performance for rates close to the channel capacity. However, for low rates the sliding window approach exceeds Hashimoto's error exponent. Comparing this to the block coding case the result may be unexpected, because in the block code case all decision feedback error exponents are smaller or equal to Forney's. In fact, Hashimoto's result is directly related to Forney's through the so-called *inverse concatenation construction* [15]. The improvements compared with Hashimoto's bound can be explained due to the different code ensembles under consideration. We considered unit memory codes while Hashimoto considered ordinary (multimemory) convolutional codes. Yet, it was shown by Thommesen and Justesen [8] that unit memory codes have better distance properties and better error exponents with maximum-likelihood decoding. Hence, Hashimoto's composite scheme could possibly be improved by using unit memory codes, however with the expense of a much higher decoding complexity. The complexity of Hashimoto's scheme increases like the decoding complexity of Viterbi decoding with exponential order $2^{Rn+\nu} = 2^{Rn(m+1)}$, where $\nu = km = Rnm$ denotes the overall constraint length. For ordinary convolutional codes with a fixed value of Rn the complexity is determined by the term 2^{ν}. For Viterbi decoding of unit memory codes we have a complexity order $2^{Rn+\nu} = 2^{2\nu}$. On the other hand, if we employ bounded distance decoding as in [17] (cf. [20]) the decoding complexity with Algorithm 1 increases asymptotically like $2^{Rn(1-C(\epsilon))} = 2^{\nu(1-C(\epsilon))}$, where $C(\epsilon) = 1 - h_2(\epsilon)$ is the channel capacity of the binary symmetrical channel with crossover probability ϵ. The reduction in complexity depends on the channel. In the worst case, that is for rates close to capacity, the complexity order is $2^{\nu(1-R)}$. This complexity order is for all rates $R > 0$ smaller than the complexity order of Viterbi decoding with ordinary convolutional codes.

References

[1] T. Hashimoto and M. Taguchi, "Performance of explicit error detection and threshold decision in decoding with erasures," *IEEE Trans. Inf. Theory*, vol. 43, pp. 1650–1655, Sep. 1997.

[2] T. Hashimoto, "Composite scheme LR + Th for decoding with erasures and its effective equivalence to Forney's rule," *IEEE Trans. Inf. Theory*, vol. 45, pp. 78–93, Jan. 1999.

[3] H. Yamamoto and K. Itoh, "Viterbi decoding algorithm for convolutional codes with repeat request," *IEEE Trans. Inf. Theory*, vol. IT-26, pp. 540–547, Sep. 1980.

[4] G. D. Forney Jr, "Exponential error bounds for erasure, list, and decision feedback schemes," *IEEE Trans. Inf. Theory*, vol. IT-14, pp. 206–220, Mar. 1968.

[5] J. M. Wozencraft and I. M. Jacobs, *Principles of Communication Engineering*. New York: Wiley, 1965.

[6] L. N. Lee, "Real-time minimal-bit-error probability decoding of convolutional codes," *IEEE Trans. Commun.*, vol. COM-22, pp. 146–151, Feb. 1974.

[7] ——, "Short unit-memory byte-oriented binary convolutional codes having maximum free distance," *IEEE Trans. Inf. Theory*, vol. IT-22, pp. 349–352, May 1976.

[8] C. Thommesen and J. Justesen, "Bounds on distances and error exponents of unit memory codes," *IEEE Trans. Inf. Theory*, vol. IT-29, pp. 637–649, Sep. 1983.

[9] K. Abdel-Ghaffar, R. J. McEliece, and G. Solomon, "Some partial-unit-memory convolutional codes," in *JPL TDA Progr. Rep.*, 1991.

[10] S. Höst, R. Johannesson, K. Zigangirov, and V. Zyablov, "Active distances for convolutional codes," *IEEE Trans. Inf. Theory*, vol. 45, pp. 658–669, Mar. 1999.

[11] S. Shamai and I. Sason, "Variations on the Gallager bounds, connections, and applications," *IEEE Trans. Inf. Theory*, vol. 48, pp. 3029–3051, Dec. 2002.

[12] R. Johannesson and K. S. Zigangirov, *Fundamentals of Convolutional Coding*. Piscataway, NJ: IEEE, 1999.

[13] E. L. Blokh and V. V. Zyablov, *Linear Concatenated Codes* (in Russian). Moscow, U.S.S.R.: Nauka, 1982.

