SAMPLING DISCRETE-TIME PIECEWISE BANDLIMITED SIGNALS

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ABSTRACT

We consider sampling discrete-time periodic signals which are *piecewise* bandlimited, that is, a signal that is the sum of a bandlimited signal with a piecewise polynomial signal containing a finite number of transitions. These signals are not bandlimited and thus the Shannon[1] sampling theorem for bandlimited signals can not be applied. In this paper, we derive sampling and reconstruction schemes based on those developed in [1, 6, 7] for piecewise polynomial signals which take into account the extra degrees of freedom due to the bandlimitedness.

1. INTRODUCTION

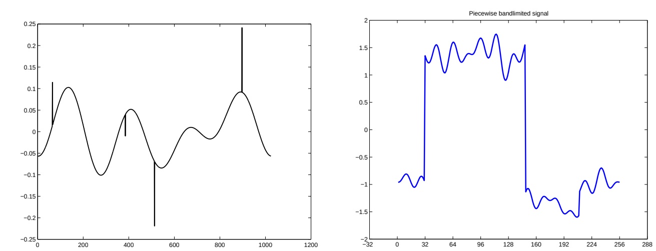
Sampling of bandlimited signals has been a subject of interest to the sampling community for more than half a century [4]. The well-known sampling theorem [2] states that a continuous-time signal $x(t)$ bandlimited to $[-\omega_m, \omega_m]$ is uniquely represented by a uniform set of samples $x[n] = x(nT)$ taken T seconds apart, if the sampling rate is greater or equal to the bandwidth of the signal, that is, $2\pi/T \geq 2\omega_m$. But not all signals are bandlimited. Recall the definition of bandlimited.

Definition 1 *$B-$bandlimited signal.*
A discrete-time periodic signal $x[n]$ with period N is $B-$bandlimited if the discrete-time Fourier series coefficients $X[k]$ are nonzero inside the band $[-B, B]$ and zero outside the band $[-B, B]$, that is,

$$X[k] = \begin{cases} \sum_{n=0}^{N-1} x[n]\, e^{-i2\pi nk/N} & k \in [-B, B] \\ 0 & k \notin [-B, B] \end{cases} \quad (1)$$

with $k \in \mathbb{Z}, B \in \mathbb{N}$

In [1, 6, 7] sampling theorems for particular non-bandlimited signals namely streams of Diracs and piecewise polynomial signals were given. These signals belong to a certain class of signals which have a finite rate of innovation ρ, that is, the number of degrees of freedom per unit of time is finite. Discrete-time periodic bandlimited signals of period N also have a finite rate of innovation, that is, $\rho = (2B + 1)/N$.
In this paper we are interested in signals that would have been bandlimited if it were not for spikes or discontinuities in certain locations, see Figure 1.



(a) (b)

Figure 1: (a) Bandlimited signal with 4 spikes; (b) Bandlimited signal with 3 discontinuities.

These signals are defined as being the sum of a bandlimited signal with a stream of Diracs, in the simplest case, or with a piecewise polynomial signal. Formally we have the following

Definition 2 *Discrete-time piecewise bandlimited signal.*
Let $\mathbf{x}_B$ be a discrete-time periodic $B-$bandlimited signal of period N with corresponding DTFS coefficients $\mathbf{X}_B$ such that $X_B[m] = 0 \quad \forall m \notin [-B, B]$. Let $\mathbf{x}_{PP}$ be a discrete-time periodic piecewise polynomial signal of period N of zero mean and with K pieces each piece of maximum degree R. Then a piecewise bandlimited signal $\mathbf{x}$ is defined by

$$\mathbf{x} = \mathbf{x}_B + \mathbf{x}_{PP} \quad (2)$$

with corresponding DTFS coefficients defined by

$$X[m] = \begin{cases} X_B[m] + X_{PP}[m] & \text{if } m \in [-B, B] \\ X_{PP}[m] & \text{if } m \notin [-B, B] \end{cases} . \quad (3)$$

[1]also due to Kotelnikov, Whittaker

The bandlimited signal with 3 discontinuities illustrated in Figure 1 (b) is obtained by adding a piecewise constant signal with 3 discontinuities in Figure 2 (c) to a bandlimited signal in Figure 2 (a). From Figure 2 (f) the piecewise bandlimited signal is evidently not bandlimited.

