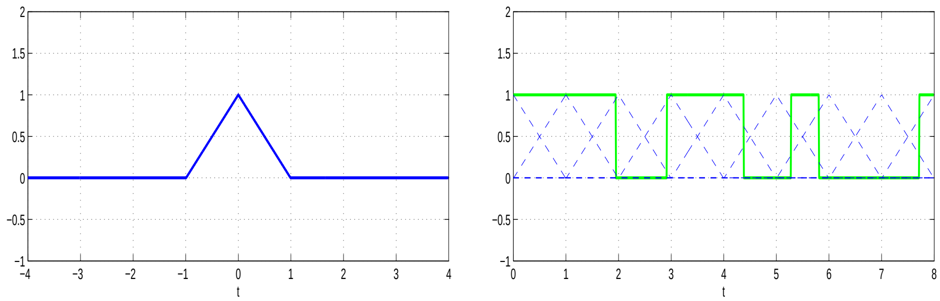
leads to sample values defined by $y_n = < x(t), \varphi_1(t/T - n) >$ or

$$y_n = \int_{(n-1)T}^{nT} x(t)\big(1 + t/T - n\big)\, dt + \int_{nT}^{(n+1)T} x(t)\big(1 - (t/T - n)\big)\, dt. \tag{107}$$

From Figure 13 we can see that there are two sample values covering the interval $[nT, (n+1)T]$. We will show next that in this case we can uniquely determine the signal.

Proposition 4: An infinite length bilevel signal $x(t)$, with initial condition $x(0) = 1$ is uniquely determined from the samples defined by

$$y_n = < x(t), \varphi_1(t/T - n) > \tag{108}$$



(a) (b)

Fig. 13. (a) Hat spline sampling kernel, $\varphi(t/T)$, $T = 1$. (b) Bilevel signal with two transitions in an interval $[n, n+1]$ sampled with a hat sampling kernel.

where $\varphi_1(t)$ is the hat sampling kernel if and only if there are at most two transitions $t_k \neq t_j$ in each interval $[nT, (n+2)T]$.

Proof: Again, for simplicity let $T = 1$ and suppose the signal is known for $t \leq n$ and $x(t)\big|_{t=n} = 1$.

First we show sufficiency by showing the existence and uniqueness of a solution. Then we show necessity by a counterexample.

$\Leftarrow$: Similar to the box sampling kernel the sample values will depend on the configuration of the transitions in the interval $[n, n+2]$. If there are at most 2 transitions in the interval $[n, n+2]$ then the possible configurations are

$$(0,0), (0,1), (0,2), (1,0), (1,1), (2,0)$$

where the first and second component indicate the number of transitions in the intervals $[n, n+1]$, $[n+1, n+2]$ respectively, see Figure 14.

Furthermore since the hat sampling kernel is of degree one we obtain for each configuration a quadratic system of equations with variables t_0, t_1.

$$y_n = \int_{n-1}^{n} x(t)\big(1+t-n\big)\,dt + \int_{n}^{n+1} x(t)\big(1-(t-n)\big)\,dt \tag{109}$$

$$y_{n+1} = \int_{n}^{n+1} x(t)\big(1+t-(n+1)\big)\,dt + \int_{n+1}^{n+2} x(t)\big(1-(t-(n+1))\big)\,dt. \tag{110}$$

First we show that the quadratic system of equations admits a solution and then that it is unique.

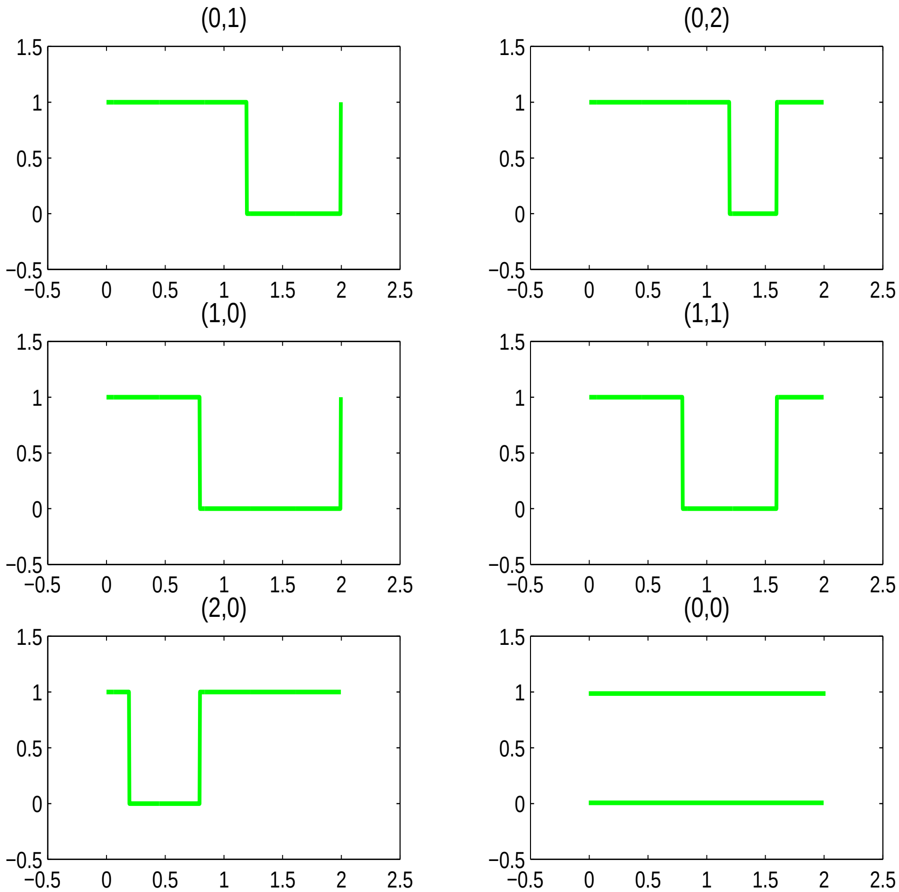


Fig. 14. Bilevel signal containing at most 2 transitions in the interval $[0, 2]$: All possible configurations.

(a) Existence.

Take $n = 0$ and so the moving interval is $[0, 2]$.

The configuration $(0, 0)$ will lead to sample values $y_0 = 1$, $y_1 = 1$.

