DOMAIN DECOMPOSITION METHODS FOR THE COUPLING OF SURFACE AND GROUNDWATER FLOWS

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To Luisa and Giulio,
MariaCarla and GianPaolo

Abstract

The purpose of this thesis is to investigate, from both the mathematical and numerical viewpoint, the coupling of surface and porous media flows, with particular concern on environmental applications.

Domain decomposition methods are applied to set up effective iterative algorithms for the numerical solution of the global problem. To this aim, we reformulate the coupled problem in terms of an interface (Steklov-Poincaré) equation and we investigate the properties of the Steklov-Poincaré operators in order to characterize optimal preconditioners that, at the discrete level, yield convergence in a number of iterations independent of the mesh size h.

We consider a new approach to the classical Robin-Robin method and we reinterpret it as an alternating direction iterative algorithm. This allows us to characterize robust preconditioners for the linear Stokes/Darcy problem which improve the behaviour of the classical Dirichlet-Neumann and Neumann-Neumann ones. Several numerical tests are presented to assess the convergence properties of the proposed algorithms.

Finally, the nonlinear Navier-Stokes/Darcy coupling is investigated and a general nonlinear domain decomposition strategy is proposed for the solution of the interface problem, extending the usual Newton or fixed-point based algorithms.

Version abrégée

L'objet de cette thèse est l'étude du point de vue mathématique et numérique du couplage d'un écoulement fluide de surface et d'un écoulement en milieu poreux, pour des applications à l'environnement.

Les méthodes de décomposition de domaines sont appliquées afin de caracteriser des algorithmes itératifs efficaces pour résoudre numériquement le problème global. Pour cela, nous écrivons le problème couplé sous la forme d'une équation d'interface (dite de Steklov-Poincaré) et nous étudions les propriétés des opérateurs de Steklov-Poincaré afin de définir des préconditionneurs optimaux qui, au niveau discret, garantissent une vitesse de convergence indépendante de la taille h du maillage considéré.

Nous nous proposons de réinterpreter la méthode classique de Robin-Robin comme un algorithme des directions alternées (*alternating direction iterative method*). Ceci nous permet de caracteriser un préconditionneur robuste pour le problème linéaire de Stokes/Darcy, qui améliore le comportement des préconditionneurs de type Dirichlet-Neumann et Neumann-Neumann. Des résultats numériques sont présentés pour valider les propriétés de convergence de la méthode proposée.

Finalement, nous étudions le problème non linéaire de Navier-Stokes/Darcy et nous proposons un cadre général des possibles méthodes itératives issues de la décomposition de domaines qui étendent les méthodes de Newton et du point fixe.

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I wish them all the best.

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If anyone supposes he knows something,
he does not yet know as he ought to know.
(St. Paul, 1Cor 8,2)

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Introduction

The filtration of fluids through porous media is a very interesting subject with relevant applications. To quote some examples, these phenomena occur in physiology like the filtration of blood through vessel walls, in industrial processes involving e.g. air or oil filters, in the environment concerning the waters of an hydrological basin which can percolate through rocks and sand.
In this thesis we address this last application from both a mathematical and a numerical viewpoint.
Our computational domain will be a region naturally split into two parts: one occupied by the fluid, the other by the porous media, as represented in Fig. 0.1. In each subregion we consider two different mathematical models, typically the Navier-Stokes and Darcy equations in Ω_f and Ω_p, respectively. These equations, linked through suitably chosen conditions which describe the fluid flows across the upper surface of the porous media Γ (hereafter called *interface*), give rise to a global differential heterogenous model.

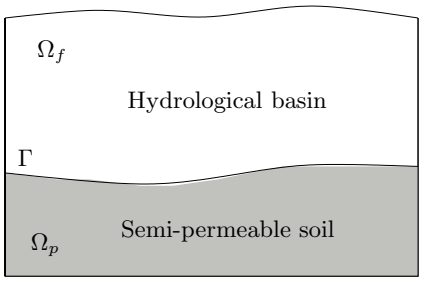


Fig. 0.1. Computational domain configuration.

A Galerkin discretization of this coupled problem based e.g. on conforming, or mixed finite elements, or discontinuous Galerkin methods would lead to represent it as a linear system with a large, sparse and ill-conditioned matrix, that requires an effective preconditioning strategy to be solved using iterative methods.
Moreover, based on the naturally decoupled structure of the fluid-porous media problem, it would be interesting to reduce the size of the global problem by keeping separated the fluid and the porous media parts and exchanging information between surface and groundwater flows only through boundary conditions at the interface. Such a strategy would also permit to reuse existing codes specifically implemented for fluid or groundwater flow simulations.

Domain decomposition methods seem to fulfill both these requests. In fact, the basic idea of a domain decomposition approach is to split the computational domain, say Ω, into $M \geq 2$ subdomains Ω_i, $i = 1, \ldots, M$, such that $\overline{\Omega} = \bigcup_{i=1}^{M} \overline{\Omega}_i$. The intersection $\Omega_i \cap \Omega_j$ $(i \neq j)$ may be empty: if this occurs, we speak of *nonoverlapping* domain decomposition, *overlapping* otherwise. Then, the original problem

$$P(u) = 0 \text{ in } \Omega \tag{0.1}$$

can be reformulated as a family of subproblems of reduced size $P_i(u) = 0$, $i = 1, \ldots, M$, within each subdomain Ω_i. If the differential operator is the same in all subdomains then (0.1) is said to define a *homogeneous* domain decomposition problem, otherwise we say that (0.1) is a *heterogeneous* domain decomposition problem.
Finally, each subproblem is coupled to the others through the values of the unknowns across the interfaces.
This coupling is then removed introducing an iterative scheme among subdomains which permit to recover the solution of the original problem (0.1) by independently solving the subproblems $P_i(u)$ featuring a lower complexity over each subdomain.

In view of our application, let us consider more in details the nonoverlapping (or *Schur*) domain decomposition. In this case, we can associate to the original problem (0.1) an equivalent interface problem which solely involves the unknowns at the interface Γ:

$$S(u_\Gamma) = 0\,. \tag{0.2}$$

S is a pseudo-differential operator called *Steklov-Poincaré operator* and it is composed of local operators S_i (also called *Dirichlet-to-Neumann* maps). In the case of two subdomains ($M = 2$), we can write

$$S = S_1 + S_2, \tag{0.3}$$

where each S_i is associated to the subproblem $P_i(u)$ in Ω_i and it inherits the properties of the differential operator which models $P_i(u)$.
Steklov-Poincaré operators are named in this way since the pioneering work of Agoshkov and Lebedev in the years 1981-1983 (see, e.g., [AL85, Ago88, AL90a, AL90b]), while addressing iterative methods for solving the interface problem (0.2). More precisely, they have introduced the inverse operators S_i^{-1} and called them Poincaré-Steklov operators.
Indeed, we are interested in using iterative methods to solve (0.2) since they would require to compute at each step k the application of S to a given value u_Γ^k. Then, owing to (0.3), this would imply to apply independently each S_i, that is to solve separately the subproblems $P_i(u) = 0$ in Ω_i with suitable boundary data on the interface.
In order to increase the convergence rate of the iterative method, we introduce a *preconditioner* or, more generally, a *scaling operator*, say $\mathcal{P}$.
At the stage of choosing $\mathcal{P}$, the analysis of the Steklov-Poincaré operators is crucial to identify a preconditioner which would be spectrally equivalent to S and, therefore, would serve to achieve convergence in a number of steps independent of the physical quantities characterizing S (or,

equivalently, $P(u)$) and, at the discrete level, of the size of the original problem. A preconditioner $\mathcal{P}$ which fulfill these requests will be said *optimal*.
In the literature of homogeneous nonoverlapping domain decomposition, basic classical optimal preconditioners are the so-called Neumann-Neumann preconditioners which involve a weighted combination of inverses of the local Steklov-Poincaré operators. For the case of two subdomains we have:

$$\mathcal{P}^{-1} = \theta_1 S_1^{-1} + \theta_2 S_2^{-1}$$

with at least one $\theta_i \neq 0$ (see [SBG96, QV99, TW04]).
Concerning heterogeneous domain decomposition, the issue of characterizing optimal preconditioners is not straightforward because of the difficulty of finding a preconditioning operator which can account for the different models defined in each subdomain.
An overview on the literature about heterogeneous domain decomposition can be found, e.g., in [QV99] chapter 8.

The aim of this work is to interpret the coupled fluid-porous media problem in terms on an interface equation analogous to (0.2) and to study it in order to devise effective solution strategies for its numerical approximation. In particular, we would like to characterize ad-hoc preconditioners with optimality properties not only with respect to the grid size h, but also to the relevant physical quantities of this problem, such as the fluid viscosity and the conductivity of the porous media.
To our knowledge this thesis constitutes the first attempt in literature to carry out such an analysis. In fact, although the mathematical and numerical analysis of the single fluid and porous media problems are well-developed and the literature on their numerical approximation is really broad, the study of the global model is far less standard due to its intrinsic complexity. The first rigorous mathematical studies are those by Jäger and Mikelić [JM96, JM00, JMN01] concerning the characterization of proper coupling conditions, but well-posedness analysis of the linear Stokes/Darcy coupling and possible discrete approximations can be found only in the very recent works [BH02, DMQ02, DQ03, DQ04, LSY03, MQS03, Riv03, RY03].
Hints to iterative substructuring methods for the numerical solution of the linear fluid-porous media problem can be found also in the works by I. Yotov and B. Rivière, however a thorough investigation of such methods based on the domain decomposition theory is presented only in [DQ03, DQ04].
Finally, we point out that no result about the nonlinear coupled problem involving the full Navier-Stokes equations in the fluid domain has been published yet.

Thesis Outline

In **chapter 1** we introduce the setting of the coupled Navier-Stokes/Darcy problem. In particular, we discuss the issue of matching conditions between the two subproblems and we briefly present the approach based on homogenization theory that has been used in the literature to derive them.

Chapter 2 is devoted to the analysis of the linear Stokes/Darcy problem. We address the well-posedness of this coupled model and then we rewrite it in terms of Steklov-Poincaré interface problems. We consider two possible choices of the interface variables and we analyze the related pseudo-differential operators. This allows us to characterize in both cases two optimal preconditioners to solve the interface problems.

A Galerkin finite element approximation of the coupled problem is presented in **chapter 3**, taking conforming grids across the interface. We guarantee the existence and uniqueness of the discrete solution and we replicate at the discrete stage the analysis developed in chapter 2. In particular, we write the Schur complement systems associated to the vector unknowns at the interface. Then, we define the optimal algebraic preconditioners which can be used in the framework of Krylov type methods. Finally, we set up and analyze two iterative substructuring algorithms to compute the problem solution.

Chapter 4 presents some numerical results obtained using the algorithms devised in chapter 3. We show the optimality properties of Dirichlet-Neumann methods with respect to grid parameters, and we point out some difficulties encountered in handling realistic physical parameters.
In order to study possible improvement strategies, in **chapter 5** we analyze an operator-splitting based method for the solution of the Steklov-Poincaré equation associated to a generic elliptic problem. Precisely, we prove the equivalence between the so-called Robin-Robin method and the alternating direction iterations (ADI) to solve the interface problem, and we discuss the issue of accelerating the convergence rate of this method by suitably chosen relaxation parameters.

Based on this theory, we can set up an ADI method for the Stokes/Darcy coupling which, provided a good choice is made for the acceleration coefficients, yields satisfactory convergence results as illustrated in **chapter 6**.

Finally, **chapter 7** focuses on the nonlinear coupled Navier-Stokes/Darcy problem. We write the nonlinear Steklov-Poincaré equation associated to this coupling and we prove its well-posedness. Then, we set up a domain decomposition framework for its solution which extends the classical fixed point or Newton based algorithms.

1. Coupling of Surface and Subsurface Flow

In this chapter we introduce the Navier-Stokes and Darcy equations that we shall extensively use in the following. In particular, we shall discuss the derivation and physical meaning of these equations and we shall address the issue of finding suitable coupling conditions to describe the filtration processes between free fluids and porous media.

1.1 Introduction

We consider a bounded domain Ω of $\mathbb{R}^d$ $(d = 2, 3)$ composed of two subdomains Ω_f and Ω_p such that $\overline{\Omega} = \overline{\Omega}_f \cup \overline{\Omega}_p$, $\Omega_f \cap \Omega_p = \emptyset$ and $\overline{\Omega}_f \cap \overline{\Omega}_p = \Gamma$. The hypersurface Γ (a line if $d = 2$, a surface if $d = 3$) is the interface separating the domain Ω_f filled by an incompressible fluid from the domain Ω_p formed by a porous medium. We assume that the fluid has a prescribed upper surface; however, at the end of this chapter we shall give some guidelines to extend our considerations to the more general case of free surface fluids. We denote by $\mathbf{n}_f$ the unit outward normal direction on $\partial\Omega_f$, and by $\mathbf{n}_p$ the normal direction on $\partial\Omega_p$, oriented outward. Then $\mathbf{n}_f = -\mathbf{n}_p$ on the interface Γ and we shall indicate $\mathbf{n} = \mathbf{n}_f$ on Γ. Throughout this work we shall always suppose the boundaries $\partial\Omega_f$, $\partial\Omega_p$ to be Lipschitz continuous.
We adopt the Navier-Stokes equations to describe the flow field in the domain Ω_f and Darcy equations in the porous part Ω_p.

1.1.1 The Navier-Stokes Problem

The Navier-Stokes equations provide a model for the flow motion of a homogeneous incompressible Newtonian fluid. A rigorous derivation of Navier-Stokes equations can be found, e.g., in [Gal94]. In the steady case they read:

$$-\nu\triangle\mathbf{u}_f + (\mathbf{u}_f \cdot \nabla)\mathbf{u}_f + \nabla p_f = \mathbf{f} \quad \text{in}\,\Omega_f \tag{1.1}$$

$$\nabla \cdot \mathbf{u}_f = 0 \quad \text{in}\,\Omega_f \tag{1.2}$$

where $\mathbf{u}_f$ denotes the velocity of the fluid, p_f the ratio between its pressure and density ρ_f, $\mathbf{f}$ is the external force field and $\nu > 0$ is the kinematic viscosity.
We have indicated by ∇ the gradient operator for vector functions:

$$(\nabla \mathbf{v})_{ij} = \frac{\partial v_i}{\partial x_j}$$

while $\nabla\cdot$ is the divergence operator:

$$\nabla \cdot \mathbf{v} = \sum_{i=1}^{d} \frac{\partial v_i}{\partial x_i}\,.$$

Finally, $\triangle$ is the Laplace operator

$$(\triangle \mathbf{v})_i = \sum_{j=1}^{d} \frac{\partial^2 v_i}{\partial x_j^2}$$

and

$$(\mathbf{v} \cdot \nabla)\mathbf{w} = \sum_{i=1}^{d} v_i \frac{\partial \mathbf{w}}{\partial x_i}$$

for all vector functions $\mathbf{v} = (v_1, \ldots, v_d)$, $\mathbf{w} = (w_1, \ldots, w_d)$.

