WAVELETS GENERATED BY IIR FILTER BANKS

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ABSTRACT

The relation between orthogonal finite impulse response filter banks and orthonormal bases of compactly supported wavelets has been established by Daubechies. Building on this result we use infinite impulse response filter banks to construct more general orthonormal wavelet bases, which have infinite support, but rapid decay. We give a complete constructive method which gives all rational orthogonal two channel filters banks. We develop a family of wavelets which have similar smoothness and moment properties to those of Daubechies. Finally we derive wavelet bases for the space of piecewise polynomial functions, which are alternatives to the Battle-Lemarié bases, and have the desirable property of being realizable. We present relevant design examples.

1 INTRODUCTION

The discrete wavelet transform uses a set of basis functions which are discrete scales and translates of a single basis wavelet. It was shown by Mallat and Meyer that the dyadic wavelet transform can be generated by a multiresolution analysis scheme [5]. Daubechies derived orthonormal bases of compactly supported wavelets using filter banks with finite impulse response (FIR) filters [2]. Details on wavelets and filter banks can be found in [5, 2, 10].

In this paper we use infinite impulse response (IIR) filter banks to derive orthonormal bases of wavelets. First we derive a complete constructive method to generate all rational orthogonal two channel filter banks. We examine the whole family of filters of minimal degree with a maximum number of zeros at $z = -1$, which contains Daubechies, Butterworth and intermediate solutions. These give wavelets with desirable smoothness properties. Finally we show how to construct alternative bases to the Battle and Lemarié bases for the spline spaces [1, 6]; these enjoy the same properties as those of Battle and Lemarié, except for symmetry, but have the property that they are constructed from realizable IIR filter banks. This last result has also been derived independently by Unser and Aldroubi [9].

2 FILTER BANKS AND WAVELETS

In the filter bank iteration scheme to generate the wavelet transform the expression for the scaling function (in the Fourier domain) is:

$$\Phi(w) = \prod_{i=1}^{\infty} H_0(w/2^i), \qquad (1)$$

where $H_0(z)$ is the lowpass filter of a perfect reconstruction filter bank (PRFB).

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For perfect reconstruction it is necessary and sufficient that the synthesis filters, $G_i(z)$, be related to the analysis filters, $H_i(z)$, as follows:

$$[G_0(z)\, G_1(z)] = C(z)[H_1(-z) \; - H_0(-z)], \qquad (2)$$

where $C(z)$ is uniquely determined as [8, 10]:

$$1/C(z) = H_0(z)H_1(-z) - H_0(-z)H_1(z). \qquad (3)$$

If we observe that $C(z) = -C(-z)$ and define $P(z) = H_0(z)H_1(-z)C(z)$, we can write that the necessary and sufficient condition for perfect reconstruction is:

$$P(z) + P(-z) = 1. \qquad (4)$$

Since this condition plays an important role in what follows, we will refer to any function having this property as *valid*.

A central point that we would wish to emphasize is that to a large degree it is the function $P(z)$ which determines the properties of the filter bank and the wavelet. Different factorizations $P(z) = H_0(z)H_1(-z)C(z)$ give different wavelets, but when dealing with orthogonal filter banks, the spectrum and the regularity, for example, do not depend on the particular factorization chosen; it is the same for all.

3 STRUCTURE OF SOLUTIONS

We have just seen that the problem of constructing a PRFB can be reduced to that of finding a function $P(z)$ that satisfies (4). In this section we investigate how to construct such functions.

Lemma 3.1 *If a valid rational function $P(z)$ has no common factors between the numerator and denominator, then the denominator is one of the two polyphase components of the numerator.*

Proof: The constraint that $P(z)$ be valid gives: $P(z) + P(-z) = \sum_{n=-\infty}^{\infty} 2p(2n)z^{-2n} = 1$, so $p(2n) = \delta_n$. Hence $P(z) = 1 + z^{-1}F(z^2)$ for some rational function $F(z)$. If $F(z^2)$ has no common factors between its numerator and denominator, then they must each be functions of z^2. That is $F(z^2) = N(z^2)/D(z^2)$. So we have:

$$P(z) = \frac{D(z^2) + z^{-1}N(z^2)}{D(z^2)}.$$

Clearly the numerator and denominator of $P(z)$ are coprime if and only if $N(z)$ and $D(z)$ are. □

