A Theory of Multirate Filter Banks

MARTIN VETTERLI

Abstract—Multirate filter banks produce multiple output signals by filtering and subsampling a single input signal, or conversely, generate a single output by upsampling and interpolating multiple inputs. Two of their main applications are subband coders for speech processing and transmultiplexers for telecommunications. Below, we derive a theoretical framework for the analysis, synthesis, and computational complexity of multirate filter banks. The use of matrix notation leads to basic results derived from properties of linear algebra. Using rank and determinant of filter matrices, it is shown how to obtain aliasing/crosstalk-free reconstruction, and when perfect reconstruction is possible. The synthesis of filters for filter banks is also explored, three design methods are presented, and finally, the computational complexity is considered.

I. Introduction

A filter bank is a signal processing device that produces M signals from a single signal by means of filtering by M simultaneous filters. In the multirate case [4], the M signals are subsequently subsampled by a factor N. While the above-described device performs an analysis of the input signal, a filter bank can also be used to synthesize a single signal from M input signals (upsampled by N in the multirate case).

As will be shown, this simultaneity of the filtering and sampling rate change has profound consequences on the properties of the filter bank, both from a theoretical and a computational complexity point of view.

Concerning the theory, it turns out that in a multirate filter bank, there is no need to meet the sampling theorem on a channel-by-channel basis, since it is sufficient to meet it on the sum of the channels. Therefore, ideal band-pass filters (which are unrealizable) are not necessary anymore, and a new theory can be developed which looks at all channels simultaneously rather than considering each one separately (as in single filter signal processing).

This simultaneity was implicitly used in the quadrature mirror filter (QMF) approach first introduced in [5], where nonideal half-band filters precede a subsampling by 2. Nevertheless, a clever synthesis annihilates the aliasing completely, showing that even if the sampling theorem was violated in each single channel, the filter bank as a whole did not violate it. The QMF's have then been studied extensively and applied successfully to subband coding of speech [3], [6], [7], [8], [11]. Recently, they have been extended to the two-dimensional case as well [33], [40].

Concerning the complexity of filter banks, a breakthrough was obtained with the polyphase/FFT implementation of uniformly modulated filter banks [1] which are used in TDM to FDM transmultiplexing [4]. Again, the simultaneity of the processing (together with the fact that the various filters are derived from a single prototype) is central to this powerful result. This polyphase/FFT approach has also been used in spectrum analysis [10], [31].

While there is obviously a relationship between subband coding and transmultiplexing since both use multirate filter banks, the two fields long evolved independently. The first attempt to use the results on complexity of transmultiplexers in the context of subband coding was the introduction of the pseudo-QMF filters [14], an approach that has been further studied and implemented [22], [17], [18], [2], [13]. That the result on aliasing cancellation from subband coders could be used to cancel crosstalk in transmultiplexers as well was shown theoretically in [35] and [36].

While quadrature mirror filters annihilate aliasing perfectly, they allow only approximate reconstruction of the original signal. The first solution allowing perfect reconstruction was shown in [24] for two channel systems, and was extended in [39]. For an arbitrary number of channels, perfect reconstruction appears in [35]. While these methods use FIR filters only, perfect reconstruction with IIR filters has also been proposed [26], [30], [38], but the stability of the filters is difficult to achieve.

In parallel to these developments, an effort was put into trying to formalize the various results within a common framework. The main result was certainly the introduction of matrix notations for the analysis of multirate filter banks [21], [25], [26], [34], [35]. Thanks to this formalism, it has been shown that aliasing in subband coders can be cancelled in general, a fact known previously only for two channel systems. The power of the matrix approach to multirate filter banks is shown by the numerous results that can be obtained with it [25], [27], [37], [38], and the conciseness of some of the proofs below should be convincing.

The paper starts out in Section II with some considerations on linear, periodically time-varying systems, a class of systems to which multirate filter banks belong to. After reviewing the basic operations (up- and subsampling), we consider the two generic filter banks, namely, the

Manuscript received June 16, 1986; revised September 8, 1986. This work was supported in part by the Fonds National Suisse de la Recherche Scientifique, and in part by the American National Science Foundation under Grant CDR-84-21402.

The author was with the Ecole Polytechnique Federale de Lausanne, Chemin de Bellerive 16, CH-1007 Lausanne, Switzerland. He is now with the Center for Telecommunications Research, Columbia University, New York, NY 10027.

IEEE Log Number 8611976.

analysis and the synthesis filter bank. The mathematical treatment of these banks leads naturally to the definition of filter matrices (of size M by N) which will be central to our further developments.

Section III gives the analysis of the two major physical systems that use multirate filter banks, that is, the subband coder and the transmultiplexer. Thanks to the matrix notation, the developments are succinct, and the important conditions of aliasing/crosstalk-free reconstruction as well as of perfect reconstruction can be clearly stated.

Section IV looks at the fundamental properties inherent to filter banks. After a closer look at filter matrices (factorization, inverse), a number of results is proven on aliasing-free reconstruction in subband coders and crosstalk-free reconstruction in transmultiplexers, as well as on perfect reconstruction. Basically, the rank of the filter matrix has to be equal to the sampling rate change in order to allow aliasing/crosstalk-free reconstruction. Perfect reconstruction is related to the zero locations of the determinant of the filter matrix. The duality of subband coding and transmultiplexing is demonstrated. It is also shown that there exist minimum delay solutions (delay of $N - 1$ samples in an N channel system) as well as perfect reconstruction using linear phase filters only. Results on modulated filters are also derived.

Section V explores the synthesis of filters for filter banks, showing the possible tradeoff between filter quality, reconstruction quality, and input–output delay. Three filter design methods are also described, two of them guaranteeing perfect reconstruction. The last section is concerned with the computational complexity of filter banks. After showing potential problems linked to aliasing cancellation (requiring more complex synthesis filters in general), we review some methods which yield substantial reductions in complexity.

Before proceeding, we indicate some notational conventions that will be used in the following. Bold face italic letters indicate matrices (upper case) and vectors (lower case). Note that the numbering of lines or columns always starts with 0. Diag $[\cdots]$ refers to a diagonal matrix whose elements are listed between the brackets (the elements can be given in vector form as well). Det $[\cdot\,\cdot]$ and Co $[\cdot\,\cdot]$ stand for determinant and cofactor matrix, respectively. The letter N is used for sampling rate change, M for the number of filters in a filter bank, and L for the length of FIR filters. Implicitly, W stands for the Nth root of unity ($W = e^{-j2\pi/N}$), where N is given in the context. The z-transform [19], [20] of signals and filters will be used extensively and indicated by the letter z. As a simple example, $\boldsymbol{H}(z)$ is a matrix of rational functions in z, that is, its elements are z-transforms of signals or filters.

II. Basic Operations in Multirate Filter Banks

Three basic operations are used in multirate filter banks: linear filtering, subsampling, and upsampling. Since subsampling by a constant factor belongs to the class of linear and periodically time-varying systems, we first consider the analysis of such systems. Next, we review the basic relations defining up- and subsampling. Finally, the two generic filter banks, that is, analysis and synthesis filter banks, are defined and analyzed.

A. Polyphase and Modulation Representation of Periodically Time-Varying Linear Systems

Multirate filter banks belong to the class of linear periodically time-varying systems [4], since they contain linear filters as well as time-varying operations (subsampling by N). Such systems can be modeled with N impulse responses corresponding to the system response to impulses at time $0, 1, \cdots, N-1$. Assume that we know the z-transforms of these impulse responses and call them $T_0(z)$ to $T_{N-1}(z)$. In vector form, we can write

$$\boldsymbol{t}(z) = \left[T_0(z), T_1(z), \cdots, T_{N-1}(z)\right]^T. \quad (1)$$

Now, we introduce the *polyphase decomposition* (of size N) of the input signal (in the z-transform domain). This decomposition, which groups all input samples having the same phase (modulo a period N) into a single element of a size N vector, can be written in vector form as

$$\boldsymbol{x}_p(z) = \left[X_{p0}(z), X_{p1}(z), \cdots, X_{pN-1}(z)\right]^T \quad (2)$$

where

$$X_{pi}(z) = z^{-i} \sum_{n=-\infty}^{\infty} z^{-nN} \cdot x(nN + i). \quad (3)$$

The polyphase decomposition is known to be fundamental in the transmultiplexer case [1], and its importance for filter banks in general will be shown below. Using (1) and (2), it is easy to write the output of a linear system varying with a period N as (we assume here that input and output have the same sampling frequency):

$$Y(z) = \left[\boldsymbol{t}(z)\right]^T \cdot \boldsymbol{x}_p(z). \quad (4)$$

Another possible approach uses the *modulation decomposition* (of size N) of the input, which is obtained from the signal and its $N - 1$ versions modulated by the roots of unity of order N (except the root equal to 1). This leads to the following size N vector (in the z-transform domain):

$$\boldsymbol{x}_m(z) = \left[X(z), X(Wz), \cdots, X(W^{N-1}z)\right]^T$$
$$W = e^{-j2\pi/N}. \quad (5)$$

As one can verify, the following relation relates the polyphase to the modulation representation of a signal [38]:

$$\boldsymbol{x}_p(z) = 1/N\, \boldsymbol{F} \cdot \boldsymbol{x}_m(z) \quad (6)$$

where $\boldsymbol{F}$ is the usual Fourier matrix of size $N \times N$ whose elements are defined as

$$\boldsymbol{F}_{ij} = W^{ij}. \quad (7)$$

Using (4) and (6), one can write the output of a linear and periodically time-varying system as

$$Y(z) = 1/N\,[t(z)]^T \cdot F \cdot x_m(z). \tag{8}$$

Thus, the output is a linear combination of filtered versions of $X(z)$, $X(Wz)$ up to $X(W^{N-1}z)$. The polyphase and the modulation representation are two fundamental ways of looking at linear, periodically time-varying systems. By analogy with conventional system analysis, one could call the polyphase representation a time domain view of the system behavior, while the modulation representation could be considered a frequency view. It is thus quite natural that the two representations are related through a Fourier transform, as expressed by (6). In the following, we will use either one of these two representation modes, depending on which one is more convenient for the specific problem under consideration.

B. Basic Operations

Besides linear filtering, the two fundamental operations in multirate filter banks are *subsampling* and *upsampling* by a factor N. We will only consider integer sampling rate changes in the following, since rational ones can be obtained by cascading integer ones [4]. If a signal $x(n)$ with z-transform $X(z)$ is subsampled by N to yield an output $Y(z)$, then the latter can be represented as [23], [4]

$$Y(z) = 1/N \sum_{k=0}^{N-1} X(W^k z^{1/N}). \tag{9}$$

Using vector notation and relations (2)–(6), this can be written as

$$Y(z) = 1/N[1 \;\; 1 \cdots 1] \cdot x_m(z^{1/N}) \tag{10a}$$

$$= [1 \;\; 0 \cdots 0] \cdot x_p(z^{1/N}). \tag{10b}$$

If a signal $y(n)$ is obtained from upsampling the signal $x(n)$ by a factor N, that is, stuffing $N - 1$ zeros between each sample of $x(n)$, then their z-transforms are related by [23], [4]

$$Y(z) = X(z^N). \tag{11}$$

The two operations corresponding to (9) and (11) are shown in Fig. 1 together with some useful combinations of them. Table I gives the corresponding input–output relations that we briefly consider below. The case c) of Table I [Fig. 1(c)] is trivially obtained by replacing (11) into (9) or (10), and case e) is obvious. In the case d), where filtering is placed between the sub- and the upsampling, the filter appears simply as a multiplicative factor (but with Nth powers of z). In case f) finally, where filtering is placed between the up- and the subsampling, the filter appears also in aliased versions at the output.