Adimensional Form of the Navier-Stokes Equations. After introducing suitable adimensional variables for the velocity and pressure, it is well-known that the Navier-Stokes equations can be rewritten in the adimensional form

$$-\frac{1}{Re_f}\triangle \mathbf{u}_f + (\mathbf{u}_f \cdot \nabla)\mathbf{u}_f + \nabla p_f = \mathbf{f} \quad \text{in } \Omega_f \tag{1.3}$$

$$\nabla \cdot \mathbf{u}_f = 0 \quad \text{in } \Omega_f \tag{1.4}$$

where we have introduced the Reynolds number

$$Re_f = \frac{L_f U_f \rho_f}{\mu} \tag{1.5}$$

L_f being a characteristic length of the domain Ω_f and U_f a characteristic velocity of the fluid, while $\mu = \nu\rho_f$ is the fluid dynamic viscosity. Notice that, for the sake of simplicity, we have used the same notations as in (1.1), (1.2), but all the variables in (1.3), (1.4) are to be intended as adimensional variables.

1.1.2 Darcy Equations

The filtration of an incompressible fluid through porous media is often described using Darcy's law. The latter provides the simplest linear relation between velocity and pressure in the porous domain under the physically reasonable assumption that fluid flows in porous media are usually very slow and all the inertial (nonlinear) terms may be neglected.
Groundwater flows could be treated microscopically by the laws of hydrodynamics if the granular skeleton of the porous medium were a simple geometrical assembly of unconnected tubes. However, the seepage path is tortuous and it branches into a multitude of tributaries. Darcy's law avoids the insurmountable difficulties of the hydrodynamic microscopic picture by introducing a macroscopic concept. In fact, it considers a fictitious flow velocity, the *Darcy velocity* or

specific discharge $\mathbf{q}$ through a given cross section of the porous medium, rather than the true velocity $\mathbf{u}_p$ with respect to the porous matrix:

$$\mathbf{u}_p = \frac{\mathbf{q}}{n} \tag{1.6}$$

with n being the *volumetric porosity*, defined as the ratio between the volume of void space and the total volume of the porous medium.
This simplifying concept was introduced by the nature of Darcy's experiment (see [Dar56]) which only permitted the measurement of averaged hydraulic values from the percolation of water through a column of horizontally stratified beds of sand in a cylindrical pipe.
To introduce Darcy's law, we define a scalar quantity φ called *piezometric head* which essentially represents the fluid pressure in Ω_p:

$$\varphi = z + \frac{p_p}{g} \tag{1.7}$$

where z is the elevation from a reference level, accounting for the potential energy per unit weight of fluid, p_p is the ratio between the fluid pressure in Ω_p and its viscosity ρ_f, and g is the gravity acceleration.
Then, Darcy's law can be written as

$$\mathbf{q} = -\mathsf{K}\nabla\varphi, \tag{1.8}$$

where K is a symmetric positive definite tensor $\mathsf{K} = (\mathsf{K}_{ij})_{i,j=1,\dots,d}$, $\mathsf{K}_{ij} \in L^\infty(\Omega_p)$, $\mathsf{K}_{ij} > 0$, $\mathsf{K}_{ij} = \mathsf{K}_{ji}$, called *hydraulic conductivity tensor*, which depends on the properties of the fluid as well as on the characteristics of the porous medium. In fact, its components are proportional to the *intrinsic permeability* k of the porous medium:

$$\mathsf{K} = \frac{k\rho_f g}{\mu} \tag{1.9}$$

and k is equal to $n\varepsilon^2$ (times a multiplicative adimensional constant), ε being the characteristic length of the pores; then, $\mathsf{K} \propto \varepsilon^2$. The hydraulic conductivity K is therefore a macroscopic quantity characterizing porous media and in table 1.1 we report some typical values that it may assume (see [Bea79]).

<table>
<tr><td>K (m/s): 1.e−</td><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td><td>8</td><td>9</td><td>10</td><td>11</td><td>12</td></tr>
<tr><td>Permeability</td><td colspan="4">Pervious</td><td colspan="4">Semipervious</td><td colspan="5">Impervious</td></tr>
<tr><td rowspan="2">Soils</td><td colspan="2">Clean gravel</td><td colspan="3">Clean sand or sand and gravel</td><td colspan="3">Very fine sand, silt, loam</td><td colspan="5"></td></tr>
<tr><td colspan="4"></td><td colspan="3">Peat</td><td colspan="2">Stratified clay</td><td colspan="5">Unweathered clay</td></tr>
<tr><td>Rocks</td><td colspan="4"></td><td colspan="4">Oil rocks</td><td colspan="2">Sandstone</td><td colspan="2">Good limestone, dolomite</td><td colspan="2">Breccia, granite</td></tr>
<tr><td>k (m^2): 1.e−</td><td>7</td><td>8</td><td>9</td><td>10</td><td>11</td><td>12</td><td>13</td><td>14</td><td>15</td><td>16</td><td>17</td><td>18</td><td>19</td></tr>
</table>

Table 1.1. Typical values of hydraulic conductivity K and permeability k.

Finally, we notice that the hydraulic conductivity tensor K can be diagonalized by introducing three mutually orthogonal axes called *principal directions of anisotopy*. In the following, we will

always suppose that the principal axes are in the x, y and z directions so that the tensor will be considered diagonal: $\mathsf{K} = diag(\mathsf{K}_1, \mathsf{K}_2, \mathsf{K}_3)$.

In conclusion, the motion of an incompressible fluid through a saturated porous medium is described by the following equations:

$$\mathbf{u}_p = -\frac{\mathsf{K}}{n}\nabla\varphi \quad \text{in } \Omega_p \tag{1.10}$$

$$\nabla \cdot \mathbf{u}_p = 0 \quad \text{in } \Omega_p \tag{1.11}$$

where (1.10) is Darcy's law, while (1.11) states the conservation of the mass.

Adimensional Form of Darcy Equations. An adimensional form of Darcy equations may be written as well. If we set for simplicity $z = 0$ in (1.7), we can write

$$\mathbf{u}_p = -\frac{1}{n} \cdot \frac{k\rho_f g}{\mu} \cdot \frac{1}{\rho_f g}\nabla p_p$$

and since $k \simeq n\varepsilon^2$, we have

$$\mathbf{u}_p = -\frac{\varepsilon}{\mu}\nabla p_p \,.$$

After introducing adimensional variables as for the Navier-Stokes case, we obtain the adimensional form of Darcy's law:

$$\mathbf{u}_p = -\delta Re_p \nabla p_p \tag{1.12}$$

(here again we use the same notation for the adimensional and dimensional variables) where

$$\delta = \frac{\varepsilon}{L_p}$$

is the ratio between the pore size (microscale) and the characteristic length of the porous medium (macroscale), while Re_p is the adimensional Reynolds number

$$Re_p = \frac{\varepsilon U_p \rho_f}{\mu}. \tag{1.13}$$

Usually, it is assumed that Darcy's law is valid as long as $Re_p < 10$. For higher Reynolds number, a common extension to Darcy's law is the Forchheimer equation:

$$-\mathsf{K}\nabla\varphi = \mathbf{q} + \beta\rho_f|\mathbf{q}|\mathbf{q}$$

where β is a constant possibly depending on the geometry of the pores (see [For01, Dup63, Gio97, MPM00]), or the Brinkman equation

$$n\mathbf{u}_p + \mu'\mathsf{K}\triangle\mathbf{u}_p = -\mathsf{K}\nabla\varphi$$

where μ' is the so-called *effective porosity* (see [Bri47]).

1.2 Coupling Conditions

We consider now the issue of finding effective coupling conditions across the interface Γ which separates the channel flow and the porous medium. This is a classical problem which has been investigated both from a physical and from a rigorous mathematical point of view.
A mathematical difficulty arises from the fact that we need to couple two different systems of partial differential equations: Darcy equations (1.10), (1.11) are second order for the pressure and first order for the velocity, while in the Navier-Stokes system it is the opposite.
Three conditions are to be prescribed on Γ.

1. The obvious condition to assign at a permeable interface is the continuity of the normal velocity, which is a consequence of the incompressibility.
2. Moreover, a suitable condition relating the pressures of the two fluids across Γ has to be prescribed.
3. Finally, in order to have a completely determined flow of the free fluid, we have to specify some condition on the tangential component for the fluid velocity at the interface.

Concerning 3., a classically used condition for the free fluid is the vanishing of the tangential velocity at the interface. However, this condition, which is correct in the case of a permeable surface, is not completely satisfactory for a permeable interface. Beavers and Joseph proposed a new condition postulating that the difference between the slip velocity of the free fluid and the tangential component of the seepage velocity is proportional to the shear rate of the free fluid (see [BJ67]). They verified this law experimentally and found that the proportionality constant depends linearly on the square root of the permeability. Precisely, the coupling condition that they advocated reads:

$$\boldsymbol{\tau}_j \cdot \frac{\partial \mathbf{u}_f}{\partial \mathbf{n}} = \frac{\alpha_{BJ}}{\sqrt{\mathsf{K}}}(\mathbf{u}_f - \mathbf{u}_p) \cdot \boldsymbol{\tau}_j \qquad \text{on } \Gamma \tag{1.14}$$

where α_{BJ} is a dimensionless constant which depends only on the structure of the porous medium; $\boldsymbol{\tau}_j$ $(j = 1, \ldots, d-1)$ are linear independent unit tangential vectors to the boundary Γ. This experimental coupling condition was further studied by Saffman who pointed out that the velocity $\mathbf{u}_p$ was much smaller than the other quantities appearing in the law of Beavers and Joseph (1.14) and that, in fact, it could be dropped. Therefore, he proposed to consider the interface condition (see [Saf71]):

$$\boldsymbol{\tau}_j \cdot \frac{\partial \mathbf{u}_f}{\partial \mathbf{n}} = \frac{\alpha_{BJ}}{\sqrt{\mathsf{K}}}\mathbf{u}_f \cdot \boldsymbol{\tau}_j + O(\sqrt{\mathsf{K}}) \qquad \text{on } \Gamma. \tag{1.15}$$

This problem was later reconsidered in [ESP75] and [LSP75] using an asymptotic expansion argument and distinguishing two cases. First, the authors considered the case of a pressure gradient on the side of the porous medium normal to the interface (see Fig. 1.1 a)); they found that the flow is balanced on both sides of the interface and that the velocities across Γ are of the same order. Then, using an asymptotic expansion, they obtained the following interface laws:

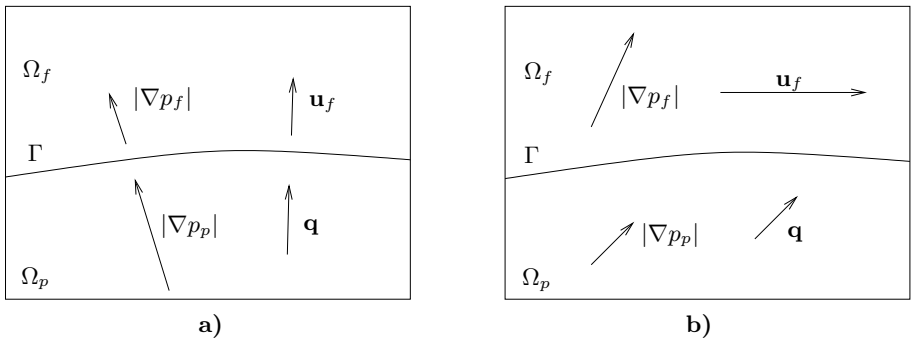


Fig. 1.1. Two configurations for the gradient of pressure: **a)** normal to the interface Γ; **b)** not normal to Γ.

$$\mathbf{u}_f \cdot \mathbf{n} = \mathbf{u}_p \cdot \mathbf{n}, \qquad p_f = const \qquad \text{on } \Gamma.$$

Secondly, they studied the case of pressure gradient not normal to the interface (see Fig. 1.1 b)). In this case, they found that the velocity $\mathbf{u}_f$ is much larger than the filtration velocity in the porous body and, in the first approximation, the flow around the porous medium is the same as if the body were impervious. Then, they conducted a local study in the vicinity of Γ leading to the existence of an intermediate layer, of characteristic thickness ε (the representative length of the porous matrix), which allows the asymptotic matching of the free fluid with the flow in the porous body. The free fluid contains a Prandtl's type boundary layer near Γ if the Reynolds number $Re_f \gg 1$ (see Fig. 1.2). Finally, they concluded that, in the first approximation, the suitable boundary condition at Γ is the continuity of the pressure.

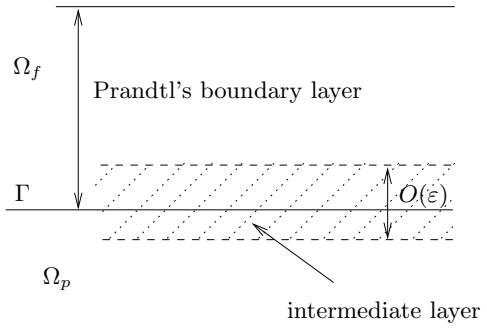


Fig. 1.2. The intermediate layer of thickness $O(\varepsilon)$ and the Prandtl's boundary layer if $Re_f \gg 1$.

In practice this approach leads to some mathematical difficulties in solving the effective equations, since the boundary conditions given on Γ are not enough to guarantee the well-posedness of the fluid problem. On the contrary, the law of Beavers and Joseph leads to a well-posed problem in the free fluid domain.

A first attempt towards an analytical study of the interface conditions between a free fluid and a porous medium can be found in [PS98]; however, a rigorous mathematical investigation has

been conducted by Jäger and Mikelić using homogenization theory (see [JM96, JM00, JMN01]). For completeness, we briefly recall their approach and the main results they achieved.
They considered a porous medium containing a large number of periodically distributed channels of characteristic size ε small if compared with the characteristic length L_p of the porous domain, as represented in Fig. 1.3.

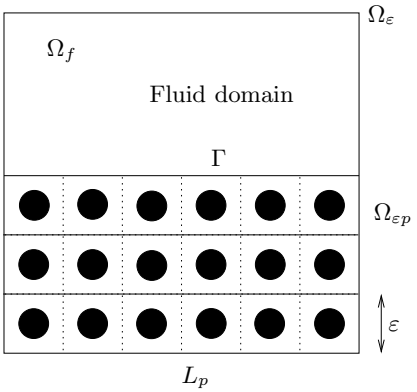


Fig. 1.3. Schematic representation of the domain Ω_ε, with porous matrix of width ε and characteristic length L_p.

As done by Beavers and Joseph, Jäger and Mikelić considered a uniform pressure gradient in the longitudinal direction of $\Omega_\varepsilon = \Omega_f \cup \Gamma \cup \Omega_{\varepsilon p} \subset \mathbb{R}^2$ and, for a fixed $\varepsilon > 0$, they looked for two functions $\mathbf{u}^\varepsilon$ and p^ε satisfying the Navier-Stokes equations:

$$\begin{aligned} -\nu \triangle \mathbf{u}^\varepsilon + (\mathbf{u}^\varepsilon \cdot \nabla)\mathbf{u}^\varepsilon + \nabla p^\varepsilon &= \mathbf{0} \quad \text{in } \Omega_\varepsilon \\ \nabla \cdot \mathbf{u}^\varepsilon &= 0 \quad \text{in } \Omega_\varepsilon \end{aligned}$$

with suitable boundary conditions (see [JM00]).