Our main interest is in orthogonal filter banks. By this we mean that the filter impulse responses obey the following orthogonality conditions:

$$\langle h_i(n), h_i(n-2k) \rangle = \delta_k \qquad i \in \{0,1\} \qquad (5)$$

$$\langle h_0(n), h_1(n-2k) \rangle = 0. \qquad (6)$$

Taking the z-transform of (5) gives that every second value of the autocorrelation functions of both $h_0(n)$ and $h_1(n)$ must equal zero, except at the the origin:

$$H_i(z)H_i(z^{-1}) + H_i(-z)H_i(-z^{-1}) = 1 \quad i \in \{0,1\}. \quad (7)$$

So clearly $H_i(z)H_i(z^{-1})$ must be valid. Similarly taking the z-transform of (6) gives that every second sample of the crosscorrelation of $h_0(n)$ and $h_1(n)$ must equal zero:

$$H_0(z)H_1(z^{-1}) + H_0(-z)H_1(-z^{-1}) = 0.$$

It is easily shown [4] that this forces:

$$H_1(z) = z^{2k-1}H_0(-z^{-1})Q(z^2), \quad (8)$$

for some $Q(z)$. It then follows immediately from (7) that $Q(z)Q(z^{-1}) = 1$, that is $Q(z)$ is an allpass function, and that $H_1(z)H_1(z^{-1}) = H_0(-z)H_0(-z^{-1})$. We now have the necessary material for the central result.

Theorem 3.2 *All orthogonal rational two channel filter banks can be formed as follows:*
(i) Choosing an arbitrary polynomial $R(z)$, form:

$$P(z) = \frac{R(z)R(z^{-1})}{R(z)R(z^{-1}) + R(-z)R(-z^{-1})}, \quad (9)$$

(ii) Factor as $P(z) = H(z)H(z^{-1})$,
(iii) Form the filter $H_0(z) = A_0(z)H(z)$, where $A_0(z)$ is an arbitrary allpass,
(iv) Choose $H_1(z) = z^{2k-1}H_0(-z^{-1})A_1(z^2)$, where $A_1(z)$ is again an arbitrary allpass,
(v) Choose $G_0(z) = H_0(z^{-1})$, and $G_1(z) = -H_1(z^{-1})$.

Proof: We have already seen that to get an orthogonal filter bank it is necessary and sufficient to find a valid $P(z)$ which is an autocorrelation. If an autocorrelation is a rational function, then both numerator and denominator must be autocorrelations also; so the numerator must indeed be of the form $R(z)R(z^{-1})$ for some FIR function $R(z)$. Since the numerator is symmetric and of odd length one of its polyphase components is symmetric of even length, and therefore has a zero at $z = -1$. Since zeros on the unit circle must be avoided in the denominator (9) is the only possible choice.

If the numerator and denominator are not co-prime then $P(z) = H(z)H(z^{-1})a(z)a(z^{-1})/a(z)a(z^{-1}) = A_0(z)H(z)H(z^{-1})A_0(z^{-1})$, where $A_0(z)A_0(z^{-1}) = 1$. Hence $P(z) = H_0(z)H_0(z^{-1})$, where $H_0(z) = A_0(z)H(z)$ is the most general case.
We hence use (8) to get the filter $H_1(z)$ in *(iv)*. Finally it is readily verified that $C(z) = A_1(z^2)$, so that using (2) gives the filters in *(v)*. □

Note that the IIR structure presented in [3, 4] which achieved linear phase and orthogonality is a special case.

3.1 Closed form factorization

Theorem 3.2 gives a constructive method for finding orthogonal filter banks; step *(ii)* however involves a numerical factorization, which is dependent on the accuracy of a root extraction procedure. Observe however that if $R(z)$ is of even length $N+1$, and is symmetric, then its polyphase components are related: $R_1(z) = R_0(z^{-1})z^{-(N-1)/2}$. So we can write: $R(z) = R_0(z^2) + z^{-N}R_0(z^{-2})$, from which it follows that:

$$R(z)R(z^{-1}) + R(-z)R(-z^{-1}) = 2R_0(z^2)R_0(z^{-2}).$$

This gives that:

$$P(z) = \frac{R(z)R(z^{-1})}{2R_0(z^2)R_0(z^{-2})},$$

and one factorization is immediate:

$$H(z) = \frac{R(z)}{\sqrt{2}R_0(z^2)}. \quad (10)$$

4 WAVELETS WITH MOMENT PROPERTIES

That the limit of orthogonal filter banks lead to orthonormal bases of wavelets has been firmly established by Daubechies for the case where FIR filters and compactly supported wavelets are involved [2]. That orthogonal IIR filter banks lead to infinitely supported bases follows as an essentially obvious extension [4]. The sufficient condition given by Daubechies to guarantee convergence to a continuous function was that the iterated lowpass filter, $H_0(z)$ should contain an adequate number of zeros at $z = -1$. In fact the design strategy followed in [2] was to find FIR orthogonal filters with the maximum number of such zeros: start with $B(z) = (1+z^{-1})^N(1+z)^N$ and then find the least degree FIR function $F(z)$ such that $P(z) = B(z)F(z)$ is valid; clearly $F(z)$ contains only zeros. These give rise to very smooth wavelets; *i.e.* $\psi(x) \in C^k$ where k grows linearly with the length of the filters.

4.1 Butterworth solutions

Designing orthogonal IIR filter banks with properties similar to those of Daubechies, is now very simple with the aid of theorem 3.2: start with $B(z)$ as before and find the least degree all-pole function $F(z)$ such that $B(z)F(z)$ is valid. This is equivalent to choosing $R(z) = (1+z^{-1})^N$. Hence we factor:

$$\frac{(1+z^{-1})^N(1+z)^N}{(z^{-1}+2+z)^N + (-z^{-1}+2-z)^N} = H_0(z)H_0(z^{-1}). \quad (11)$$

The wavelets have moment properties identical to those of Daubechies, but are smoother. It is worth pointing out that this particular choice gives the halfband digital Butterworth filters. That these filters obey the requirements for orthogonality has been previously pointed out by Smith [8], and that they were useful in a wavelet context by Lemarié and Malgouyres [7].

Example: Choosing $N = 5$ we find that $R(z)$ is of even length, and of course symmetric, so we can use the closed form factorization of (10). The filters are:

$$H_0(z) = \frac{(1+5z^{-1}+10z^{-2}+10z^{-3}+5z^{-4}+z^{-5})}{\sqrt{2}\cdot(1+10z^{-2}+5z^{-4})}$$

$$H_1(z) = z^{-1}\frac{(1-5z^1+10z^2-10z^3+5z^4-z^5)}{\sqrt{2}\cdot(1+10z^2+5z^4)}.$$

Figure 1.a. shows the wavelet, and figure 1.b. the associated spectrum.

4.2 Intermediate solutions

Both the construction in [2], and that given in section 4.1 above involved finding $F(z)$ such that $B(z)F(z)$ is valid. Between the two extremes where $F(z)$ has only zeros (Daubechies' solution), and where it has only poles (Butterworth solution) there are others where $F(z)$ has both poles and zeros. At first this may appear puzzling, since if $F(z) = F_N(z)/F_D(z)$ the numerator of the product $B(z)F(z)$ is increased in length by the degree of $F_N(z)$; so lemma 3.1 would appear to require that adding zeros would increase the number of poles also. Essentially the idea is that the whole sample symmetric polyphase component of $B(z)F_N(z)$ should have certain of its endterms set to zero. We omit the details, however solutions are found simply by solving a set of linear equations [4].

Example: Again considering the case $N = 5$ we can choose:

$$F(z) = \frac{(z^2 - 10z^1 + 34z - 10z^{-1} + z^{-2})}{1792z^2 + 4608 + 1792z^{-2}}.$$

Note that here $F(z)$ has 4 poles and 4 zeros, compared with 8 poles for the Butterworth case, and 8 zeros for the Daubechies case. Figure 2 shows the associated wavelet and spectrum.

4.3 Spline space bases

If the lowpass filter used in the iteration (1) has N zeros at $z = -1$ it can be written as $H_0(z) = (1 + z^{-1})^N K(z)$, for some $K(z)$. Hence

$$\Phi(w) = \prod_{i=1}^{\infty}(1+e^{-jw2^{-i}})^N \prod_{i=1}^{\infty} K(w/2^i) = G(w)\cdot\prod_{i=1}^{\infty} K(w/2^i).$$