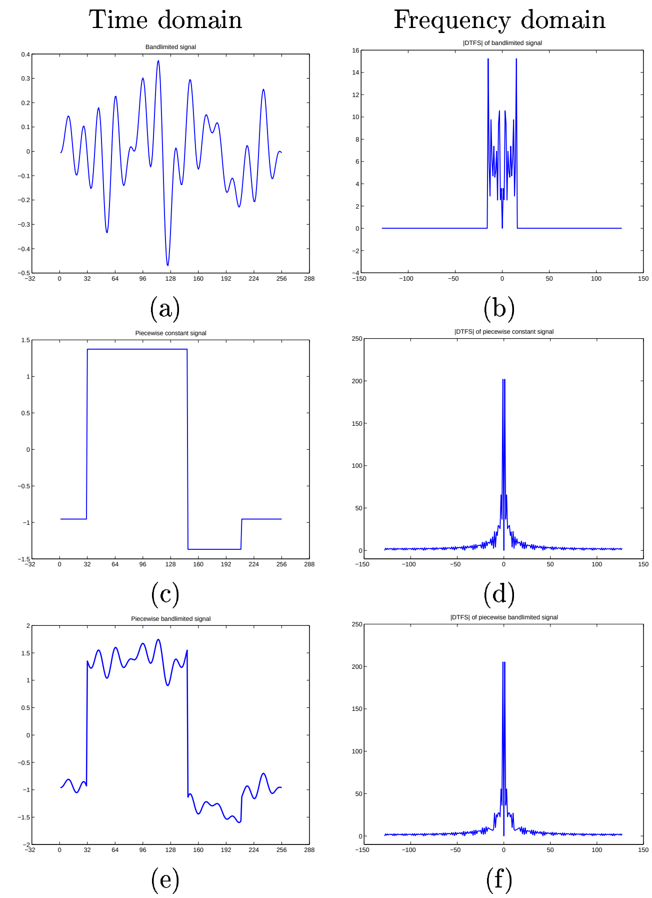


Figure 2: (a) Discrete-time periodic bandlimited signal $\mathbf{x}_B$ of period $N = 256$; (b) $|DTFS|$ of bandlimited signal with $B = 15$ (c) Discrete-time periodic piecewise constant signal with $K = 3$ pieces of period $N = 256$; (d) $|DTFS|$ of piecewise constant signal; (e) Discrete-time periodic bandlimited piecewise constant signal; (f) $|DTFS|$ of bandlimited piecewise constant signal.

It follows that discrete-time piecewise bandlimited signals also have a finite rate of innovation and sampling methods will be given in Section 3. We begin by introducing sampling theorems for streams of Diracs and piecewise polynomial signals in Section 2.

2. SAMPLING DISCRETE-TIME STREAMS OF DIRACS AND PIECEWISE POLYNOMIAL SIGNALS

Consider a discrete-time piecewise polynomial signal where each piece is of maximum degree R. By differentiating $R+1$ times we obtain a stream of Diracs. We begin by recalling the results given in [1, 6, 7] on sampling streams of Diracs, followed by those on sampling piecewise polynomial signals.

2.1. Discrete-time stream of Diracs

Consider a discrete-time periodic stream of K weighted Diracs at locations $\{n_0, n_1, \ldots, n_{K-1}\}$, with respective weights $\{c_0, c_1, \ldots, c_{K-1}\}$ and of period N

$$x_D[n] \;=\; \sum_{k=0}^{K-1} c_k\, \delta[n-n_k], \quad n = 0, \ldots, N-1 \tag{4}$$

where $\delta[n]$ is the Kronecker delta and equal to 1 if $n = 0$ and 0 if $n \neq 0$. The corresponding discrete-time Fourier series (DTFS) coefficients are defined by

$$X_D[m] \;=\; \sum_{k=0}^{K-1} c_k\, W_N^{n_k m}, \quad m = 0, \ldots, N-1 \tag{5}$$

where $W_N = e^{-i2\pi/N}$ is the $N-$th root of unity. From Figure 3 (b) the stream of $K = 15$ weighted Diracs with period $N = 256$ illustrated in Figure 3 (a) is evidently not bandlimited. Consider filtering the stream