Remark 1.2.1. Adopting the Navier-Stokes equations not only in Ω_f but also in $\Omega_{\varepsilon p}$ is motivated by the fact that Darcy's law can be obtained from the (Navier-)Stokes equations through homogenization, at least in the interior of $\Omega_{\varepsilon p}$. A proof can be found, e.g., in [Tar80] where it is shown that the sequences of functions (depending on ε) $\mathbf{u}^\varepsilon$ and p^ε in $\Omega_{\varepsilon p}$, with homogeneous Dirichlet boundary conditions $\mathbf{u}^\varepsilon \cdot \mathbf{n} = 0$ on $\partial\Omega_{\varepsilon p}$, tend to the asymptotic velocity $\mathbf{u}_p^0$ and pressure p_p^0:

$$\begin{aligned} \frac{\mathbf{u}^\varepsilon}{\varepsilon^2} &\rightharpoonup \mathbf{u}_p^0 \qquad \text{weakly in } L^2(\Omega_p) \\ p^\varepsilon &\to p_p^0 \qquad \text{strongly in } L^2(\Omega_p) \end{aligned}$$

where $\mathbf{u}_p^0$ and p_p^0 satisfy the boundary value problem

$$\begin{aligned} \mathbf{u}_p^0 &= -\mathsf{K}\nabla p_p^0 \\ \nabla \cdot \mathbf{u}_p^0 &= 0 \end{aligned} \quad \text{in } \Omega_p$$

with $\mathbf{u}_p^0 \cdot \mathbf{n}_p = 0$ on $\partial\Omega_p$. From the convergence proof it can be seen that $\mathsf{K} \propto \varepsilon^2/\nu$, according to (1.9). □

Jäger and Mikelić proved that, consistently with the considerations by Ene and Sanchez-Palencia, the velocity field is of order $O(1)$ in Ω_f, of order $O(\varepsilon^2)$ in $\Omega_{\varepsilon p}$, and that there is a boundary layer of thickness $O(\varepsilon)$ for the velocity at the interface, while the pressure fields are of order $O(1)$ in both media. In particular, the effective velocity field in Ω_f is described by the solution $\mathbf{u}_f^0$ of Stokes equations with the no-slip condition $\mathbf{u}_f^0 = 0$ on Γ, giving an L^2-approximation of order $O(\varepsilon)$ for the velocity $\mathbf{u}^\varepsilon$.
However, this approximation is too rough since it cannot account for the velocity in the porous medium which is $O(\varepsilon^2)$. Therefore, they considered higher order terms in ε for the velocity introducing a boundary layer problem across Γ whose solution decays exponentially away from Γ and which accounts for the shear effects near the interface.
This correction led to introduce two positive non-null constants C_1^{bl} and C_2^{bl} (bl stands for boundary layer) and to characterize the following interface conditions for the macroscale problem:

$$\mathbf{u}_f \cdot \boldsymbol{\tau}_j - \varepsilon C_1^{bl} \boldsymbol{\tau}_j \cdot \frac{\partial \mathbf{u}_f}{\partial \mathbf{n}} = 0 \qquad \text{on } \Gamma \tag{1.16}$$

and

$$p_p = p_f - \nu C_2^{bl} \mathbf{n} \cdot \frac{\partial \mathbf{u}_f}{\partial \mathbf{n}} \qquad \text{on } \Gamma\,. \tag{1.17}$$

Finally, the following estimates hold (see [JM00]):

$$\begin{aligned} \|\nabla(\mathbf{u}^\varepsilon - \mathbf{u}_f)\|_{L^1(\Omega_f)} &\leq C\varepsilon|\log\varepsilon| \\ \|\mathbf{u}^\varepsilon - \mathbf{u}_f\|_{H^{1/2-\gamma}(\Omega_f)} &\leq C'\varepsilon^{3/2}|\log\varepsilon| \quad 0<\gamma<1/2 \end{aligned}$$

(the log term is due to the presence of corners in the domain).
Notice that (1.16) is exactly Saffman's modification of Beavers and Joseph's law with $\sqrt{\mathsf{K}}/\alpha = \varepsilon C_1^{bl}$, while condition (1.17) shows that, contrary to intuition, the effective pressure in the system channel flow/porous medium is not always continuous and, therefore, the continuity assumption of Sanchez-Palencia is not generally correct.
The constants C_1^{bl}, C_2^{bl} have been computed for some configurations of porous media and, on the base of the results reported in [JMN01], we shall assume $C_1^{bl}, C_2^{bl} \sim 1$.

Strictly speaking, (1.16) is not a coupling condition in the sense that it does not relate quantities from the two subdomains Ω_f and Ω_p, but it is actually a boundary condition on Γ for the fluid problem. Moreover, the term in (1.16) involving the normal derivative of $\mathbf{u}_f$ is multiplied by ε and the velocity itself can be supposed at least of order $O(\varepsilon)$ in the neighborhood of Γ; therefore, this term is small and, actually, we will set it equal to zero in chapters 2 and 3 in order to simplify our analysis.

We point out that the conditions studied by Jäger and Mikelić have been adopted also in [LSY03, RY03, BH02].

Finally, the condition $\boldsymbol{\tau}_j \cdot \nabla \mathbf{u}_f \cdot \mathbf{n} = 0$ can be regarded as a simplified form of (1.16); in fact, although not completely precise from the physical point of view, it is perfectly acceptable from the mathematical viewpoint since it allows to write a well-posed problem for the fluid part, and, in this sense, it could be adopted as well.

1.3 Boundary Conditions and Problem Setting

In order to complete the definition of our coupled problem, we have to introduce suitable boundary conditions.

Concerning Darcy's equation, we split the boundary $\partial\Omega_p$ as $\partial\Omega_p = \Gamma \cup \Gamma_p \cup \Gamma_p^b$ as shown in Fig. 1.4 and we assign the piezometric head $\varphi = \varphi_p$ on the bottom surface Γ_p^b. Moreover, we require that the normal component of the velocity vanishes on the lateral surfaces, that is, $\mathbf{u}_p \cdot \mathbf{n}_p = 0$ on Γ_p.

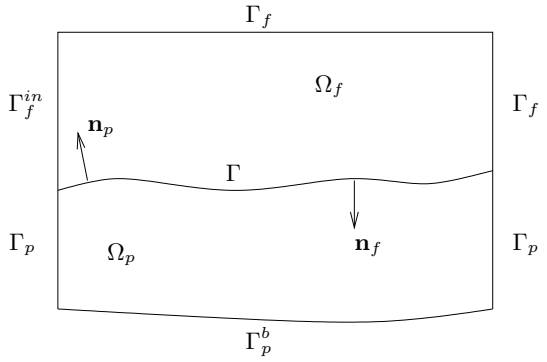


Fig. 1.4. Schematic representation of a 2D vertical section of the computational domain.

For the Navier–Stokes problem in Ω_f, several combinations of boundary conditions could be considered, representing different kinds of flow problem; we indicate some of them and we refer to Fig. 1.5 for the notation that we adopt hereafter. A comprehensive description of possible boundary conditions for the Navier-Stokes equations can be found in [QV94] chapter 10, [Pir88] chapter 4, and references therein.

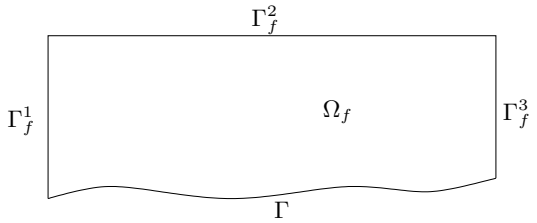


Fig. 1.5. Schematic representation of a 2D vertical section of the computational domain Ω_f.

A first possibility is to assign the velocity vector $\mathbf{u}_f = \mathbf{0}$ on $\Gamma_f^1 \cup \Gamma_f^3$ and a natural boundary condition $\mathsf{T}(\mathbf{u}_f, p_f) \cdot \mathbf{n}_f = \mathbf{g}$ on Γ_f^2 (a fictitious boundary). Here,

$$\mathsf{T}(\mathbf{u}_f, p_f) = \nu \nabla \mathbf{u}_f - p_f \mathsf{I}$$

is the stress tensor, and $\mathbf{g}$ a given vector function, representing the flux across Γ_f^2 of the fluid column standing above.
Alternatively, we can prescribe a non-null inflow $\mathbf{u}_f = \mathbf{u}_{in}$ on the left-hand boundary Γ_f^1, a slip condition $\mathbf{u}_f \cdot \mathbf{n}_f = 0$ and $(\mathsf{T}(\mathbf{u}_f, p_f) \cdot \mathbf{n}_f) \cdot \boldsymbol{\tau}_j = 0$ on Γ_f^2, and an outflow $\mathsf{T}(\mathbf{u}_f, p_f) \cdot \mathbf{n}_f = 0$ on the right-hand boundary Γ_f^3.
A third possibility consists in assigning again a non-null inflow $\mathbf{u}_f = \mathbf{u}_{in}$ on the left-hand boundary Γ_f^1 and a no-slip condition $\mathbf{u}_f = \mathbf{0}$ on the remaining boundary $\Gamma_f^2 \cup \Gamma_f^3$.
In the next chapters we shall consider the last choice we have indicated, but, as we shall see, our analysis could be modified to accommodate the other boundary conditions as well. From now on, we denote Γ_f^1 as Γ_f^{in} (standing for Γ_f^{inflow}) and the remaining boundary $\Gamma_f^2 \cup \Gamma_f^3$ simply by Γ_f (see Fig. 1.4).

Using Darcy's law (1.10), we can rewrite the system (1.10), (1.11) as an elliptic equation for the scalar unknown φ:

$$-\nabla \cdot (\mathsf{K} \nabla \varphi) = 0 \quad \text{in } \Omega_p. \tag{1.18}$$

Therefore, the differential formulation of the coupled Navier-Stokes/Darcy problem we consider reads:

$$-\nu \triangle \mathbf{u}_f + (\mathbf{u}_f \cdot \nabla)\mathbf{u}_f + \nabla p_f = \mathbf{f} \quad \text{in } \Omega_f \tag{1.19}$$

$$\nabla \cdot \mathbf{u}_f = 0 \quad \text{in } \Omega_f \tag{1.20}$$

$$-\nabla \cdot (\mathsf{K} \nabla \varphi) = 0 \quad \text{in } \Omega_p \tag{1.21}$$

$$\mathbf{u}_f = \mathbf{u}_{in} \quad \text{on } \Gamma_f^{in} \tag{1.22}$$

$$\mathbf{u}_f = \mathbf{0} \quad \text{on } \Gamma_f \tag{1.23}$$

$$-\mathsf{K} \nabla \varphi \cdot \mathbf{n}_p = 0 \quad \text{on } \Gamma_p \tag{1.24}$$

$$\varphi = \varphi_p \quad \text{on } \Gamma_p^b \tag{1.25}$$

and it must be completed with the interface conditions on Γ:

$$\mathbf{u}_f \cdot \mathbf{n} = -\frac{\mathsf{K}}{n} \frac{\partial \varphi}{\partial \mathbf{n}} \tag{1.26}$$

$$-\nu \mathbf{n} \cdot \frac{\partial \mathbf{u}_f}{\partial \mathbf{n}} + p_f = g\varphi \tag{1.27}$$

$$\mathbf{u}_f \cdot \boldsymbol{\tau}_j - \varepsilon \boldsymbol{\tau}_j \cdot \frac{\partial \mathbf{u}_f}{\partial \mathbf{n}} = 0. \tag{1.28}$$

We will assume that $\mathbf{u}_{in}$ is null in a neighborhood of the intersection $\overline{\Gamma} \cap \overline{\Gamma}_f^{in}$.

The global problem is then nonlinear. A linearization can be obtained by replacing the Navier-Stokes momentum equation (1.19) with the Stokes one:

$$-\nu\triangle\mathbf{u}_f + \nabla p_f = \mathbf{f} \quad \text{in } \Omega_f \tag{1.29}$$

i.e. dropping the nonlinear convective terms. This replacement is justified when the Reynolds number Re_f of the fluid is low, i.e. in case of slow motion of fluids with high viscosity. This linearized problem is also interesting since a steady Stokes problem can be generated when considering a semi-implicit time advancement of the Navier-Stokes equations where all terms but the nonlinear convective one have been dealt with implicitely.

1.4 Some Extensions

We would like to introduce two generalizations of the Navier-Stokes/Darcy coupling we have just presented. In particular, we want to consider the case with a free surface fluid in the upper domain Ω_f and a similar problem encountered in external aerodynamics.

1.4.1 The Free Surface Case

In the case of a free surface fluid, we would like to replace the Navier-Stokes equations by a simpler model based on the so-called Shallow Water equations. With this aim we characterize the domain Ω_f as follows. Let $\hat{\Omega}$ be a bounded domain of $\mathbb{R}^2$ representing the undisturbed free surface of the fluid, while $z = h(x, y)$ and $z = \eta(x, y, t)$ are two functions describing respectively the bathymetry and the free surface with respect to a reference level $z = 0$. Ω_f is therefore the normal domain with respect to the z axis defined as

$$\Omega_f = \{\mathbf{x} = (x, y, z) | (x, y) \in \hat{\Omega},\ z \in (h, \eta)\}$$

(see Fig. 1.6).
We describe the motion of the free surface fluid in Ω_f by the 3D non-hydrostatic Shallow Water equations with constant density. The total pressure is the sum of a hydrostatic part and a hydrodinamic correction: $p_f = \rho g(\eta - z) + q$. We consider therefore the following model: $\forall t > 0$,

$$\frac{D\mathbf{u}_f}{Dt} - \frac{\partial}{\partial z}\left(\nu_v \frac{\partial \mathbf{u}_f}{\partial z}\right) + \nabla q + diag(g, g, 0) \cdot \nabla\eta = \hat{\mathbf{f}} \quad \text{in } \Omega_f \tag{1.30}$$

$$\nabla \cdot \mathbf{u}_f = 0 \quad \text{in } \Omega_f \tag{1.31}$$

$$\frac{\partial \eta}{\partial t} + \nabla \cdot \int_h^\eta (u_f^1, u_f^2)^T = \tilde{Q} \quad \forall (x, y) \in \hat{\Omega} \tag{1.32}$$

where g is the gravity acceleration, $\hat{\mathbf{f}} = (f^1, f^2, 0)^T$ is the external force vector, q is the hydrodynamic pressure and ν_v is the vertical viscosity coefficient. $\tilde{Q}$ is equal to the normal component of the velocity $\mathbf{u}_f$ and is equal to zero when the bottom surface is impermeable. Finally, $D/Dt = \partial/\partial t + (\mathbf{u}_f \cdot \nabla)$ denotes the Lagrangian derivative.