C. Basic Filter Banks

Filter banks appear in two basic configurations. The first one, called *analysis filter bank*, divides the signal into M filtered and subsampled versions. Such a filter bank is depicted in Fig. 2. The second configuration, called *synthesis filter bank*, generates a single signal from M upsampled and interpolated signals. Fig. 3 shows such a synthesis filter bank.

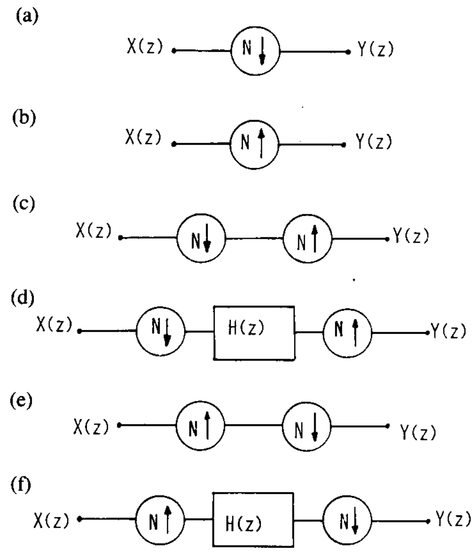


Fig. 1. Schematics of the basic operations in multirate filter banks (see Table I for the input–output relations).

TABLE I
BASIC OPERATIONS IN MULTIRATE FILTER BANKS WITH THEIR INPUT–OUTPUT RELATIONS (SEE FIG. 1 FOR THE SCHEMATICS)

operation	input-output relation
a) subsampling by N	$Y(z) = 1/N \sum_{k=0}^{N-1} X(W^k z^{1/N})$
b) upsampling by N	$Y(z) = X(z^N)$
c) sub- followed by upsampling by N	$Y(z) = 1/N \sum_{k=0}^{N-1} X(W^k z)$
d) same as c) but with filtering in between	$Y(z) = 1/N\; H(z^N) \sum_{k=0}^{N-1} X(W^k z)$
e) up- followed by sub-sampling by N (note that the sub-sampling has to be done in phase with the upsampling)	$Y(z) = X(z)$
f) same as e) but with filtering in between	$Y(z) = 1/N\; X(z) \sum_{k=0}^{N-1} H(W^k z^{1/N})$

We start by analyzing the filter bank of Fig. 2. The size of that filter bank is M, while the subsampling factor at the output is N. Call $h(z)$ the vector of size M containing the filters $H_0(z)$ to $H_{M-1}(z)$, and $x(z)$ the vector of the outputs of these filters. Obviously,

$$x(z) = h(z) \cdot X(z). \tag{12}$$

The vector of subsampled outputs, called $y(z)$, is equal to [from (9) and (12)]

$$y(z) = 1/N \sum_{k=0}^{N-1} x(W^k z^{1/N})$$

$$= 1/N \sum_{k=0}^{N-1} h(W^k z^{1/N}) \cdot X(W^k z^{1/N}). \tag{13}$$

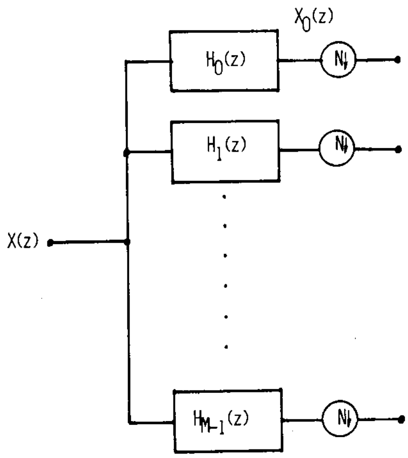


Fig. 2. Analysis filter bank of size M with outputs subsampled by N.

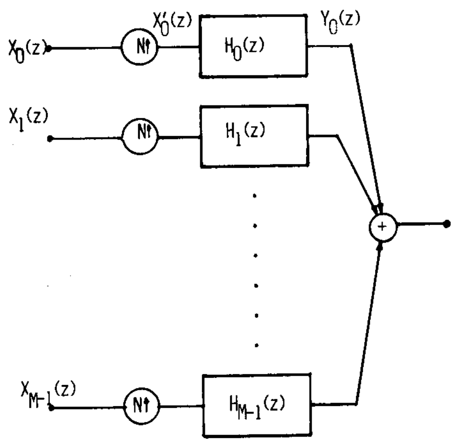


Fig. 3. Synthesis filter bank of size M with inputs upsampled by N.

Similarly to what was done in (10), the relation (13) is more conveniently expressed with matrix notation. To this end, we introduce the *modulated filter matrix* $\boldsymbol{H}_m(z)$, defined as follows:

$$\boldsymbol{H}_m = [\boldsymbol{h}(z), \boldsymbol{h}(Wz), \cdots, \boldsymbol{h}(W_z^{N-1})] \tag{14a}$$

$$= \begin{bmatrix} H_0(z) & H_0(Wz) & \cdots & H_0(W^{N-1}z) \\ H_1(z) & H_1(Wz) & \cdots & H_1(W^{N-1}z) \\ \vdots & \vdots & \cdots & \vdots \\ H_{M-1}(z) & H_{M-1}(Wz) & \cdots & H_{M-1}(W^{N-1}z) \end{bmatrix}. \tag{14b}$$

Then, the relation (13) becomes equal to [using (5)]

$$\boldsymbol{y}(z) = 1/N\, \boldsymbol{H}_m(z^{1/N}) \cdot \boldsymbol{x}_m(z^{1/N}). \tag{15}$$

The equivalent representation in the polyphase plane is obtained by reversing (6) and introducing it into (15). Then, $\boldsymbol{y}(z)$ is equal to

$$\boldsymbol{y}(z) = \boldsymbol{H}_m(z^{1/N}) \cdot \boldsymbol{F}^{-1} \cdot \boldsymbol{x}_p(z^{1/N}). \tag{16}$$

We can now define a *polyphase filter matrix* $\boldsymbol{H}_p(z)$. In order to have a similar relation between the matrices as between the vectors [see relation (6)], we write

$$\boldsymbol{H}_p(z) = 1/N\, \boldsymbol{H}_m(z)\, \boldsymbol{F}. \tag{17}$$

Note that both matrices will be analyzed in detail later on. Recall simply that their sizes are both $M \times N$. In order to simplify (16), we rewrite $\boldsymbol{F}^{-1}$ as

$$\boldsymbol{F}^{-1} = 1/N\, \boldsymbol{F} \boldsymbol{J} \tag{18}$$

where

$$\boldsymbol{J} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & & 0 \\ 0 & 0 & 0 & & \cdots & 0 & 1 \\ 0 & 0 & 0 & & \cdots & 1 & 0 \\ 0 & & & & & & 0 \\ 0 & 0 & 1 & 0 & & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \tag{19}$$

is simply a permutation matrix that exchanges line or column i with line or column $N - i$ $(i = 1 \cdots N - 1)$. Using (17) and (18) allows one to transform (16) into

$$\boldsymbol{y}(z) = \boldsymbol{H}_p(z^{1/N}) \cdot \boldsymbol{J} \cdot \boldsymbol{x}_p(z^{1/N}). \tag{20}$$

The meaning of (20) is the following: the output $\boldsymbol{y}(z)$ is made up from samples appearing at time instants which are multiples of N only (because of the subsampling). The ith column of $\boldsymbol{H}_p(z)$ produces implicitly a delay of i samples, while the ith line of $\boldsymbol{x}_p(z)$ represents a delay of i samples as well. Thus, thanks to the permutation $\boldsymbol{J}$, the product in (20) has overall delays which are multiples of N only. Thus, with the relations (15) and (20), we have precisely characterized the output of a subsampled analysis filter bank, and this both in the modulation and in the polyphase plane.

Turning to the synthesis filter bank of Fig. 3, we note that both $\boldsymbol{x}(z)$ and $\boldsymbol{h}(z)$ are size M vectors. It is easy to see from relation (11) and Fig. 3 that the output of such a filter bank can be expressed as

$$Y(z) = [\boldsymbol{h}(z)]^T \cdot \boldsymbol{x}(z^N). \tag{21}$$

In conclusion to this section, we recall that two basic representation modes are possible for multirate filter banks (which are linear periodically time-varying systems): the polyphase representation (similar to a time domain view) and the modulation representation (equivalent to a frequency representation). Then, we expressed concisely the output of the two basic filter banks, and this with relations (15) and (20) for the analysis and (21) for the synthesis filter bank. The matrix notation that was used so far mainly for convenience will prove to be extremely useful (actually necessary in our opinion) in order to prove basic results in later sections.

III. Subband Coders and Transmultiplexers

In the following, we will consider filter banks whose original signal (or signals) are reconstructed from the out-

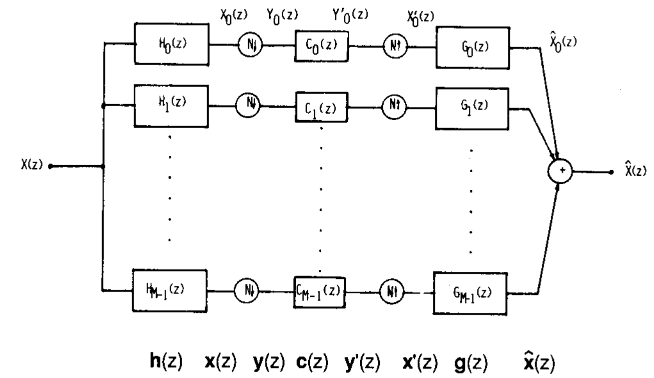


Fig. 4. Subband coder with M channels subsampled by N.

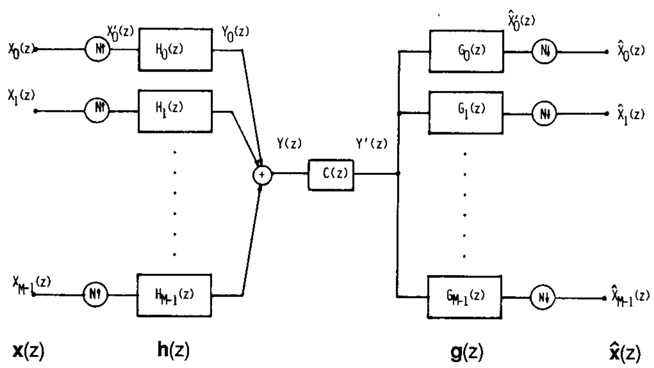


Fig. 5. TDM-FDM transmultiplexer (with signal recovery) with M inputs and a channel with N times higher sampling frequency.

put of an analysis or a synthesis bank. The first such system, where the analysis bank precedes a synthesis bank, is generally referred to as a *subband coder*, since subband coding is its most well-known application [4]. Such a subband coder is shown in Fig. 4. The second system, where the analysis bank follows the synthesis bank, corresponds to *TDM-FDM transmultiplexing* (time division to frequency division transmultiplexing [1], [4]), and is shown in Fig. 5. Below, both systems will be analyzed by using the formalism developed in Section II.