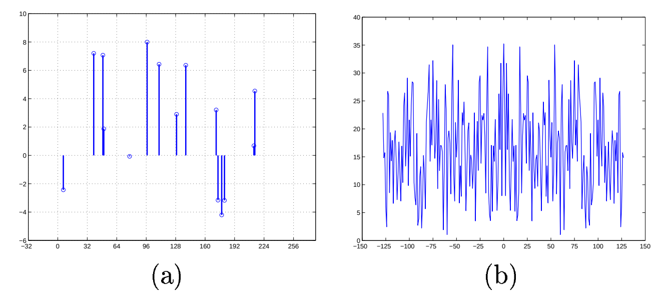


Figure 3: (a) Discrete-time periodic stream of $K = 15$ weighted Diracs; (b) —DTFS— of stream of Diracs.

of Diracs $x_D[n]$ with a lowpass filter $\tilde{\varphi}_K[n] = \varphi_K[-n]$ whose bandwidth is $[-K, K]$

$$y[n] = x_D[n] *_c \tilde{\varphi}_K[n], \quad n = 0, \ldots, N-1 \tag{6}$$

and then sample the filtered signal by an integer value M which is also a divisor of N. In [1, 6, 7] we have shown that we can uniquely recover the stream of Diracs from the samples $y_s[l] = y[lM]$. We recall the following

Theorem 1 *Consider a discrete-time periodic signal $x_D[n]$ of period N containing K weighted Diracs. Let M be an integer divisor of N satisfying $N/M \geq 2K+1$. Consider the discrete-time periodic sinc sampling kernel $\varphi_K[n] = \frac{1}{N}\sum_{m=-K}^{K} W_N^{-mn}$, that is, the inverse DTFS of the $Rect_{[-K,K]}$. Then the $N/M \in \mathbb{N}$ samples defined by*

$$y_s[l] =< x_D[n], \varphi_K[n-lM] >_{circ}, \quad l = 0, \ldots, N/M-1 \tag{7}$$

are a sufficient representation of the signal.

The proof of Theorem 1 consists in first showing that $2K$ contiguous DTFS coefficients $X_D[m]$ with $m \in [-K, K]$ are sufficient to determine the stream of K weighted Diracs and then showing that the N/M samples $y_s[l]$ are a sufficient representation of the $2K$ spectral values $X_D[m]$ with $m \in [-K, K]$. Here we just summarize the important steps in the proof.
The stream of weighted Diracs is determined by its K locations and associated weights. To determine the locations it is sufficient to find an annihilating filter [3], that is, a filter $\mathbf{H} = (1, H[1], \ldots, H[K])$ such that

$$(H * X_D)[m] = 0, \quad \forall m = 0, \ldots, N-1. \tag{8}$$

The $z-$transform of the annihilating filter is $H(z) = \sum_{l=0}^{K} H[l]\, z^{-l}$ with $H[0] = 1$ and can be factored as follows

$$H(z) = \prod_{k=0}^{K-1}(1 - z^{-1} W_N^{n_k}). \tag{9}$$

The locations $\{n_k\}_{k=0}^{K-1}$ of the Diracs are given by the roots of $H(z)$. The weights $\{c_k\}_{k=0}^{K-1}$ on the other hand are obtained by solving the Vandermonde $K \times K$ system of equations in (5) with $m = 0, \ldots, K-1$.
The $2K$ DTFS coefficients $X_D[m], m \in [-K, K]$ of the stream of Diracs are given by

$$Y_s[m] = \frac{1}{M} X_D[m], \quad m = 0, \ldots, N/M-1 \tag{10}$$

where

$$Y_s[m] = \sum_{l=0}^{N/M-1} y_s[l]\, W_{N/M}^{lm}, \quad m = 0, \ldots, N/M-1 \tag{11}$$

are the DTFS coefficients of the sample values $y_s[l], l = 0, \ldots, N/M-1$ with $N/M \geq 2K$, see Figure 4. Hence we are able to sample and reconstruct a discrete-time periodic stream of Diracs.