The configuration $(0, 1)$ will lead to sample values

$$y_0 = 1/2 + \int_1^{t_0} (t-1)\,dt = \frac{1}{2}{t_0}^2 + 1 - t_0 \tag{111}$$

$$y_1 = 1/2 + \int_1^{t_0} (2-t)\,dt = -\frac{1}{2}{t_0}^2 - 1 + 2t_0 \tag{112}$$

$\Rightarrow t_0 = y_0 + y_1 = 1 + \sqrt{-1 + 2y_0} = 2 - \sqrt{2 - 2y_1}$.

The configuration $(0, 2)$ will lead to sample values

$$y_0 = \frac{1}{2}{t_0}^2 + 1 - t_0 + t_1 - \frac{1}{2}{t_1}^2 \tag{113}$$

$$y_1 = -\frac{1}{2}\,{t_0}^2 + 1 + 2\,t_0 + \frac{1}{2}\,{t_1}^2 - 2\,t_1 \tag{114}$$

$\Rightarrow t_0 = \frac{(-2-2y_1+y_0^2+2y_0y_1+y_1^2)}{2(-2+y_0+y_1)}, t_1 = \frac{-(10-6y_1-8y_0+y_1^2+y_0^2+2y_0y_1)}{2(-2+y_0+y_1)}$.

The configuration $(1,0)$ will lead to sample values

$$y_0 = -\frac{1}{2}t_0^2 + t_0 \quad (115)$$

$$y_1 = \frac{1}{2}t_0^2 \quad (116)$$

$\Rightarrow t_0 = y_0 + y_1 = 1 - \sqrt{1-2y_0} = \sqrt{2y_1}$.

The configuration $(1,1)$ will lead to sample values

$$y_0 = -\frac{1}{2}t_0^2 + t_0 - \frac{1}{2}t_1^2 + t_1 \quad (117)$$

$$y_1 = \frac{1}{2}t_0^2 + 2 + \frac{1}{2}t_1^2 - 2t_1 \quad (118)$$

$\Rightarrow t_0 = \frac{y_0+y_1+\sqrt{-y_1^2-2y_0y_1+4y_1-y_0^2}}{2}, t_1 = \frac{-y_1-y_0+4+\sqrt{-y_1^2-2y_0y_1+4y_1-y_0^2}}{2}$

The configuration $(2,0)$ will lead to sample values

$$y_0 = -\frac{1}{2}t_0^2 + 1 + t_0 + \frac{1}{2}t_1^2 - t_1 \quad (119)$$

$$y_1 = \frac{1}{2}t_0^2 + 1 - \frac{1}{2}t_1^2 \quad (120)$$

$\Rightarrow t_0 = \frac{2-2y_1+y_1^2+2y_0y_1+y_0^2-4y_0}{2(-2+y_1+y_0)}, t_1 = -\frac{6-4y_0-6y_1+y_1^2+2y_0y_1+y_0^2}{2(-2+y_1+y_0)}$.

(b) Uniqueness.

If $y_n = 1$ and $y_{n+1} = 1$ then this implies configuration $(0,0)$.

If $y_n = 1$ and $1/2 \leq y_{n+1} \leq 1$ then the possible configurations are $(0,1), (0,2)$.

By hypothesis, there are at most two transitions in the interval $[n+1, n+3]$ therefore if $y_{n+2} \leq 1/2$ then the configuration in the interval $[n, n+2]$ is $(0,1)$ otherwise if $y_{n+2} \geq 1/2$ then the configuration is $(0,2)$.

If $1/2 \leq y_n \leq 1$ and $1/2 \leq y_{n+1} \leq 1$ then this implies configuration $(2,0)$.

If $1/2 \leq y_n \leq 1$ and $0 \leq y_{n+1} \leq 1/2$ then this implies configuration $(1,0)$.

$\Rightarrow$: Necessity is shown by counterexample.

Consider a bilevel signal with three transitions in the interval $[0,2]$ but with all three in the interval $[0,1]$, see Fig. 15.

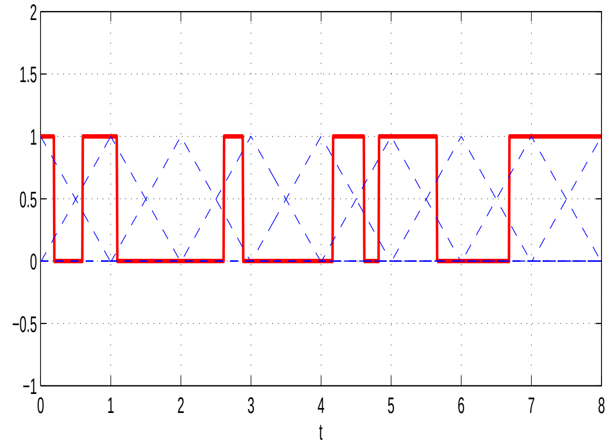


Fig. 15. Bilevel signal containing three transitions in an interval $[4, 5]$, sampled with the hat sampling kernel $\varphi_1(t)$.

Then the sample values in this case are are equal to

$$y_0 = 1/2 + \int_0^{t_0} (1-t)\,dt + \int_{t_1}^{t_2} (1-t)\,dt \tag{121}$$

$$= 1/2 + t_0 - t_1 + t_2 - t_0^2/2 + t_1^2/2 - t_2^2/2 \tag{122}$$

$$y_1 = \int_0^{t_0} t\,dt + \int_{t_1}^{t_2} t\,dt \tag{123}$$

$$= t_0^2/2 - t_1^2/2 + t_2^2/2. \tag{124}$$

There is no unique solution for this quadratic system of equations. Therefore there must be at most 2 transitions in an interval $[0, 2]$.

■

Once again if there is an unknown shift in the bilevel signal then there may be three transitions in an interval $[nT, (n+1)T]$ and so we increase the number of samples by sampling with $\varphi_1(t/(T/2))$. The pseudo-code for sampling bilevel signals using the box and hat functions are given in full detail in Section VI-C.1.

When going to higher order splines, necessity carries over. Sufficiency is more tedious since we must solve a system of higher order polynomial equations.