A. Subband Coder Analysis

Consider the system of Fig. 4. The channel signals (the outputs of the analysis bank) are modified by linear filters $C_i(z)$ before entering the synthesis bank whose output is the reconstructed signal $\hat{X}(z)$. We assume that sizes (given by M) and sampling rate changes (given by N) are the same in the analysis and synthesis bank. While $y(z)$ is given by (15), the modified channel signals are equal to

$$y'(z) = C_s(z) \cdot y(z) \tag{22}$$

where $C_s(z)$ is an $M \times M$ diagonal matrix (the subscript "s" stands for subband coding) and is given by

$$C_s(z) = \text{Diag}\left[C_0(z) \quad C_1(z) \quad \cdots \quad C_{N-1}(z)\right]. \tag{23}$$

For the subsequent synthesis filter bank, we can use relation (21). Using (15) and (22) in relation (21) leads to the following formula for the output of the subband coder (where $g(z)$ is the vector of output filters):

$$\hat{X}(z) = 1/N\,[g(z)]^T \cdot C_s(z^N) \cdot H_m(z) \cdot x_m(z). \tag{24a}$$

This relation can also be expressed in terms of the polyphase decomposition of the input signal. In that case, by using (6) and (17), $\hat{X}(z)$ can be written as

$$\hat{X}(z) = 1/N\,[g(z)]^T \cdot C_s(z^N) \cdot H_p(z) \cdot J \cdot x_p(z). \tag{24b}$$

Using relations (24a–b), it is now easy to state fundamental properties of subband coders like the one in Fig. 4 (where the filters are assumed to be time invariant).

i) Aliasing-free output is achieved if

$$[g(z)]^T C_s(z^N)\,H_m(z) = [F(z) \quad 0 \quad 0 \quad \cdots \quad 0]^T \tag{25a}$$

where $F(z)$ is an arbitrary transmission filter.

ii) Perfect reconstruction is obtained if

$$[g(z)]^T C_s(z^N)\,H_m(z) = [z^{-k} \quad 0 \quad 0 \quad \cdots \quad 0] \tag{25b}$$

where z^{-k} is an arbitrary delay. Relations equivalent to (25a–b), but using polyphase filter matrices, can easily be derived by using (17) or (24b). The means to achieve the above properties will be explored in detail in the next section. Note that nonlinear effects (like the quantization of the channels) have not been considered.

B. Transmultiplexer Analysis

In the TDM-FDM transmultiplexer depicted in Fig. 5, we assume that the upsampling at the input and the subsampling at the output are done in phase. Violation of this assumption has been considered in [36] and can be modeled by an additional phase in the channel. The channel itself can be modeled by a linear filter $C(z)$. The input to the channel, $Y(z)$, is given by (21) and, thus, the output $Y'(z)$ equals

$$Y'(z) = C(z)\,Y(z) = C(z)\,[h(z)]^T \cdot x(z^N). \tag{26}$$

To keep notations simple, we replace z by z^N, which means that the reference frequency is now the channel sampling frequency (rather than the input sampling frequency). Thus, and following (15), the output $\hat{x}(z^N)$ equals

$$\hat{x}(z^N) = 1/N\,G_m(z) \cdot y'_m(z) \tag{27}$$

where $y'_m(z)$ is, from (26), equal to

$$y'_m(z) = C_t(z) \cdot [H_m(z)]^T \cdot x(z^N). \tag{28}$$

The size $N \times N$ matrix $C_t(z)$ is diagonal (the subscript "t" stands for transmultiplexing) and is given by

$$C_t(z) = \text{Diag}\left[C(z) \quad C(Wz) \quad \cdots \quad C(W^{N-1}z)\right]. \tag{29}$$

Combining (27) with (28) leads to the following result for the vector of outputs:

$$\hat{x}(z^N) = 1/N\, G_m(z) \cdot C_t(z) \cdot [H_m(z)]^T \cdot x(z^N). \tag{30a}$$

The equivalent relation using polyphase filter matrices is, using (17),

$$\hat{x}(z^N) = 1/N\, G_p(z)\, F\, C_t(z)\, F[H_p(z)]^T \cdot x(z^N). \tag{30b}$$

Again, it is now easy to state basic properties of transmultiplexers.

i) Reconstruction without crosstalk (assuming perfect phase recovery) is achieved if the matrix $T_t(z^N)$, defined by

$$T_t(z^N) = G_m(z) \cdot C_t(z) \cdot [H_m(z)]^T \tag{31}$$

is diagonal. Note that it can be checked that $T_t(z^N)$ is a function of z^N.

ii) Perfect reconstruction is achieved if $T_t(z^N)$ is a diagonal matrix of delays.

The next section will show how to achieve these goals. Furthermore, the close relationship that can already be noticed between subband coders and transmultiplexers will be highlighted.

IV. Fundamental Properties of Multirate Filter Banks

The basic question that is explored is: given an input filter bank of a subband coder or a transmultiplexer, how should one choose the output filter bank in order to achieve aliasing (crosstalk)-free or even perfect reconstruction of the input signal (signals). We will see that this leads basically to the problem of inverting (partially) the filter matrices that were introduced above, since we have to solve a linear system of equations as given by (25) or (31). The notion of critical sampling (number of channels equal to the sampling rate change) is clarified, since it is the limiting case where reconstruction without aliasing/crosstalk can be achieved.

Thus, we first look at some properties of the filter matrices $H_m(z)$ and $H_p(z)$, and then we give conditions and methods for aliasing/crosstalk-free reconstruction as well as for perfect reconstruction. Minimum delay and linear phase solutions are also described, and the important case of modulated filter banks is considered in more detail.

A. Properties of Filter Matrices

Because of their importance, we will consider $H_m(z)$ and $H_p(z)$ in more detail, especially their possible factorizations and their inverses. First, we give some definitions. As with matrices of scalars, one can define determinants and cofactors of filter matrices. The *rank* is then defined as the size of the largest square submatrix with nonzero determinant. Since the determinant is a rational function in z, a zero determinant is one that vanishes for all values of z (and not only for some isolated values). We call a filter matrix *stable* if all its elements correspond to stable filters. Note that the cofactor matrix of a stable matrix is stable (since it is obtained by sum and products of stable filters). Similarly, one can speak of a *causal* filter matrix (all elements are causal filters) and note that its cofactor matrix is causal as well [38].

Now, we will explore the structure of the filter matrices $H_m(z)$ and $H_p(z)$, since they are not general matrices of rational functions in z, but rather a subset of them with a well-defined structure. This will help the analysis and the inversion of the filter matrices. First, we recall that any infinite impulse response (IIR) filter can be replaced by an equivalent filter (in the sense of its transfer function) but having a denominator in z^N. This is detailed in [1] and [4], and we give only a brief derivation below. The IIR filter $H(z)$ can be written as

$$H(z) = N(z)/D(z) = (N(z)\,N'(z))/D(z^N) \tag{32a}$$

where

$$N'(z) = D(z^N)/D(z) \tag{32b}$$

is a finite impulse response (FIR) filter as can be verified. Actually, if p_i is a root of $D(z)$, then $W^k p_i (k = 0 \cdots N-1)$ are roots of $D(z^N)$. Thus, the roots of $N'(z)$ are of the form $W^k p_i (k = 1 \cdots N-1)$. From now on, we assume that all filters $H_i(z)$ are in the canonical form

$$H_i(z) = N_i(z)/D_i(z^N). \tag{33}$$

If this is not the case, they can first be transformed following (32). The filter $H_i(z)$ can be further decomposed into polyphase components, similarly to what was done for the input signal in (1)–(3):

$$H_i(z) = 1/D_i(z)\, z^{-j} \sum_{j=0}^{N-1} N_{i,j}(z^N) \tag{34a}$$

where $N_{i,j}(z^N)$, the jth polyphase component of the numerator $N_i(z)$, is given by

$$N_{i,j}(z^N) = \sum_{l=0}^{L} n_i(lN + j)\, z^{-lN}$$
$$L = \lfloor (L_i - j)/N \rfloor,$$
$$L_i = \text{length of } N_i(z) \tag{34b}$$

where $\lfloor x \rfloor$ means biggest integer $\leq x$. With the filters in the form given by (33), we can rewrite the modulated filter matrix $H_m(z)$ as

$$H_m(z) = D(z^N) \cdot N_m(z) \tag{35a}$$

where

$$D(z^N) = \text{Diag}\,[1/D_0(z^N) \quad 1/D_1(z^N) \quad \cdots \quad 1/D_{M-1}(z^N)] \tag{35b}$$

is an $M \times M$ diagonal matrix of denominators and

$$N_m(z) = \begin{bmatrix} N_0(z) & N_0(Wz) & \cdots & N_0(W^{N-1}z) \\ N_1(z) & N_1(Wz) & \cdots & N_1(W^{N-1}z) \\ \vdots & \vdots & \cdots & \vdots \\ N_{M-1}(z) & N_{M-1}(Wz) & \cdots & N_{M-1}(W^{N-1}z) \end{bmatrix} \tag{35c}$$

is an $M \times N$ matrix of modulated numerators. Similarly, the polyphase filter matrix can be written in factorized form by using (17), (34), and (35):

$$\boldsymbol{H}_p(z) = \boldsymbol{D}(z^N) \cdot \boldsymbol{N}_p(z^N) \cdot \boldsymbol{I}_d(z) \tag{36a}$$

where $N_p(z^N)$ is made of the N polyphase components of the M numerators $N_i(z)$ [see (34)]:

$$N_p(z^N) = \begin{bmatrix} N_{0,0}(z^N) & N_{0,1}(z^N) & \cdots & N_{0,N-1}(z^N) \\ N_{1,0}(z^N) & N_{1,1}(z^N) & \cdots & N_{1,N-1}(z^N) \\ \vdots & \vdots & \cdots & \vdots \\ N_{M-1,0}(z^N) & \cdots & \cdots & N_{M-1,N-1}(z^N) \end{bmatrix} \tag{36b}$$

and $\boldsymbol{I}_d(z)$ is a diagonal matrix of increasing delays

$$\boldsymbol{I}_d(z) = \text{Diag}\,[1 \quad z^{-1} \quad z^{-2} \cdots z^{-N+1}]. \tag{36c}$$

Note the generalization of the polyphase concept that is done above. In the classical transmultiplexer case with a single prototype filter [1], there are only N polyphase components (plus a DFT) while here each of the M filters has N polyphase components, that is a total of $N \cdot M$ different polyphase components. From (17), (18), and (36), we note that $\boldsymbol{H}_m(z)$ can be factored as

$$\boldsymbol{H}_m(z) = \boldsymbol{D}(z^N) \cdot \boldsymbol{N}_p(z^N) \cdot \boldsymbol{I}_d(z) \cdot \boldsymbol{F} \cdot \boldsymbol{J}. \tag{37}$$

The factorizations in (36a) and (37) are important because they highlight the structure and facilitate the analysis of the filter matrices.