2.2. Discrete-time piecewise polynomial signals

Consider a discrete-time periodic piecewise polynomial signal $x_{PP}[n]$ of period N with K pieces each of maximum degree R and with zero mean. Suppose a discrete-time difference operator $d[n] = \delta[n] - \delta[n-1]$ is applied $R+1$ times to the piecewise polynomial signal. The DTFS of the differentiated signal $x_{PP}^{R+1}[n]$ is

$$X_{PP}^{(R+1)}[m] = (D[m])^{R+1} X_{PP}[m], \quad m = 0, \ldots, N-1 \tag{12}$$

where $D[m] = 1 - W_N^m$ is the DTFS of the discrete-time difference operator. This results in putting to zero all the polynomial pieces. Suppose there are discontinuities between the pieces, then K transitions can lead to at most $K(R+1)$ weighted Diracs. From Theorem 1 we can uniquely recover the $K(R+1)$ Diracs from $2K(R+1)$ contiguous DTFS coefficients $X^{(R+1)}[k]$ of the differentiated signal (which is a stream of Diracs). The piecewise polynomial signal is then reconstructed by applying the inverse discrete-time difference operator $R+1$ times on the stream of weighted Diracs. The discrete-time difference operator $d[n]$ is a singular operator (since $D[0] = 0$) and so we define the inverse discrete-time difference operator as $D^{-1}[m] = 0$ for $m = 0$ and $D^{-1}[m] = (1 - W_N^m)^{-1}$ for $m = 1, \ldots, N-1$. Hence instead of using the sinc sampling kernel $\varphi_K[n]$ we will use the differentiated sinc sampling kernel defined by

$$\psi[n] = (\underbrace{d * d * \cdots * d}_{R+1} * \varphi_{K(R+1)})[n] \tag{13}$$

which has at least $R+1$ zeros at the origin $z = 1$ and a larger bandwidth. The DTFS coefficients of $\psi[n]$ are

$$\Psi[m] = (1 - W_N^m)^{R+1}\, \Phi[m], \quad m = 0, \ldots, N-1 \tag{14}$$

where $\Phi[m]$ is the $Rect_{[-K(R+1), K(R+1)]}$ function. This is summarized in the following

Theorem 2 *Consider a discrete-time periodic piecewise polynomial signal $x_{PP}[n]$ of period N with K pieces of degree R and with zero mean. Let M be an integer and a divisor of N such that $N/M \geq 2K(R+1)+1$. Take a sampling kernel $\psi[n]$ with DTFS coefficients defined in (14). Then we can recover the signal from the $N/M \in \mathbb{N}$ samples*

$$y_s[l] =< x_{PP}[n], \psi[n-lM] >, \quad l = 0, \ldots, N/M-1. \tag{15}$$

The details of the proof can be found in [1, 6, 7]. Hence we are able to sample and reconstruct a discrete-time periodic piecewise polynomial signal. We are now ready to investigate piecewise bandlimited signals.