[14] A. J. Viterbi, "Error bounds for convolutional codes and an asymptotically optimum decoding algorithm," *IEEE Trans. Inf. Theory*, vol. IT-13, pp. 260–269, Apr. 1967.

[15] G. D. Forney Jr, "Convolutional codes II: Maximum likelihood decoding," *Inf. Contr.*, vol. 25, pp. 222–266, Jul. 1974.

[16] F. J. MacWilliams and N. J. A. Sloane, *The Theory of Error Correcting Codes*. Amsterdam, The Netherlands: North-Holland, 1977.

[17] J. Freudenberger and V. Zyablov, "On the complexity of suboptimal decoding for list and decision feedback schemes," *J. Appl. Discrete Mathematics*, vol. 154, pp. 294–304, Feb. 2006.

[18] J. B. Anderson, "Limited search trellis decoding of convolutional codes," *IEEE Trans. Inf. Theory*, vol. 35, pp. 944–955, Sep. 1989.

[19] T. Hashimoto, "A list-type reduced-constraint generalization of the Viterbi algorithm," *IEEE Trans. Inf. Theory*, vol. 33, pp. 866–876, Nov. 1987.

[20] J. Freudenberger, *Bounded Distance Decoding and Decision Feedback*. Düsseldorf, Germany: VDI Verlag, 2004.

Sampling and Exact Reconstruction of Bandlimited Signals With Additive Shot Noise

Pina Marziliano, *Member, IEEE*, Martin Vetterli, *Fellow, IEEE*, and Thierry Blu, *Member, IEEE*

***Abstract*—In this correspondence, we consider sampling continuous-time periodic bandlimited signals which contain additive shot noise. The classical sampling scheme does not perfectly recover these particular nonbandlimited signals but only reconstructs a lowpass filtered approximation. By modeling the shot noise as a stream of Dirac pulses, we first show that the sum of a bandlimited signal with a stream of Dirac pulses falls into the class of signals that contain a finite rate of innovation, that is, a finite number of degrees of freedom. Second, by taking into account the degrees of freedom of the bandlimited signal in the sampling and reconstruction scheme developed previously for streams of Dirac pulses, we derive a sampling and perfect reconstruction scheme for the bandlimited signal with additive shot noise.**

***Index Terms*—Annihilating filters, degrees of freedom, Dirac pulses, nonbandlimited, rate of innovation, sampling, shot noise.**

I. Introduction

Sampling of bandlimited signals has been a subject of interest for more than half a century [1], [5]. The classical sampling theorem states that a continuous-time signal $x(t)$ bandlimited to $[-\omega_m, \omega_m]$ is uniquely represented by a uniform set of samples $x[n] = x(nT)$ taken T seconds apart with $T \leq \pi/\omega_m$, that is, the sampling rate is greater than or equal to the bandwidth of the signal. If the bandlimited signal contains additive shot noise as illustrated in Fig. 1(a) then it is no longer bandlimited (see Fig. 1(b)).

Manuscript received April 28, 2002; revised February 6, 2006. The work of P. Marziliano was supported by the Swiss National Science Foundation.

P. Marziliano was with the Laboratory for Audio Visual Communication (LCAV) at the Swiss Federal Institute of Technology, Lausanne, Switzerland. She is now with the Division of Information Engineering in the School of Electrical and Electronic Engineering at the Nanyang Technological University, Singapore 639798, Singapore (e-mail: epina@ntu.edu.sg).

M. Vetterli is with the Laboratory for Audio Visual Communication in the Faculty of Information and Communication, the Swiss Federal Institute of Technology (EPFL), Lausanne CH-1015, Switzerland and also with the Electrical Engineering and Computer Science Department, University of California at Berkeley, Berkeley, CA USA 94720 (e-mail: martin.vetterli@epfl.ch).

T. Blu is with the Biomedical Imaging Group in the Institute of Applied Optics, the Swiss Federal Institute of Technology (EPFL), Lausanne CH-1015, Switzerland (e-mail: thierry.blu@epfl.ch).