In the following, we will focus on inverses of square filter matrices, that is, when $M = N$ (the filter banks are critically sampled [4], [21]). Using (37), one can rewrite the inverse of $\boldsymbol{H}_m(z)$ as

$$[\boldsymbol{H}_m(z)]^{-1} = [\boldsymbol{F} \cdot \boldsymbol{J}]^{-1} \cdot [\boldsymbol{I}_d(z)]^{-1} \cdot [\boldsymbol{N}_p(z^N)]^{-1} \cdot [\boldsymbol{D}(z^N)]^{-1}. \tag{38a}$$

The inverse of $\boldsymbol{FJ}$ is simply $1/N\boldsymbol{F}$ [using (18)]. The inverses of both $\boldsymbol{I}_d(z)$ and $\boldsymbol{D}(z^N)$ are immediate, since they are diagonal

$$[\boldsymbol{I}_d(z)]^{-1} = \text{Diag}\,[1 \quad z^1 \quad z^2 \cdots z^{N-1}] \tag{38b}$$

$$[\boldsymbol{D}(z^N)]^{-1} = \text{Diag}\,[D_0(z^N) \quad D_1(z^N) \cdots D_{N-1}(z^N)]. \tag{38c}$$

Note, however, that $[\boldsymbol{I}_d(z)]^{-1}$ is not causal. The causality of $[\boldsymbol{N}_p(z^N)]^{-1}$ is not a problem, but its stability is not clear at this point. Therefore, we write

$$[\boldsymbol{N}_p(z^N)]^{-1} = 1/\text{Det}\,[\boldsymbol{N}_p(z^N)] \cdot \text{Co}\,[\boldsymbol{N}_p(z^N)]^T. \tag{38d}$$

In (38d), note that the cofactor matrix, the determinant, and thus the inverse $[\boldsymbol{N}_p(z^N)]^{-1}$ are functions of z^N. Instead of the true inverse of $\boldsymbol{H}_m(z)$, let us introduce a partial inverse $\boldsymbol{H}_*(z)$ that will be causal and stable:

$$\boldsymbol{H}_*(z) = z^{-N}\Delta(z^N) \cdot [\boldsymbol{H}_m(z)]^{-1} \tag{39a}$$

where

$$\Delta(z^N) = \text{Det}\,[\boldsymbol{N}_p(z^N)]. \tag{39b}$$

Actually, a multiplication by z^{-N+1} would be sufficient to make $\boldsymbol{H}_*(z)$ causal [see (38b)]. It turns out that a delay of N (which is time invariant in systems varying with period N) is more convenient. This brief overview of structure, factorization, and inverse of filter matrices should be sufficient for the following sections. A more detailed treatment can be found in [38].

B. Theorems on Multirate Filter Banks

This subsection presents some basic results on multirate filter banks. Conditions under which aliasing- or cross-talk-free reconstruction is possible are given, together with results on perfect reconstruction [37], [38]. The equivalence between subband coders and transmultiplexers is also shown.

Theorem 1: Aliasing-free reconstruction in a subband coder is possible if and only if the product of the channel filter matrix $[\boldsymbol{C}_s(z^N)]$ times the analysis filter matrix $[\boldsymbol{H}_m(z)]$ has rank N (the subsampling factor).

An immediate consequence of this theorem is that the subsampling factor N has to be smaller or equal to the number of channels M, a result that is clear from a sampling theory point of view. The necessity of the rank being equal to N is proven by contradiction. Aliasing-free reconstruction was defined in (25a). Postmultiplying (25a) by $1/N \cdot \boldsymbol{F}$ leads to the following equivalent condition:

$$[\boldsymbol{g}(z)]^T \cdot \boldsymbol{C}_s(z^N) \cdot \boldsymbol{H}_p(z) = 1/N\,[F(z) \quad F(z) \cdots F(z)]. \tag{40}$$

Using the factorization of $\boldsymbol{H}_p(z)$ given in (36) and post-multiplying (40) by $[\boldsymbol{I}_d(z)]^{-1}$, we get

$$[\boldsymbol{g}(z)]^T \cdot \boldsymbol{C}_s(z^N) \cdot \boldsymbol{D}(z^N) \cdot \boldsymbol{N}_p(z^N) = 1/N\,[F(z) \quad zF(z) \cdots z^{N-1}\;F(z)]. \tag{41}$$

Using the shorthand $A(z^N) = \boldsymbol{C}_s(z^N)\boldsymbol{D}(z^N)\boldsymbol{N}_p(z^N)$, we note that if $\boldsymbol{C}_s(z^N) \cdot \boldsymbol{H}_p(z)$ has rank lower than N, then the rank of $A(z^N)$ is also lower than N. In that case, at least one column $\boldsymbol{a}_k(z^N)$ of $A(z^N)$ is either zero (in which case (41) cannot be verified) or linearly dependent of some other columns of $A(z^N)$:

$$a_k(z^N) = \sum_{\substack{i=0 \\ i \neq k}}^{N-1} w_i(z^N) \cdot a_i(z^N) \qquad (42)$$

where $w_i(z^N)$ are weighting factors. Note that all functions in (42) are in z^N. Now, because of (41), we have

$$[g(z)]^T \cdot a_k(z^N) = 1/N\, z^k \cdot F(z) \qquad (43)$$

but also, because of (42),

$$[g(z)]^T \cdot a_k(z^N) = 1/N\, F(z) \sum_{\substack{i=0 \\ i \neq k}}^{N-1} z^i \cdot w_i(z^N). \qquad (44)$$

Since the $w_i(z^N)$ are functions of z^N and $i \neq k$, relation (44) is in contradiction with (43), and the necessity of the rank being equal to N is thus proven.

That this condition is also sufficient is shown below. If $M > N$, we first reduce the system to N channels which correspond to a nonsingular $A(z^N)$ matrix. We then make the following choice of the synthesis filters [see (39)]:

$$[g(z)]^T = [1 \;\; 0 \;\; 0 \;\; \cdots \;\; 0]\, H_*(z) \operatorname{Co}[C_s(z^N)]. \qquad (45)$$

Using (39) and (45), we see that the output of the subband coder in (24) becomes

$$\hat{X}(z) = 1/N\, z^{-N} \operatorname{Det}[C_s(z^N)]\, \Delta(z^N)\, X(z). \qquad (46)$$

The aliasing has thus completely disappeared. Note that the original signal is filtered by a function in z^N in (46). Except for the compensation of the channel effects, the solution in (45) is based on the cofactor matrix of $H_m(z)$. Note that such a solution has been proposed by several authors [21], [26], [34], [35]. Next, we consider perfect reconstruction.

Corollary 1.1: For perfect reconstruction it is *sufficient* that the determinant of a nonsingular submatrix of $C_s(z^N) \cdot H_m(z)$ has all its zeros within the unit circle. In the case of critical subsampling, this condition is also *necessary* for $N = 2$ or for modulated filter banks.

This condition is sufficient, because in that case, the inverse of the determinant is a stable filter. Thus, in (46), one can use a postfilter equal to:

$$P(z) = \frac{N}{\operatorname{Det}[C_s(z^N)]\, \Delta(z^N)}. \qquad (47)$$

Therefore, the output signal is equal to the input within a delay. In order for the condition to be necessary, one has to show that an unstable pole of $P(z)$ in (47) cannot be cancelled by a zero of the cofactor matrices in (45). In that case, $P(z)$ would be unstable and thus perfect reconstruction impossible. The necessity of the condition is shown in the Appendix for two important cases, that is, for two channel systems and for the case where the filters are obtained by modulation from a single prototype filter.

An equivalent corollary holds for all-pass reconstruction as well.

Corollary 1.2: For all-pass reconstruction (no amplitude distortion), it is *sufficient* that the determinant of a nonsingular submatrix of $C_s(z^N) \cdot H_m(z)$ has no zeros on the unit circle. When the subsampling is critical, the condition becomes *necessary* for $N = 2$ as well as when the filters are modulated.

The condition is sufficient, because all unstable poles of $P(z)$ can be replaced by stable mirror poles (placed at the same angle but with one over the norm). The necessity is proved by using the fact that there cannot be cancellation of unstable poles in the case of $N = 2$ or modulated banks (see the Appendix). Therefore, zeros of the determinant on the unit circle prohibit all-pass reconstruction at least in these cases.

After these considerations on subband coders, we can similarly analyze the transmultiplexer case. The equivalent of theorem 1 is (we assume that the phase at the receiver is perfectly recovered) as follows.

Theorem 2: Crosstalk-free reconstruction in a transmultiplexer is possible if and only if both the channel filter matrix $[C_t(z^N)]$ and the synthesis filter matrix $[H_m(z)]$ have rank M (the number of input signals).

It follows immediately that N (the upsampling at the input) has to be greater or equal to M (the number of input signals), a result that is again clear from a sampling theory point of view.

The necessity for the rank size is verified as follows. From (31), we know that $T_t(z^N)$ has to be diagonal in order to cancel crosstalk. Since it is an $M \times M$ matrix, its rank therefore has to be equal to M. Since the rank of a product is upperbounded by the minimum of the ranks of the terms of a product [12], we get, with (31),

$$\operatorname{rank}[T_t(z^N)] \leq \operatorname{Min}\big[\operatorname{rank}[G_m(z)],\; \operatorname{rank}[C_t(z)],\; \operatorname{rank}[H_m(z)]\big]. \qquad (48)$$

Now, if either $C_t(z)$ or $H_m(z)$ have rank smaller than M, then $T_t(z^N)$ will also have a rank smaller than M and thus cannot be diagonal. Note that $C_t(z)$ has either rank 0 ($C(z) = 0$) or N. To show that the condition is sufficient, we assume that M is equal to N (otherwise, one can either add dummy input signals or reduce the upsampling factor). Now, using the following synthesis filters [see (39)],

$$g(z) = [H_*(z)]^T \cdot \operatorname{Co}[C_t(z)] \cdot [1 \;\; 0 \;\; 0 \;\; \cdots \;\; 0]^T \qquad (49)$$

leads to the following transmission matrix as can be verified [38]:

$$T_t(z^N) = \operatorname{Det}[C_t(z)]\, z^N \Delta(z^N) \cdot I. \qquad (50)$$

Corollaries similar to corollaries 1.1 and 1.2 hold also in the context of the above theorem and define the possibility of perfect and all-pass reconstruction in transmultiplexers. Fig. 6 shows the potential of the coherent crosstalk

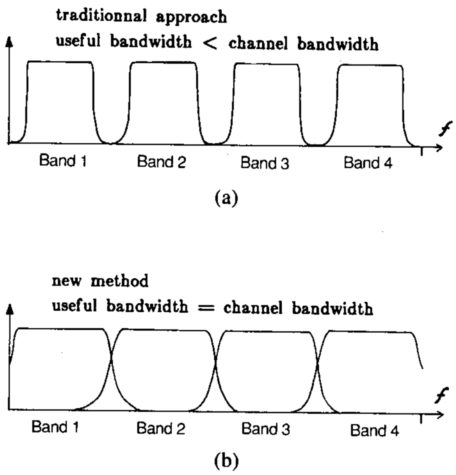


Fig. 6. Duality of subband coding and transmultiplexing allowing the use of the full channel bandwidth for transmission in transmultiplexers. (a) Crosstalk suppression using guard bands and sharp passband filters. (b) Crosstalk suppression using filters that cancel the crosstalk at the receiver.

annulation in transmultiplexers that is achieved with the filters in (49). In a conventional FDM scheme, there are guard bands between the channels in order to avoid crosstalk (thus, the channel bandwidth is not fully used). In the new scheme, the full bandwidth can be used even if the band-pass filters are not perfect, since the crosstalk will be cancelled at the receiver.

In order to explore the similarities and differences between subband coders and transmultiplexers in more detail, we assume in the following that the channels are ideal ($C_s(z^N) = C_t(z) = I$) and that the sampling rate change is critical ($M = N$). In that case, the following theorem holds.