B. Piecewise polynomials

Similar to bilevel signals we consider sampling piecewise polynomials with the box sampling kernel. Consider an infinite length piecewise polynomial signal $x(t)$ where each piece is a polynomial of degree R and defined on an interval $[t_{k-1}, t_k]$, that is,

$$x(t) = \begin{cases} x_0(t) = \sum_{m=0}^{R} c_{0m}\, t^m & t \in [0, t_0] \\ x_1(t) = \sum_{m=0}^{R} c_{1m}\, t^m & t \in [t_0, t_1] \\ \vdots & \\ x_K(t) = \sum_{m=0}^{R} c_{Km}\, t^m & t \in [t_{K-1}, t_K] \\ \vdots & \end{cases} . \qquad (125)$$

Each polynomial piece $x_k(t)$ contains $R+1$ unknown coefficients c_{km}. The transition value t_k is easily obtained once the pieces $x_{k-1}(t)$ and $x_k(t)$ are determined, thus there are $2(R+1)+1$ degrees of freedom. If there is one transition in an interval of length T the maximal local rate of innovation is $\rho_m(T) = (2(R+1)+1)/T$. Therefore in order to recover the polynomial pieces and the transition we need to have at least $2(R+1)+1$ samples per interval T. This is achieved by sampling with the following box sampling kernel $\varphi_0(t/\frac{T}{2(R+1)+1})$. For example if $x(t)$ is a piecewise linear signal with 2 pieces as illustrated in Fig. 16 then to recover the

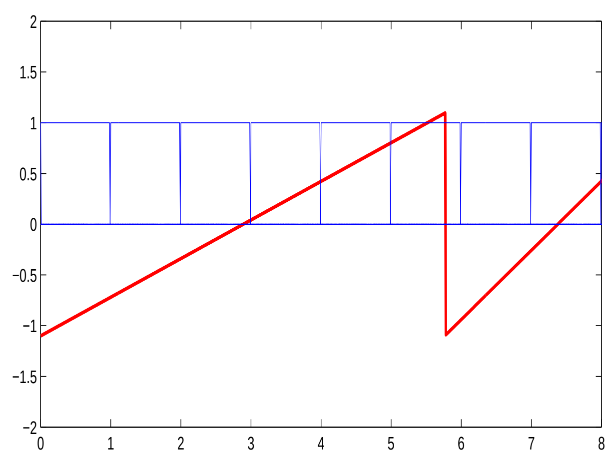


Fig. 16. Piecewise linear signal sampled with a box sampling kernel.

signal it is sufficient to take 5 samples: two before the transition, two after the transition and one sample covering the transition.

We can generalize by noting that the Rth derivative of a piecewise polynomial of degree R is a piecewise constant signal. The pseudo-code for sampling piecewise constant signals with the box sampling kernel is found in Section VI-C.2 .

C. Local reconstruction algorithms

The following algorithms have been implemented in Maple^{TM}. In all of the algorithms k is the index of a transition value and n is the index of the current interval $[n, n+1]$. We suppose that $x(t) = 1, \forall t \leq 0$.

C.1 Bilevel signals

Suppose N sample values $y_n =< x(t), \varphi_0(t-n) >$ are available.

Algorithm 3: Bilevel signal with box sampling kernel.

Require: $k \leftarrow 0$, $n \leftarrow 0$

  while $n \leq N-1$ **do**

    if $y_n = 1$ **then**

      $x(t) = 1 \quad \forall t \in [n, n+1]$.

    end if

    if $y_n = 0$ **then**

      $x(t) = 0 \quad \forall t \in [n, n+1]$.

    end if

    if $0 < y_n < 1$ **then**

      if $x(n^+) = 1$ **then**

        $t_k \leftarrow y_n + n$

      else

        $t_k \leftarrow n + 1 - y_n$

      end if

      $k \leftarrow k + 1$

    end if

    $n \leftarrow n + 1$

  end while

Next we give the pseudo-code for bilevel signals sampled with a hat sampling kernel. Suppose N sample values $y_n =< x(t), \varphi_1(t-n) >$ are available. The variable tncode is a set whose last component indicates the number of transitions in the interval $[n-1, n]$.

Algorithm 4: Bilevel signal with hat sampling kernel.

Require: $tncode \leftarrow \emptyset$, $k \leftarrow 0$, $n \leftarrow -1$

while $n \leq N-1$ **do**

if $y_n = 1$ **then**

$tncode \leftarrow 0$

$x(t) = 1 \qquad \forall t \in [n-1, n+1]$

end if

if $y_n = 0$ **then**

$tncode \leftarrow 0$

$x(t) = 0 \qquad \forall t \in [n-1, n+1]$

end if

if $0 < y_n < 1$ **then**

if 0 transitions in the interval $[n-1, n]$ **then**

if $y_n = 0.5$ **then**

$t_k \leftarrow n$

$tncode \leftarrow 0$

else

sol = solve for configuration $(1, 0) \in [n, n+2]$

if $sol \neq \emptyset$ **then**

$tncode \leftarrow 1, 0$

if $x(n^+) = 1$ **then**

$t_k \leftarrow n + 1 - \sqrt{2 - 2\,y_n}$

else

$t_k \leftarrow n + 1 - \sqrt{2}\,\sqrt{y_n}$

end if

$n \leftarrow n + 2$

$k \leftarrow k + 1$

solfound $\leftarrow$ True

else

sol = solve for configuration $(1,1) \in [n, n+2]$

if $sol \neq \emptyset$ **then**

$tncode \leftarrow 1, 1$

if $x(n^+) = 1$ **then**

$t_k \leftarrow n + 1 - \sqrt{2 - 2\,y_n}$

$t_{k+1} \leftarrow n + 2 - \sqrt{-3 + 2\,y_{n+1} + 2\,\sqrt{2 - 2\,y_n} + 2\,y_n}$

else

$t_k \leftarrow n + 1 - \sqrt{2}\,\sqrt{y_n},$

$t_{k+1} \leftarrow n + 2 - \sqrt{1 + 2\,\sqrt{2}\,\sqrt{y_n} - 2\,y_{n+1} - 2\,y_n}$

end if

$n \leftarrow n + 2$

$k \leftarrow k + 2$

solfound $\leftarrow$ True

else

solfound $\leftarrow$ False

end if

end if

end if

if not solfound **then**

sol = solve for configuration $(2,0) \in [n, n+2]$

if $sol \neq \emptyset$ **then**

$tncode \leftarrow 2, 0$

if $x(n^+) = 1$ **then**

$t_k \leftarrow \frac{1}{2}\,\frac{2 - 2\,y_{n+1} - 4\,y_n + 2\,y_{n+1}\,y_n + y_n^2 + y_{n+1}^2 - 4\,n + 2\,y_n\,n + 2\,y_{n+1}\,n}{-2 + y_{n+1} + y_n}$

