OVERCOMPLETE EXPANSIONS AND ROBUSTNESS

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ABSTRACT

Increased robustness of overcomplete expansions compared to nonredundant ones is manifested for two primary sources of degradation, white additive noise and quantization. Reconstruction from expansion coefficients adulterated by an additive noise reduces the noise effect by a factor proportional to expansion redundancy. We conjecture that the effect of quantization error can be reduced inversely to the square of expansion redundancy and prove that result in two particular cases, Weyl-Heisenberg expansions and oversampled A/D conversion.

1. INTRODUCTION

Motivation for the development of the theory of time-frequency and time-scale expansions towards wavelet and Weyl-Heisenberg frames [1] stems mainly from the design freedom which is usually attained with overcomplete expansions. Also, it has been observed that for a given accuracy of representation overcomplete expansions allow for a progressively coarser quantization provided that redundancy is increased. The fact that overcomplete expansions are less sensitive to degradations than nonredundant ones comes as no surprise considering that redundancy in engineering systems usually provides robustness. Although the principle of redundancy-robustness trade-off seems intuitive, it is not always simple to unravel underlying mechanisms and give a quantitative characterization.

The first analysis of the effect of increased robustness of overcomplete expansions to white additive noise was given by Daubechies [1]. This analysis will be reviewed in the next section. It demonstrated that signals can be reconstructed from noisy expansion coefficients with an error whose variance is inversely proportional to expansion redundancy. Daubechies further conjectured that if quantization is the source of degradation of expansion coefficients, then signals could be reconstructed with an error which decays with increased expansion redundancy faster than would be expected based on the white noise model for quantization error.

The purpose of this paper is to demonstrate that overcomplete expansions indeed exhibit a higher degree of robustness to quantization error than to a white additive noise. We conjecture that the information contained in quantized expansion coefficients allows for reconstruction with an error whose squared norm is inversely proportional to the square of the expansion redundancy. This result is proven for a fundamental instance of quantization of overcomplete expansion, namely oversampled A/D conversion, and then generalized to Weyl-Heisenberg expansions.

Notation

Convolution of two signals $f(t)$ and $g(t)$ will be written as $f(t) * g(t)$. The Fourier transform of a signal $f(t)$, $\mathcal{F}\{f(t)\}$, will be written as $\hat{f}(\omega)$.

2. GENERAL CONCEPTS

Consider a frame $\{\varphi_j\}_{j\in J}$ in a Hilbert space $\mathcal{H}$. Let $\{\tilde{\varphi}_j\}_{j\in J}$ be its minimal dual frame, and let $\tilde{F}$, $\tilde{F} : \mathcal{H} \to \ell^2(J)$, be the associated frame operator defined by $\tilde{F}f = \{c_j : c_j = \langle f, \tilde{\varphi}_j\rangle\}_{j\in J}$. Coefficients of the expansion of a signal f in $\mathcal{H}$ with respect to $\{\varphi_j\}_{j\in J}$

$$f = \sum_{j\in J} c_j \varphi_j, \tag{1}$$

are the image of f under $\tilde{F}$. If $\{\varphi_j\}_{j\in J}$ is not an exact frame (Riesz bases) in $\mathcal{H}$, vectors $\tilde{\varphi}_j$ are linearly dependent, and consequently the range of $\tilde{F}$ is a proper subspace of $\ell^2(J)$.

Assume that the expansion coefficients of f are degraded by an additive white noise $\{n_j\}_{j\in J}$. The noise can be represented as $n_j = n_j^r + n_j^o$, where $\{n_j^r\}$ is the component which is in the range of $\tilde{F}$, while $\{n_j^o\}$ is in its orthogonal complement. The *linear reconstruction* formula

$$f_{rec} = \sum_{j\in J} (c_j + n_j)\varphi_j, \tag{2}$$

implicitly reduces to zero the noise component orthogonal to $Ran(\tilde{F})$, giving $f_{rec} = \sum_{j\in J}(c_j + n_j^r)\varphi_j$, as illustrated in Figure 1. As the frame redundancy increases, the range of $\tilde{F}$ becomes more and more constrained

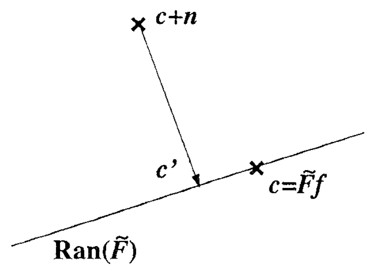

Figure 1: *Noise reduction in frames using linear reconstruction.*

so the noise reduction becomes more effective. Based on this argument it was shown in [1, 5] that in the case of tight frames in finite dimensional spaces and a white, zero-mean noise, the expected squared error norm $E(\|f - f_{rec}\|^2)$ behaves as

$$E(\|f - f_{rec}\|^2) = O(\sigma^2/r), \tag{3}$$

where σ^2 is the noise variance.