Theorem 3: Aliasing and crosstalk cancellation are equivalent if and only if the product of the analysis and synthesis filter matrix (one being transposed) is equal to a function in z^N times the identity matrix.

In order to have both aliasing and crosstalk cancellation, we require that [from (25)]

$$[G_m(z)]^T H_m(z) = \text{Diag}\,[F(z) \quad F(Wz) \quad \cdots \quad F(W^{N-1}z)] \quad (51a)$$

and [from (30)]

$$H_m(z)\,[G_m(z)]^T = \text{Diag}\,[F_0(z^N) \quad F_1(z^N) \quad \cdots \quad F_{N-1}(z^N)]. \quad (51b)$$

The necessity can be shown by contradiction: if $F(z)$ in (51a) is not a function of z^N, simple counterexamples show that the product in (51b) is not diagonal (note that $H_m(z)$ and $G_m(z)$ are never diagonal). The condition is also sufficient because in that case, the matrix product is commutative [29], and thus, (51a) and (51b) are equivalent.

An interesting point to note is in which respect the solutions for subband coders and transmultiplexers can be different. Choosing $g(z)$ as in (45) or (49),

$$g(z) = [H_*(z)]^T \cdot [1 \quad 0 \quad 0 \quad \cdots \quad 0]^T \quad (52a)$$

gives the desired equivalence since

$$[G_m(z)]^T \cdot H_m(z) = H_m(z) \cdot [G_m(z)]^T = z^{-N}\Delta(z^N)\, I. \quad (52b)$$

Now, the set of all other solutions achieving aliasing cancellation in subband coders [given a certain $H_m(z)$] are obtained from

$$\{g_s(z)\} = S(z) \cdot g(z) \quad (53)$$

where $S(z)$ is an arbitrary filter. The set of all solutions achieving crosstalk cancellation in transmultiplexers [given $H_m(z)$] is

$$\{g_s(z)\} = \text{Diag}\,[S_0(z^N) \quad S_1(z^N) \quad \cdots \quad S_{N-1}(z^N)] \cdot g(z). \quad (54)$$

This highlights the difference between the reconstruction filters for subband coders and transmultiplexers. In the first case, an arbitrary filter can be put in cascade with all reconstruction filters, while in the second case, each output can have a different filter in cascade, but only a filter in z^N. Nevertheless, in both cases, the basic solution is obtained by evaluating the partial inverse $H_*(z)$, that is, mainly inverting $D(z^N)$ and evaluating the cofactor matrix of $N_p(z^N)$ in (36) or (37).

Both theorems 1 and 2 have used the rank of the filter matrix, and we will give another example to show the fundamental nature of the notion of rank. If the rank of a filter matrix corresponding to an analysis bank subsampled by N (for example, used in short time spectrum analysis) is equal to N, then two different input signals produce different output signals. This is equivalent to say that the application realized by the filter bank is an injection [28]. Otherwise, if the rank is lower than N, whole classes of input signals will produce the same outputs, as can be verified.

In order to summarize this important subsection, we recall that, given a filter matrix with rank equal to the sampling rate change, one can always annihilate crosstalk or aliasing in transmultiplexers and subband coders. Both problems are basically equivalent, since differences are reduced to some possibly different "scaling" factors (postfilters). Perfect or all-pass reconstruction was shown to depend on the location of the zeros of the determinant. Thus, and quite naturally, the two basic characteristics of a multirate filter bank are the rank and the determinant of the associated filter matrix.

C. Minimum Delay Solutions

Both in subband coders and in transmultiplexers, the delay from input to output due to the filtering (which is often done with linear phase filters) can be problematic. It is thus interesting to note that the *minimum delay* associated with an N channel system is equal to $N - 1$ *samples* (of the high sampling rate). Of course, we restrict ourselves to causal filters only.

The above assertion is easily proved by looking at (39). As already noted, a multiplication by z^{-N+1} in (39) would be sufficient to obtain a causal filter matrix $\boldsymbol{H}_*(z)$. In that case and assuming perfect channels, the output of the subband coder becomes [following (46)]

$$\hat{X}(z) = 1/N \cdot z^{-N+1} \cdot \Delta(z^N) \cdot X(z). \quad (55)$$

If $\Delta(z^N)$ is made equal to one (and we will see later how to achieve this goal), then we obtain a minimum delay system. If $\Delta(z^N)$ is minimum phase, it can be cancelled out and the resulting system has also minimum delay. These results carry over to transmultiplexers as well (following theorem 3), but the subsampling at the receiver has to be advanced by one sample in order to be in phase with the reconstructed signals. That the delay cannot be lower than $N - 1$ follows from the fact that otherwise $\boldsymbol{H}_*(z)$ in (39) becomes noncausal, a fact that can be verified on simple examples. Note that we proved a lower bound for the delay of multirate filter banks, but that nothing was said about the quality of the resulting filters.

D. Finite Impulse Response Solutions

FIR filters are often desirable because of three major reasons: they are always stable, their numerical properties are good, and they can achieve linear phase behavior. In the case of filter banks with reconstruction, FIR filters have also the advantage that they do not realize implicit pole/zero cancellation between physically distinct filters [35]. We talk about FIR solutions when all involved filters are FIR, and thus, an FIR filter matrix is one in which all elements correspond to FIR filters.

For perfect reconstruction, it is sufficient that the determinant $\Delta(z^N)$ is a pure delay (we still assume perfect channels). This condition becomes necessary in the case of $N = 2$ and when the filters are modulated (see the Appendix). That the condition is sufficient is clear from (46) and (50) for subband coders and transmultiplexers, respectively. In the next section, we will show how to achieve the condition of a determinant being equal to a delay.

That linear phase solutions allowing perfect reconstruction exist was shown in [35] and [38]. However, restrictions are put on the filter lengths as well as on the symmetries of the filters involved. For example, in the important case of two channel systems, it is shown that linear phase filters for perfect reconstruction exist only if the filter length is even and if the two filters have different symmetries.

E. Modulated Filter Banks

An important special case appears when the filters of a filter bank are obtained by modulation from a single prototype filter, that is,

$$H_i(z) = H_\pi(W^i z) \quad (56)$$

where $H_\pi(z)$ is the prototype filter. In that case, the filter matrix is circulant and of the form (assuming critical subsampling)

$$\boldsymbol{H}_m(z) = \begin{bmatrix} H_\pi(z) & H_\pi(Wz) & \cdots & H_\pi(W^{N-1}z) \\ H_\pi(Wz) & H_\pi(W^2z) & \cdots & H_\pi(z) \\ \vdots & \vdots & \ddots & \vdots \\ H_\pi(W^{N-1}z) & \cdots & & H_\pi(W^{N-2}z) \end{bmatrix}. \quad (57)$$

Because of its circulant form, $\boldsymbol{H}_m(z)$ can be diagonalized with Fourier matrices, and this in the following way [30], [35], [38] [where we assume that the prototype filter $H_\pi(z)$ is written as $N_\pi(z)/D(z^N)$ following (32)]:

$$\boldsymbol{H}_m(z) = 1/D(z^N) \cdot \boldsymbol{F}^{-1} \cdot \boldsymbol{L}(z) \cdot \boldsymbol{F} \quad (58a)$$

where

$$\boldsymbol{L}(z) = N \cdot \boldsymbol{J} \cdot \text{Diag}\,[N_{\pi 0}(z^N) \;\; z^{-N+1}N_{\pi N-1}(z^N) \;\; \cdots \;\; z^{-1}N_{\pi 1}(z^N)]. \quad (58b)$$

Note that $N_{\pi i}(z^N)$ is the ith polyphase component of the prototype filter numerator [after expansion to the canonical form in (33)]. Because of this factorization, it is shown in the Appendix that the zeros of the determinant $\Delta(z^N)$ are equal to the zeros of the various polyphase components $N_{\pi i}(z^N)$, because

$$\Delta(z^N) = N^N \prod_{i=0}^{N-1} N_{\pi i}(z^N). \quad (59)$$

Thus, $\boldsymbol{H}_m(z)$ has rank N if no polyphase component is equal to zero, a result which is quite clear. Similarly, the inverse of $\boldsymbol{H}_m(z)$ is stable if and only if all polyphase components are minimum phase (see corollaries 1.1 and 1.2 as well as the Appendix).

As a conclusion to this section, we first note that a useful factorization of the filter matrix was given in (36) and (37). Then, basic properties of multirate filter banks were derived by using the rank of the filter matrices, a notion that is quite fundamental. The basic result is that aliasing and crosstalk can always be cancelled with stable filters, while perfect or all-pass reconstruction is dependent on the zero locations of the determinant. It was then shown that the minimum delay of an N channel system is $N - 1$ samples. FIR and linear phase solutions were addressed next, and finally, it was shown how the filter matrix could be diagonalized when the filter bank is made up of modulated filters.

V. Synthesis of Filters for Filter Banks

The framework developed so far will be used below to investigate the design of filters in the context of filter banks. As it turns out, this is a different problem than the design of a single filter, especially due to the central importance of the determinant of the filter matrix. In its most general form and for an analysis/synthesis system with M channels, one has to design $2M$ filters under the constraint of achieving a certain input/output relation (usually as close to a certain delay as possible).

The size of the problem can be reduced in certain particular cases, for example, in modulated filter banks (only one prototype has to be designed) or if only the analysis filters are considered. Nevertheless, this can lead to drawbacks as shown, for example, in [26] where passband analysis filters lead to synthesis filters with large out-of-band components. While this causes no problem when the channels are perfect, a deterioration of the channel characteristics (like quantization in subband coding) will produce poor reconstruction if the synthesis is not passband as well (especially, aliasing will appear [38]).

Below, three design methods will be briefly presented. The first one, called the *factorization method* [39], [34], [35] is suited for two channel FIR systems and achieves perfect reconstruction. The other two methods can be used for arbitrary size FIR or IIR filter banks. The *complementary filter method* [35], [38] allows perfect reconstruction by choosing the last of the M filters appropriately and finally, the *simultaneous optimization method* [11], [38] is a more classical approach based on the simultaneous design of the filters and the resulting determinant. Note that, in the following, we consider only filter banks which are critically sampled (that is, $M = N$).

A. Factorization Method

In the two channel FIR case, the filter matrix $\boldsymbol{H}_m(z)$ can be written as

$$\boldsymbol{H}_m(z) = \begin{bmatrix} H_{00}(z^2) & H_{01}(z^2) \\ H_{10}(z^2) & H_{11}(z^2) \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{60}$$

When the synthesis filters are chosen so as to cancel aliasing, that is, following (52), the input–output relation equals

$$\begin{aligned} \hat{X}(z) &= \tfrac{1}{2}\left[H_{00}(z^2)\, H_{11}(z^2) - H_{01}(z^2)\, H_{10}(z^2)\right] \cdot z^{-2} \\ &= -\tfrac{1}{4}\left[H_0(z)\, H_1(-z) - H_0(-z)\, H_1(z)\right] \cdot z^{-1}. \end{aligned} \tag{61}$$

Introducing an auxiliary product filter $P(z) = H_0(z) H_1(-z)$, it is easy to see that the reconstruction in (61) will be perfect if (and only if when IIR filters are excluded) $P(z)$ has arbitrary coefficients for the even powers of z, but only a single nonzero coefficient for the odd powers of z. The design method consists in choosing $P(z)$ with the given properties and then factorize it into $H_0(z)$ and $H_1(-z)$. Given that $P(z)$ is a half-band low-pass filter, the factorization is done such that $H_0(z)$ and $H_1(z)$ are good low-pass and high-pass half-band filters, respectively. Note that the first perfect reconstruction solution proposed in [24] is a factorization into maximum and minimum phase components.