$t_{k+1} \leftarrow -\frac{1}{2}\,\frac{6 - 6\,y_{n+1} + 4\,n - 4\,y_n - 2\,y_n\,n + y_n^2 + 2\,y_{n+1}\,y_n - 2\,y_{n+1}\,n + y_{n+1}^2}{-2 + y_{n+1} + y_n}$

else

$t_k \leftarrow -\frac{1}{2}\,\frac{-2\,y_{n+1} - 2\,y_n\,n + y_n^2 + 2\,y_{n+1}\,y_n - 2\,y_{n+1}\,n + y_{n+1}^2}{y_n + y_{n+1}}$

$t_{k+1} \leftarrow \frac{1}{2}\,\frac{2\,y_{n+1} + 2\,y_n\,n + y_n^2 + 2\,y_{n+1}\,y_n + 2\,y_{n+1}\,n + y_{n+1}^2}{y_n + y_{n+1}}$

```
            end if
            n ← n + 2
            k ← k + 2
          end if
        end if
      else if 1 transition in the interval [n − 1, n] then
        sol = solve for configuration (1, 0) ∈ [n − 1, n + 1] given t_{k−1} ∈ [n − 1, n]
        if sol ≠ ∅ then
          tncode ← 0
          n ← n + 1
        else
          sol = solve for configuration (1, 1) ∈ [n − 1, n + 1] given t_{k−1} ∈ [n − 1, n]
          if sol ≠ ∅ then
            tncode ← 1
            if x((n − 1)^+) = 1 then
              t_k ← n + √(2 − 2 y_{n+1})
            else
              t_k ← n + √2 √(y_{n+1})
            end if
            n ← n + 1
            k ← k + 1
          end if
        end if
      else
        { 2 transitions in the interval [n − 1, n]}
        tncode ← 0
      end if
    end if
  end while
```

C.2 Piecewise constant signal

We consider sampling a piecewise constant signal with the box sampling kernel. doubling the sampling rate is sufficient to recover the signal, thus we suppose $2N$ sample values y_n are available.

Algorithm 5: Piecewise constant signal with box sampling kernel.

Require: , $k \leftarrow 0$,$n \leftarrow 0$ $y_n, n = 0 \ldots 2N-1$

while $n \leq 2N-2$ **do**

if $|y_{n+1} - y_n| = 0$ **then**

$n = n+1$

else

$c_k = y_{n-1}$

$c_{k+1} = y_{n+1}$

$t_k = \frac{y_n + nc_k - (n+1)c_{k+1}}{c_k - c_{k+1}}$

$n \leftarrow n+2$

$k \leftarrow k+1$

end if

end while

VII. Applications

The applications we consider involve the discrete-time periodic stream of Diracs and piecewise polynomial signals. It is well known that a bandlimited signal can be perfectly recovered from its samples by sampling it at twice the maximum frequency. What if the bandlimited signal has a jump or a discontinuity then the signal is no longer bandlimited and the usual method is not valid. These are what we call piecewise bandlimited signals. Another type of non-bandlimited signal which we may come across in nature is a signal which is obtained from a system with a certain frequency response. The output of the system is a filtered signal. We will look at filtered stream of Diracs and filtered piecewise polynomials.

A. Piecewise bandlimited signals

A discrete-time periodic piecewise bandlimited signal is the sum of a bandlimited signal with a stream of Diracs in the simplest case or with a piecewise polynomial signal. An example is illustrated in Figure 17(e) and is obviously not bandlimited from Figure 17(f). Formally, we have the following

Definition 4: Piecewise bandlimited signals.

Let $\mathbf{x}_{BL}$ be a discrete-time periodic $L-$bandlimited signal of period N with corresponding DTFS coefficients $\mathbf{X}_{BL}$ such that $X_{BL}[m] = 0 \quad \forall m \notin [-L, L]$. Let $\mathbf{x}_{PP}$ be a zero mean discrete-time piecewise polynomial signal of period N with K pieces and with each piece of maximum degree R. Then a piecewise bandlimited signal $\mathbf{x}$ is defined by

$$\mathbf{x} = \mathbf{x}_{BL} + \mathbf{x}_{PP} \tag{126}$$

with corresponding DTFS coefficients $\mathbf{X}$ defined by

$$X[m] = \begin{cases} X_{BL}[m] + X_{PP}[m] & \text{if } m \in [-L, L] \\ X_{PP}[m] & \text{if } m \notin [-L, L] \end{cases} . \tag{127}$$

First consider a stream of K weighted Diracs, $\mathbf{x}_{PP}$. From Section III-A, we can recover the K weighted Diracs from $2K$ contiguous frequency values $\mathbf{X}_{PP}$. Since the DTFS coefficients of the bandlimited signal, $\mathbf{X}_{BL}$, are equal to zero outside of the band $[-L, L]$, we have that the DTFS coefficients of the signal outside of the band $[-L, L]$ are exactly equal to the DTFS coefficients of the piecewise polynomial, that is, $\mathbf{X}[m] = \mathbf{X}_{PP}[m], \quad \forall |m| > L$. Therefore it is sufficient to take the $2K$ DTFS coefficients of the outside of the band $[-L, L]$, for instance in $[L+1, L+2K]$. Suppose we have the DTFS of the signal $X[m]$, with $m \in [-(L+2K), L+2K]$ then the DTFS of the bandlimited signal are obtained by subtracting $X_{PP}[m]$ from $X[m]$ for $m \in [-L, L]$.

Recall that the piecewise polynomial has $2K(R+1)$ degrees of freedom and the bandlimited signal has $2L + 1$. It follows that we can sample the signal using a discrete-time periodized differentiated sinc sampling kernel bandlimited to $2K(R+1) + L$.