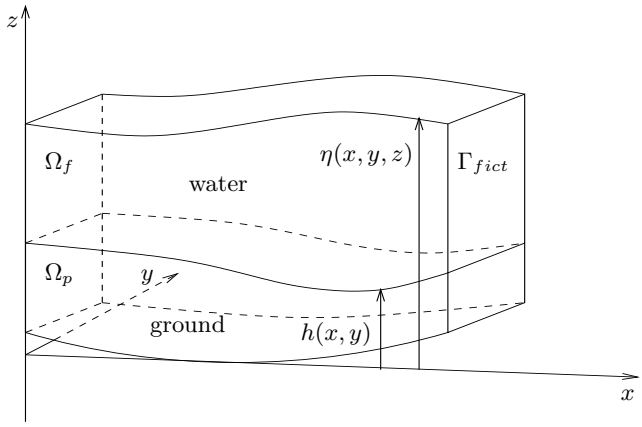


Fig. 1.6. Schematic representation of the domain of free surface/Darcy problem.

For the coupled model free surface fluid/Darcy we propose the following interface conditions on Γ (analogous to the ones for the channel fluid case):

$$\mathbf{u}_f \cdot \mathbf{n} = \mathbf{u}_p \cdot \mathbf{n}, \tag{1.33}$$

$$\boldsymbol{\tau}_j \cdot \frac{\partial \mathbf{u}_f}{\partial \mathbf{n}} = \frac{\alpha_{BJ}}{\sqrt{\mathsf{K}}}(\mathbf{u}_f - \mathbf{u}_p) \cdot \boldsymbol{\tau}_j, \tag{1.34}$$

$$\rho_f g \varphi = \rho_f g H + p_p = p_f \tag{1.35}$$

where $H = \eta - h$ is the total height of the fluid in Ω_f.
We observe that condition (1.33) imposes the continuity of the normal component of the velocity, however it allows a discontinuity of its tangential components; the pressure can be discontinuous across the interface Γ.
For a more detailed presentation of the free surface/Darcy problem we refer, e.g., to [Mig00, MQS03, DMQ02] and references therein.

To resume, the two problems Navier-Stokes/Darcy and free surface/Darcy can be seen in abstract form as follows: we have a problem

$$P_f(\xi_f) = 0 \text{ in the domain } \Omega_f, \tag{1.36}$$

where ξ_f indicates all the unknowns therein (velocity, pressure and free surface location in the case of free surface flow), and a problem

$$P_p(\xi_p) = 0 \text{ in } \Omega_p, \tag{1.37}$$

where ξ_p represents the unknowns of Darcy's problem. Finally, we have three interface conditions involving pressure and velocity on Γ that we can indicate in a compact form as follows:

$$\mathbf{u}_p \cdot \mathbf{n} = \mathbf{u}_f \cdot \mathbf{n} \tag{1.38}$$

$$\Psi_f(\mathbf{u}_f, p_f) = \Psi_p(\mathbf{u}_p) \tag{1.39}$$

$$\Phi_f(\mathbf{u}_f, p_f) = \Phi_p(p_p) \tag{1.40}$$

with $\Psi_.$ and $\Phi_.$ suitable functions of the velocities and pressures in the two subdomains.

1.4.2 A Stokes/Laplace Problem

Models similar to the one we have introduced in Sect. 1.3 can be used in external aerodynamics to describe the motion of an incompressible fluid around a body such as, for example, a ship, a boat or a submerged body in a water basin. In fact, such problems can be studied by decomposing the computational domain into two parts: a region Ω_{int} close to the body where, due to the viscosity effects, all the interesting features of the flow occur, and an outer region Ω_{ext} far away from the body where one can neglect the viscosity effects.
Therefore, suitable heterogeneous differential models comprising Navier-Stokes equations, Euler equations, potential flows and other models from fluid dynamics could be envisaged (see, e.g., [IC03]).
Here, we present a simple model where in Ω_{int} we consider the full Navier-Stokes equations, while in Ω_{ext} we adopt a Laplace equation for the velocity potential.
A coupled heterogeneous model of this kind has been studied in [SH94] considering a computational domain as in Fig. 1.7 and the following generalized Stokes problem:

$$\beta\mathbf{u}_\epsilon - \mu_\epsilon \triangle \mathbf{u}_\epsilon + \nabla p_\epsilon = \tilde{\mathbf{f}} \quad \text{in } \widetilde{\Omega} \tag{1.41}$$

$$\nabla \cdot \mathbf{u}_\epsilon = 0 \quad \text{in } \widetilde{\Omega} \tag{1.42}$$

$$\mathbf{u}_\epsilon = \mathbf{0} \quad \text{on } \Gamma_w \tag{1.43}$$

with suitable boundary conditions on the outer boundary Γ_∞. The viscosity is $\mu_\epsilon = \mu$ in Ω_{int}, while $\mu_\epsilon = \epsilon$ in Ω_{ext}.

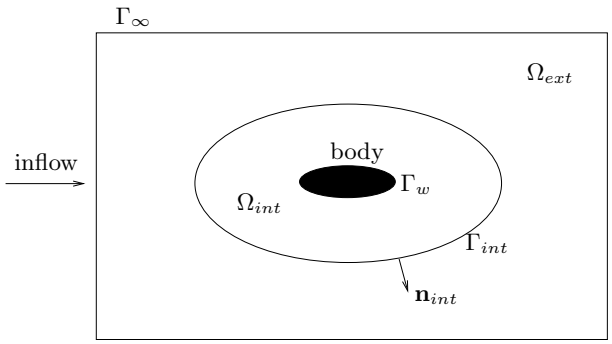


Fig. 1.7. Schematic representation of the domain computational domain for an external aerodynamics problem.

Then, they applied a vanishing viscosity argument letting $\epsilon \to 0$ in Ω_{ext} in order to set up a suitable global model and to define the correct interface conditions across Γ_{int}. Precisely, they found the following limit coupled problem:

$$\beta \mathbf{u} - \mu \triangle \mathbf{u} + \nabla p = \tilde{\mathbf{f}} \quad \text{in } \Omega_{int} \tag{1.44}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega_{int} \tag{1.45}$$

$$\triangle q = \nabla \cdot \tilde{\mathbf{f}} \quad \text{in } \Omega_{ext} \tag{1.46}$$

with suitable boundary conditions and the coupling conditions across the interface Γ_{int}

$$-\mu \frac{\partial \mathbf{u}}{\partial \mathbf{n}_{int}} + p\mathbf{n}_{int} = q\mathbf{n}_{int} \quad \text{on } \Gamma \tag{1.47}$$

$$\frac{\partial q}{\partial \mathbf{n}_{int}} = (\tilde{\mathbf{f}} - \beta \mathbf{u}) \cdot \mathbf{n}_{int} \quad \text{on } \Gamma\,. \tag{1.48}$$

$\mathbf{n}_{int}$ denotes the unit normal vector on Γ_{int} directed from Ω_{int} to Ω_{ext}. We remark that, apart from the physical meaning of the variables, these coupling conditions are similar in their structure to those for the Navier-Stokes/Darcy case (1.26)-(1.28). In fact, (1.48) corresponds to (1.26), and in (1.47) we allow again the pressure to be discontinuous across Γ_{int}, even if we do not distinguish between the normal and the tangential components of the stress tensor as in (1.27), (1.28).

Because of these similarities, the analysis we shall develop for the Navier-Stokes/Darcy problem could be accommodated with minor changes to account also for the heterogenous coupling (1.44)-(1.48).

2. Mathematical Analysis of the Coupled Problem: the Linear Case

In this chapter we analyze the linear coupled problem formed by Stokes and Darcy equations. We write it as a saddle-point problem, then we prove an existence and uniqueness result. Moreover, after introducing appropriate interface variables, we rewrite the coupled problem in terms of equations solely defined on the interface.
The results presented in this chapter extend those published in [DQ03].

2.1 Introduction

We begin our analysis of the coupled problem (1.19)-(1.28) considering the linear Stokes problem (1.29). Our goal is to guarantee the well-posedness of this problem and to reformulate it in terms of its unknowns across the interface Γ. This re-interpretation of the global problem in terms of the interface unknowns will be crucial to set up iterative substructuring procedures to solve it as we shall see also in chapter 3.
Before starting the analysis, we introduce some functional analysis tools that we need for the following sections and chapters. We refer to [Ada75, Bre83, LM68, Yos74] for a rigorous and exhaustive presentation of these results.

2.1.1 Preliminary Notations and Results

***L^p* Spaces.** Let $\mathcal{D}$ be an open set contained in $\mathbb{R}^d$ $(d = 2, 3)$ and consider in $\mathcal{D}$ the Lebesgue measure. We consider the set of real measurable functions v on $\mathcal{D}$ and we introduce the equivalence relation:

$$v \equiv w \text{ iff } meas(\{\mathbf{x} \in \mathcal{D} | v(\mathbf{x}) \neq w(\mathbf{x})\}) = 0. \tag{2.1}$$

Then, for $1 \leq p < \infty$, we define $L^p(\mathcal{D})$ as the space of classes of equivalence of measurable functions with respect to (2.1) such that

$$\int_{\mathcal{D}} |v(\mathbf{x})|^p < \infty \qquad 1 \leq p < \infty.$$

This is a Banach space endowed with the norm

$$\|v\|_{L^p(\mathcal{D})} = \left(\int_{\mathcal{D}} |v(\mathbf{x})|^p \right)^{1/p} \qquad 1 \leq p < \infty.$$

If $p = 2$ we have an Hilbert space with the scalar product

$$(v, w)_{L^2(\mathcal{D})} = \int_{\mathcal{D}} v(\mathbf{x})w(\mathbf{x}) \ .$$

Finally, if $p = \infty$, we say that $v \in L^\infty(\mathcal{D})$ if

$$\inf\{M \geq 0 | \, |v(\mathbf{x})| \leq M \text{ almost everywhere (a.e.) in } \mathcal{D}\} < \infty$$

and we define its norm as

$$\|v\|_{L^\infty(\mathcal{D})} = \inf\{M \geq 0 | \, |v(\mathbf{x})| \leq M \text{ a.e. in } \mathcal{D}\}.$$

We recall that in the Hilbert space $L^2(\mathcal{D})$ the Cauchy-Schwarz inequality holds:

$$|(v, w)_{L^2(\mathcal{D})}| \leq \|v\|_{L^2(\mathcal{D})}\|w\|_{L^2(\mathcal{D})} \ . \tag{2.2}$$

Sobolev Spaces $H^k(\mathcal{D})$. The Sobolev space $H^k(\mathcal{D})$, with k a non-negative integer, is the space of functions $v \in L^2(\mathcal{D})$ such that all the distributional derivatives of v of order up to k are a function of $L^2(\mathcal{D})$:

$$H^k(\mathcal{D}) = \{v \in L^2(\mathcal{D}) | D^{\boldsymbol{\alpha}} v \in L^2(\mathcal{D}) \ \forall \text{multi-index } \boldsymbol{\alpha} \geq 0 : \ |\boldsymbol{\alpha}| \leq k\},$$

where

$$D^{\boldsymbol{\alpha}} v = \frac{\partial^{|\boldsymbol{\alpha}|} v}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} \ .$$

$H^k(\mathcal{D})$ is a Hilbert space endowed with the norm

$$\|v\|_{H^k(\mathcal{D})} = \left(\sum_{|\boldsymbol{\alpha}| \leq k} \|D^{\boldsymbol{\alpha}} v\|^2_{L^2(\mathcal{D})} \right)^{1/2} ,$$

the seminorm

$$|v|_{H^k(\mathcal{D})} = \left(\sum_{|\boldsymbol{\alpha}| = k} \|D^{\boldsymbol{\alpha}} v\|^2_{L^2(\mathcal{D})} \right)^{1/2} ,$$

and the scalar product

$$(v, w)_{H^k(\mathcal{D})} := \sum_{|\boldsymbol{\alpha}| \leq k} (D^{\boldsymbol{\alpha}} v, D^{\boldsymbol{\alpha}} w)_{L^2(\mathcal{D})} \, .$$

Remark 2.1.1. In the following, for simplicity of notation, we shall indicate by $\| \cdot \|_0$ and $\| \cdot \|_1$, respectively, the norm in $L^2(\mathcal{D})$ and in $H^1(\mathcal{D})$ (analogous notations will be adopted for seminorms and scalar products). In particular, we shall distinguish the norms in Ω_f and in Ω_p by a lower index f or p, for example, $\| \cdot \|_{1,f}$, $\| \cdot \|_{1,p}$. □

Some Results about Sobolev Spaces. We recall some properties enjoyed by the functions belonging to Sobolev spaces and, in particular, we consider some results on traces. We begin by the following theorem.

Theorem 2.1.1 (Trace theorem). *Let $\mathcal{D}$ be a bounded open set in $\mathbb{R}^d$ with Lipschitz continuous boundary $\partial\mathcal{D}$.*

1. *There exists a unique linear continuous surjective map $\gamma_0 : H^1(\mathcal{D}) \to H^{1/2}(\partial\mathcal{D})$ such that $\gamma_0 v = v_{|\partial\mathcal{D}}$ for each $v \in H^1(\mathcal{D}) \cap C^0(\overline{\mathcal{D}})$.*
2. *There exists a linear continuous injective extension operator $R_0 : H^{1/2}(\partial\mathcal{D}) \to H^1(\mathcal{D})$ such that $\gamma_0 R_0 \eta = \eta$ for each $\eta \in H^{1/2}(\partial\mathcal{D})$.*

Analogous results hold if we consider the trace γ_Σ over a Lipschitz continuous subset Σ of the boundary $\partial\mathcal{D}$ of positive measure.