Noise reduction in Weyl-Heisenberg and wavelet frames in $L^2(\mathbf{R})$ was also studied by Daubechies [1] in the case of signals which are well-localized in a bounded region of the time-frequency plane, and can be well approximated using a finite number of expansion terms. If a signal f is reconstructed from noisy coefficients as

$$f_{rec} = \sum_{j \in B \subset J} (\langle f, \tilde{\varphi}_j \rangle + n_j)\varphi_j,$$

the reconstruction error can be bounded as

$$E(\|f - f_{rec}\|^2) = \epsilon \|f\|^2 + O(\sigma^2/r). \tag{4}$$

The $\epsilon\|f\|^2$ component in (4) is the result of the approximation of f using the finite set of the expansion terms, $\{\varphi_j\}_{j \in B \subset J}$. A rigorous proof of this result was given by Munch [2] for the case of tight Weyl-Heisenberg frames and integral frame redundancy factors.

Based on these results it may be conjectured that the $O(1/r)$ noise reduction property has a wider scope than discussed here. However, no more general results have been proven yet.

Quantization error is commonly modeled as a white additive noise, and this approach gives satisfactory results when applied to orthogonal expansions. However, in the case of overcomplete expansions quantization error exhibits certain structure which is obscured by the statistical analysis. Here we give a different explanation for the robustness to quantization which reveals more about the underlying principles.

Quantization of expansions in a Hilbert space $\mathcal{H}$, with respect to a given frame $\{\varphi_j\}_{j \in J}$, is a many-to-one mapping from $\mathcal{H}$ to $\mathcal{H}$. It defines a partition of $\mathcal{H}$ into disjoint quantization cells $\{C_i\}_{i \in \mathcal{I}}$. In the case of uniform scalar quantization each of the cells is defined by a set of convex constraints of the type

$$C_i = \{f : (n_{ij} - 1/2)q \leq \langle f, \tilde{\varphi}_j \rangle < (n_{ij} + 1/2)q, \; j \in J\}, \tag{5}$$

where q is the quantization step, and n_{ij}'s are integer numbers. For each of the cells, the quantization maps all the signals in the cell to a single signal in its interior, usually its centroid. Roughly speaking, the expected value of quantization error is proportional to the average quantization cell size. The error can be reduced by decreasing the quantization step. Alternatively, the partition can be further refined by adding new vectors to the family $\{\varphi_j\}$. Increased redundancy of the expansion induces subdivisions of the quantization cells (see Figure 2) which are result of additional constraints of the form given in (5). This gives another explanation of the error reduction property, this time for the quantization error. If the signal f is reconstructed from quantized coefficients $\{\hat{c}_j\}_{j \in J}$ using the linear reconstruction formula $f_{rec} = \sum_{j \in J} \hat{c}_j \varphi_j$, then according to the white noise model the expected value of the reconstruction error behaves as $E(\|f - f_{rec}\|^2) = O(q^2/r)$. This result agrees with experimental data [5] for moderate frame redundancies. However, with high expansion redundancies the deterministic nature of the quantization error is more pronounced and the white noise model is inappropriate. Besides, if expansions in infinite dimensional spaces, (*e.g.* $\ell^2(\mathbf{Z})$ or $L^2(\mathbf{R})$), are considered the statistical approach is not convenient for estimating the error norm since expansion coefficients are generally not square summable after a white noise is added.

In addition to the inadequacy of the statistical approach, it turns out that linear reconstruction is suboptimal since it does not necessarily give a signal which lies in the same quantization cell as the original. It can be expected that a reconstruction algorithm which always yields a signal in the quantization cell of the original better exploits information contained in the quantized coefficients and thus reduces the quantization error more than the linear algorithm. Such a reconstruction strategy is called *consistent reconstruction*.