Corollary 1: Consider a piecewise bandlimited signal $\mathbf{x}$ as defined in Definition. 4. Let $\psi[n]$ be the $(R+1)$th differentiated sinc sampling kernel with DTFS

$$\Psi[m] = (D[m])^{R+1}\, Rect_{[-(2K(R+1)+L), 2K(R+1)+L]}. \tag{128}$$

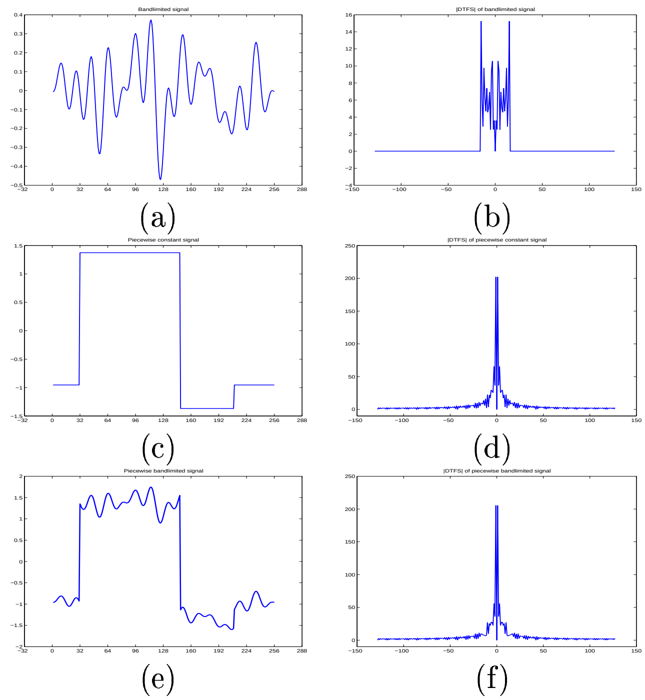


Fig. 17. (a) Bandlimited signal of length $N = 256$; (b) DTFS of Bandlimited signal, $L = 15$ (c) Piecewise constant signal with $K = 3$ pieces; (d) DTFS of piecewise constant signal; (e) Bandlimited piecewise constant signal; (f) DTFS of bandlimited piecewise constant signal.

Let M be an integer divisor of N, and let $N/M \geq 2(2K(R+1)+L)$ then the samples

$$y_s[l] = < x[n], \psi[n - lM] >, \quad l = 0, \dots, N/M - 1 \tag{129}$$

are a sufficient representation of $\mathbf{x}$.

Proof: The proof is exactly the same as in Theorem 1 until equation (43)

$$Y_s[k] = \frac{1}{M}(1 - W_N^{-k})^{R+1}\, X[k], k = 0, \dots, 2K(R+1) + L \tag{130}$$

$$= \begin{cases} \frac{1}{M}(1 - W_N^{-k})^{R+1}\left(X_{BL}[k] + X_{PP}[k]\right) & \text{if } k = 0, \dots, L \\ \frac{1}{M}(1 - W_N^{-k})^{R+1}\, X_{PP}[k] & \text{if } k = L+1, \dots, 2K(R+1) + L \end{cases} \tag{131}$$

Therefore $2K(R+1)$ values of

$$X_{PP}[k] = \frac{M}{(1 - W_N^{-k})^{R+1}}\, Y_s[k], \quad k \in [L+1, 2K(R+1) + L] \tag{132}$$

are sufficient to recover the piecewise polynomial $\mathbf{x}_{PP}$. From these we can recover the L spectral components of the bandlimited signal since

$$X_{BL}[k] = \frac{1}{(1 - W_N^{-k})^{R+1}}(Y_s[k] - X_{PP}[k]), \quad k = 0, \dots, L. \tag{133}$$

This gives us the the bandlimited signal $\mathbf{x}_{BL}$ and thus the piecewise bandlimited signal as defined in Definition. 4 is recovered $\mathbf{x} = \mathbf{x}_{BL} + \mathbf{x}_{PP}$. ■

Figure 18 the illustrates the reconstruction of a bandlimited plus a piecewise constant signal using the following reconstruction scheme:

Algorithm 6: Reconstruction of piecewise bandlimited signals.

Require: $N, M, N/M \geq 2(2K(R+1)+L)+1$;

Calculate the samples $y_s[l] =< x[n], \psi[n-lM] >$, $l = 0, \ldots, N/M-1$;

Calculate the DTFS $X[m], m \in [-(2K(R+1)+L), (2K(R+1)+L)]$ from the DTFS of samples $y_s[l] \longrightarrow X_{PP}[m] = X[m], m \in [L+1, (2K(R+1)+L)]$;

Solve $\mathbf{h} * X_{PP}[m] = 0, m \in [L+1, (2K(R+1)+L)] \longrightarrow \mathbf{x}_{PP}$;

Calculate $X_{BL}[m] = X[m] - X_{PP}[m], \quad m \in [-L, L] \longrightarrow \mathbf{x}_{BL}$;

The reconstruction is $\mathbf{x} = \mathbf{x}_{BL} + \mathbf{x}_{PP}$.

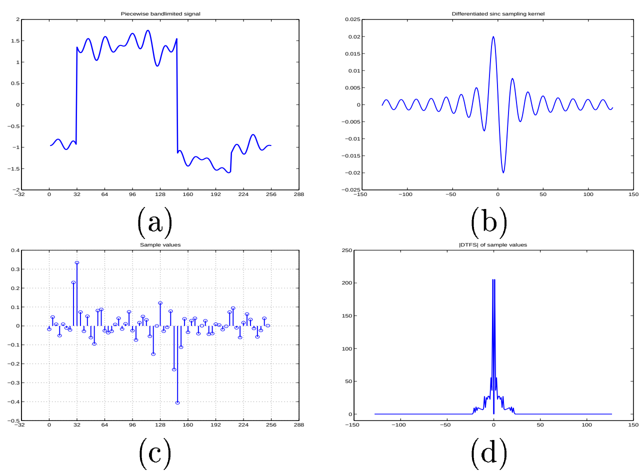


Fig. 18. (a) Bandlimited piecewise constant signal,$x[n]$, with $K = 3$, $L = 15$,$R = 0$,$N = 256$; (b) Differentiated sinc sampling kernel, $\psi[n] = d[n] * \varphi[n]$, bandlimited to $2K(R+1)+1+L = 22$ (c) Sample values $y_s[l] =< x[n], \psi[n-lM] >, l = 0, \ldots, N/M-1$, $M = 4$; (d) $|DTFS|$ of sample values; Reconstruction error is 10^{-13}.