$H^{1/2}(\partial\mathcal{D})$ is the space of traces of functions in $H^1(\mathcal{D})$.
By means of these trace operators it is possible to characterize the spaces

$$H_0^1(\mathcal{D}) = \{v \in H^1(\mathcal{D}) | \gamma_0 v = 0\} \quad \text{and} \quad H_\Sigma^1(\mathcal{D}) = \{v \in H^1(\mathcal{D}) | \gamma_\Sigma v = 0\}.$$

The traces over Σ of functions in $H^1_{\partial\mathcal{D}\setminus\Sigma}(\mathcal{D})$ belong to the trace space $H_{00}^{1/2}(\Sigma)$ which is strictly included in $H^{1/2}(\Sigma)$ and is endowed with a norm which is larger that the norm of $H^{1/2}(\Sigma)$. In particular, $H_{00}^{1/2}(\Sigma)$ is the completion of the smooth functions with compact support in Σ with respect to the norm

$$\|\mu\|_{H^{1/2}(\partial\mathcal{D})} = \left(\|\mu\|^2_{L^2(\partial\mathcal{D})} + \int_{\partial\mathcal{D}} \int_{\partial\mathcal{D}} \frac{|\mu(t_1) - \mu(t_2)|^2}{|t_1 - t_2|^d} ds_{t_1} ds_{t_2} \right)^{1/2}.$$

Any function $\mu \in H_{00}^{1/2}(\Sigma)$ has the property that its extension by zero to $\partial\mathcal{D}$ gives a function $\tilde{\mu} \in H^{1/2}(\partial\mathcal{D})$ with

$$\|\tilde{\mu}\|_{H^{1/2}(\partial\mathcal{D})} \leq C \|\mu\|_{H_{00}^{1/2}(\Sigma)}.$$

The following trace inequalities hold:

$$\|v_{|\Sigma}\|_{L^2(\Sigma)} \leq C_{tr}^1 \|v\|_1 \qquad \forall v \in H^1(\mathcal{D}); \tag{2.3}$$

$$\|v_{|\Sigma}\|_{H^{1/2}(\Sigma)} \leq C_{tr}^2 \|v\|_1 \qquad \forall v \in H^1(\mathcal{D}); \tag{2.4}$$

$$\|v_{|\Sigma}\|_{H_{00}^{1/2}(\Sigma)} \leq C_{tr}^3 \|v\|_1 \qquad \forall v \in H^1(\mathcal{D}). \tag{2.5}$$

Finally, an important result that we will often use in our proofs is the so-called *Poincaré inequality.*

Theorem 2.1.2 (Poincaré inequality). *Let $\mathcal{D}$ be a bounded connected open set of $\mathbb{R}^d$ and Σ a (non-empty) Lipschitz continuous subset of the boundary $\partial\mathcal{D}$. There exists a constant $C_\mathcal{D} > 0$ such that*

$$\int_\mathcal{D} v^2(\mathbf{x}) \leq C_\mathcal{D} \int_\mathcal{D} |\nabla v(\mathbf{x})|^2 \tag{2.6}$$

for each $v \in H_\Sigma^1(\mathcal{D})$.

2.2 Weak Formulation of the Stokes/Darcy Problem

To write the weak form of the Stokes/Darcy problem as a saddle-point problem, we need to introduce the following functional spaces:

$$H_{\Gamma_f} = \{v \in H^1(\Omega_f) | v = 0 \text{ on } \Gamma_f\} , \tag{2.7}$$

$$H_{\Gamma_f \cup \Gamma_f^{in}} = \{v \in H_{\Gamma_f} | v = 0 \text{ on } \Gamma_f^{in}\} , \tag{2.8}$$

$$H_f = (H_{\Gamma_f \cup \Gamma_f^{in}})^d , \tag{2.9}$$

$$\widetilde{H}_f = \{\mathbf{v} \in (H^1(\Omega_f))^d | \, \mathbf{v} = \mathbf{0} \text{ on } \Gamma_f \cup \Gamma\} , \tag{2.10}$$

$$H_f^0 = \{\mathbf{v} \in H_f | \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma\} , \tag{2.11}$$

$$Q = L^2(\Omega_f) , \tag{2.12}$$

$$H_p = \{\psi \in H^1(\Omega_p) | \psi = 0 \text{ on } \Gamma_p^b\} , \tag{2.13}$$

$$H_p^0 = \{\psi \in H_p | \psi = 0 \text{ on } \Gamma\} , \tag{2.14}$$

and the trace spaces

$$\Lambda := H_{00}^{1/2}(\Gamma) \qquad \text{and} \qquad \Lambda_\dagger = H^{1/2}(\Gamma). \tag{2.15}$$

Finally, we consider the Hilbert space $W = H_f \times H_p$ with norm

$$\|\underline{w}\|_W = \left(\|\mathbf{w}\|_{1,f}^2 + \|\psi\|_{1,p}^2\right)^{1/2} \qquad \forall \underline{w} = (\mathbf{w}, \psi) \in W.$$

We introduce a continuous extension operator

$$E_f \colon (H^{1/2}(\Gamma_f^{in}))^d \to \widetilde{H}_f . \tag{2.16}$$

Then $\forall \mathbf{u}_{in} \in (H_{00}^{1/2}(\Gamma_f^{in}))^d$ we can construct a vector function $E_f \mathbf{u}_{in} \in \widetilde{H}_f$ such that $E_f \mathbf{u}_{in|\Gamma_f^{in}} = \mathbf{u}_{in}$.

Remark 2.2.1. Alternatively, we could consider a divergence free extension $\widetilde{E}_f \mathbf{u}_{in}$ of $\mathbf{u}_{in}$. To this aim, let $\mathcal{E}_f \mathbf{u}_{in} \in (H_{\Gamma_f})^d$ such that $\mathcal{E}_f \mathbf{u}_{in} = \mathbf{u}_{in}$ on Γ_f^{in}. Then, we construct a function $\mathbf{w}_{in}$ which is the solution of the following problem: find $\mathbf{w}_{in} \in H_f$ such that for all $q \in Q$

$$-\int_{\Omega_f} q \, \nabla \cdot \mathbf{w}_{in} = \int_{\Omega_f} q \, \nabla \cdot (\mathcal{E}_f \mathbf{u}_{in}) . \tag{2.17}$$

The solvability of (2.17) is guaranteed by the inf-sup condition: there exists a constant $\beta^* > 0$ such that

$$\forall q \in Q \quad \exists \mathbf{v} \in H_f, \ \mathbf{v} \neq \mathbf{0} : \quad -\int_{\Omega_f} q \, \nabla \cdot \mathbf{v} \geq \beta \|\mathbf{v}\|_{1,f} \|q\|_{0,f} \tag{2.18}$$

(see, e.g., [QV99] p. 158–159). Finally, we indicate by $\widetilde{E}_f \mathbf{u}_{in} = \mathcal{E}_f \mathbf{u}_i + \mathbf{w}_{in}$ the divergence-free extension of $\mathbf{u}_{in}$. We remark that $\widetilde{E}_f \mathbf{u}_{in} = \mathbf{u}_{in}$ on Γ_f^{in}, $\widetilde{E}_f \mathbf{u}_{in} = \mathbf{0}$ on Γ_f and that, thanks to (2.17), it holds

$$\int_{\Omega_f} q \, \nabla \cdot (\widetilde{E}_f \mathbf{u}_{in}) = 0 \quad \forall q \in Q .$$

We point out that the extension $\widetilde{E}_f \mathbf{u}_{in}$ cannot satisfy the additional constraint $\widetilde{E}_f \mathbf{u}_{in} \cdot \mathbf{n}_f = 0$ on Γ, except for the special case of $\mathbf{u}_{in}$ such that $\int_{\Gamma_f^{in}} \mathbf{u}_{in} \cdot \mathbf{n}_f = 0$. □

We introduce another continuous extension operator:

$$E_p\colon H^{1/2}(\Gamma_p^b) \to H^1(\Omega_p), \quad \text{such that } E_p\varphi_p = 0 \text{ on } \Gamma. \tag{2.19}$$

Then, for all $\varphi \in H^1(\Omega_p)$ we define the function $\varphi_0 = \varphi - E_p\varphi_p$.

Finally, we define the following bilinear forms:

$$a_f(\mathbf{v},\mathbf{w}) = \int_{\Omega_f} \nu\nabla\mathbf{v}\cdot\nabla\mathbf{w} \qquad \forall \mathbf{v},\mathbf{w} \in (H^1(\Omega_f))^d \tag{2.20}$$

$$b_f(\mathbf{v},q) = -\int_{\Omega_f} q\,\nabla\cdot\mathbf{v}\,, \qquad \forall \mathbf{v} \in (H^1(\Omega_f))^d, \quad \forall q \in Q \tag{2.21}$$

$$a_p(\varphi,\psi) = \int_{\Omega_p} \nabla\psi\cdot\mathsf{K}\nabla\varphi\,, \quad \forall \varphi,\psi \in H^1(\Omega_p)\,. \tag{2.22}$$

Now, if we multiply (1.29) by $\mathbf{v} \in H_f$ and integrate by parts we obtain

$$a_f(\mathbf{u}_f,\mathbf{v}) + b_f(\mathbf{v},p_f) + \int_\Gamma \left(-\nu\frac{\partial\mathbf{u}_f}{\partial\mathbf{n}} + p_f\mathbf{n}\right)\mathbf{v} = \int_{\Omega_f} \mathbf{f}\,\mathbf{v}\,.$$

Notice that we can write

$$\int_\Gamma \left(-\nu\frac{\partial\mathbf{u}_f}{\partial\mathbf{n}} + p_f\mathbf{n}\right)\mathbf{v} = \int_\Gamma \left[\left(-\nu\frac{\partial\mathbf{u}_f}{\partial\mathbf{n}} + p_f\mathbf{n}\right)\cdot\mathbf{n}\right]\mathbf{v}\cdot\mathbf{n} + \int_\Gamma \sum_{j=1}^{d-1}\left[\left(-\nu\frac{\partial\mathbf{u}_f}{\partial\mathbf{n}} + p_f\mathbf{n}\right)\cdot\boldsymbol{\tau}_j\right]\mathbf{v}\cdot\boldsymbol{\tau}_j$$

so that we can incorporate in weak form the interface conditions (1.27) and (1.28) as follows:

$$\int_\Gamma \left(-\nu\frac{\partial\mathbf{u}_f}{\partial\mathbf{n}} + p_f\mathbf{n}\right)\mathbf{v} = \int_\Gamma g\varphi(\mathbf{v}\cdot\mathbf{n}) + \int_\Gamma \sum_{j=1}^{d-1}\frac{\nu}{\varepsilon}(\mathbf{u}_f\cdot\boldsymbol{\tau}_j)(\mathbf{v}\cdot\boldsymbol{\tau}_j)\,.$$

Finally, we consider the lifting $E_f\mathbf{u}_{in}$ of the boundary datum and we split $\mathbf{u}_f = \mathbf{u}_f^0 + E_f\mathbf{u}_{in}$ with $\mathbf{u}_f^0 \in H_f$; we recall that $E_f\mathbf{u}_{in} = 0$ on Γ and we get

$$a_f(\mathbf{u}_f^0,\mathbf{v}) + b_f(\mathbf{v},p_f) + \int_\Gamma g\varphi(\mathbf{v}\cdot\mathbf{n}) + \int_\Gamma \sum_{j=1}^{d-1}\frac{\nu}{\varepsilon}(\mathbf{u}_f\cdot\boldsymbol{\tau}_j)(\mathbf{v}\cdot\boldsymbol{\tau}_j) = \int_{\Omega_f}\mathbf{f}\,\mathbf{v} - a_f(E_f\mathbf{u}_{in},\mathbf{v}). \tag{2.23}$$

From (1.20) we find

$$b_f(\mathbf{u}_f^0,q) = -b_f(E_f\mathbf{u}_{in},q) \qquad \forall q \in Q. \tag{2.24}$$

On the other hand, if we multiply (1.21) by $\psi \in H_p$ and integrate by parts we get

$$a_p(\varphi,\psi)+\int_\Gamma \mathsf{K}\frac{\partial\varphi}{\partial\mathbf{n}}\psi=0\,.$$

Now we incorporate the interface condition (1.26) in weak form as

$$a_p(\varphi,\psi)-\int_\Gamma n(\mathbf{u}_f\cdot\mathbf{n})\psi=0$$

and, considering the splitting $\varphi=\varphi_0+E_p\varphi_p$ we obtain

$$a_p(\varphi_0,\psi)-\int_\Gamma n(\mathbf{u}_f\cdot\mathbf{n})\psi=-a_p(E_p\varphi_p,\psi). \tag{2.25}$$

We multiply (2.23), (2.24) by n, and (2.25) by g and sum up; then, we define

$$\begin{aligned}\mathcal{A}(\underline{v},\underline{w}) &= n\,a_f(\mathbf{v},\mathbf{w})+g\,a_p(\varphi,\psi)\\ &\quad+\int_\Gamma ng\,\varphi(\mathbf{w}\cdot\mathbf{n})-\int_\Gamma ng\,\psi(\mathbf{v}\cdot\mathbf{n})\\ &\quad+\int_\Gamma n\sum_{j=1}^{d-1}\frac{\nu}{\varepsilon}(\mathbf{w}\cdot\boldsymbol{\tau}_j)(\mathbf{v}\cdot\boldsymbol{\tau}_j)\end{aligned} \tag{2.26}$$

$$\mathcal{B}(\underline{w},q) = n\,b_f(\mathbf{w},q) \tag{2.27}$$

for all $\underline{v}=(\mathbf{v},\varphi)$, $\underline{w}=(\mathbf{w},\psi)\in W$, $q\in Q$. Finally, we define the following linear functionals:

$$\langle\mathcal{F},\underline{w}\rangle=\int_{\Omega_f} n\mathbf{f}\,\mathbf{w}-n\,a_f(E_f\mathbf{u}_{in},\mathbf{w})-g\,a_p(E_p\varphi_p,\psi) \tag{2.28}$$

$$\langle\mathcal{G},q\rangle=-nb_f(E_f\mathbf{u}_{in},q) \tag{2.29}$$

for all $\underline{w}=(\mathbf{w},\psi)\in W$, $q\in Q$.
Adopting these notations, the weak formulation of the Stokes/Darcy coupled problem reads:

find $\underline{u}=(\mathbf{u}_f^0,\varphi_0)\in W$, $p_f\in Q$ such that

$$\mathcal{A}(\underline{u},\underline{v})+\mathcal{B}(\underline{v},p_f) = \langle\mathcal{F},\underline{v}\rangle \quad\forall\underline{v}=(\mathbf{v},\psi)\in W \tag{2.30}$$

$$\mathcal{B}(\underline{u},q) = \langle\mathcal{G},q\rangle \quad\forall q\in Q. \tag{2.31}$$

The interface conditions (1.26)-(1.28) have been incorporated in the above weak model as natural conditions on Γ: in particular, (1.27) and (1.28) are natural conditions for Stokes problem, while (1.26) becomes a natural condition for Darcy's problem.

2.3 Well-Posedness of the Coupled Problem

To prove existence and uniqueness we apply the abstract theory of saddle-point problems developed in [Bre74].

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed real Hilbert spaces, and let X' and Y' be their dual spaces (i.e. the spaces of linear and continuous functionals on X and Y, respectively). We introduce two bilinear forms

$$a(\cdot,\cdot) : X \times X \to \mathbb{R}, \quad b(\cdot,\cdot) : X \times Y \to \mathbb{R} \tag{2.32}$$

such that

$$|a(v,w)| \le c_1 \|v\|_X \|w\|_X, \quad |b(w,q)| \le c_2 \|w\|_X \|q\|_Y, \qquad \forall v, w \in X,\ q \in Y. \tag{2.33}$$

We consider the following constrained problem:

find $(u, \eta) \in X \times Y$ such that

$$\begin{array}{rcll} a(u,v) + b(v,\eta) & = & \langle l, v\rangle & \forall v \in X \\ b(u,q) & = & \langle \sigma, q\rangle & \forall q \in Y \end{array} \tag{2.34}$$

where $l \in X'$ and $\sigma \in Y'$, and $\langle\cdot,\cdot\rangle$ denotes the duality pairing between X' and X, or Y and Y'. Then, the following result can be proved (see [Bre74, BF91]).