Communicated by G. Battail, Associate Editor At Large.
Digital Object Identifier 10.1109/TIT.2006.872844

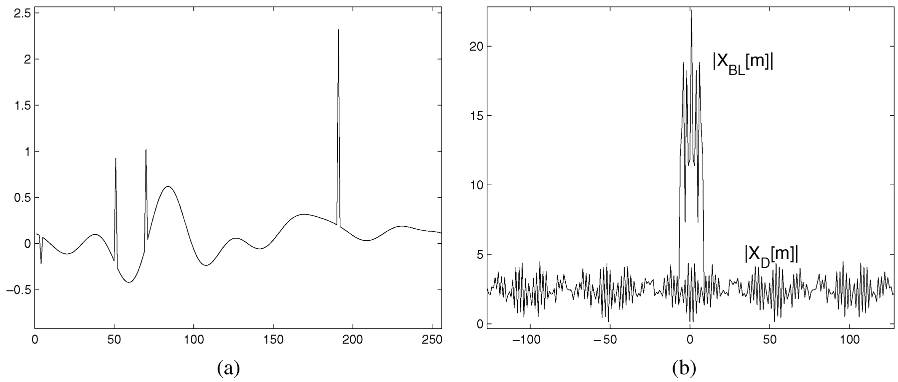

Fig. 1. (a) Periodic bandlimited signal with additive shot noise, $x(t) = x_{\rm BL}(t) + x_{\rm D}(t), \tau = 256, L = 7, K = 4$; (b) Spectrum of periodic bandlimited signal with additive shot noise, $|X[m]| = |X_{\rm BL}[m] + X_{\rm D}[m]|$.

The classical sampling theorem cannot be applied anymore, and the standard practice is to filter the noisy data with a sinc function of bandwidth $2\omega_m$, which yields only an approximation of the initial bandlimited signal. Thus a new sampling and perfect reconstruction scheme must be developed.

In [4], [7], [8] sampling theorems for particular nonbandlimited signals, namely discrete-time Kronecker pulses and continuous-time streams of Dirac pulses, nonuniform splines and piecewise polynomial signals, were given. All of these signals belong to a certain class of signals which have a finite rate of innovation ρ, where the rate of innovation is defined as the number of degrees of freedom per unit of time. Bandlimited signals also have a finite rate of innovation, $\rho = \frac{\omega_m}{\pi}$. Thus it follows that the sum of a bandlimited signal with a stream of Dirac pulses also belongs to the class of signals with a finite rate of innovation. A sampling and perfect reconstruction scheme for such a signal will be exhibited in this paper.

The paper is organized as follows. Section II recalls the sampling and exact reconstruction method for periodic streams of Dirac pulses as given in [6]; Section III presents a sampling theorem for the sum of a bandlimited signal and streams of Dirac pulses and to conclude we mention that the results can be extended to the sum of bandlimited signals with piecewise polynomials in Section IV.

II. Periodic Streams of Dirac Pulses

Shot noise can be modeled as a stream of Dirac pulses. Thus to remove shot noise from a bandlimited signal we first show how to sample and perfectly reconstruct a periodic stream of Dirac pulses.

Consider the signal, $x_{\rm D}(t)$, which is a stream of K weighted Dirac pulses periodized at τ

$$x_{\rm D}(t) = \sum_{n\in\mathbb{Z}} c_n\,\delta(t-t_n) \tag{1}$$

where $t_{n+K} = t_n + \tau$ and $c_{n+K} = c_n$

$$= \sum_{k=0}^{K-1} c_k \sum_{n\in\mathbb{Z}} \delta(t-t_k-n\tau)$$

$$= \sum_{k=0}^{K-1} c_k\,\frac{1}{\tau}\sum_{m\in\mathbb{Z}} e^{i(2\pi m(t-t_k)/\tau)} \tag{2}$$

from Poisson's summation formula

$$= \sum_{m\in\mathbb{Z}} X_{\rm D}[m]\,e^{i2\pi mt/\tau} \tag{3}$$

where

$$X_{\rm D}[m] = \frac{1}{\tau}\int_0^\tau x_{\rm D}(t)e^{-i\frac{2\pi mt}{\tau}}\,dt = \frac{1}{\tau}\sum_{k=0}^{K-1} c_k\,e^{-i\frac{2\pi mt_k}{\tau}},\ m\in\mathbb{Z} \tag{4}$$

are the corresponding continuous-time Fourier series (CTFS) which completely define the periodic signal $x_{\rm D}(t)$. From (4), the number of degrees of freedom, in one period τ, is K from the locations $\{t_k\}_{k=0,\ldots,K-1}$ and K from the weights $\{c_k\}_{k=0,\ldots,K-1}$ therefore the rate of innovation of $x_{\rm D}(t)$ is $\rho = 2K/\tau$. In the proof of the following theorem we show that $2K$ contiguous CTFS $X_{\rm D}[m]$ are sufficient to perfectly reconstruct the periodic stream of Dirac pulses. We will also show that these $2K$ contiguous CTFS can be obtained from a uniform set of samples of the lowpass approximation of the periodic stream of Dirac pulses.