B. Filtered piecewise polynomials

Another application of sampling piecewise polynomial signals consists in sampling their filtered output. Figure 19 illustrates that a filtered stream of Diracs is not bandlimited. These signals are formally defined in the following

Definition 5: Filtered piecewise polynomials.

Let $\mathbf{x}_{PP}$ be a zero mean discrete-time periodic piecewise polynomial signal of period N with K pieces of maximum degree R. Let $\mathbf{g}$ be a filter with DTFS $\mathbf{G}$. Then a filtered piecewise polynomial $\mathbf{x}$ is defined by

$$\mathbf{x} = \mathbf{g} * \mathbf{x}_{PP} \tag{134}$$

and the corresponding DTFS coefficients $\mathbf{X}$ are defined by

$$X[m] = G[m] \cdot X_{PP}[m], \quad m = 0, \ldots, N-1. \tag{135}$$

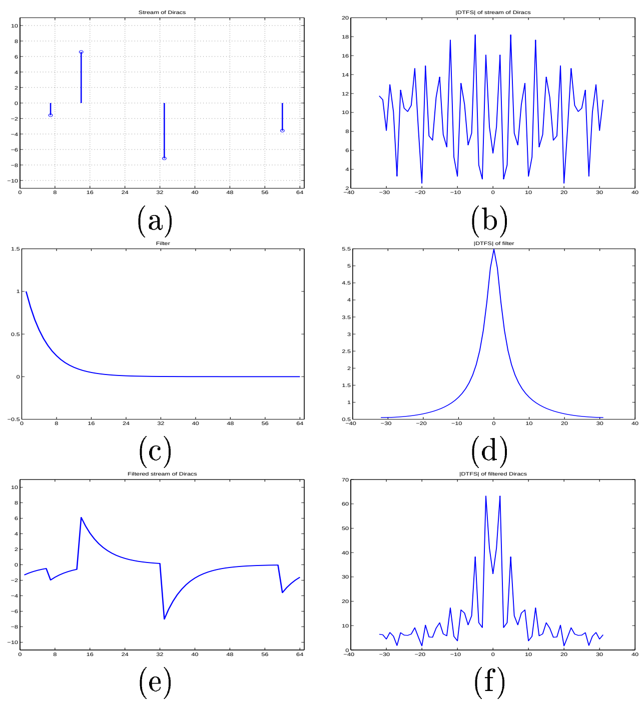


Fig. 19. (a) Stream of $K = 4$ Diracs with period $N = 64$; (b) $|DTFS|$ of stream of Diracs (c) Known filter $g[n] = \alpha^n, n = 0, \ldots, N-1$, $\alpha = 0.4$; (d) $|DTFS|$ of filter; (e) Filtered stream of Diracs; (f) $|DTFS|$ of filtered stream of Diracs.

Suppose $\mathbf{x}_{PP}$ is a stream of K Diracs. If the filter has $2K$ contiguous nonzero frequency values $G[m]$ then $2K$ frequency values of the signal $X[m]$ will be enough to determine $2K$ frequency values of the stream of Diracs, since $X_{PP}[m] = X[m]/G[m]$, and from Proposition 1 these are sufficient to recover the stream of Diracs .

Corollary 2: Consider a filtered piecewise polynomial signal $\mathbf{x}$ as defined in Definition. 5 with $G[m] \neq 0, m \in [-K(R+1), K(R+1)]$. Consider an $(R+1)$ differentiated sinc sampling kernel $\psi[n]$ with DTFS

$$\Psi[m] = (D[m])^{R+1}\, Rect_{[-K(R+1),K(R+1)]}. \tag{136}$$

Let M be an integer divisor of N such that $N/M \geq 2K(R+1)+1$. Then the filtered piecewise polynomial signal can be recovered from the N/M samples

$$y_s[l] =< x[n], \psi[n-lM] >, \quad l = 0, \dots, N/M-1. \tag{137}$$

Proof: Similar to the proof of piecewise bandlimited signals, we have that the DTFS coefficients of the samples $y_s[l]$ are equal to

$$\begin{aligned} Y_s[k] &= \frac{1}{M}(1-W_N^{-k})^{R+1}\, X[k], \quad k \in [-K(R+1), K(R+1)] & (138)\\ &= \frac{1}{M}(1-W_N^{-k})^{R+1}\, \big(G[k]\, X_{PP}[k]\big). & (139)\end{aligned}$$

Since $G[k] \neq 0$ for $k \in [-K(R+1), K(R+1)]$ we have $2K(R+1)$ values of the DTFS of the piecewise polynomial

$$X_{PP}[k] = \frac{M}{(1-W_N^{-k})^{R+1}\, G[k]}\, Y_s[k], \quad k \in [-K(R+1), K(R+1)] \tag{140}$$

which are sufficient to recover $\mathbf{x}_{PP}$ and which leads to the filtered signal by Definition. 5. ■

The reconstruction scheme is described in the following algorithm and an example of the reconstruction is illustrated in Figure 20.

Algorithm 7: **Require:** $N, M, N/M \geq 2K(R+1)+1$;

Calculate the samples $y_s[l] =< x[n], \psi[n-lM] >$, $l = 0, \dots, N/M-1$;

Calculate $\mathbf{Y}_s = \mathbf{DFT}_{N/M} \cdot \mathbf{y}_s \longrightarrow X[m], m \in [-K(R+1), K(R+1)]$;

Calculate $X_{PP}[m] = X[m]/G[m], m \in [-K(R+1), K(R+1)]$;

Solve $\mathbf{h} * X_{PP}[m] = 0, m \in [-K(R+1), K(R+1)] \longrightarrow \mathbf{x}_{PP}$;

The reconstruction is $\mathbf{x} = \mathbf{g} * \mathbf{x}_{PP}$.

We have seen that the crux of the proof relies on the fact that the filter is known and is invertible over the number of degrees of freedom of the problem. What if the filter has a finite rate of innovation but is unknown? This is more complex and remains to be investigated.