Theorem 2.3.1. *Assume that the following hypotheses hold true:*

1. *the bilinear form $a(\cdot,\cdot)$ satisfies (2.33) and there exists a positive constant $c_3 > 0$ such that*

$$a(v,v) \ge c_3 \|v\|_X^2 \quad \forall v \in X^0 \tag{2.35}$$

   where

$$X^0 = \{v \in X |\ b(v,q) = 0 \quad \forall q \in Y\}; \tag{2.36}$$

2. *the bilinear form $b(\cdot,\cdot)$ satisfies (2.33);*
3. *the following compatibility condition (*inf-sup *or Ladyzhenskaya-Babuška-Brezzi (LBB) condition) holds: there exists a constant $\beta^* > 0$ such that*

$$\forall q \in Y \quad \exists v \in X,\ v \ne 0 : \quad b(v,q) \ge \beta^* \|v\|_X \|q\|_Y . \tag{2.37}$$

Then, for each $l \in X'$, $\sigma \in Y'$, there exists a unique solution $(u, \eta) \in X \times Y$ to (2.34); furthermore, the map $(l, \sigma) \to (u, \eta)$ is an isomorphism from $X' \times Y'$ onto $X \times Y$ and

$$\|u\|_X \le \frac{1}{c_3}\left(\|l\|_{X'} + \frac{c_1 + c_3}{\beta^*}\|\sigma\|_{Y'}\right) \tag{2.38}$$

$$\|\eta\|_Y \le \frac{1}{\beta^*}\left[\left(1 + \frac{c_1}{c_3}\right)\|l\|_{X'} + \frac{c_1(c_1+c_3)}{c_3\beta^*}\|\sigma\|_{Y'}\right] . \tag{2.39}$$

In order to apply Theorem 2.3.1 in our case, we need to prove some preliminary results concerning the bilinear forms $\mathcal{A}$ and $\mathcal{B}$ and the functionals $\mathcal{F}$ and $\mathcal{G}$.

Lemma 2.3.1. *The following results hold:*

1. $\mathcal{A}(.,.)$ *is continuous and coercive on* W *and, in particular, it is coercive on the space*

$$W^0 = \{\underline{v} \in W \,|\, \mathcal{B}(\underline{v}, q) = 0 \quad \forall q \in Q\} \ ;$$

2. $\mathcal{B}(.,.)$ *is continuous on* $W \times Q$ *and satisfies the following inf-sup condition: there exists a positive constant* $\beta > 0$ *such that* $\forall q \in Q \ \exists \underline{w} \in W$ *such that*

$$\mathcal{B}(\underline{w}, q) \geq \beta \|\underline{w}\|_W \|q\|_{0,f} \ . \tag{2.40}$$

3. $\mathcal{F}$ *is a continuous linear functional on* W.
4. $\mathcal{G}$ *is a continuous linear functional on* Q.

Proof. 1. Thanks to the Cauchy-Schwarz inequality and to the trace inequality (2.3) we have

$$\begin{aligned} |\mathcal{A}(\underline{v}, \underline{w})| \ \leq \ & n\nu \|\mathbf{v}\|_{1,f} \|\mathbf{w}\|_{1,f} + g \max_j \|\mathsf{K}_j\|_{\infty,p} \|\psi\|_{1,p} \|\varphi\|_{1,p} \\ & + ngC^1_{tr,f} C^1_{tr,p} \|\varphi\|_{1,p} \|\mathbf{w}\|_{1,f} + ngC^1_{tr,f} C^1_{tr,p} \|\psi\|_{1,p} \|\mathbf{v}\|_{1,f} \\ & + n(d-1)(\nu/\varepsilon)(C^1_{tr,f})^2 \|\mathbf{v}\|_{1,f} \|\mathbf{w}\|_{1,f}. \end{aligned}$$

We define

$$\gamma = \max\{\gamma_1, \gamma_2\} \tag{2.41}$$

where

$$\begin{aligned} \gamma_1 &= \max\{n\nu + n(d-1)(C^1_{tr,f})^2(\nu/\varepsilon), ngC^1_{tr,f} C^1_{tr,p}\}, \\ \gamma_2 &= \max\{g \max_j \|\mathsf{K}_j\|_{\infty,p}, ngC^1_{tr,f} C^1_{tr,p}\}, \end{aligned}$$

so that

$$\begin{aligned} |\mathcal{A}(\underline{v}, \underline{w})| \ &\leq \ \gamma(\|\mathbf{v}\|_{1,f} + \|\varphi\|_{1,p})(\|\mathbf{w}\|_{1,f} + \|\psi\|_{1,p}) \\ &\leq \ 2\gamma \|\underline{v}\|_W \|\underline{w}\|_W \end{aligned} \tag{2.42}$$

where (2.42) follows from the inequality

$$(x + y) \leq \sqrt{2}(x^2 + y^2)^{1/2}, \quad \forall x, y \in \mathbb{R}^+ \, .$$

The coercivity is a consequence of the Poincaré inequality; in fact we have, for all $\underline{v} = (\mathbf{v}, \varphi) \in W$,

$$\begin{aligned} \mathcal{A}(\underline{v}, \underline{v}) \ &= \ n\, a_f(\mathbf{v}, \mathbf{v}) + g\, a_p(\varphi, \varphi) + \int_\Gamma n \sum_j^{d-1} \frac{\nu}{\varepsilon} (\mathbf{v} \cdot \boldsymbol{\tau}_j)^2 \\ &\geq \ n\, a_f(\mathbf{v}, \mathbf{v}) + g\, a_p(\varphi, \varphi) \\ &\geq \ n\nu \min\left(1, \frac{1}{C_{\Omega_f}}\right) \|\mathbf{v}\|^2_{1,f} + g m_K \min\left(1, \frac{1}{C_{\Omega_p}}\right) \|\varphi\|^2_{1,p} \\ &\geq \ \alpha \|\underline{v}\|^2_W, \end{aligned}$$

where

$$\alpha = \min\left\{ n\nu \min\left(1, \frac{1}{C_{\Omega_f}}\right), g m_K \min\left(1, \frac{1}{C_{\Omega_p}}\right)\right\} , \tag{2.43}$$

$$m_K = \min_{i=1,\dots,d} \inf_{\mathbf{x}\in\Omega_p} \mathsf{K}_i(\mathbf{x}), \quad (m_K > 0) \tag{2.44}$$

and C_{Ω_f}, C_{Ω_p} are the constants from the Poincaré inequality. Finally, since $W^0 \subset W$, the thesis follows.

2. Concerning continuity, thanks to the Cauchy–Schwarz inequality, we have

$$|\mathcal{B}(\underline{w}, q)| \le n\|q\|_{0,f}\|\underline{w}\|_W, \qquad \text{for all} \quad \underline{w} \in W,\, q \in Q\ .$$

Moreover, thanks to (2.18), there exists a constant $\beta_* > 0$ such that $\forall q \in Q \quad \exists \mathbf{w} \in H_f$, $\mathbf{w} \neq 0$, such that

$$-\int_{\Omega_f} q\, \nabla\cdot \mathbf{w} \ge \beta_* \|\mathbf{w}\|_{1,f}\|q\|_{0,f}\ . \tag{2.45}$$

Then, considering $\underline{w} = (\mathbf{w}, 0) \in H_f \times H_p$, the result follows with $\beta = n\beta_* > 0$.

3. Thanks to the Cauchy–Schwarz inequality, the trace inequality (2.3) and the continuity of the extension operators E_f and E_p, whose continuity constants are denoted hereafter by C_1 and C_2, respectively, we have

$$\begin{aligned} |\langle \mathcal{F}, \underline{w}\rangle| &\le n\|\mathbf{f}\|_{0,f}\|\mathbf{w}\|_{1,f} + n\nu C_1 \|\mathbf{u}_{in}\|_{H^{1/2}(\Gamma_f^{in})}\|\mathbf{w}\|_{1,f} \\ &\quad + g \max_j \|\mathsf{K}_j\|_{\infty,p} C_2 \|\psi\|_{1,p}\|\varphi\|_{H^{1/2}(\Gamma_p^b)} \\ &\le C_{\mathcal{F}}(\|\mathbf{w}\|_{1,f} + \|\varphi\|_{1,p}) \\ &\le \sqrt{2} C_{\mathcal{F}} \|\underline{w}\|_W\ , \end{aligned}$$

where

$$C_{\mathcal{F}} = \max\{ n\|\mathbf{f}\|_{0,f} + C_1 n\nu \|\mathbf{u}_{in}\|_{H^{1/2}(\Gamma_f^{in})}, g C_2 \max_j \|\mathsf{K}_j\|_{\infty,p}\|\varphi_p\|_{H^{1/2}(\Gamma_p^b)}\}. \tag{2.46}$$

4. The continuity of the functional $\mathcal{G}$ follows from the Cauchy-Schwarz inequality and of the continuity of the extension operator E_f, in fact it holds:

$$|\langle \mathcal{G}, q\rangle| \le C_{\mathcal{G}} \|q\|_{0,f}\,, \tag{2.47}$$

with $C_{\mathcal{G}} = C_1 n \|\mathbf{u}_{in}\|_{H^{1/2}(\Gamma_f^{in})}$. □

We can now prove the main result of this section.

Proposition 2.3.1. *The Stokes/Darcy coupled problem (2.30), (2.31) admits a unique solution* $(\mathbf{u}_f^0, p_f, \varphi_0) \in H_f \times Q \times H_p$ *which satisfy the following a-priori estimates:*

$$\|(\mathbf{u}_f^0, \varphi_0)\|_W \le \frac{1}{\alpha}\left(\sqrt{2} C_{\mathcal{F}} + \frac{\alpha + 2\gamma}{\beta} C_{\mathcal{G}}\right),$$

$$\|p_f\|_{0,f} \leq \frac{1}{\beta}\left[\left(1+\frac{2\gamma}{\alpha}\right)\sqrt{2}C_{\mathcal{F}} + \frac{2\gamma(\alpha+2\gamma)}{\alpha\beta}C_{\mathcal{G}}\right],$$

where β, γ, α, $C_{\mathcal{F}}$ and $C_{\mathcal{G}}$ are the constants defined in (2.40), (2.41), (2.43), (2.46) and (2.47), respectively.

Proof. It is a straightforward consequence of Theorem 2.3.1, whose hypotheses are satisfied thanks to Lemma 2.3.1. □

Remark 2.3.1. The analysis we have just presented can be replicated with minor changes if we consider as interface condition

$$\boldsymbol{\tau}_j \cdot \nabla \mathbf{u}_f \cdot \mathbf{n} = 0 \text{ on } \Gamma$$

instead of (1.28) as already pointed out at the end of Sect. 1.2. In that case we need to slightly modify the definition of the functional spaces and, in particular, we should replace the space $\widetilde{H}_f$ introduced in (2.10) by the space

$$H'_f = \{\mathbf{v} \in (H^1(\Omega_f))^d |\ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_f,\ \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \Gamma\}.$$

Then, the structure of the analysis remains essentially the same. For all the details we refer to [DMQ02, DQ03]. □

Remark 2.3.2. In our approach we have chosen to rewrite Darcy's equation in form of the Poisson problem (1.21). Should we keep the mixed formulation (1.10), (1.11) a well-posedness analysis can be developed as well; we refer to the recent work [LSY03]. The authors study a Stokes/Darcy coupling analogous to ours, still adopting the interface conditions proposed by Jäger and Mikelić, however they use the mixed form of Darcy's equations and realize the coupling via Lagrange multipliers. In particular, they introduce the Lagrange multiplier $\ell \in \Lambda$:

$$\ell = -\nu\frac{\partial \mathbf{u}_f}{\partial \mathbf{n}}\cdot\mathbf{n} + p_f = p_p \quad \text{on } \Gamma,$$

and the dual pairing

$$b_\Gamma(\underline{v}, \ell) = \langle \mathbf{v}_1\cdot\mathbf{n} + \mathbf{v}_2\cdot\mathbf{n}, \ell\rangle\ :\ (H_f \times X_2)\times\Lambda \to \mathbb{R}$$

where X_2 is a suitable subspace of $H(div;\Omega_p)$ accounting for the boundary conditions and we have denoted $\underline{v} = (\mathbf{v}_1, \mathbf{v}_2)$.
Then, they guarantee existence and uniqueness of the solution of the global mixed problem: find $\underline{u} = (\mathbf{u}_f, \mathbf{u}_p) \in H_f \times X_2$, $\underline{p} = (p_f, p_p) \in M$, $\ell \in \Lambda$:

$$a(\underline{u},\underline{v}) + b(\underline{v},\underline{p}) + b_\Gamma(\underline{v},\ell) = f(\underline{v}) \qquad \forall \underline{v} \in H_f \times X_2 \tag{2.48}$$

$$b(\underline{u},\underline{q}) = g(\underline{q}) \qquad \forall \underline{q} \in M \tag{2.49}$$

$$b_\Gamma(\underline{u},\sigma) = 0 \qquad \forall \sigma \in \Lambda \tag{2.50}$$

with

$$a(\underline{u},\underline{v}) = a_f(\mathbf{u}_f,\mathbf{v}_f) + \int_\Gamma \sum_{j=1}^{d-1} \frac{\nu}{\varepsilon}(\mathbf{u}_f\cdot\boldsymbol{\tau}_j)(\mathbf{v}_f\cdot\boldsymbol{\tau}_j) + \int_{\Omega_p} \mathsf{K}^{-1}\mathbf{u}_p\,\mathbf{v}_p$$

$$b(\underline{v},\underline{p}) = b_f(\mathbf{v}_f,p_f) - \int_{\Omega_p} p_p\nabla\cdot\mathbf{v}_p$$

and f, g are suitably defined linear continuous functionals. Finally, M is a subspace of $Q\times L^2(\Omega_p)$. If the computational domain is such that $\overline{\Gamma}\cap\partial\Omega = \emptyset$, i.e. if the porous medium is entirely enclosed in the fluid region, then (2.48)-(2.50) can be equivalently restated on the subspace of $H_f\times X_2$ with trace continuous normal velocities: $\{\underline{v}\in H_f\times X_2 | b_\Gamma(\underline{v},\sigma) = 0\ \forall\sigma\in\Lambda\}\subset H_f\times X_2$. □

Remark 2.3.3. An alternative "global" approach could be adopted to treat the Stokes/Darcy coupling by considering only (1.27) and (1.28) as interface conditions and introducing a suitable product space, say H, on Ω endowed with the following norm

$$\|(\mathbf{u},p)\|_H = \left(\int_\Omega \tilde{\nu}|\nabla\mathbf{u}|^2 + \frac{\tilde{\mu}}{\varepsilon}|\mathbf{u}|^2 + |\nabla\cdot\mathbf{u}|^2 + |p|^2\right)^{1/2},$$

where $\tilde{\nu} = \nu$ in Ω_f, $\tilde{\nu} = 0$ in Ω_p, and $\tilde{\mu} = 0$ in Ω_f, $\tilde{\mu} = \mu$ in Ω_p. Notice that in this case the continuity of the normal velocities across Γ would be guaranteed by the definition of the norm $\|\cdot\|_H$ ([Ago04]). □

2.4 Multidomain Formulation of the Coupled Problem

After proving the well-posedness of the Stokes/Darcy problem, we aim at setting up effective methods to compute numerically its solution. As we shall illustrate in chapter 3, a discretization of this problem using e.g. finite elements leads to a large sparse ill-conditioned linear system which requires a suitable preconditioning strategy to be solved.
Moreover, we would like to exploit the intrinsic decoupled structure of the problem at hand to design an iterative procedure requiring at each step to compute independently the solution of the fluid and of the groundwater problems.
Therefore, in the next sections we shall apply a domain decomposition technique at the differential level to study the Stokes/Darcy coupled problem. Our aim will be to introduce and analyze the Steklov-Poincaré interface equation associated to our problem, in order to reformulate it solely in terms of interface unknowns. This re-interpretation will be crucial to set up iterative procedures between the subdomains Ω_f and Ω_p, that will be later replicated at the discrete level.