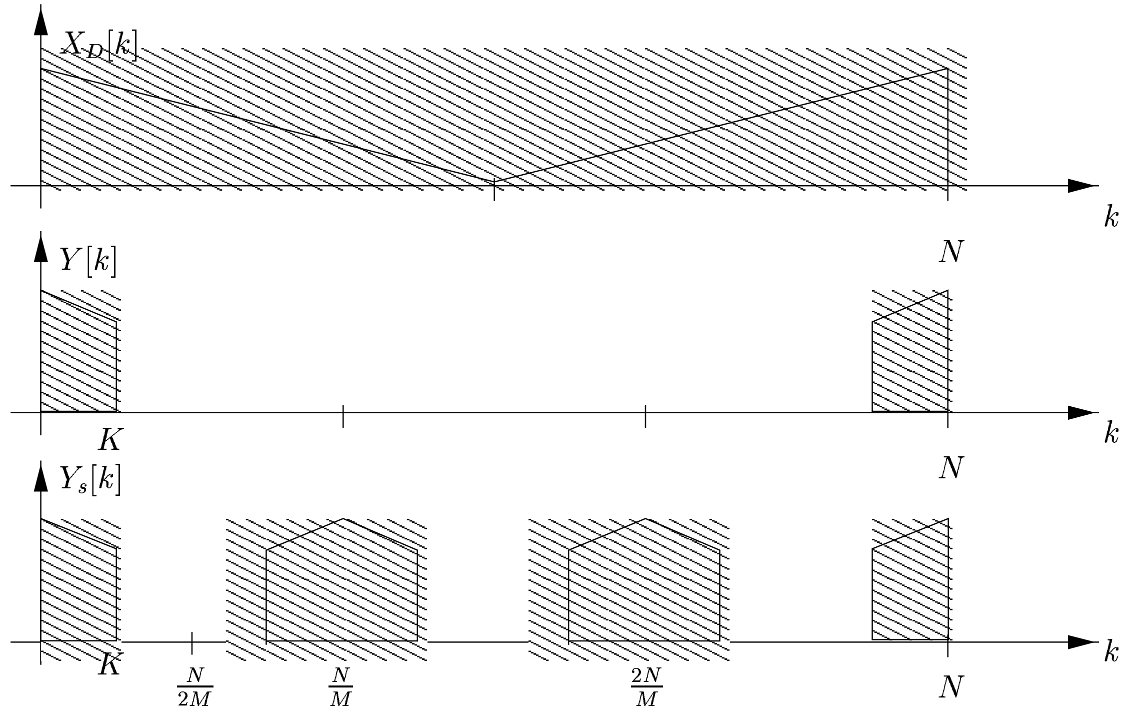


Figure 4: $X[k]$ is the DTFS of stream of Diracs $\mathbf{x}_D$, $Y[k]$ is the DTFS of filtered signal, $\mathbf{y} = \mathbf{x}_D *_c \tilde{\varphi}_K$, $Y_s[k]$ is the DTFS of sample values $y_s[l] =< x[n], \varphi_K[n - lM] >, l = 0, \ldots, N/M - 1$.

3. SAMPLING DISCRETE-TIME PIECEWISE BANDLIMITED SIGNALS

In this section we derive a sampling and reconstruction scheme for piecewise bandlimited signals. We show that these are simply applications of the methods described for streams of Diracs and piecewise polynomial signals in Section 2. We look in detail the discrete-time case and give the result for the continuous-time case.

3.1. Discrete-time periodic piecewise bandlimited signals

First we consider sampling a discrete-time periodic $B-$bandlimited signal of period N with K weighted Diracs, that is,

$$\mathbf{x} = \mathbf{x}_B + \mathbf{x}_D. \tag{16}$$

From Section 2.1 we can recover the K weighted Diracs from $2K$ contiguous frequency values $\mathbf{X}_D$. Since the DTFS coefficients of the bandlimited signal, $\mathbf{X}_B$, are equal to zero outside of the band $[-B, B]$, we have that the DTFS coefficients of the signal outside of the band $[-B, B]$ are exactly equal to the DTFS coefficients of the stream of Diracs, that is,

$$X[m] = X_D[m], \quad m \notin [-B, B]. \tag{17}$$

Therefore in order to recover the stream of Diracs it is sufficient to take $2K$ DTFS coefficients $X[m]$ outside of the band $[-B, B]$, for instance in $[B+1, B+2K]$ see Figure 5. Suppose we have the DTFS of the signal $X[m]$, with $m \in [-2K - B, B + 2K]$ then the DTFS of the bandlimited signal are obtained by subtracting $X_D[m]$ from $X[m]$ for $m \in [-B, B]$ and thus giving the bandlimited signal $\mathbf{x}_B$.

Corollary 1 *Consider a discrete-time periodic $B-$bandlimited signal of period N with K weighted Diracs, $\mathbf{x} = \mathbf{x}_B + \mathbf{x}_D$. Let $\varphi_{B+2K}[n]$ be discrete-time periodic sinc sampling kernel. Let M be an integer divisor of N, and let $N/M \geq 2(B + 2K) + 1$ then the samples*

$$y_s[l] =< x[n], \varphi_{B+2K}[n - lM] >, \quad l = 0, \ldots, N/M - 1 \tag{18}$$

are a sufficient representation of $\mathbf{x}$.

Proof: The proof is done in two steps. First we show that the DTFS coefficients $X[m], m \in [-2K - B, B + 2K]$ are sufficient to determine the bandlimited signal with Diracs. Second we must show that N/M samples are sufficient to determine the DTFS coefficients $X[m], m \in [-2K - B, B + 2K]$.