VIII. Conclusion

- We derived sampling theorems for periodic signals in particular streams of weighted Diracs and piecewise polynomials. These signals have a finite rate of inno-

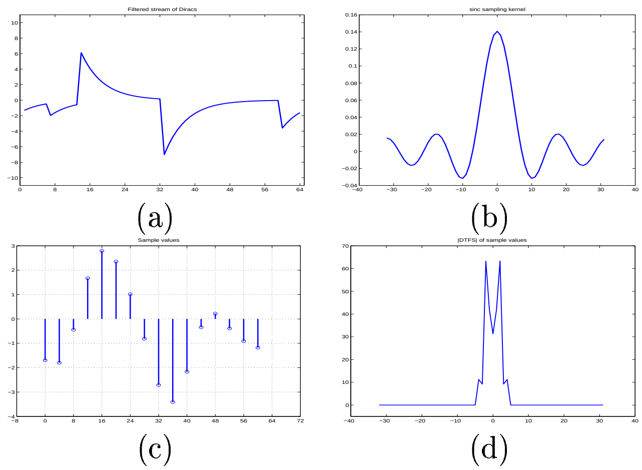


Fig. 20. (a) Filtered stream of Diracs, $x[n]$, $N = 64$; (b) Sinc sampling kernel $\varphi[n]$ bandlimited to $K = 4$ (c) Sample values $y_s[l] = < x[n], \varphi[n - lM] >, l = 0, \ldots, 15$, $M = 4$; (d) $|DTFS|$ of sample values; Reconstruction error is 10^{-13}.

vation ρ which is equal to the number of degrees of freedom per period.

- The samples are obtained by taking the inner product of the signal with a shifted version of the periodized sinc kernel or differentiated sinc kernels. The bandwidth of these kernels must be greater or equal to the degrees of freedom of the signal.
- The discrete-time periodic signals are perfectly recovered when the sampling rate $1/M$ is greater or equal to the rate of innovation $\rho = 2K/N$ in the case of streams of weighted Diracs or $\rho = 2K(R+1)/N$ in the case of a piecewise polynomial signal with K pieces and maximum degree R.
- The continuous-time periodic streams of Diracs and piecewise polynomial signals are perfectly recovered when the sampling rate $1/T$ is greater or equal to the rate of innovation $\rho = 2K/\tau$ since we assumed that the piecewise polynomial signal belonged to $\mathcal{C}^{R-1}$.
- The sampling and reconstruction scheme is illustrated in Figure 21.
- A finite length stream of K Diracs can be recovered from N samples y_n obtained as the inner product between the signal and shifted versions of the sinc and Gaussian sampling kernel, when $N \geq 2K$.
- For both types of sampling kernels two systems of equations must be solved: the first system is to find the locations of the Diracs and the second is to find the

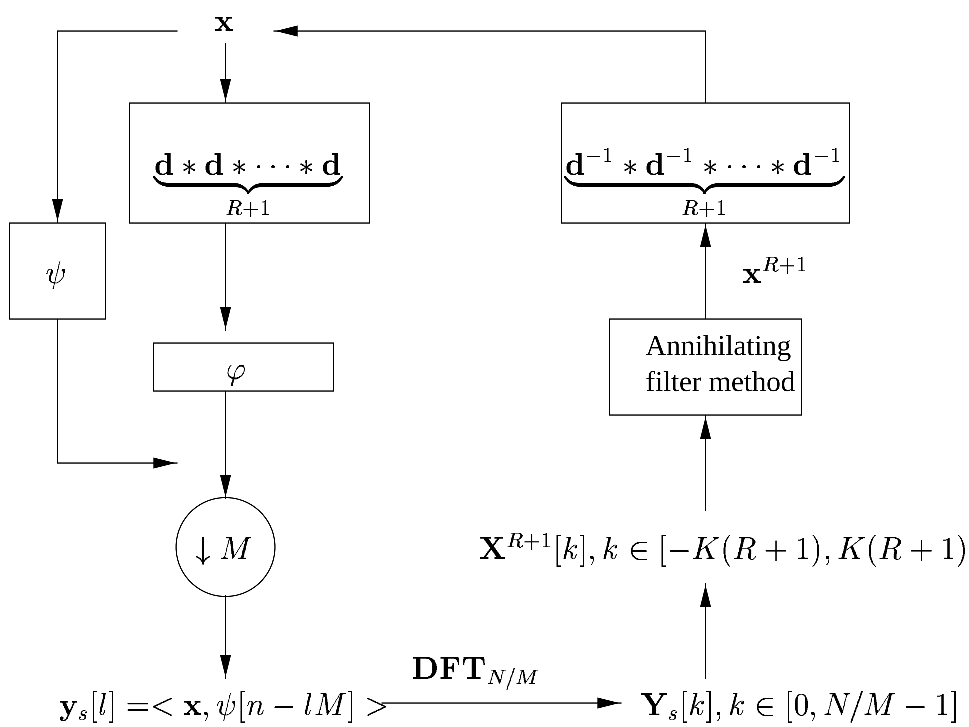


Fig. 21. Sampling and reconstruction scheme for discrete-time piecewise polynomial signals with K pieces of maximum degree R; N/M is the number of samples; $2K(R+1)+1$ is the bandwidth of the sampling kernel.

weights of the Diracs.

- When sampling a randomly spaced stream of Diracs with the *sinc* kernel the system leading to the transitions may be ill-conditioned if the sampling interval T is not chosen appropriately. It is illustrated that at critical sampling, that is, when we have $N = 2K$ sample values, the optimal sampling interval obtained for these type of signals is $T = 0.5$.
- When sampling a randomly spread stream of Diracs with the *Gaussian* kernel the conditioning of both systems depends also on the value of the variance σ^2 in the Gaussian kernel.
- The sampling schemes using the sinc and the Gaussian kernels can be generalized to both continuous-time and discrete-time piecewise polynomial signals.
- Infinite length signals were sampled using a compact support sampling kernel.
- Bilevel signals can be recovered using a *Box* sampling kernel $\varphi_0(t/T)$ if and only if there is at most one transition in each interval $[n, (n+1)T]$.

- Bilevel signals can be recovered using a *Hat* sampling kernel $\varphi_1(t/T)$ if and only if there is at most two transitions in each interval $[n, (n+2)T]$.
- In general, to recover the infinite length piecewise polynomials with K pieces of of maximum degree R using a box sampling kernel, the sampling rate must be greater than the maximum local rate of innovation

$$\rho_m(T) = (2(R+1)+1)/T.$$

- Sampling and reconstruction algorithms were given for each problem in their respective sections.