This observations were made first by Thao and one of us in the context of oversampled A/D conversion [3]. It was also shown in [3] that in the case of the conversion of periodic bandlimited signals (trigonometric polynomials), consistent reconstruction gives an error $\|f - f_{rec}\|^2 = O(1/r^2)$, where r is the oversampling ratio, provided that f has a sufficient number of quantization threshold crossings. The question which naturally arises is whether this result has a wider scope. So far it has been numerically verified for frames in $\mathbf{R}^n$ [5]. In the remainder of this paper we prove under certain assumptions the error behaves as $O(1/r^2)$ in the case of Weyl-Heisenberg expansions. This result comes as an implication of the deterministic analysis of over-

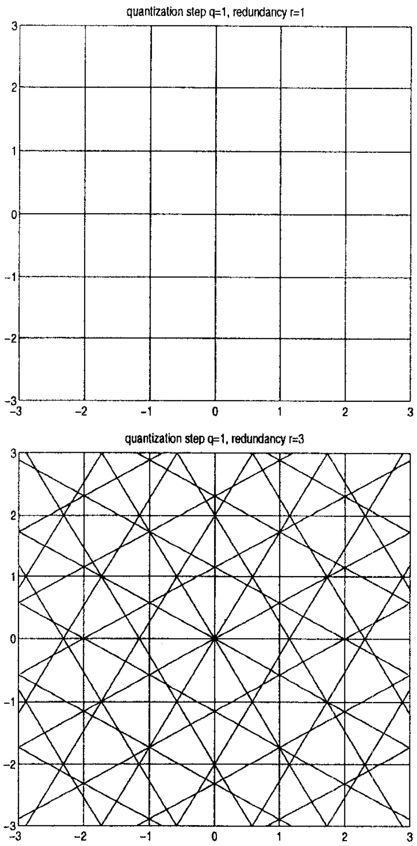

Figure 2: *Partitions of* $\mathbf{R}^2$ *induced by quantization of expansions with respect to an orthonormal basis (top) and with respect to a frame of redundancy* $r = 3$ *(bottom).*

sampled A/D conversion of signals in $L^2(\mathbf{R})$, which is hereafter studied in more detail.

3. OVERSAMPLED A/D CONVERSION

Oversampled analog-to-digital conversion of an analog σ-bandlimited signal $f(t)$ consists of sampling with some sampling interval $\tau < \pi/\sigma$, followed by quantization with a quantization step q. Note that we say that $f(t)$ is σ-bandlimited if $f(t) \in L^2(\mathbf{R})$ and $\hat{f}(\omega) = 0$ for $\omega > \sigma$. The sampling gives expansion coefficients of $f(t)$ with respect to a family of *sinc* functions, $\{sinc_\sigma(t-n\tau)\}_{n\in\mathbf{Z}}$, which is a tight frame for the space of σ-bandlimited signals. Therefore, oversampled A/D conversion amounts to quantization of a tight frame expansion.

Let $g(t)$ be a consistent estimate of $f(t)$, that is

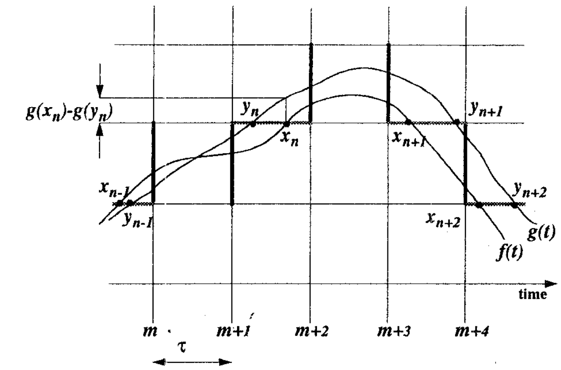

Figure 3: *Quantization threshold crossings of an analog signal* $f(t)$ *and its consistent estimate* $g(t)$.

a signal which shares the same quantization cell with $f(t)$, and assume that the sampling interval is small enough so that all quantization threshold crossings of $f(t)$ occur in distinct sampling intervals. Then whenever $f(t)$ goes through a quantization threshold at some point x_n, $g(t)$ passes through the same quantization threshold at a point y_n which is in the same sampling interval with x_n, i.e. $|x_n - y_n| < \tau$ (see Figure 3). At the point x_n, the difference between $f(t)$ and $g(t)$ is bounded as $|f(x_n) - g(x_n)| = |g(x_n) - g(y_n)| \leq c_n \cdot \tau$, which follows from the bandlimitedness of $g(t)$. Hence, at the quantization threshold crossings of $f(t)$, $\{x_n\}$, the difference between $f(t)$ and its consistent estimates tends to zero as the oversampling ratio $r = \pi/\sigma\tau$ tends to infinity. If $\{x_n\}$ is a *sequence of stable sampling* for the space of σ-bandlimited signals [6], then the reconstruction error also tends to zero when $r \to \infty$, and the following theorem describes error behavior. Note that as in Landau's work [6] which introduced the notion of *sequence of stable sampling* we assume that $\{x_n\}$ is a uniformly discrete set. Such a sequence of points will be also called here a *frame sequence* for the space of σ-bandlimited signals.