Problems with the factorization method are of two kinds. First, the number of possible factorizations of $P(z)$ into $H_0(z)$ and $H_1(z)$ grows exponentially with the number of zeros [38], thus making it rapidly impossible to check the quality of the various factorizations. Assume a length-L FIR filter $P(z)$, L being odd. The number of possible factorizations of its $L - 1$ zeros into equal size groups of $(L - 1)/2$ zeros is equal to [38]:

$$n_i = \frac{2^{(L-1)}}{\sqrt{(2\pi(L - 1))}} \tag{62}$$

where we used the Stirling formula for approximating factorials. For example, if $L = 31$, the number of length-16 filters that can be obtained is over 6000 (assuming that zeros appear in conjugate pairs that cannot be split in order to keep the filters real). If we consider only linear phase filters (zeros appear in groups of 4), there are still 35 solutions.

The second problem appears when the size of $P(z)$ is small. In this case, it is difficult to obtain good factorizations, especially when one desires linear phase solutions. Fig. 7 shows two possible factorizations of a length-15 filter $P(z)$ shown in part (a). Parts (b) and (c) give a factorization into minimum/maximum phase components, respectively, while parts (d) and (e) show the linear phase factorization which yields poor filters as can be seen. The intrinsic problem is that while the product of two good low-pass filters yields a good low-pass filter, the reciprocal statement is not true. This remark explains why the factorization of $P(z)$ can be problematic.

B. Complementary Filter Method

In this method [35], [38], after choosing the $N - 1$ first filters of a size N bank, we calculate the last filter in such a way that the determinant $\Delta(z^N)$ equals a pure delay, thus guaranteeing perfect reconstruction. The method is suited for both FIR and IIR banks of arbitrary size, but we will only detail the FIR case here. Assuming that all filters are FIR and of length-L, one can verify that $\Delta(z^N)$ has at most $L - N + 1$ nonzero coefficients. After a reasonable choice of the $N - 1$ first filters, the constraint of the determinant being equal to a delay leads to $L - N + 1$ equations for the coefficients of the last filter. One can therefore put $N - 1$ additional constraints on this last filter. Note that the resulting set of equations is, in general, solvable [35]. This method can also be applied to linear phase filters, in which case the number of equations is halved.

While perfect reconstruction is again guaranteed, the problem with this method lies in the fact that the last, complementary filter can be of poor quality due to the fact that it is obtained by solving a system of equations. A typical example is given in Fig. 8, where the complementary filter to the 32-tap low-pass filter from [11] was calculated. While the reconstruction is now perfect (which

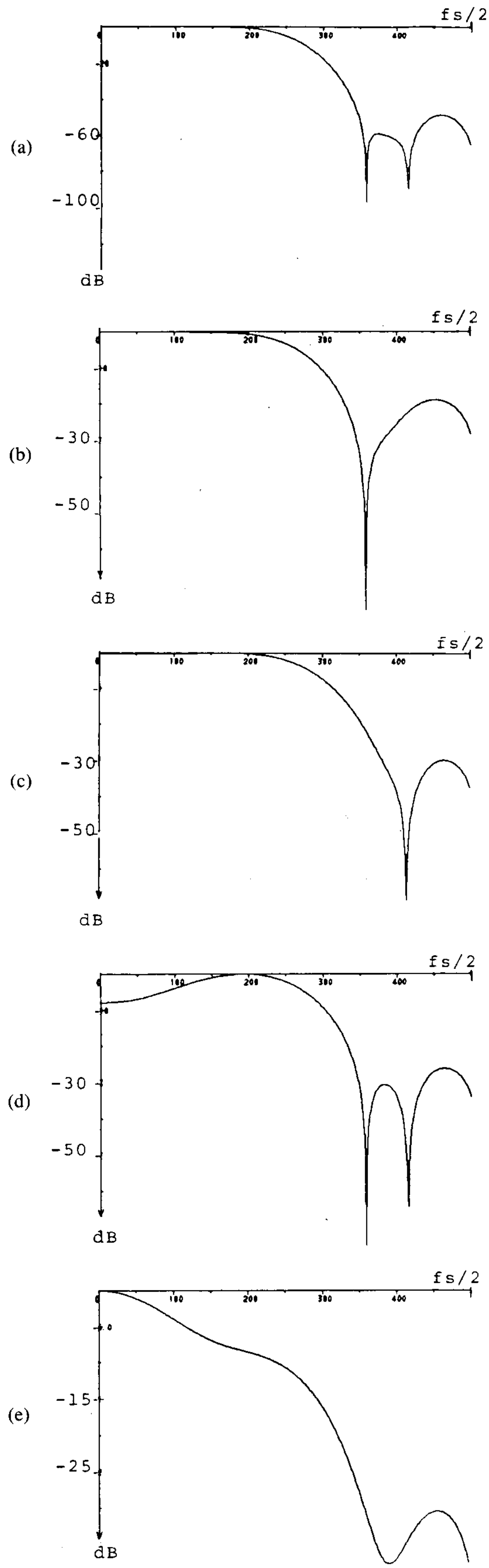


Fig. 7. Amplitude of the transfer functions on the unit circle of $P(z)$, $H_0(z)$, and $H_1(-z)$ in the factorization methods where $P(z) = H_0(z)\, H_1(-z)$. (a) Initial length-15 filter $P(z)$ to be factored. (b) Minimum phase filter $H_0(z)$. (c) Maximum phase filter $H_1(-z)$. (d) Linear phase filter $H_0(z)$. (e) Linear phase filter $H_1(-z)$.

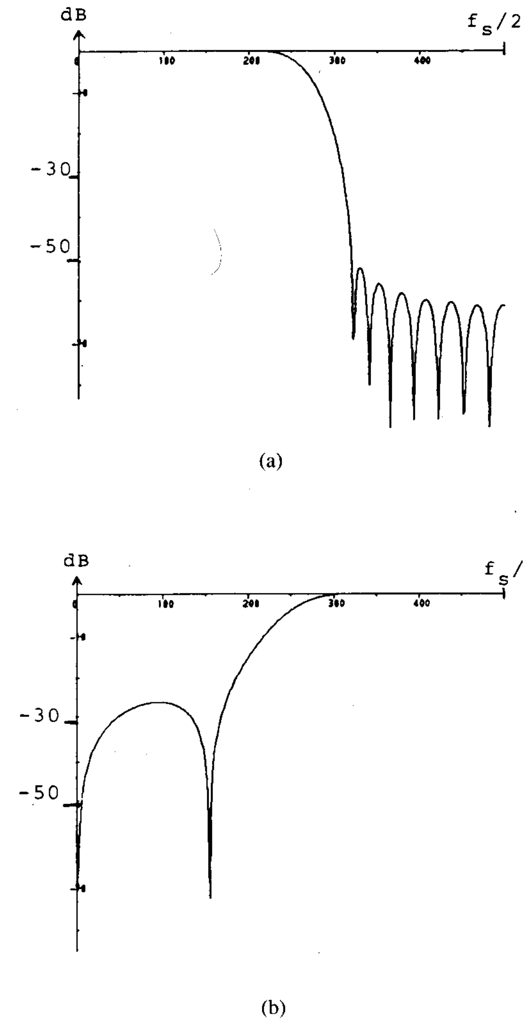


Fig. 8. Example of a complementary filter. (a) Original 32-taps low-pass filter from [11]. (b) Complementary high-pass filter allowing perfect reconstruction.

was not the case with the original QMF scheme, where the high-pass filter is simply the low-pass filter modulated by $(-1)^n$), we see that the complementary high-pass filter is of poor quality.

To obtain better complementary filters, one can either relax the constraint of the determinant being equal to a delay, or take a longer complementary filter (in which case, more constraints can be imposed). Note that, often, the complementary filter corresponds to a channel which is not very important (for example, in subband coding, the channel corresponding to the highest frequencies) and thus, its poor quality is not of great consequence. Finally, this method allows to easily obtain minimum delay solutions (see Section IV-C). When choosing the coefficients of the determinant (actually, choosing which one should be different from zero), it is sufficient to take the first coefficient equal to a constant (and all others equal to zero) in order to obtain a minimum delay solution. The resulting complementary filter is obviously modified and usually worse when compared to a solution producing a higher delay.

C. Simultaneous Optimization Method

This is simplest and most flexible method and it is also applicable to arbitrary size FIR or IIR filter banks. One optimizes simultaneously the various filters as well as the determinant by minimizing a cost function. This cost function is the squared error when comparing the current filters and determinant to the desired filters and determinant. The flexibility is obtained by weighting the errors from passbands, stopbands, and determinant independently. Note that the desired determinant can be chosen so as to reduce the input–output delay to a minimum. While this method does not give an analytical solution to the filter design problem and that perfect reconstruction is usually only approached, it does not have any of the drawbacks of the other methods presented.

As a conclusion to this section on filter design for filter banks, we recall once again the importance of the determinant of the filter matrix. In a first approach, a product filter yielding a determinant equal to a delay is factored to produce the two filters of the bank. The second approach calculates the Nth filter from the given $N - 1$ first one in such a way that the determinant reduces to a pure delay. Finally, in the last method presented, the determinant is one of the parameters to be optimized.

VI. Computational Complexity of Filter Banks

This section overviews briefly the computational complexity issue in filter banks, especially in the analysis/synthesis case. Several methods for complexity reduction in the case of filter trees and modulated filters are indicated. Most of the discussion is qualitative for conciseness, and we refer the reader to the literature for more details. Again, we restrict ourselves to the important case of critically sampled banks ($M = N$).

A. Analysis versus Synthesis Complexity

Given a certain analysis bank, we have seen that the synthesis filters in a subband coder are basically chosen from the cofactor matrix of the analysis filter matrix [see (45)], and this in order to cancel aliasing. The same holds for transmultiplexers, with the role of analysis and synthesis filters simply reversed. In the following, we will only consider the subband coder case, since all results carry over to transmultiplexers by duality.

The basic problem, from a computational complexity point of view, is that the filters obtained from the cofactor matrix are in general much more complex than those of the original matrix. As a simple example, assume that all analysis filters are FIR and of length L_a. In that case, the length L_s of the synthesis filters derived from the cofactor matrix is upperbounded by

$$L_s \leq (N - 1) L_a. \tag{63}$$

Even if some of the coefficients are zero (or can be neglected as being too small), the complexity of the synthesis filters is in general higher than that of the analysis filters when $N > 2$. A notable exception appears when the filters are modulated and have minimum phase polyphase components (see also the Appendix). In that case, these polyphase components can be inverted [35], [38], and the complexity of the synthesis is equal to that of the analysis. However, if the polyphase components are not minimum phase, one has to use the cofactor matrix for deriving the synthesis filters. These are also modulated, but the length of the prototype filter is increased when compared to the analysis filters. Another exception is the so-called pseudo-QMF filters [22], [17], [18], [2], [13], which achieve good aliasing cancellation while leading to equally complex analysis and synthesis filters.