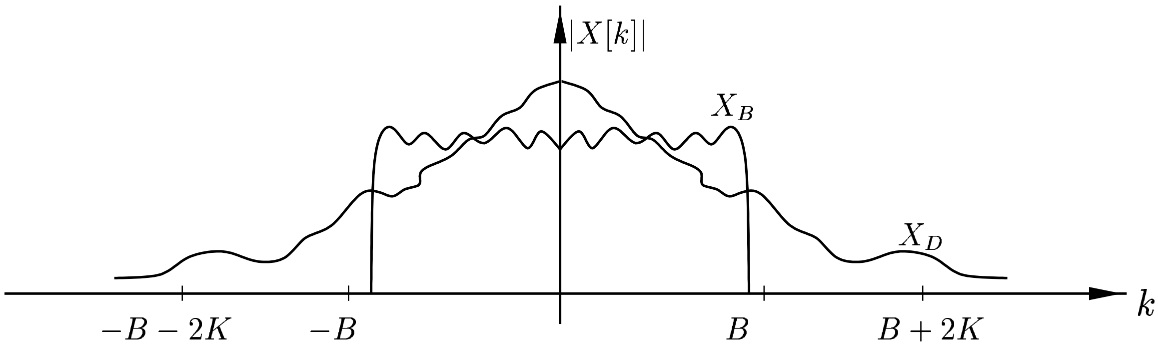


Figure 5: X_B is the DTFS of B-bandlimited signal x_B, X_D is the DTFS of the discrete-time periodic stream of K weighted Diracs x_D.

1. Consider the following spectral values $X[m], m \in [-2K-B, B+2K]$ then from Definition 2

$$X[m]=\begin{cases} X_B[m]+X_D[m] & m \in [0,B] \\ X_D[m] & m \in [B+1, B+2K] \end{cases} \quad (19)$$

From Theorem 1 the stream of K weighted Diracs $\mathbf{x}_D$ are perfectly recovered from $2K$ contiguous DTFS $X_D[m]=X[m], m \in [B+1, B+2K]$. On the other hand the $2B+1$ spectral components of the bandlimited signal are given by

$$X_B[k]=(X[m]-X_D[k]), \quad k \in [-B,B]. \quad (20)$$

giving the bandlimited signal $\mathbf{x}_B$. Thus the bandlimited signal with spikes is $\mathbf{x}=\mathbf{x}_B+\mathbf{x}_D$.

2. In the same way as Theorem 1 the spectral components of the signal are obtained from the spectral components of the sample values $Y_s[m]$,

$$X[m]=M\,Y_s[m], \quad m \in [-2K-B, B+2K]. \quad (21)$$

■

The result found in Corollary 1 can be generalized to discrete-time periodic piecewise bandlimited signals. Recall that to reconstruct a discrete-time periodic piecewise polynomial signal we needed $2K(R+1)$ contiguous spectral values. It follows that we can sample the discrete-time periodic piecewise bandlimited signal using a discrete-time periodic differentiated sinc sampling kernel ψ as defined in (13) with bandwidth $[-2K(R+1)-B, B+2K(R+1)]$.

Corollary 2 *Consider a piecewise bandlimited signal* $\mathbf{x}$ *as defined in Definition 2. Let* $\psi[n]$ *be the* $(R+1)$*th differentiated sinc sampling kernel with bandwidth* $[-2K(R+1)-B, B+2K(R+1)]$. *Let* M *be an integer divisor of* N, *and let* $N/M \geq 2(B+2K(R+1))+1$ *then the samples*

$$y_s[l]=<x[n], \psi[n-lM]>, \quad l=0,\dots,N/M-1 \quad (22)$$

are a sufficient representation of $\mathbf{x}$.