The sampling and reconstruction method for bandlimited signals with additive shot noise will be based on this theorem and it is thus worthwhile for the reader to go through the proof.

Theorem 1: Consider a τ–periodic stream of K weighted Dirac pulses $x_{\rm D}(t)$ as defined in (1) with rate of innovation $\rho = \frac{2K}{\tau}$. Consider a sinc[1] sampling kernel $h_B(t) = B\,\mathrm{sinc}(Bt)$ with bandwidth $2B\pi$ where B is greater than or equal to the rate of innovation ρ, $B \geq \rho$. If the lowpass filtered signal, $y(t) = (h_B * x_{\rm D})(t)$ is sampled at N uniform locations $t = nT, n = 0,\ldots,N-1$, where $T = \frac{\tau}{N}$, $N \geq 2M+1$ and $M = \lfloor\frac{B\tau}{2}\rfloor$, then the samples of the uniform set

$$y(nT) = y_n = < h_B(t-nT), x_{\rm D}(t) >, \quad n = 0,\ldots,N-1 \tag{5}$$

where $< \cdot,\cdot >$ denotes the inner product,[2] are sufficient to perfectly reconstruct $x_{\rm D}(t)$.

Proof: The proof is done in two steps. First, we must show that $N = 2M+1$ samples $y_n, n = 0,\ldots,N-1$, are sufficient to determine the CTFS coefficients $X_D[m], m \in [-M, M]$ and second we show that $X_{\rm D}[m], m\in[-M,M]$ are sufficient to determine the K locations $\{t_k\}_{k=0,\ldots,K-1}$ and K weights $\{c_k\}_{k=0,\ldots,K-1}$ of the signal $x_{\rm D}(t)$.

Letting $B = \rho = 2K/\tau$ then $M = \lfloor\frac{B\tau}{2}\rfloor = K$ and substitute (3) into (5), we obtain

$$y_n = \sum_{m\in\mathbb{Z}} X_{\rm D}[m] < h_B(t-nT), e^{i\frac{2\pi mt}{\tau}} > \tag{6}$$

$$= \sum_{m\in\mathbb{Z}} X_{\rm D}[m]\,H_B\left(\frac{2\pi m}{\tau}\right)e^{i\frac{2\pi mnT}{\tau}} \tag{7}$$

$$= \sum_{m=-K}^{K} X_{\rm D}[m]\,e^{i\frac{2\pi mn}{N}} \tag{8}$$

where $H_B(\omega) = \mathrm{Rect}(\frac{\omega}{2\pi B}) = \begin{cases}1 & \text{if } |\omega| \leq \pi B\\ 0 & \text{else}\end{cases}$ is the Fourier transform of $h_B(t)$. This Vandermonde system of equations is invert-

[1]The sinc definition used here is $\mathrm{sinc}(t) = \sin(\pi t)/\pi t$.

[2]Note that the inner product is defined by

$$< f(t), g(t) >= \int_{-\infty}^{\infty} f(t)g^*(t)\,dt.$$

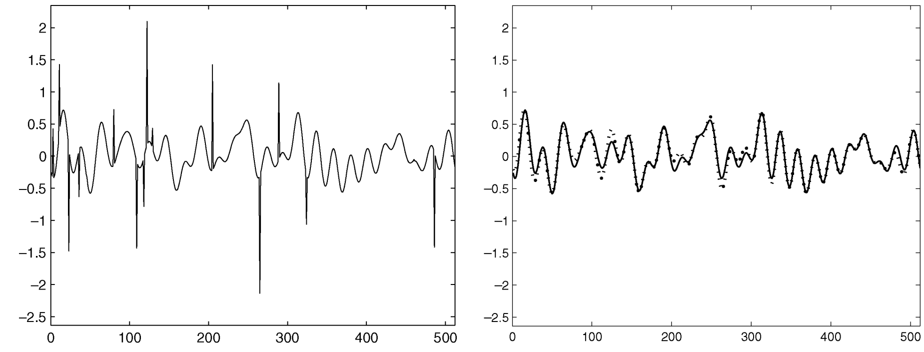

Fig. 2. (a) Periodic bandlimited signal with additive shot noise, $x(t) = x_{\rm BL}(t) + x_{\rm D}(t), \tau = 512, L = 25, K = 16$; (b) Reconstructed bandlimited signal after filtering $x(t)$ with a sinc filter of bandwidth $2L+1$ (dotted line), relative MSE $= 0.2452$, and our method (solid line) relative MSE $= 6.0734 \times 10^{-6}$.