Theorem 1 *[4] If the sequence of quantization threshold crossings of* $f(t)$ *forms a sequence of stable sampling for the space of* σ*-bandlimited signals, then for a sufficiently large oversampling ratio and any consistent estimate of* $f(t)$, $g(t) \in C^1$,

$$||f(t) - g(t)||^2 \leq k||f(t)||^2/r^2, \tag{6}$$

where k *is a constant which does not depend on* r.

This theorem can be easily extended to complex bandlimited signals, which is necessary for the generalization to Weyl-Heisenberg expansions.

4. QUANTIZATION OF WEYL-HEISENBERG EXPANSIONS

Let $\{\varphi_{m,n}(t) : \varphi_{m,n}(t) = \varphi(t - nt_0)exp(jm\omega_0 t)\}_{m,n\in\mathbf{Z}}$ be a Weyl-Heisenberg frame in $L^2(\mathbf{R})$ derived from a window $\varphi(t)$. Frame coefficients $\{c_{m,n} : c_{m,n} = \langle\varphi_{m,n}, f\rangle\}$ of a signal f can be expressed in the Fourier domain as

$$c_{m,n} = \int_{-\infty}^{\infty} \hat{f}(\omega - m\omega_0)\hat{\varphi}^*(\omega)e^{j\omega n t_0}d\omega. \qquad (7)$$

For fixed m, the coefficients $c_{m,n}$ are samples of the signal $f_m(t) = [f(t)exp(jm\omega_0 t)] * \varphi(-t)$, which will be called the *m-th subband component* of $f(t)$. This interpretation of Weyl-Heisenberg frame coefficients of the signal $f(t)$ implies that their quantization amounts to quantization of samples of the subband components of $f(t)$. Note that these coefficients are in general complex and it is assumed here that the real and imaginary parts are quantized separately. If the frame window $\varphi(t)$ is a σ-bandlimited function, each of the subband components is also a σ-bandlimited signal. In this context, a signal $g(t)$ is said to be a consistent estimate of $f(t)$ if they have the same quantized values of the frame coefficients and each subband component of $g(t)$ is continuously differentiable, $g_m(t) \in C^1$. Note that the subband signals, being bandlimited, are continuously differentiable almost everywhere.

According to Theorem 1 we can expect that if the frame redundancy is increased by decreasing the time step t_0 for a fixed ω_0, the quantization error of consistent reconstruction should decay as $O(t_0^2)$. This result is established by the following corollary of Theorem 1.

Corollary 1 *[4] Let $\{\varphi_{m,n}(t)\}$ be a Weyl-Heisenberg frame in $L^2(\mathbf{R})$, with time step t_0 and frequency step ω_0, derived from a σ-bandlimited window function $\varphi(t)$. Consider quantization of the frame coefficients of a signal $f(t) \in L^2(\mathbf{R})$ and suppose that for a certain ω_0 the following hold:*

i) the quantization threshold crossings of both the real and imaginary parts of all the subband components $f_m(t) = \left(f(t)e^{jm\omega_0 t}\right) * \varphi(-t)$ form frame sequences for the space of σ-bandlimited signals, with frame bounds $0 < \alpha_m^r \leq \beta_m^r < \infty$ and $0 < \alpha_m^i \leq \beta_m^i < \infty$;

ii)

$$\sup_{m\in\mathbf{Z}} \max\left(\frac{\beta_m^r}{\alpha_m^r}, \frac{\beta_m^i}{\alpha_m^i}\right) = M < \infty.$$

Then there exists a constant δ, such that if $t_0 < \delta$, the reconstruction error satisfies for any consistent estimate $g(t)$ of $f(t)$,

$$\|f(t) - g(t)\|^2 \leq k\|f(t)\|^2 t_0^2, \qquad (8)$$

where k is a constant which does not depend on t_0.

Since we consider the case when ω_0 is constant, this result can be expressed in terms of the oversampling ratio, $r = 2\pi/\omega_0 t_0$, as $\|e\|^2 = O(1/r^2)$.

Along the same lines it can be shown that analogous results hold in cases when the window function is time-limited or if the signal has a compact support in either time or frequency.

5. CONCLUSION

In this paper we studied the effect of increased robustness of overcomplete expansions to quantization. The effect of quantization is commonly analyzed using the white noise model, which indicates that the quantization error decays inversely to the expansion redundancy. Overcomplete expansions, however, exhibit a higher degree of robustness to quantization than to the white noise degradation. We demonstrated that in the case of Weyl-Heisenberg expansions and oversampled A/D conversion the quantization error is inversely proportional to the square of the redundancy factor.

6. REFERENCES

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