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Linear Phase Wavelets: Theory and Design [1]

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Abstract

Recent work has shown the strong connection between FIR perfect reconstruction filter banks and bases of compactly supported wavelets. The compactly supported wavelets that have been presented to date have been very constrained, and correspond to filter banks where very little freedom in the design of the filters is allowed. In this paper we present new theoretical results on FIR filter banks based on Diophantine equations and continued fraction expansions; and use them to show how more general wavelets may be designed. We further show that by considering a noncausal IIR structure it is possible to have a linear phase paraunitary solution. A number of design examples illustrating the advantages of the new results are presented.

1 Introduction

The discrete wavelet transform uses a set of basis functions which are discrete scales and translates of a single basis wavelet:

$$h_{mn}(t) = a_0^{-m/2} \cdot h(t/a_0^m - nb_0), \;\; (m,n \in \mathcal{Z}), \;\; a_0 > 1, \;\; b_0 \neq 0 \tag{1}$$

The wavelet transform gives a time-scale representation which is sharp in time at small scales, and sharp in scale at large scales. It has recently been shown that a dyadic discrete wavelet scheme (i.e. $a_0 = 2$) may be realised using a multirate filter bank, and that a paraunitary FIR filter bank gives rise to an orthonormal basis of compactly supported wavelets [2] under certain conditions. It has also been demonstrated that more general perfect reconstruction filter banks generate biorthogonal systems of compactly supported wavelets [5]. While orthonormality is lost, considerable freedom in the design of the wavelets is gained; in particular it is possible to achieve linear phase [1, 5, 6].

The conditions for the convergence of the infinitely iterated and subsampled lowpass filter in the filter bank realisation of the discrete wavelet scheme were considered in [2]. A condition sufficient to ensure convergence to a continuous function was found to be that the filter possess a sufficient number of zeros at π to adequately attenuate the supremum of the magnitude of the Fourier transform of the remaining factor. It was for this reason that the filters in [2] and [5] were designed to have the maximum possible number of zeros at π.

In this paper we first present new theoretical results on filter banks which allow us to design filter banks that still have a fixed number of zeros at π but have greater freedom than those of [2] and [5]. These generate orthonormal or biorthogonal systems of compactly supported wavelets. We present a novel IIR structure which satisfies the requirements for both linear phase and losslessness. Finally relevant design examples are given.

2 Bezout's identity

It is well known that in a two-channel maximally decimated filter bank, the necessary and sufficient condition to achieve perfect reconstruction with an FIR synthesis section after an FIR analysis with filters $H_0(z)$ and $H_1(z)$ can be written in two equivalent forms [6]:

$$H_{00}(z)H_{11}(z) - H_{01}(z)H_{10}(z) = z^{-l} \tag{2}$$
$$H_0(z)H_1(-z) - H_0(-z)H_1(z) = 2z^{-2l-1} \tag{3}$$

In (2) we have introduced the polyphase notation [3, 4] for the filters $H_0(z)$ and $H_1(z)$:

$$H_i(z) = H_{i0}(z^2) + z^{-1}H_{i1}(z^2)$$

The conditions (2) and (3) of course greatly constrain the possible solutions. We will refer to any filter $H_1(z)$ such that (3) is true as a complementary filter to $H_0(z)$.

A polynomial Bezout identity is an equation of the form $a(z)p(z) + b(z)q(z) = 1$ where all quantities are polynomials. It is well known that given $a(z)$ and $b(z)$ there is a solution $[p(z), q(z)]$ if and only if $a(z)$ and $b(z)$ are coprime. The fact that (2) and (3) have the Bezout identity form gives us the following two facts [6].

Fact 2.1 *Assume that the filters $H_0(z)$ and $H_1(z)$ are both FIR and causal. Then given one of the pairs $[H_{00}(z), H_{01}(z)]$, $[H_{10}(z), H_{11}(z)]$, $[H_{00}(z), H_{10}(z)]$ or $[H_{01}(z), H_{11}(z)]$ in order to calculate the other pair necessary to achieve perfect reconstruction it is necessary and sufficient that the given pair be coprime (except for possible zeros at $z = \infty$ or $z = 0$).*

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Fact 2.2 *A filter $H_0(z)$ has a complementary filter if and only if it has no zeros in pairs at $z = \alpha$ and $z = -\alpha$.*

Fact 2.2 tells us when $H_0(z)$ will have a complement; if it exists it can be found by solving a set of linear equations [4], or by using Euclid's algorithm [1, 6]. Proofs and further implications are given in [6]. That solutions can generate biorthogonal bases of compactly supported wavelets is shown in [1, 6].