ible since the $N = 2M+1$ equations are of maximal rank $2M+1$. Solving the linear system of equations in (8) will lead us to the $2K+1$ contiguous CTFS $X_{\rm D}[m], m = -K, \ldots, K$ of which $2K$ contiguous values are sufficient. Consider a filter $A[m]$ whose z–transform has K zeros at $u_k = e^{-i\frac{2\pi t_k}{\tau}}$, that is

$$A(z) = \prod_{k=0}^{K-1} (1 - u_k\, z^{-1}). \tag{9}$$

Since the CTFS of the signal $x_{\rm D}(t)$ is a linear combination of K complex exponentials u_k, that is

$$X_{\rm D}[m] = \frac{1}{\tau} \sum_{k=0}^{K-1} c_k\, u_k^m \tag{10}$$

it follows that $A[m]$ is an annihilating filter[3] and satisfies the following condition:

$$A[m] * X_{\rm D}[m] = 0. \tag{11}$$

The coefficients of the annihilating filter are found solving (11) which is equivalent to the following Toeplitz linear system of equations:

$$\sum_{k=0}^{K} A[k]\, X_{\rm D}[m-k] = 0, m = -K, \ldots, K. \tag{12}$$

Thus once the annihilating filter coefficients $A[m], m = 0, \ldots, K$, are found, the locations $\{t_k\}_{k=0,\ldots,K-1}$ of the Dirac pulses are given by the roots of $A(z)$ and the associated weights $\{c_k\}_{k=0,\ldots,K-1}$ are obtained by solving the Vandermonde linear system of equations in (10) for $m = 1, \ldots, K$.

□

III. Bandlimited Signals With Additive Shot Noise

Bandlimited signals with additive shot noise will be modeled as the sum of a bandlimited signal with a stream of Dirac pulses. In the previous section it was shown that a periodic stream of K Dirac pulses can be perfectly reconstructed from $2K$ contiguous CTFS coefficients that were obtained from a uniform set of samples of the lowpass filtered approximation of the stream of Dirac pulses. In a similar way, next we present a sampling and perfect reconstruction method for bandlimited signals with additive shot noise.

Consider a τ-periodic signal, $x(t)$, defined as the sum of a τ-periodic L-bandlimited signal, $x_{\rm BL}(t)$, with a τ-periodic stream of K Dirac pulses $x_{\rm D}(t)$, that is

$$x(t) = x_{\rm BL}(t) + x_{\rm D}(t) \tag{13}$$

[3]This is also known as the error locator polynomial in error correction coding [2].

where an L-bandlimited signal $x_{\rm BL}(t)$ is such that its CTFS $X_{\rm BL}[m] = 0, \forall m \notin [-L, L]$ and the stream of K Dirac pulses, $x_{\rm D}(t)$, is defined in (1). The corresponding CTFS coefficients of the bandlimited signal with additive shot noise $x(t)$ are defined by

$$X[m] = \begin{cases} X_{\rm BL}[m] + X_{\rm D}[m] & \text{if } m \in [-L, L] \\ X_{\rm D}[m] & \text{if } m \notin [-L, L] \end{cases} \tag{14}$$

and the rate of innovation is given by

$$\rho = \frac{2L+1+2K}{\tau} \tag{15}$$

where $(2L+1)/\tau$ and $2K/\tau$ are the number of degrees of freedom of the bandlimited signal and the stream of K Dirac pulses, respectively.