In this section we start by rewriting the Stokes/Darcy problem in a multidomain formulation and, in particular, we prove the following result.

Proposition 2.4.1. *Let Λ be the space of traces (2.15). Problem (2.30), (2.31) can be reformulated in an equivalent way as follows: find $\mathbf{u}_f^0 \in H_f$, $p_f \in Q$, $\varphi_0 \in H_p$ such that*

$$a_f(\mathbf{u}_f^0 + E_f\mathbf{u}_{in}, \mathbf{w}) + b_f(\mathbf{w}, p_f) + \int_\Gamma n \sum_{j=1}^{d-1} \frac{\nu}{\varepsilon}(\mathbf{u}_f^0 \cdot \boldsymbol{\tau}_j)(R_1\mu \cdot \boldsymbol{\tau}_j) = \int_{\Omega_f} \mathbf{f}\,\mathbf{w} \qquad \forall \mathbf{w} \in H_f^0 \tag{2.51}$$

$$b_f(\mathbf{u}_f^0 + E_f\mathbf{u}_{in}, q) = 0 \qquad \forall q \in Q \tag{2.52}$$

$$a_p(\varphi_0 + E_p\varphi_p, \psi) = 0 \qquad \forall \psi \in H_p^0 \tag{2.53}$$

$$\int_\Gamma n\,(\mathbf{u}_f^0 \cdot \mathbf{n})\,\mu = a_p(\varphi_0 + E_p\varphi_p, R_2\mu) \qquad \forall \mu \in \Lambda \tag{2.54}$$

$$\int_\Gamma g\varphi_0\mu = \int_{\Omega_f} \mathbf{f}\,(R_1\mu) - a_f(\mathbf{u}_f^0 + E_f\mathbf{u}_{in}, R_1\mu) - b_f(R_1\mu, p_f) - \int_\Gamma \sum_{j=1}^{d-1} \frac{\nu}{\varepsilon}(\mathbf{u}_f^0 \cdot \boldsymbol{\tau}_j)(R_1\mu \cdot \boldsymbol{\tau}_j) \quad \forall \mu \in \Lambda\,, \tag{2.55}$$

where R_1 is any possible extension operator from Λ to H_f, i.e., a continuous operator from Λ to H_f such that $(R_1\mu)\cdot\mathbf{n} = \mu$ on Γ for all $\mu \in \Lambda$, and R_2 is any possible continuous extension operator from $\Lambda_\dagger$ to H_p such that $R_2\mu = \mu$ on Γ for all $\mu \in \Lambda_\dagger$.

Proof. Let $(\underline{u}, p) \in W \times Q$ be the solution to (2.30), (2.31). Considering in (2.30) as test functions $(\underline{w}, \psi) \in H_f^0 \times H_p^0$, we obtain (2.51) and (2.53). Moreover, (2.31) implies (2.52).
Now let $\mu \in \Lambda$, $R_1\mu \in H_f$, and $R_2\mu \in H_p$. From (2.30) we have:

$$\begin{aligned} &n\,a_f(\mathbf{u}_f^0 + E_f\mathbf{u}_{in}, R_1\mu) - \int_{\Omega_f} n\mathbf{f}(R_1\mu) + g\,a_p(\varphi_0 + E_p\varphi_p, R_2\mu) \\ &- \int_\Gamma ng(\mathbf{u}_f^0 \cdot \mathbf{n})\mu + nb_f(R_1\mu, p_f) \\ &+ \int_\Gamma n \sum_{j=1}^{d-1} \frac{\nu}{\varepsilon}(\mathbf{u}_f^0 \cdot \boldsymbol{\tau}_j)(R_1\mu \cdot \boldsymbol{\tau}_j) = -\int_\Gamma ng\varphi_0\mu, \end{aligned}$$

so that (2.54) and (2.55) are satisfied.

Consider now two arbitrary functions $\mathbf{w} \in H_f$, $\psi \in H_p$ and let us indicate by μ the normal trace of $\mathbf{w}$ on Γ, i.e. $\mathbf{w}\cdot\mathbf{n}_{|\Gamma} = \mu \in \Lambda$, and by η the trace of ψ on Γ, that is $\psi_{|\Gamma} = \eta \in H^{1/2}(\Gamma)$. Then $(\mathbf{w} - R_1\mu) \in H_f^0$ and $(\psi - R_2\eta) \in H_p^0$. Setting $\underline{u} = (\mathbf{u}_f^0, \varphi_0)$ and $\underline{v} = (\mathbf{w}, \psi)$ we have:

$$
\begin{aligned}
\mathcal{A}(\underline{u},\underline{v})+\mathcal{B}(\underline{v},p) \;=\;& n\,a_f(\mathbf{u}_f^0,\mathbf{w}-R_1\mu)+nb_f(\mathbf{w}-R_1\mu,p_f)\\
&+\int_\Gamma n\sum_{j=1}^{d-1}\frac{\nu}{\varepsilon}(\mathbf{u}_f^0\cdot\boldsymbol{\tau}_j)((\mathbf{w}-R_1\mu)\cdot\boldsymbol{\tau}_j)\\
&+ga_p(\varphi_0,\psi-R_2\eta)+\int_\Gamma ng\varphi_0(\mathbf{w}-R_1\mu)\cdot\mathbf{n}\\
&-\int_\Gamma ng(\psi-R_2\eta)(\mathbf{u}_f^0\cdot\mathbf{n})\\
&+na_f(\mathbf{u}_f^0,R_1\mu)+nb_f(R_1\mu,p_f)\\
&+\int_\Gamma n\sum_{j=1}^{d-1}\frac{\nu}{\varepsilon}(\mathbf{u}_f^0\cdot\boldsymbol{\tau}_j)(R_1\mu\cdot\boldsymbol{\tau}_j)\\
&+\int_\Gamma ng\varphi_0(R_1\mu\cdot\mathbf{n})+ga_p(\varphi_0+E_p\varphi_p,R_2\eta)\\
&-ga_p(E_p\varphi_p,R_2\mu)-\int_\Gamma ng(R_2\eta)(\mathbf{u}_f^0\cdot\mathbf{n})\;.
\end{aligned}
$$

Then, using (2.51) and (2.53)-(2.55) we obtain:

$$
\begin{aligned}
\mathcal{A}(\underline{u},\underline{v})+\mathcal{B}(\underline{v},p) \;=\;& \int_{\Omega_f} n\mathbf{f}\,(\mathbf{w}-R_1\mu)-na_f(E_f\mathbf{u}_{in},\mathbf{w}-R_1\mu)\\
&-ga_p(E_p\varphi_p,\psi-R_2\mu)+\int_{\Omega_f} n\mathbf{f}\,(R_1\mu)-na_f(E_f\mathbf{u}_{in},R_1\mu)\\
&+\int_\Gamma ng(\mathbf{u}_f^0\cdot\mathbf{n})\eta-\int_\Gamma ng(\mathbf{u}_f^0\cdot\mathbf{n})\eta\\
&-ga_p(E_p\varphi_p,R_2\eta)
\end{aligned}
$$

and, recalling the definition (2.28) of the functional $\mathcal{F}$, we find that $\underline{u}=(\mathbf{u}_f^0,\varphi_0)$ and p_f satisfy (2.30), for all $\mathbf{w}\in H_f$, $\psi\in H_p$.
The proof is completed by observing that (2.31) follows from (2.52). □

Now we have to choose a suitable governing variable on the interface Γ. Considering the interface conditions (1.26) and (1.27) which couple the Stokes and Darcy subproblems, we can foresee two different strategies to select the interface variable:

1. we can set the interface variable λ as the trace of the normal velocity on the interface:

$$\lambda=\mathbf{u}_f\cdot\mathbf{n}=-\frac{\mathsf{K}}{n}\frac{\partial\varphi}{\partial\mathbf{n}} \tag{2.56}$$

2. we can define the interface variable σ as the trace of the piezometric head on Γ:

$$\sigma=\varphi=\frac{1}{g}\left(-\nu\frac{\partial\mathbf{u}_f}{\partial\mathbf{n}}+p_f\mathbf{n}\right)\cdot\mathbf{n}. \tag{2.57}$$

Both choices are suitable from the mathematical viewpoint since they guarantee well-posed subproblems in the fluid and the porous part. We shall analyze the interface equations corresponding to both λ and σ and the correlated preconditioned substructuring methods.

The role played in this context by the interface variables λ and σ is quite different than the classical cases encountered in domain decomposition. We clarify this point on a test example. Consider the Poisson problem $-\triangle u = f$ on a domain split into two nonoverlapping subdomains. The interface conditions are

$$u_1 = u_2 \quad \text{and} \quad \frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} \qquad \text{on the interface.}$$

We have therefore two possible choices of the interface variable, say $\tilde{\lambda}$:

1. $\tilde{\lambda} = u_1 = u_2$ on the interface: this is the classical approach (see [QV99] chapter 1) which gives the usual Steklov-Poincaré equation in $\tilde{\lambda}$ featuring the so-called Dirichlet-to-Neumann maps. Note that $\tilde{\lambda}$ provides a Dirichlet boundary condition on the interface for both subproblems.
2. $\tilde{\lambda} = \partial u_1/\partial n = \partial u_2/\partial n$: this is the so-called FETI approach (see [TW04] chapter 1) which can be seen as dual to the one recalled in 1. In this case the value of $\tilde{\lambda}$ provides a Neumann boundary condition on the interface for the two subproblems.

After computing $\tilde{\lambda}$, we have to solve in both cases the same kind of boundary value problem in the subdomains to recover the global solution.
For the Stokes/Darcy problem this is no longer true: in fact, should we know λ on Γ, then we would have to solve a "Dirichlet" problem in Ω_f and a Neumann problem in Ω_p. On the other hand, choosing σ as interface variable would lead to consider a Stokes problem in Ω_f with a Neumann boundary condition on Γ, and a Darcy problem in Ω_p with a Dirichlet boundary condition on Γ.
This behaviour is due to the heterogeneity of the coupling itself and it will strongly influence the construction of the Steklov-Poincaré operators that will not play the role of Dirichlet-to-Neumann maps for both subdomains as in the Laplace case.
We have encountered an analogous asymmetry in the interface conditions when dealing with an heterogeneous fluid-structure coupling (see [DDQ04]).

2.5 Interface Equation for the Normal Velocity

We consider as governing variable on the interface Γ the normal component of the velocity field $\lambda = \mathbf{u}_f \cdot \mathbf{n}$ as indicated in (2.56).
Should we know a priori the value of λ on Γ, from (2.56) we would obtain a Dirichlet boundary condition for the Stokes system in Ω_f ($\mathbf{u}_f \cdot \mathbf{n} = \lambda$ on Γ) and a Neumann boundary condition for the Darcy equation in Ω_p ($-(\mathsf{K}\nabla\varphi \cdot \mathbf{n})/n = \lambda$ on Γ).
Joint with (1.28) for the fluid problem, these conditions allow us to recover (independently) the solutions $(\mathbf{u}_f, p_f)$ of the Stokes problem in Ω_f and the solution φ of the Darcy problem in Ω_p.

For simplicity, from now on we consider the following condition on the interface:

$$\mathbf{u}_f \cdot \boldsymbol{\tau}_j = 0 \quad \text{on } \Gamma \tag{2.58}$$

instead of (1.28). This simplification is acceptable from the physical viewpoint as discussed in Sect. 1.2 and it does not dramatically influence the coupling of the two subproblems since, as we have already pointed out, condition (1.28) is not strictly a coupling condition but only a boundary condition for the fluid problem in Ω_f.

Remark 2.5.1. Using the simplified condition (2.58), the multidomain formulation of the Stokes/ Darcy problem (2.51)-(2.52) becomes:

find $\mathbf{u}_f^0 \in H_f^\tau$, $p_f \in Q$, $\varphi_0 \in H_p$ such that

$$a_f(\mathbf{u}_f^0 + E_f\mathbf{u}_{in}, \mathbf{w}) + b_f(\mathbf{w}, p_f) = \int_{\Omega_f} \mathbf{f}\,\mathbf{w} \qquad \forall \mathbf{w} \in (H_0^1(\Omega_f))^d \tag{2.59}$$

$$b_f(\mathbf{u}_f^0 + E_f\mathbf{u}_{in}, q) = 0 \qquad \forall q \in Q \tag{2.60}$$

$$a_p(\varphi_0 + E_p\varphi_p, \psi) = 0 \qquad \forall \psi \in H_p^0 \tag{2.61}$$

$$\int_\Gamma n(\mathbf{u}_f^0 \cdot \mathbf{n})\mu = a_p(\varphi_0 + E_p\varphi_p, R_2\mu) \qquad \forall \mu \in \Lambda \tag{2.62}$$

$$\begin{aligned}\int_\Gamma g\varphi_0\mu = &\int_{\Omega_f} \mathbf{f}\,(R_1^\tau\mu) - a_f(\mathbf{u}_f^0 + E_f\mathbf{u}_{in}, R_1^\tau\mu) \\ &-b_f(R_1^\tau\mu, p_f) \qquad \forall \mu \in \Lambda\end{aligned} \tag{2.63}$$

with R_2 defined as in Proposition 2.4.1, and $R_1^\tau : \Lambda \to H_f^\tau$ is any possible continuous extension operator from Λ to H_f^τ such that $R_1^\tau\mu \cdot \mathbf{n} = \mu$ on Γ for all $\mu \in \Lambda$, with

$$H_f^\tau = \{\mathbf{v} \in H_f |\, \mathbf{v} \cdot \boldsymbol{\tau}_j = 0 \text{ on } \Gamma\}. \tag{2.64}$$

□

We define the continuous extension operator

$$E_\Gamma : H^{1/2}(\Gamma) \to H_f^\tau, \quad \eta \to E_\Gamma\eta \quad \text{s.t.} \quad E_\Gamma\eta \cdot \mathbf{n} = \eta \quad \text{on } \Gamma. \tag{2.65}$$