3 Diophantine equations

The equation

$$ax + by = c \tag{4}$$

where all quantities are integers is known as a basic diophantine equation. It is obvious that the solution (x, y) is not unique; for example if solutions to the equation $aw + bz = 0$ are available we can add them to (x, y) to generate new solutions $(x', y') = (x, y) + k(w, z)$ where k is any constant. The analogy between the equations (4) and (3) allows us to create new perfect reconstruction solutions to (3) based on the fact that $H_1'(z) = E(z)H_0(z)$ sets the right hand side of (3) to zero provided that $E(z) = E(-z)$.

In fact it turns out that all valid complementary filters must have this form. The result makes use of an important restriction on the lengths of the possible solutions: that a general length N filter has $N-2$ length $N-2$ complementary filters.

Lemma 3.1 *All filters of length $N + 2m - 2$ which are complementary to a length N filter $H_0(z)$ have the form:*

$$H_1'(z) = z^{-2k} H_1(z) + E(z)H_0(z)$$

where $E(z) = E(-z)$ is a polynomial of degree $2(m-1)$, $k \epsilon (0, 1, \cdots m)$ and $H_1(z)$ is a length $N-2$ complementary filter.

A case of very particular interest is when both $H_0(z)$ and $H_1(z)$ are linear phase, and we desire alternative linear phase complementary filters. Here we make use of the fact that for a linear phase length N filter there is a unique length $N-2$ filter if N is odd. It suffices to consider the odd length linear phase case only, since the even length case can always be brought to this form.

Lemma 3.2 *Given a linear phase $H_0(z)$ of odd length N, and its length $N-2$ linear phase complement $H_1(z)$, all higher degree linear phase filters complementary to $H_0(z)$ are of the form:*

$$H_1'(z) = z^{-2m} H_1(z) + E(z)H_0(z)$$

where

$$E(z) = \sum_{i=1}^{m} \alpha_i (z^{-2(i-1)} + z^{-(4m-2i)})$$

The case where $H_0(z)$ is chosen to be an even power of the binomial is treated in detail by I. Daubechies [2]. Proofs of the above lemmas, with additional background material are in [6].

4 Continued fraction expansions

We now show that different solutions to (3) are related by the canonic continued fraction expansion (CFE) of the ratio of their polyphase components.

We define $D_{-1} = H_{00}$, $D_0 = H_{01}$, $A_{-1} = H_{10}$ and $A_0 = H_{11}$. For the sake of simplicity we remove the phase factor in (2). In this notation (2) becomes:

$$D_{-1}(z)A_0(z) - A_{-1}(z)D_0(z) = 1 \tag{5}$$

Now use Euclid's algorithm starting with the pair $D_{-1}(z), D_0(z)$. The general step is:

$$D_{j-1}(z) = b_j(z)D_j(z) + D_{j+1}(z) \qquad \deg D_j > \deg D_{j+1}$$

so we get the sequences $b_j(z)$ and $D_j(z)$. Similarly by using Euclid's algorithm on the pair $A_{-1}(z), A_0(z)$ we get the sequences $a_j(z)$ and $A_j(z)$. It is shown in [6] that (5) implies $a_0(z) = b_0(z)$, and hence

$$D_0(z)A_1(z) - A_0(z)D_1(z) = -1$$

Since this is of the same form as but of lower degree than (5), we can continue, and it is easily shown that we find a succession of Bezout identities:

$$D_{j-1}(z)A_j(z) - D_j(z)A_{j-1}(z) = (-1)^j \tag{6}$$

which are of decreasing degree. The equations given by (6) imply in turn that $a_0(z) = b_0(z), a_1(z) = b_1(z), \cdots, a_j(z) = b_j(z), \cdots a_N(z) = b_N(z)$. But it is well known that the outputs of Euclid's algorithm (the a_j and b_j) are the partial denominators of the CFE. Hence:

$$\frac{D_{-1}(z)}{D_0(z)} = \frac{H_{00}(z)}{H_{01}(z)} = b_0(z) + \cfrac{1}{b_1(z) + \cfrac{1}{b_2(z) + \cfrac{1}{\ddots + \frac{1}{b_N(z)}}}}$$

$$= [b_0(z);\ b_1(z), b_2(z), \cdots b_N(z)]$$

where we have used the standard notation $[b_0;\ b_1, b_2, \cdots b_N]$ to denote a terminating CFE. Hence it follows from the equalities $a_j = b_j$ that the CFE's of $D_{-1}(z)/D_0(z)$ and $A_{-1}(z)/A_0(z)$ are identical for the first $N+1$ terms. The terminal equation for $j = N$ gives: $a_{N+1}(z) = -(-1)^N A_N(z) D_N$. Note that D_N is scalar, being the last divisor in the algorithm, and $D_0(z)$ and $D_1(z)$ being coprime by assumption. In summary:

$$\frac{H_{00}(z)}{H_{01}(z)} = [b_0(z);\ b_1(z), b_2(z), \cdots b_N(z)]$$

$$\frac{H_{10}(z)}{H_{11}(z)} = [b_0(z);\ b_1(z), b_2(z), \cdots b_N(z), -(-1)^N A_N(z) D_N]$$