Similarly to Section II the bandlimited signal with additive shot noise, $x(t)$, will be reconstructed from a contiguous set of its CTFS coefficients, $X[m], m \in [-M, M]$, that will be obtained from a uniform set of samples of the low-pass approximation of the bandlimited with additive shot noise signal, $x(t)$. Recall that the periodic stream of K Dirac pulses $x_{\rm D}(t)$ is perfectly recovered from any $2K$ contiguous frequency values $X_{\rm D}[m]$. Since the CTFS coefficients of the bandlimited signal, $X_{\rm BL}[m]$, are equal to zero outside of the band $[-L, L]$, it follows that the CTFS coefficients of the bandlimited signal with additive shot noise, $X[m]$, outside of the band $[-L, L]$ are exactly equal to the CTFS coefficients of the stream of K Dirac pulses, that is

$$X[m] = X_{\rm D}[m], m \notin [-L, L]. \tag{16}$$

Therefore, in order to recover the stream of Dirac pulses it is sufficient to take $2K$ contiguous CTFS coefficients $X[m]$ outside of the band $[-L, L]$, for instance in $[L+1, L+2K]$. Once we have the CTFS of the signal $X[m]$, with $m \in [-2K-L, L+2K]$ then the CTFS coefficients of the bandlimited signal are obtained by subtracting $X_{\rm D}[m]$ from $X[m]$ for $m \in [-L, L]$ and thus perfectly recovering the bandlimited signal $x_{\rm BL}(t)$. The next theorem follows from the result in Theorem 1.

Theorem 2: Consider a τ–periodic L–bandlimited signal added to a periodic stream of K weighted Dirac pulses, $x(t) = x_{\rm BL}(t) + x_{\rm D}(t)$ with rate of innovation $\rho = \frac{2L+1+2K}{\tau}$. Take a sinc sampling kernel $h_B(t) = B\,\text{sinc}(Bt)$ such that

$$B \geq \rho = \frac{2L+1+2K}{\tau}. \tag{17}$$

If the low-pass filtered signal $y(t) = (h_B * x)(t)$ is sampled at N uniform locations $t = nT, n = 0, \ldots, N-1$, where $T = \frac{\tau}{N}$, $N \geq 2M+1$ and $M = \lfloor \frac{B\tau}{2} \rfloor$, then the samples of the uniform set

$$y(nT) = y_n = < h_B(t - nT), x(t) >, \quad n = 0, \ldots, N-1 \tag{18}$$

are sufficient to perfectly reconstruct $x(t)$.

Proof: The first step of the proof is exactly the same as the one in Theorem 1.

Let $B = \frac{2(L+2K)}{\tau}$ then $M = \lfloor \frac{B\tau}{2} \rfloor = L + 2K$ and similarly to (8) we solve the following Vandermonde system of equations:

$$y_n = \sum_{m=-(L+2K)}^{L+2K} X[m]\, e^{i\frac{2\pi mn}{N}} \tag{19}$$

which will lead to the CTFS coefficients $X[m], m \in [-(L+2K), L+2K]$.

Second, we show that $X[m], m \in [-(L+2K), L+2K]$ are sufficient[4] to reconstruct the bandlimited with additive shot noise signal $x(t)$. Consider the spectral values $X[m], m \in [-(L+2K), L+2K]$ then we have

$$X[m] = \begin{cases} X_{\rm BL}[m] + X_{\rm D}[m] & m \in [-L, L] \\ X_{\rm D}[m] & m \in [L+1, L+2K] \end{cases}. \tag{20}$$

From Theorem 1 the periodic stream of K Dirac pulses $x_{\rm D}(t)$ is perfectly recovered from $2K$ contiguous CTFS $X_{\rm D}[m] = X[m], m \in [L+1, L+2K]$. Then, the $2L+1$ spectral components of the bandlimited signal are given by

$$X_{\rm BL}[m] = X[m] - X_{\rm D}[m], m \in [-L, L] \tag{21}$$

which uniquely defines the bandlimited signal $x_{\rm BL}(t)$. Thus the bandlimited signal with additive shot noise $x(t) = x_{\rm BL}(t) + x_{\rm D}(t)$ is perfectly recovered. □

As an example of application of Theorem 2, we show how to recover a bandlimited signal corrupted by additive shot noise and compare our method to the standard approach. Consider a $\tau = 512$-periodic bandlimited signal with bandwidth $2L+1$, where $L = 25$ with additive shot noise made up of $K = 16$ Dirac pulses as illustrated in Fig. 2(a). Fig. 2(b) illustrates the reconstructed bandlimited signal after filtering the noisy data with a sinc function of bandwidth $2L+1$ (dotted line) and our reconstruction scheme (solid line). The relative mean squared errors are $= 0.2452$ and 6.0734×10^{-6}, respectively.