TABLE II
Number of Multiplications and Additions for Each New Input Sample in an Elementary 2 Filter Bank (Subsampled by 2)

RIF filter	length L	$H_0(z)$	$H_1(z)$	mults.	adds.
1) arbitrary	arbitrary	$H_0(z)$	$H_1(z)$	L	L
2) linear phase	arbitrary	$H_0(z)$	$H_1(z)$	L/2	L
3) modulated	arbitrary	H(z)	H(-z)	L/2	L/2
4) mod. lin. phase	even	H(z)	H(-z)	L/2	L/2
5) mod. lin. phase	odd	H(z)	H(-z)	L/4	L/2
6) mod. lin. phase half band	odd	H(z)	H(-z)	L/8	L/4
7) min/max phase	even	H(z)	$z^{-L+1}H(z^{-1})$	3L/4	5L/4

Remarks:
4-6) from [8]
6) the complexity of the correction filter [8] has been neglected.
7) from [9]

B. Methods to Reduce the Computational Complexity of Filter Banks

Interestingly, the number of operations used in filter banks is usually of the same order as the number used by a single filter, and not at all N times higher as could be expected. There are two main reasons that contribute to this fact. First, the outputs of the filter bank are subsampled by N, and therefore, only one out of N outputs has to be computed. This subsampling can be used, both in time and frequency domain implementations, in order to reduce the number of required operations. Second, most filter banks have some special structure that can be taken advantage of, the two most well-known cases being the tree-structured and the modulated filter bank.

In the first case, the tree-structured filter bank, common pole, and zeros are computed simultaneously for the various filters [32]. Usually, the elementary block of a tree is made up of two filters, and Table II gives the number of operations for an elementary two filter block (subsampled by 2) using different filters. Since the length of the filters is normally halved at each stage of the tree, a conventional QMF tree with $N = 2^p$ channels and an initial filter length of L uses, for each new input sample, about [8]:

$$L(1 - 2^p) \text{ multiplications and additions.} \tag{64}$$

When computing a filter tree in the Fourier domain, one can use both the subsampling and the modulation (in the QMF case) in order to reduce the computational complexity, but at the price of a more involved structure. As an example, take a binary tree of depth 4 (16 filters) with QMF filters of length 128, 64, 32, and 16, respectively. While a time domain approach takes 120 multiplications and additions for each new input [from (64)], a DFT approach (with FFT's of length 1024) needs only 21 multiplications and 54 additions [38].

In the second important case—the modulated filter bank—it is well known that the computational complexity can be drastically reduced by the method of the polyphase network combined with a fast transform [1]. Instead of using the classical method, one can evaluate the polyphase network with Fourier transforms [16], or calculate the whole filter bank in the transform domain [38]. Both of these methods lead to two-dimensional transforms which can be evaluated very efficiently [15]. Therefore, they lead to the lowest known number of operations for filter banks, but at the cost of rather complex structures and large computational delays.

For the sake of comparison, consider a critically sampled bank of 16 length-128 complex FIR filters. The filters are obtained from a single prototype by modulation with the 16 roots of unity. For each new complex input sample, the classical polyphase/FFT approach uses about 25 multiplications and 47 additions to produce the outputs. Assume now an evaluation using blocks of 256 input samples. The method from [16] requires about 11 multiplications and 53 additions, while the complete evaluation in the Fourier domain [38] takes about the same number of multiplications but slightly more additions. Note that a single length-128 complex filter, subsampled by 16, takes already 24 multiplications per new input sample, thus showing the great efficiency of the above described methods. Note that the pseudo-QMF filters also belong to the class of efficient filter banks, since they can be implemented with a polyphase network and a fast transform (a discrete cosine transform).

In conclusion to this brief overlook on the computational complexity, we first recall that aliasing cancellation can require more complex filters for the synthesis than for the analysis (except for $N = 2$ or for special cases like the pseudo-QMF filters). Concerning the computational complexity itself, it was indicated that the number of operations required for a given bank can usually be brought down to the order of a single filter. This is possible by taking advantage of the subsampling and the structure of the filter bank.

VII. Conclusion

First, the two generic multirate filter banks used for the analysis and synthesis of signals were analyzed. This leads naturally to the introduction of filter matrices of size M by N, where M is the number of filters and N the subsampling factor (which leads to N modulated versions of the original filters and signals). Using these filter matrices, it was easy to express the output of the two physical systems that use multirate filter banks (subband coder and transmultiplexer). The conditions for reconstruction (aliasing/crosstalk-free or perfect) have then been stated.

Using properties of the filter matrices (rank, determinant, stability, causality), fundamental properties of subband coders and transmultiplexers were demonstrated. Essentially, the rank of the filter matrix has to be equal to the sampling rate change in order to allow aliasing/crosstalk free reconstruction. For perfect reconstruction, it is further needed that the determinant is minimum phase, a property proven to be necessary in two important cases (two channel bank and modulated filter bank). Note the following analogy: in the single filter case, the filter itself has to be minimum phase if one wants to reconstruct the original signal from the output. Similarly, in the multirate filter bank case, the same role is played by the determinant of the filter matrix. The central role of the determinant appears thus quite clearly. A further important property is that the minimum delay of an N-channel system is only $N - 1$ samples. Linear phase banks allowing perfect reconstruction and modulated filter banks were also considered.

Concerning the synthesis of filters for filter banks, it was shown that the design is a tradeoff between individual filter quality, reconstruction quality, and input–output delay. Three design methods were presented: the factorization method (suited for $N = 2$, FIR banks), the complementary filter method, and the simultaneous optimization method (both for arbitrary size FIR or IIR banks).

Finally, the computational complexity of multirate filter banks was considered. It was noted that the aliasing/crosstalk cancellation requires in general more complex output filters (except in special cases like the two channel bank or the pseudo-QMF filters). Then, different methods for complexity reduction were reviewed, showing that the number of required operations is in general comparable to the one required by a single filter.

In conclusion, it was shown that multirate filter banks are quite different from single filters, thus requiring a novel analysis method (based on matrices). Basic results can be proven by using well-known tools from linear algebra. The synthesis of filters and the reduction in computational complexity call as well for original solutions. Finally, it seems that the approaches and results presented in this paper and other related publications set a clear basis for filter bank problems, on the one hand, and open, on the other hand, a large horizon for further developments.

Appendix

In this Appendix, we prove that perfect reconstruction is possible if and only if the determinant of the filter matrix is minimum phase, and this in the two channel case and in the modulated filter case. The fact that the condition is sufficient was shown in Section IV. The necessity is shown by proving that, when evaluating the inverse that is required for perfect reconstruction and given below (causality is neglected here):

$$[\boldsymbol{H}_m(z)]^{-1} = \frac{1}{\text{Det}\,[\boldsymbol{H}_m(z)]} \text{Co}\,[\boldsymbol{H}_m(z)]^T, \quad \text{(A1)}$$

there cannot be a pole/zero cancellation between the elements of the cofactor matrix and the inverse of the determinant. Therefore, all zeros of the determinant will be poles of the synthesis filters, and thus, a stable synthesis requires a minimum phase determinant (if the determinant has poles, they are within the unit circle since the analysis filters are assumed to be stable).

A. Two Channel Case

In (38), we have seen that the only inverse that can be unstable is the inverse of $\boldsymbol{N}_p(z^2)$:

$$[\boldsymbol{N}_p(z^2)]^{-1} = 1/\Delta(z^2) \cdot \begin{bmatrix} H_{11}(z^2) & -H_{10}(z^2) \\ -H_{01}(z^2) & H_{00}(z^2) \end{bmatrix} \quad \text{(A2)}$$

where

$$\Delta(z^2) = H_{00}(z^2)\,H_{11}(z^2) - H_{01}(z^2)\,H_{10}(z^2). \quad \text{(A3)}$$

We assume now that there are no common zeros between the different elements of $\boldsymbol{N}_p(z^2)$. Otherwise, these common zeros can be factored out and will appear in the inverse [38]. Consider now the first element e_{00} of the inverse:

$$e_{00} = H_{11}(z^2)/(H_{00}(z^2)\,H_{11}(z^2) - H_{01}(z^2)\,H_{10}(z^2)). \quad \text{(A4)}$$

In order to cancel an unstable pole p_i of the inverse of the determinant (thus, $-p_i$ will also be a pole), this pole has to be a factor of $H_{11}(z^2)$, and thus, e_{00} can be rewritten as

$$e_{00} = \frac{(1 - z^{-2}p_i^2)\,H'_{11}(z^2)}{((1 - z^{-2}p_i^2)\,H_{00}(z^2)\,H'_{11}(z^2) - H_{01}(z^2)\,H_{10}(z^2))}. \quad \text{(A5)}$$

Since $(1 - z^{-2}p_i^2)$ is also a factor of the denominator of e_{00}, it is thus also a factor of $H_{01}(z^2)\,H_{10}(z^2)$, that is, either of $H_{01}(z^2)$ or $H_{10}(z^2)$. But this is in contradiction with our assumption that the various elements had no common factors. Therefore, a pole/zero cancellation in (A4) is impossible, and the necessity of the minimum phase condition for the determinant is thus proven.

B. Modulated Filter Bank Case

When inverting the filter matrix in (58), it can be verified that the prototype filter for the synthesis is given by

$$G_\pi(z) = D(z^N)\left[\frac{z^{-N}}{N_{\pi 0}(z^N)} + \frac{z^{-N+1}}{N_{\pi 1}(z^N)} + \cdots + \frac{z^{-1}}{N_{\pi N-1}(z^N)}\right]. \quad \text{(A6)}$$

We will prove that all the zeros of the various polyphase components $N_{\pi i}(z^N)$ are poles of $G_\pi(z)$. To this end, we analyze the poles of the following function $f(x)$:

$$f(x) = \frac{x^{-N+1}}{N_{\pi 0}(x^N)} + \frac{x^{-N+2}}{N_{\pi 1}(x^N)} + \cdots + \frac{1}{N_{\pi N-1}(x^N)}. \quad \text{(A7)}$$

Except for a pole at the origin, $G_\pi(z)$ and $f(x)$ have the same poles. We did not consider pole/zero cancellations due to $D(z^N)$ because these poles would be inside the unit circle (the prototype filter $H_\pi(z)$ in (56) is stable) and do not influence our stability analysis. Assume first that the various $N_{\pi i}(x^N)$ have no common or multiple zeros and that there are no zeros at the origin (this last case does not cause stability problems anyway). We can write $N_{\pi i}(x^N)$ as

$$N_{\pi i}(x^N) = \prod_{j=0}^{k_i-1} \prod_{l=0}^{N-1} (W^l p_{ij} x^{-1} - 1) \quad \text{(A8)}$$

where $W = e^{-j2\pi/N}$ and k_i is the length of the ith polyphase filter (divided by N). A rational function with a denominator degree strictly greater than the numerator degree has a unique partial fraction expansion. Thus, $f(x)$ can be uniquely written as

$$f(x) = \sum_{i=0}^{N-1} \sum_{j=0}^{k_i-1} \sum_{l=0}^{N-1} \frac{\mu_{ijl}}{(W^l p_{ij} x^{-1} - 1)}. \quad \text{(A9)}$$

Since all poles p_{ij} are different from each other, there cannot be annulation between two terms, and it is thus sufficient to prove that all μ_{ijl}'s are different from zero in order to show that all $W^l p_{ij}$'s are poles of $f(x)$. The expression of the μ_{ijl}'s is given by

$$\mu_{ijl} = \frac{[W^l p_{ij}]^i}{\prod_{m=0, \neq l}^{N-1} ([W^l p_{ij}]^{-1} W^m p_{ij} - 1) \prod_{n=0, \neq j}^{k_i-1} ([p_{ij}]^{-n}[p_{il}]^n - 1)}. \quad \text{(A10)}$$

Since we assumed all p_{ij}'s different from each other and from zero, it follows that the μ_{ijl}'s are all different from zero, and thus, each p_{ij} is a pole of $f(x)$. It is easy to check that if the pole p_{ij} has a multiplicity K in a certain polyphase filter, it will lead to the equivalent number of poles in $f(x)$.