Different choices $A_N(z)$ give different complementary filters.

5 Linear phase paraunitary IIR solutions

We now show that it is possible to achieve linear phase and losslessness in the IIR case; in the FIR case this is possible only with the filters that generate the Haar basis or trivial variations thereof.

If we consider the most general form of the rational polyphase matrix [3, 4] and impose the conditiion for losslessness:

$$\mathbf{H}_p^T(z^{-1}) = [\mathbf{H}_p(z)]^{-1}$$

we find that $\mathbf{H}_p(z)$ must necessarily be of the form:

$$\mathbf{H}_p(z) = \begin{bmatrix} A(z) & B(z) \\ -B(z^{-1})\Delta_p(z) & A(z^{-1})\Delta_p(z) \end{bmatrix} \quad (7)$$

where:

$$A(z)A(z^{-1}) + B(z)B(z^{-1}) = 1 = \Delta_p(z)\Delta_p(z^{-1}) \quad (8)$$

and $\Delta_p(z) = \det \mathbf{H}_p(z)$.

In the following we shall refer to any symmetric filter that has a central term as having whole sample symmetry, and one that does not have a central term as having half sample symmetry. For FIR filters the first case corresponds to filters of odd length, and the second to those of even length.

First take the case where $H_0(z) = A(z^2) + z^{-1}B(z^2)$ is to be half sample symmetric then the polyphase components must be reversed versions of each other:

$$\mathbf{H}_p(z) = \begin{bmatrix} H_{00}(z) & z^{-l}H_{00}(z^{-1}) \\ H_{10}(z) & -z^{-m}H_{10}(z^{-1}) \end{bmatrix} \quad (9)$$

On equating (7) and (9) we get that the following polyphase matrix exhibits linear phase and losslessness:

$$\mathbf{H}_p(z) = \begin{bmatrix} A(z) & z^{-l}A(z^{-1}) \\ -z^{l-n}A(z) & z^{-n}A(z^{-1}) \end{bmatrix} \quad (10)$$

where $A(z)$ is any allpass function. For example choosing $l = n = 0$, we get:

$$H_0(z) = A(z^2) + z^{-1}A(z^{-2}) \quad (11)$$

$$H_1(z) = -A(z^2) + z^{-1}A(z^{-2}) \quad (12)$$

Next suppose $H_0(z)$ is to be even sample symmetric. In this case one of the polyphase components must be half sample symmetric, the other whole sample symmetric, and both must be either symmetric or antisymmetric. Since antisymmetric filters always have a zero at $z = 1$ the latter case can never satisfy (8).

It is also implied by (8) that the denominators of $A(z)$ and $B(z)$ are equal, so we must solve:

$$N_A(z)N_A(z^{-1}) + N_B(z)N_B(z^{-1}) = D_A(z)D_A(z^{-1})$$

where $N_A(z)$ and $N_B(z)$ are the denominators and $D_A(z)$ is the common denominator.

Since an IIR filter is symmetric if and only if both numerator and denominator are, we need consider only the symmetry of $N_A(z)$, $N_B(z)$ and $D_A(z)$. There are four cases that give that $A(z)$ and $B(z)$ have the whole/half sample symmetries described above. These are that $D_A(z)$,$N_A(z)$ and $N_B(z)$ are all symmetric and have lengths that are respectively (odd,odd,even), (odd,even,odd), (even,even,odd) and (even,odd,even).

For example for the (odd,odd,even) case $A(z)$ has whole sample symmetry, $B(z)$ has half sample symmetry, and (7) is lossless and gives filters that have whole sample symmetry. It is not clear whether rational solutions exist or not.

6 Wavelet design results

Filters of the types described above are now used to generate wavelets. Figure 1 shows the time function and spectrum of a compactly supported wavelet from an orthonormal basis. It is a generalisation of those described in [2]; the extra freedom has improved the frequency selectivity. A paraunitary filter bank with filters of length 22 was used. Figure 2 gives the time functions and spectra of the linear phase compactly supported wavelets belonging to a biorthogonal system. They were generated by filters of length 19 and 25.