IV. Conclusion

As shown in [6], Theorem 1 can be extended to continuous-time periodic nonuniform splines and piecewise polynomials. These results could be further generalized to the sum of bandlimited and nonuniform splines or piecewise polynomial signals by modifying the condition in (17). In [3], these results were extended to the sum of bandlimited and nonuniform spline signals and applied to compression of ECG signals. We are currently investigating sampling and reconstruction of other electrobiomedical signals, for example EEG and ENG signals.

Acknowledgment

The authors would like to thank the reviewers and, in particular, the Associate Editor whose remarks improved the quality of this manuscript.

References

[1] *Modern Sampling Theory: Mathematics and Applications*, J. J. Benedetto and P. J. S. G. Ferreira, Eds., Birkhäuser, Boston, 2001.
[2] R. E. Blahut, *Theory and Practice of Error Control Codes*. Reading, MA: Addison-Wesley, 1983.
[3] Y. Hao, P. Marziliano, M. Vetterli, and T. Blu, "Compression of ECG as signal with finite rate of innovation," in *Proc. IEEE 27th Annu. Int. Conf. Eng., Med. Biol.*, Shanghai, China, Sep. 2005.
[4] P. Marziliano, "Sampling Innovations," Ph.D. dissertation, Swiss Federal Institute of Technology, Audiovisual Communications Laboratory, DSC, EPFL, Lausanne, Switzerland, 2001.
[5] M. Unser, "Sampling–50 years after Shannon," *Proc. IEEE*, vol. 88, pp. 569–587, Apr. 2000.
[6] M. Vetterli, P. Marziliano, and T. Blu, "Sampling discrete-time piecewise bandlimited signals," in *Proc. Sampl. Theory Appl. Workshop*, Orlando, FL, May 2001, pp. 97–102.
[7] ——, "A sampling theorem for periodic piecewise polynomial signals," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Processing*, Salt Lake City, UT, May 2001.
[8] ——, "Sampling signals with finite rate of innovation," *IEEE Trans. Signal Process.*, pp. 1417–1428, Jun. 2002.

[4]Note that it would be sufficient to have the values $PX[m], m \in [-L, L+2K]$. For simplicity, we work with a symmetric spectrum, or $X[m], m \in [-L-2K, L+2K]$, thus sampling above the rate of innovation by a factor of $2K/\tau$. Usually, $L \gg K$, so this is negligible, otherwise, a more complex, critical sampling can be developed.

Explicit Correlation Coefficients Among Random Variables, Ranks, and Magnitude Ranks

Jinsoo Bae, *Senior Member, IEEE*,
Hyoungmoon Kwon, *Student Member, IEEE*,
So Ryoung Park, *Member, IEEE*, Jumi Lee, *Student Member, IEEE*,
and Iickho Song, *Senior Member, IEEE*

***Abstract*—In this correspondence, we address the derivation of joint distributions and correlation coefficients for seven pairs of statistics used commonly in a number of signal detection schemes. The upper and lower bounds of the correlation coefficients are obtained, and relationships among the correlation coefficients are derived. Explicit values of the correlation coefficients evaluated for some specific noise distributions are shown for easy reference.**

***Index Terms*—Correlation coefficient, magnitude, magnitude rank, rank, sign.**

I. Introduction

A statistic is to convey (hopefully desirable) information from original observations. It is well-known that using only partial information via statistics is sometimes beneficial for signal detection. Clearly, several statistics have been used and explored for signal detection in a large volume of investigations, where it has been shown that signal detectors

Manuscript received February 13, 2005; revised February 2, 2006. This work was supported by the Ministry of Science and Technology under the National Research Laboratory Program of Korea Science and Engineering Foundation. The material in this paper was presented in part at the International Symposium on Intelligent Signal Processing and Communication Systems, Seoul, Korea, November 2004.

J. Bae is with the Department of Information and Communication Engineering, Sejong University, Seoul 143-747, Korea (e-mail: baej@sejong.ac.kr).
H. Kwon, J. Lee, and I. Song are with the Department of Electrical Engineering and Computer Science, Korea Advanced Institute of Science and Technology, Daejeon 305-701, Korea (e-mail: kwon@sejong.kaist.ac.kr; jmlee@sejong.kaist.ac.kr; i.song@ieee.org).
S. R. Park is with the School of Information, Communications, and Electronics Engineering, the Catholic University of Korea, Bucheon 420-743, Korea (e-mail: srpark@catholic.ac.kr).

Communicated by X. Wang, Associate Editor for Detection and Estimation.
Digital Object Identifier 10.1109/TIT.2006.872852