Proof: The proof follows from Corollary 1 and Theorem 2. In this case $2K(R+1)$ spectral values

$$X_{PP}[m]=X[m], \quad m \in [B+1, B+2K(R+1)] \quad (23)$$

are sufficient to recover the piecewise polynomial $\mathbf{x}_{PP}$. The bandlimited signal is recovered from the $2B+1$ spectral components

$$X_B[m]=X[m]-X_{PP}[m], \quad m \in [-B,B]. \quad (24)$$

Thus the piecewise bandlimited signal as defined in Definition 2 is recovered $\mathbf{x}=\mathbf{x}_B+\mathbf{x}_{PP}$. We are able to determine the $2(B+2K(R+1))+1$ spectral values of the signal from the DTFS of the sample values since

$$Y_s[m]=\frac{1}{M}(1-W_N^{-m})^{R+1}\,X[m], m \in [0, B+2K(R+1)]. \quad (25)$$

■

Figure 6 illustrates the different steps in the sampling and the reconstruction of a bandlimited plus a piecewise constant signal using the following

Algorithm 1 *Reconstruction of piecewise bandlimited signals.*

Require: $N, M, N/M, B, R \in \mathbb{N}$;
Calculate the samples

$$y_s[l]=<x[n], \psi[n-lM]>, l=0,\dots,N/M-1,$$

with $N/M \geq 2(2K(R+1)+B)+1$;

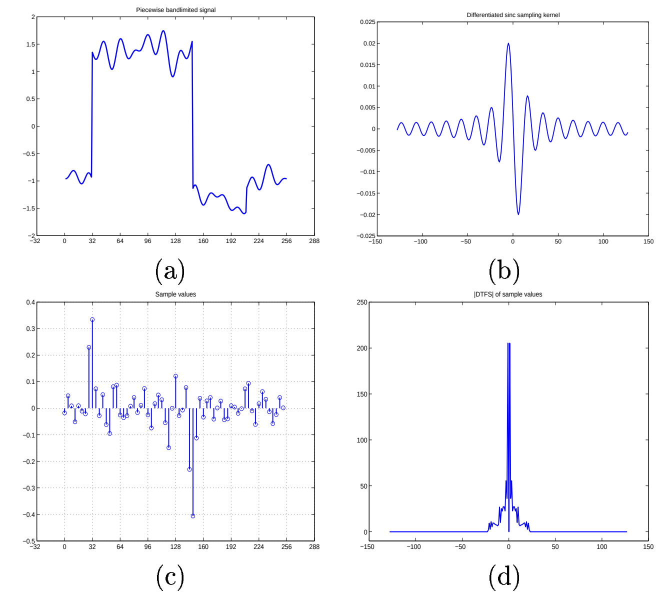


Figure 6: (a) Piecewise bandlimited signal, $\mathbf{x} = \mathbf{x}_{PP} + \mathbf{x}_B$, with $K = 3$, $B = 15$, $R = 0$, $N = 256$; (b) Differentiated sinc sampling kernel, $\psi[n] = d[n] * \varphi_{B+2K(R+1)}[n]$, bandlimited to $2K(R+1) + B = 21$ (c) Sample values $y_s[l] =< x[n], \psi[n - lM] >, l = 0, \dots, N/M - 1$, $M = 4$; (d) $|DTFS|$ of sample values; Reconstruction error is 10^{-13}.

Calculate the DTFS

$$X[m], m \in [-2K(R+1) - B, B + 2K(R+1)]$$

from the DTFS of samples $y_s[l]$

$$\longrightarrow X_{PP}[m] = X[m], m \in [B+1, B+2K(R+1)];$$

Solve for $\mathbf{H}$ *the linear system of equations*

$$(H * X_{PP})[m] = 0, m \in [B+1, B+2K(R+1)];$$

Find $\mathbf{x}_{PP}$ *using* $\mathbf{H}$*;*
Calculate

$$X_B[m] = X[m] - X_{PP}[m], \quad m \in [-B, B];$$

Find $\mathbf{x}_B$ *from* $\mathbf{X}_B$*;*
The reconstruction is

$$\mathbf{x} = \mathbf{x}_B + \mathbf{x}_{PP}.$$

4. CONCLUSION

In this paper, we recalled sampling theorems for a specific class of non bandlimited signals in particular, discrete-time periodic streams of Diracs and discrete-time periodic piecewise polynomials. We extended these sampling results to discrete-time piecewise bandlimited signals, that is, signals composed of a bandlimited signal with a stream of Diracs and the more general case with a piecewise polynomial signal. Sampling theorems for continuous-time piecewise bandlimited signals will naturally follow and is a topic under investigation.

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