It can be shown that $g(x)$, the inverse Fourier transform of $G(w)$, is a spline function, made up of pieces that are polynomials of degree $N - 1$ [1, 6]. But note that if $K(z) = E(z)/E(z^2)$ for some $E(z)$ successive numerators and denominators in the second infinite product cancel and we get:

$$\prod_{i=1}^{\infty} K(w/2^i) = \frac{E(0)}{E(w)},$$

which is a 2π periodic function. Hence:

$$\phi(x) = \sum_{k=-\infty}^{\infty} f(k)g(x - k),$$

that is $\phi(x)$ is a linear combination of splines. So our goal now is to find $E(z)$ such that:

$$P(z) = \frac{(1+z)^N(1+z^{-1})^N E(z)E(z^{-1})}{E(z^2)E(z^{-2})},$$

is valid. It works out that finding such an $E(z)$ is not difficult [4]. Once this is done we can factor as before and both $\phi(x)$ and $\psi(x)$ are piecewise polynomial as explained above. This is most easily seen for the $N = 2$ case; figure 3.a. shows $\psi(x)$, and figure 3.b. shows $\phi(x)$, both of which are piecewise linear.

It is clear therefore that we have constructed wavelet bases for the spline function spaces; an alternative to the wavelet bases derived by Battle [1] and by Lemarié [6]. In fact, although we have approached the construction differently, if we factored $P(z) = \sqrt{P(z)} \cdot \sqrt{P(z)}$ we would obtain precisely the same basis as given in their construction. In other words the different orthogonal bases correspond to different factorizations of $P(z)$. It is worth emphasizing that wavelets generated by different orthogonal factorizations of the same $P(z)$ do not in general span the same spaces. All such bases enjoy similar properties however. While the original construction gives symmetric basis functions, it is based on irrational filters and is unrealizable. The basis we present uses easily implementable IIR filters: no approximation is necessary. This fact has also been noted in [9]. The result is an example of a more general property of orthogonalizing compactly supported bases [4].

4.4 Summary

In all of the above designs the construction of $P(z)$ was the kernel of the technique. If we desire wavelets with a maximum number of disappearing moments we hence design $P(z)$ with a maximum number of zeros at $z = -1$. Those minimum degree $P(z)$'s with this property are easily listed. To make this plain we illustrate the $N = 5$ case in table 1. There are only the three cases already discussed: Daubechies, Butterworth and the intermediate case. Even though it is not of minimal degree, we tabulate here also the $P(z)$ corresponding to the orthogonal bases for the Spline spaces. We also tabulate an estimate of r such that $\phi(x), \psi(x) \in C^r$. The estimation method for the first three cases is quite crude, but suffices to show the advantage of the IIR solutions. The spline space wavelets enjoy a considerable advantage with respect to the others in that no estimation is necessary; the value is known exactly $r = N - 1$. The following shorthand notation for causal FIR functions is used in the table: $\sum_{n=0}^{\infty} a_n z^{-n} = (a_0, a_1, a_2, \cdots)$. For the spline case $E(z)E(z^{-1}) = (1, 502, 14608, 88234, 15190, 88234, 14608, 502, 1)$.

5 CONCLUSION

We have presented a family of IIR filter banks leading to wavelets with good regularity properties. The family spans the range from the Daubechies to the Butterworth wavelets, and includes intermediate solutions. For spline spaces a wavelet basis with an IIR implementation has also been described.

REFERENCES

[1] G. Battle. A block spin construction of ondelettes. Part I: Lemarié functions. *Commun. Math. Phys.*, 110:601–615, 1987.

[2] I. Daubechies. Orthonormal bases of compactly supported wavelets. *Communications on Pure and Applied Mathematics*, XLI:909–996, 1988.

[3] C. Herley and M. Vetterli. Linear phase wavelets: theory and design. In *Proc. IEEE Int. Conf. ASSP*, pages 2017–2020, Toronto, May 1991.

[4] C. Herley and M. Vetterli. Recursive filter banks and wavelets. *IEEE Trans. on Signal Proc.*, 1992. Submitted.

[5] J. M. Combes et al., editor. *Wavelets, Time-Frequency Methods and Phase Space*, New York, NY, 1989. Springer-Verlag. Lecture Notes on IPTI.

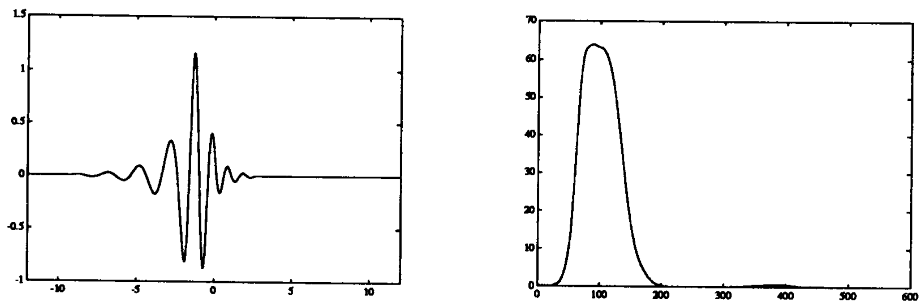
[6] P. G. Lemarié. Ondelettes à localisation exponentielle. *J. Math. pures et appl.*, 67:227–236, 1988.