Appendix

I. Annihilating filter method

The problem in spectral line analysis consists in estimating the frequencies of a sinusoidal signal from a set of values. The methods used for estimating such frequencies are known as high-resolution methods, for example MUSIC, ESPRIT and can be found in [7]. We define the following as the annihilating filter method:

Consider a filter $\mathbf{h} = (h_0, h_1, \ldots, h_K)$ characterized by its z-transform $H(z) = \sum_{l=0}^{K} h_l\, z^{-l}$.

Definition 6: Annihilating filter.

A filter h is called an annihilating filter for a signal $s[n]$ if and only if

$$(h * s)[n] = 0, \quad \forall n \in \mathbb{Z}. \tag{141}$$

Proposition 5: The filter $H(z) = 1 - u\, z^{-1}$ annihilates the exponential signal $s[n] = u^n$.

Proof: Let $y = h * s$. Then in $z-$ domain we have $Y(z) = H(z)\, S(z) = S(z) - u\, z^{-1}\, S(z)$ and thus y satisfies the difference equation

$$y[n] = s[n] - u\, s[n-1]. \tag{142}$$

Substitute $s[n] = u^n$ in (142) we obtain

$$y[n] = u^n - u\, u^{n-1} = 0, \quad \forall n \in \mathbb{Z} \tag{143}$$

■

Consider a signal $s[n], n \in \mathbb{Z}$ defined as a finite linear combination of K exponentials u_k^n,

$$s[n] = \sum_{k=0}^{K-1} c_k\, u_k^n \tag{144}$$

where c_k are real and u_k are real or complex valued. In the context of spectral line analysis $u_k = e^{i\omega_k}$ where ω_k is the kth frequency component of the signal $s[n]$.

Corollary 3: The filter

$$H(z) = \prod_{k=0}^{K-1} (1 - u_k\, z^{-1}) \tag{145}$$

annihilates the signal $s[n]$ defined in (144).

Proof: Let $y = h * s$. Denote the kth annihilating filter by $H_k(z) = (1 - u_k\, z^{-1})$ and let $S_k(z)$ be the z-transform of the kth exponential signal $s_k[n] = u_k^n$. Then we have

$$
\begin{aligned}
Y(z) &= H(z)\, S(z) && (146)\\
&= \prod_{l=0}^{K-1} (1 - u_l\, z^{-1}) \sum_{k=0}^{K-1} c_k\, S_k(z) && (147)\\
&= \sum_{k=0}^{K-1} c_k \big(\prod_{l=0}^{K-1} (1 - u_l\, z^{-1})\, S_k(z) \big) && (148)\\
&= \sum_{k=0}^{K-1} c_k \big(\prod_{l=0, l\neq k}^{K-1} (1 - u_l\, z^{-1})\, \underbrace{(1 - u_k\, z^{-1})\, S_k(z)}_{=0} \big) && (149)\\
&= 0 \text{ from Proposition 5}
\end{aligned}
$$

Therefore to find the values u_k we need to find the filter coefficients h_l in

$$H(z) = \sum_{l=0}^{K} h_l\, z^{-l} \tag{150}$$

such that

$$
\begin{aligned}
\mathbf{h} * \mathbf{s} &= \mathbf{0} && (151)\\
\Leftrightarrow \sum_{k=0}^{K} h_k\, s[n-k] &= 0, \quad \forall n. && (152)
\end{aligned}
$$

In matrix/vector form the system in (152) is equivalent to

$$
\begin{bmatrix}
\vdots & \vdots & \cdots & \vdots \\
s[0] & s[-1] & \cdots & s[-K] \\
s[1] & s[0] & \cdots & s[-(K-1)] \\
\vdots & \vdots & \ddots & \\
s[K] & s[K-1] & \cdots & s[0] \\
\vdots & \vdots & \cdots & \vdots
\end{bmatrix}
\cdot
\begin{pmatrix} h_0 \\ h_1 \\ \vdots \\ h_K \end{pmatrix}
= 0. \tag{153}
$$

Suppose a finite number of values $s[n]$ are available. Since there are $K+1$ unknown filter coefficients we need at least $K+1$ equations, and therefore we need at least $2K+1$ values of $s[n]$ to find the filter coefficients.[8] Define $\mathbf{S}$ the appropriate sub-matrix then the system $\mathbf{S} \cdot \mathbf{h} = 0$ will admit a solution when

$$rank(\mathbf{S}) < K + 1. \tag{154}$$

[8] Actually there are K unknown filter coefficients since $h_0 = 1$ and therefore we will need at least $2K$ values of $s[n]$. The system to solve in this case is known as a Yule-Walker system [2]

In pratice this system is solved using a singular value decomposition where the matrix $\mathbf{S}$ is decomposed into $\mathbf{S} = \mathbf{U}\Sigma\mathbf{V}^*$. We obtain that $\mathbf{h} = \mathbf{V} \cdot \mathbf{e}_{K+1}$ where $\mathbf{e}_{K+1}$ is a vector with 1 on position $K+1$ and 0 elsewhere. The method can be adapted to noise by minimizing $||\mathbf{S} \cdot \mathbf{h}||$ in which case $\mathbf{h}$ is given by the eigenvector associated with the smallest eigenvalue of $\mathbf{S}^T\mathbf{S}$.

Once the the filter coefficients are found then the values u_k are simply the roots of the annihilating filter $H(z)$.

To determine the weights c_k it suffices to take K equations in (144) and solve the system for c_k. Let $n = 0, \ldots, K-1$ then in matrix vector form we have the following Vandermonde system

$$\begin{bmatrix} 1 & 1 & \ldots & 1 \\ u_0 & u_1 & \ldots & u_{K-1} \\ \vdots & \vdots & \ldots & \vdots \\ u_0^{(K-1)} & u_1^{(K-1)} & \ldots & u_{K-1}^{(K-1)} \end{bmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{K-1} \end{pmatrix} = \begin{pmatrix} s[0] \\ s[1] \\ \vdots \\ s[K-1] \end{pmatrix} \tag{155}$$

and has a unique solution when

$$u_k \neq u_l, \forall k \neq l. \tag{156}$$

This concludes the annihilating filter method.

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