The interesting case appears when there are common zeros between different polyphase filters $N_{\pi i}(x^N)$ because, then, there could be cancellation by summation. That this will not happen is shown below. Assume a common zero p_c between $N_{\pi m}(x^N)$ and $N_{\pi n}(x^N)$. We consider only the two terms of $f(x)$ in which p_c appears (the others are not concerned by a possible cancellation).

$$\frac{x^{-m}}{(p_c^N x^{-N} - 1)\, N'_{\pi m}(x^N)} + \frac{x^{-n}}{(p_c^N x^{-N} - 1)\, N'_{\pi n}(x^N)} \tag{A11}$$

where $N'_{\pi m}(x^N)$ and $N'_{\pi n}(x^N)$ are the polyphase component after division by the factor corresponding to p_c. The numerator of the expression in (A11) equals

$$n(x) = x^{-m} N'_{\pi n}(x^N) + x^{-n} N'_{\pi m}(x^N). \tag{A12}$$

In order that the N poles corresponding to $W^i p_c$ ($i = 0 \cdots N - 1$) disappear, it is necessary that the following N equations are verified:

$$n(W^l p_c) = 0 \quad l = 0 \cdots N - 1 \tag{A13a}$$

$$W^{-lm} p_c^{-m} N'_{\pi n}(x^N) + W^{-ln} p_c^{-n} N'_{\pi m}(x^N) = 0. \tag{A13b}$$

Since $m \neq n$, and because of the orthogonality of the roots of unity, it is easy to show that appropriate linear combination will lead to the two equivalent equations:

$$N'_{\pi m}(p_c^N) = 0 \tag{A14a}$$

$$N'_{\pi n}(p_c^N) = 0, \tag{A14b}$$

and thus, the $W^l p_c$'s ($l = 0 \cdots N - 1$) have to be zeros of the reduced polynomials as well. By recursion, it follows that $N_{\pi m}(x^N)$ has to be equal to $N_{\pi n}(x^N)$. It is easy to verify that, in that case, the unstable poles cannot disappear (there are at least N unstable poles, and the numerator has at most $N - 1$ zeros). The proof can be easily extended to the case where a zero is common to more than 2 polyphase filters.

In conclusion, it was shown that all zeros of the polyphase components $N_{\pi i}(z^N)$ are poles of the synthesis prototype filter when the filter bank is modulated. Since the determinant is equal to the product of the polyphase components [see (59)], we can say that perfect reconstruction is possible if and only if the determinant is minimum phase.

Acknowledgment

The author is indebted to Prof. H. J. Nussbaumer for his guidance, and to R. Leonardi and N. Greene for rereading the manuscript.

References

[1] M. G. Bellanger and J. L. Daguet, "TDM-FDM transmultiplexer: Digital polyphase and FFT," *IEEE Trans. Commun.*, vol. COM-22, pp. 1199–1204, Sept. 1974.

[2] P. L. Chu, "Quadrature mirror filter design for an arbitrary number of equal bandwidth channels," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-33, pp. 203–218, Feb. 1985.

[3] R. E. Crochiere, S. A. Webber, and J. L. Flanagan, "Digital coding of speech in sub-bands," *Bell Syst. Tech. J.*, vol. 55, no. 8, pp. 1069–1085, Oct. 1976.

[4] R. E. Crochiere and L. R. Rabiner, *Multirate Digital Signal Processing*. Englewood Cliffs, NJ: Prentice-Hall, 1983.

[5] A. Croisier, D. Esteban, and C. Galand, "Perfect channel splitting by use of interpolation, decimation, tree decomposition techniques," in *Int. Conf. Inform. Sci./Syst.*, Patras, Greece, Aug. 1976, pp. 443–446.

[6] D. Esteban and C. Galand, "Application of quadrature mirror filters to split-band coding," in *ICASSP 77*, Hartford, CT, May 1977, pp. 191–195.

[7] C. Galand, "Codage en sous-bandes: Théorie et application à la compression numérique du signal de parole," thèse d'Etat, Université de Nice, France, 1983.

[8] C. R. Galand and H. J. Nussbaumer, "New quadrature mirror filter structures," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 32, pp. 522–531, June 1984.

[9] —, "Quadrature mirror filters with perfect reconstruction and reduced computational complexity," in *Proc. 1985 IEEE ICASSP*, Tampa, FL, Mar. 1985, pp. 525–529.

[10] U. Heute and P. Vary, "A digital filter bank with polyphase network and FFT hardware: Measurements and applications," *Signal Processing*, vol. 3, no. 4, pp. 307–319, Oct. 1981.

[11] J. D. Johnston, "A filter family designed for use in quadrature mirror filter banks," in *Proc. IEEE ICASSP 80*, Denver, CO, 1980, pp. 291–294.

[12] T. Kailath, *Linear Systems*. Englewood Cliffs, NJ: Prentice-Hall, 1980.

[13] J. Masson and Z. Picel, "Flexible design of computationally efficient nearly perfect QMF filter banks," in *Proc. 1985 IEEE ICASSP*, Tampa, FL, Mar. 1985, pp. 541–544.

[14] H. J. Nussbaumer, "Pseudo QMF filter bank," *IBM Tech. Disclosure Bull.*, vol. 24, no. 6, pp. 3081–3087, Nov. 1981.

[15] —, *Fast Fourier Transform and Convolution Algorithms*. Berlin, Germany: Springer-Verlag, 1982.

[16] —, "Polynomial transform implementation of digital filter banks," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-31, pp. 616–622, June 1983.

[17] H. J. Nussbaumer and M. Vetterli, "Computationally efficient QMF filter banks," in *Proc. 1984 IEEE ICASSP*, San Diego, CA, Mar. 1984, pp. 11.3.1–4.

[18] —, "Pseudo quadrature mirror filters," in *Proc. Int. Conf. Digital Signal Processing*, Florence, Italy, Sept. 1984, pp. 8–12.

[19] A. V. Oppenheim and R. W. Schafer, *Digital Signal Processing*. Englewood Cliffs, NJ: Prentice-Hall, 1975.

[20] L. R. Rabiner and B. Gold, *Theory and Application of Digital Signal Processing*. Englewood Cliffs, NJ: Prentice-Hall, 1975.

[21] T. A. Ramstad, "Analysis/synthesis filterbanks with critical sampling," in *Int. Conf. Digital Signal Processing*, Florence, Italy, Sept. 1984, pp. 130–134.

[22] J. H. Rothweiler, "Polyphase quadrature filters—A new subband coding technique," in *Proc. 1983 IEEE ICASSP*, Boston, MA, Mar. 1983, pp. 1280–1283.

[23] R. W. Schafer and L. R. Rabiner, "A digital signal processing approach to interpolation," *Proc. IEEE*, vol. 61, pp. 692–702, June 1973.

[24] M. J. T. Smith and T. P. Barnwell, "A procedure for designing exact reconstruction filterbanks for tree structured sub-band coders," in *Proc. IEEE ICASSP-84*, San Diego, CA, Mar. 1984, pp. 27.1–4.

[25] M. J. T. Smith, "Exact reconstruction analysis/synthesis systems and their application to frequency domain coding," Ph.D. dissertation, Georgia Inst. Technol., Atlanta, Dec. 1984.

[26] M. J. T. Smith and T. P. Barnwell, "A unifying framework for analysis/synthesis systems based on maximally decimated filter banks," in *Proc. IEEE ICASSP-85*, Tampa, FL, Mar. 1985, pp. 521–524.

[27] —, "A new filter bank theory for time-frequency representation," submitted for publication.

[28] H. S. Stone, *Discrete Mathematical Structure and Their Applications*. Chicago, IL: Science Research Associates, 1973.

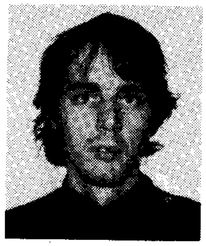
[29] G. Strang, *Linear Algebra and Its Applications*. New York: Academic, 1980.

[30] K. Swaminathan and P. P. Vaidyanathan, "Theory of uniform DFT, parallel quadrature mirror filter banks," in *Proc. ICASSP-86*, Tokyo, Japan, Apr. 1986, pp. 2543–2546.

[31] P. Vary and U. Heute, "A short-time spectrum analyzer with polyphase-network and DFT," *Signal Processing*, vol. 2, no. 1, pp. 55–65, Jan. 1980.

[32] M. Vetterli, "Tree structures for orthogonal transforms and applica-

tion to the Hadamard transform," *Signal Processing*, vol. 5, no. 6, pp. 473-484, Nov. 1983.
[33] —, "Multi-dimensional sub-band coding: Some theory and algorithms," *Signal Processing*, vol. 6, no. 2, pp. 97-112, Feb. 1984.
[34] —, "Splitting a signal into subsampled channels allowing perfect reconstruction," in *Proc. IASTED Conf. Appl. Signal Processing and Digital Filtering*, Paris, France, June 1985.
[35] —, "Filter banks allowing perfect reconstruction," *Signal Processing*, vol. 10, no. 3, pp. 219-244, Apr. 1986.
[36] —, "Perfect transmultiplexers," in *Proc. 1986 IEEE ICASSP*, Tokyo, Japan, Apr. 1986, pp. 2567-2570.
[37] —, "A theory of filter banks," in *Proc. EUSIPCO-86*, The Hague, The Netherlands, Sept. 1986, pp. 61-64.
[38] —, "Analysis, synthesis and computational complexity of digital filter banks," Ph.D. dissertation, no. 617, Ecole Polytechnique Fédérale de Lausanne, Switzerland, Apr. 1986.
[39] G. Wackersreuther, "On the design of filters for ideal QMF and polyphase filter banks," *Arch. Elekt. Übertrag*, band 39, Heft 2, pp. 123-130, 1985.
[40] J. W. Woods and S. D. O'Neil, "Subband coding of images," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-34, pp. 1278-1288, Oct. 1986.



Martin Vetterli was born in Switzerland in 1957. He received the Engineer's degree from the Eidgenössische Technische Hochschule Zürich, Switzerland, in 1981, the Master of Science degree from Stanford University, Stanford, CA, in 1982, and the Doctorat ès Science degree from the Ecole Polytechnique Fédérale de Lausanne, Switzerland, in 1986.

In 1981 he worked as a Software Engineer for Siemens, Switzerland; in 1982 he was a Research Assistant with the Computer Science Department in Stanford; and from 1983 to 1986 he was a Research Assistant at the Ecole Polytechnique. In the Fall of 1984 he was a temporary member of the Technical Staff at AT&T Bell Laboratories in Holmdel, NJ, working on VLSI design. Since October 1986 he has been a Research Associate with Columbia University's Center for Telecommunications Research in New York, NY. His research interests include algorithm design for VLSI, computational complexity, and multirate signal processing.

Dr. Vetterli was the recipient of the 1984 Best Paper Award from the European Signal Processing Society (EURASIP) and of the 1986 Brown Bovery Corporation (Switzerland) research prize.