Taking the filters in (11) and (12) and the simple allpass section:

$$A(z) = \frac{1 + az^{-1} + bz^{-2}}{b + az^{-1} + z^{-2}}$$

with $a = 7$, $b = 2.4$ we get reasonable lowpass response for $H_0(z)$. The wavelet and its spectrum are shown in figure 3.

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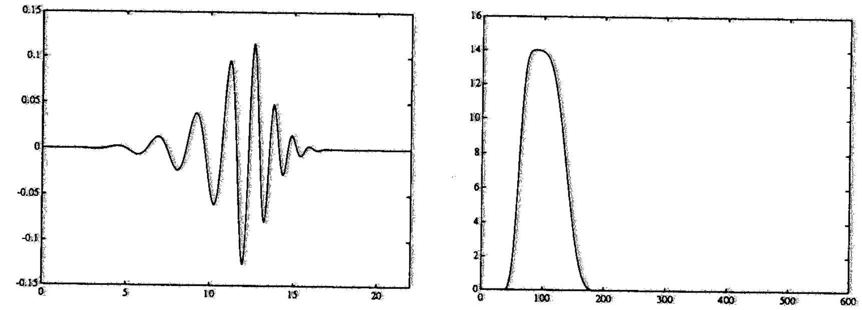

Figure 1: Time function and spectrum of wavelet from orthonormal set generated by FIR filters of length 22.

a)

b)

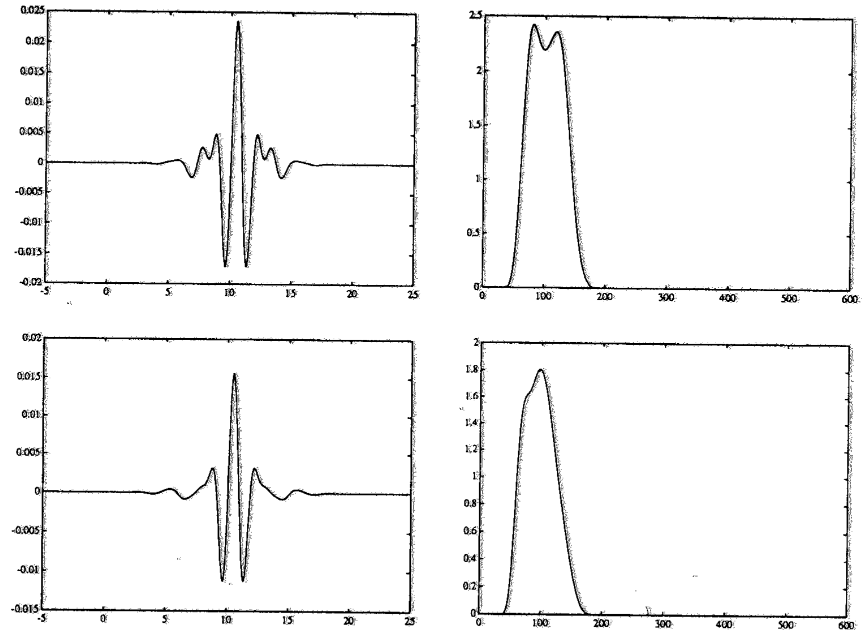

Figure 2: Analysis and synthesis biorthogonal wavelets generated by linear phase FIR filters of length 19 and 25. (a) time function and spectrum of analysing wavelet. (b) time function and spectrum of synthesis wavelet.

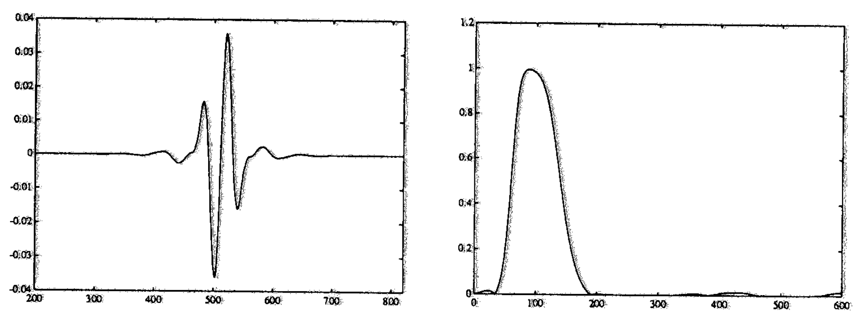

Figure 3: Time function and spectrum of wavelet generated by linear phase lossless IIR filter bank.