[7] P. G. Lemarié and G. Malgouyres. Ondelettes sans peine. Unpublished manuscript.

[8] M. J. T. Smith. *Subband Coding of Images*, chapter IIR analysis/synthesis systems. Kluwer Academic, Norwell, MA, 1991.

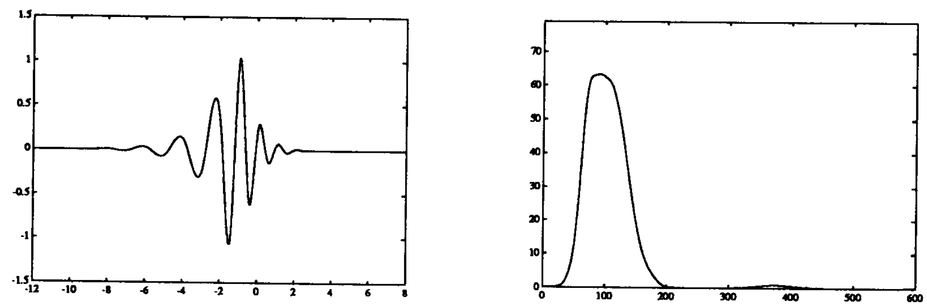
[9] M. Unser and A. Aldroubi. *Wavelets: a Tutorial*, chapter Polynomial Splines and Wavelets: a Signal Processing Perspective. Academic Press, San Diego, CA, 1992. To appear.

[10] M. Vetterli and C. Herley. Wavelets and filter banks: theory and design. *IEEE Trans. on Signal Proc.*, Sept. 1992. To appear.



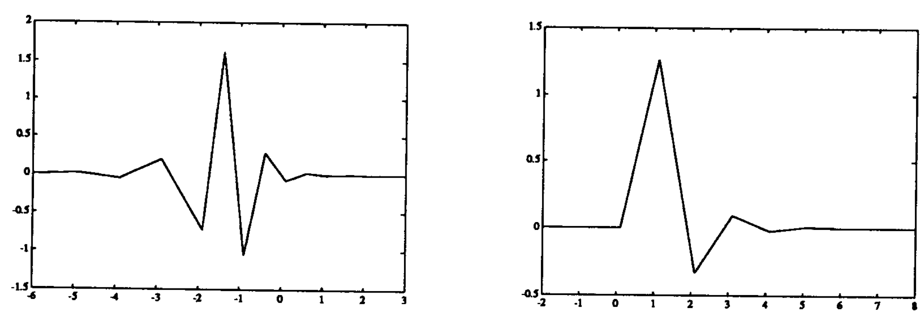
(a) (b)

Figure 1: Orthogonal filter solution derived from Butterworth $N = 5$. (a) Wavelet (b) Spectrum.



(a) (b)

Figure 2: Orthogonal filter solution derived from intermediate $N = 5$. (a) Wavelet (b) Spectrum.



(a) (b)

Figure 3: Orthogonal IIR wavelet basis for piecewise linear spline space. (a) Wavelet (b) Scaling function.

Solution	$P(z)$ for $N = 5$	Regularity
Daubechies	$(1+z)^5(1+z^{-1})^5 \cdot (35, -350, 1520, -3650, 5018, -3650, 1520, -350, 35) \cdot z^4 2^{-16}$	$r > 1.5960$
Butterworth	$(1+z)^5(1+z^{-1})^5 z^{-4} / (10, 0, 120, 0, 252, 0, 120, 0, 10)$	$r > 3.1318$
Intermediate	$(1+z)^5(1+z^{-1})^5 \cdot (1, -10, 34, -10, 1)/(1792, 0, 4608, 0, 1792)$	$r > 3.1050$
Spline	$(1+z)^5(1+z^{-1})^5 \cdot z^{-8} 2^{-9} E(z)E(z^{-1})/E(z^2)E(z^{-2})$	$r = 4.0$

Table 1: The various solutions for $N = 5$.