We consider the (unknown) interface variable $\lambda = \mathbf{u}_f \cdot \mathbf{n}$ on Γ, $\lambda \in \Lambda$, and we split it as $\lambda = \lambda_0 + \lambda_*$ where $\lambda_* \in \Lambda$ depends on the inflow data and satisfies

$$\int_\Gamma \lambda_* = -\int_{\Gamma_f^{in}} \mathbf{u}_{in} \cdot \mathbf{n}\,, \tag{2.66}$$

whereas $\lambda_0 \in \Lambda_0$, with

$$\Lambda_0 = \left\{ \mu \in \Lambda \,\middle|\, \int_\Gamma \mu = 0 \right\} \subset \Lambda\,. \tag{2.67}$$

Then, after defining the subspace of Q:

$$Q_0 = \left\{ q \in Q \,\middle|\, \int_{\Omega_f} q = 0 \right\} \tag{2.68}$$

we introduce two auxiliary problems whose solutions (which depend on the problem data) are related to that of the global problem (2.59)-(2.63), as we will see later on:

P1) find $\boldsymbol{\omega}_0^* \in (H_0^1(\Omega_f))^d$, $\pi^* \in Q_0$ such that

$$a_f(\boldsymbol{\omega}_0^* + E_f \mathbf{u}_{in} + E_\Gamma \lambda_*, \mathbf{v}) + b_f(\mathbf{v}, \pi^*) = \int_{\Omega_f} \mathbf{f}\,\mathbf{v} \quad \forall \mathbf{v} \in (H_0^1(\Omega_f))^d \tag{2.69}$$

$$b_f(\boldsymbol{\omega}_0^* + E_f \mathbf{u}_{in} + E_\Gamma \lambda_*, q) = 0 \qquad \forall q \in Q_0; \tag{2.70}$$

P2) find $\varphi_0^* \in H_p$ such that

$$a_p(\varphi_0^* + E_p \varphi_p, \psi) = \int_\Gamma n \lambda_* \psi \qquad \forall \psi \in H_p. \tag{2.71}$$

Now we define the following extension operators:

$$R_f \colon \Lambda_0 \to H_f^\tau \times Q_0, \quad \eta \to R_f \eta = (R_f^1 \eta, R_f^2 \eta)$$

such that $(R_f^1 \eta) \cdot \mathbf{n} = \eta$ on Γ and

$$a_f(R_f^1 \eta, \mathbf{v}) + b_f(\mathbf{v}, R_f^2 \eta) = 0 \qquad \forall \mathbf{v} \in (H_0^1(\Omega_f))^d \tag{2.72}$$

$$b_f(R_f^1 \eta, q) = 0 \qquad \forall q \in Q_0\ ; \tag{2.73}$$

$$R_p \colon \Lambda \to H_p\ , \quad \eta \to R_p \eta$$

such that

$$a_p(R_p \eta, R_2 \mu) = \int_\Gamma n \eta \mu \qquad \forall \mu \in \Lambda_\dagger. \tag{2.74}$$

We define the *Steklov-Poincaré* operator S as follows: for all $\eta \in \Lambda_0$, $\mu \in \Lambda$,

$$\langle S\eta, \mu \rangle = a_f(R_f^1 \eta, R_1^\tau \mu) + b_f(R_1^\tau \mu, R_f^2 \eta) + \int_\Gamma g(R_p \eta)\mu \tag{2.75}$$

which can be split as the sum of two suboperators $S = S_f + S_p$:

$$\langle S_f \eta, \mu \rangle = a_f(R_f^1 \eta, R_1^\tau \mu) + b_f(R_1^\tau \mu, R_f^2 \eta)\ , \tag{2.76}$$

$$\langle S_p \eta, \mu \rangle = \int_\Gamma g\,(R_p \eta)\mu \tag{2.77}$$

for all $\eta \in \Lambda_0$ and $\mu \in \Lambda$.

Moreover, we define the functional $\chi : \Lambda_0 \to \mathbb{R}$,

$$\begin{aligned}\langle \chi, \mu \rangle &= \int_{\Omega_f} \mathbf{f}\,(R_1^\tau \mu) - a_f(\boldsymbol{\omega}_0^* + E_f \mathbf{u}_{in} + E_\Gamma \lambda_*, R_1^\tau \mu) \\ &\quad - b_f(R_1^\tau \mu, \pi^*) - \int_\Gamma g\, \varphi_0^* \mu \end{aligned} \tag{2.78}$$

for all $\mu \in \Lambda$.

Now we can express the solution of the coupled problem in terms of the interface variable λ_0; precisely, we can prove the following result.

Theorem 2.5.1. *The solution to (2.59)-(2.63) can be characterized as follows:*

$$\mathbf{u}_f^0 = \boldsymbol{\omega}_0^* + R_f^1 \lambda_0 + E_\Gamma \lambda_*, \quad p_f = \pi^* + R_f^2 \lambda_0 + \hat{p}_f, \quad \varphi_0 = \varphi_0^* + R_p \lambda_0 \ , \tag{2.79}$$

where $\hat{p}_f = (meas(\Omega_f))^{-1} \int_{\Omega_f} p_f$ *and* $\lambda_0 \in \Lambda_0$ *is the solution of the following Steklov-Poincaré problem:*

$$\langle S\lambda_0, \mu_0 \rangle = \langle \chi, \mu_0 \rangle \qquad \forall \mu_0 \in \Lambda_0 \ . \tag{2.80}$$

Moreover, $\hat{p}_f$ *can be obtained from* λ_0 *by solving the algebraic equation*

$$\hat{p}_f = \frac{1}{meas(\Gamma)} \langle S\lambda_0 - \chi, \zeta \rangle \ , \tag{2.81}$$

where $\zeta \in \Lambda$ *is a fixed function such that*

$$\frac{1}{meas(\Gamma)} \int_\Gamma \zeta = 1 \ . \tag{2.82}$$

Proof. Thanks to the divergence theorem, for all constant functions c,

$$b_f(\mathbf{w}, c) = c \int_{\partial\Omega_f} \mathbf{w} \cdot \mathbf{n} = 0 \qquad \forall \mathbf{w} \in (H_0^1(\Omega_f))^d.$$

Then, by direct inspection, the functions defined in (2.79) satisfy (2.59), (2.61) and (2.62). Moreover (2.60) is satisfied too. Indeed, $\forall q \in Q$

$$\begin{aligned} b_f(\boldsymbol{\omega}_0^* + R_f^1 \lambda_0 + E_\Gamma \lambda_* + E_f \mathbf{u}_{in}, q) &= b_f(\boldsymbol{\omega}_0^* + R_f^1 \lambda_0 + E_\Gamma \lambda_* + E_f \mathbf{u}_{in}, q - \overline{q}) \\ &\quad + b_f(\boldsymbol{\omega}_0^* + R_f^1 \lambda_0 + E_\Gamma \lambda_* + E_f \mathbf{u}_{in}, \overline{q}) \end{aligned}$$

where $\overline{q}$ is the constant $\overline{q} = (meas(\Omega_f))^{-1} \int_{\Omega_f} q$. Still using the divergence theorem,

$$b_f(\boldsymbol{\omega}_0^* + R_f^1 \lambda_0 + E_\Gamma \lambda_* + E_f \mathbf{u}_{in}, \overline{q}) = \overline{q} \int_\Gamma \lambda_0 + \overline{q} \int_\Gamma \lambda_* + \overline{q} \int_{\Gamma_f^{in}} \mathbf{u}_{in} \cdot \mathbf{n}_f \ .$$

The right hand side is null thanks to (2.66) and since $\lambda_0 \in \Lambda_0$.
We now consider (2.63). Using (2.79) we obtain, $\forall \mu \in \Lambda$,

$$\begin{aligned}&\int_\Gamma g(R_p\lambda_0)\mu + a_f(R_f^1\lambda_0, R_1^\tau\mu) + b_f(R_1^\tau\mu, R_f^2\lambda_0)\\ &= \int_{\Omega_f} \mathbf{f}\,(R_1^\tau\mu) - \int_\Gamma g\varphi_0^*\mu\\ &\quad - a_f(\boldsymbol{\omega}_0^* + E_f\mathbf{u}_{in} + E_\Gamma\lambda_*, R_1^\tau\mu) - b_f(R_1^\tau\mu, \pi^*) - b_f(R_1^\tau\mu, \hat{p}_f)\ ,\end{aligned}$$

that is,

$$\langle S\lambda_0, \mu\rangle = \langle\chi, \mu\rangle - b_f(R_1^\tau\mu, \hat{p}_f) \qquad \forall\mu\in\Lambda\ . \tag{2.83}$$

In particular, if we take $\mu\in\Lambda_0\subset\Lambda$, we can invoke the divergence theorem and conclude that λ_0 is the solution to the Steklov-Poincaré equation (2.80).

Now any $\mu\in\Lambda$ can be decomposed as $\mu = \mu_0 + \mu_\Gamma\zeta$, with $\mu_\Gamma = (meas(\Gamma))^{-1}\int_\Gamma\mu$, so that $\mu_0\in\Lambda_0$.
From (2.83) we obtain

$$\langle S\lambda_0, \mu_0\rangle + \langle S\lambda_0, \mu_\Gamma\zeta\rangle = \langle\chi, \mu_0\rangle + \langle\chi, \mu_\Gamma\zeta\rangle + \hat{p}_f\int_\Gamma\mu \quad \forall\mu\in\Lambda\ .$$

Therefore, thanks to (2.80), we have

$$\mu_\Gamma\langle S\lambda_0 - \chi, \zeta\rangle = \hat{p}_f\int_\Gamma\mu \qquad \forall\mu\in\Lambda\ .$$

Since $\int_\Gamma\mu = \mu_\Gamma meas(\Gamma)$, we conclude that (2.81) holds. □

In next section we prove that (2.80) has a unique solution.

2.5.1 Analysis of the Steklov-Poincaré Operators S_f and S_p

We shall now prove some properties of the Steklov-Poincaré operators S_f, S_p and S.

Lemma 2.5.1. *The Steklov-Poincaré operators enjoy the following properties:*

1. *S_f and S_p are linear continuous operators on Λ_0 (i.e., $S_f\eta\in\Lambda_0'$, $S_p\eta\in\Lambda_0'$, $\forall\eta\in\Lambda_0$);*
2. *S_f is symmetric and coercive;*
3. *S_p is symmetric and positive.*

Proof. 1. S_f and S_p are obviously linear. Next we observe that for every $\mu\in\Lambda_0$ we can make the special choice $R_1^\tau\mu = R_f^1\mu$. Consequently, from (2.76) and (2.72) it follows that S_f can be characterized as:

$$\langle S_f\eta, \mu\rangle = a_f(R_f^1\eta, R_f^1\mu) \qquad \forall\eta,\mu\in\Lambda_0\ . \tag{2.84}$$

To prove continuity, we introduce the vector operator $\mathcal{H}\colon\Lambda_0\to H_f$, $\mu\to\mathcal{H}\mu$, such that

$$\begin{aligned}&\int_{\Omega_f}\nabla(\mathcal{H}\mu)\cdot\nabla\mathbf{v} = 0 \qquad \forall\mathbf{v}\in(H_0^1(\Omega_f))^d\\ &(\mathcal{H}\mu)\cdot\mathbf{n} = \mu \qquad \text{on } \Gamma\\ &(\mathcal{H}\mu)\cdot\boldsymbol{\tau}_j = 0 \qquad \text{on } \Gamma\ , \quad j = 1,\dots,d-1\\ &\mathcal{H}\mu = 0 \qquad \text{on } \partial\Omega_f\setminus\Gamma.\end{aligned} \tag{2.85}$$

By comparison with the operator R_f^1 introduced in (2.72), (2.73), we see that, for all $\mu \in \Lambda_0$, the vector function

$$\mathbf{z}(\mu) = R_f^1\mu - \mathcal{H}\mu \tag{2.86}$$

satisfies $\mathbf{z}(\mu) = 0$ on Γ; therefore $\mathbf{z}(\mu) \in (H_0^1(\Omega_f))^d$. By taking $\mathbf{v} = \mathbf{z}(\mu)$ in (2.72), in view of the definition (2.86) we have

$$|a_f(R_f^1\mu, \mathbf{z}(\mu))| = \left|b_f(\mathcal{H}\mu, R_f^2\mu)\right| \leq \|R_f^2\mu\|_{0,f}\|\mathcal{H}\mu\|_{1,f}. \tag{2.87}$$

We now consider the function $R_f^2\mu$. Since it belongs to Q_0, there exists $\mathbf{w} \in (H_0^1(\Omega_f))^d$, $\mathbf{w} \neq \mathbf{0}$, such that

$$\beta^0\|R_f^2\mu\|_{0,f}\|\mathbf{w}\|_{1,f} \leq b_f(\mathbf{w}, R_f^2\mu)$$

where $\beta^0 > 0$ is the inf-sup constant, independent of μ (see, e.g., [BF91]). Since $\mathbf{w} \in (H_0^1(\Omega_f))^d$, we can use (2.72) and obtain:

$$\beta^0\|R_f^2\mu\|_{0,f}\|\mathbf{w}\|_{1,f} \leq |a_f(R_f^1\mu, \mathbf{w})| \leq \nu\|R_f^1\mu\|_{1,f}\|\mathbf{w}\|_{1,f}\ .$$

The last inequality follows from the Cauchy–Schwarz inequality. Therefore

$$\|R_f^2\mu\|_{0,f} \leq \frac{\nu}{\beta^0}\|R_f^1\mu\|_{1,f} \qquad \forall\mu \in \Lambda_0\ . \tag{2.88}$$

Now, using the Poincaré inequality and relations (2.86)-(2.88), we obtain:

$$\begin{aligned}
\|R_f^1\mu\|_{1,f}^2 &\leq \frac{1+C_{\Omega_f}}{\nu}a_f(R_f^1\mu, R_f^1\mu) \\
&= \frac{1+C_{\Omega_f}}{\nu}\left[a_f(R_f^1\mu, \mathbf{z}(\mu)) + a_f(R_f^1\mu, \mathcal{H}\mu)\right] \\
&\leq \frac{1+C_{\Omega_f}}{\nu}\left[\|R_f^2\mu\|_{0,f}\|\mathcal{H}\mu\|_{1,f} + \nu\|R_f^1\mu\|_{1,f}\|\mathcal{H}\mu\|_{1,f}\right] \\
&\leq (1+C_{\Omega_f})\left(1+\frac{1}{\beta^0}\right)\|R_f^1\mu\|_{1,f}\|\mathcal{H}\mu\|_{1,f}
\end{aligned}$$

for all $\mu \in \Lambda_0$. Therefore

$$\begin{aligned}
\|R_f^1\mu\|_{1,f} &\leq (1+C_{\Omega_f})\left(1+\frac{1}{\beta^0}\right)\|\mathcal{H}\mu\|_{1,f} \\
&\leq \alpha^*(1+C_{\Omega_f})\left(1+\frac{1}{\beta^0}\right)\|\mu\|_\Lambda\ .
\end{aligned} \tag{2.89}$$

The last inequality follows from the observation that $\mathcal{H}\mu$ is a harmonic extension of μ; then there exists a positive constant $\alpha^* > 0$ (independent of μ) such that

$$\|\mathcal{H}\mu\|_{1,f} \leq \alpha^*\|\mathcal{H}\mu_{|\Gamma}\|_\Lambda = \alpha^*\|\mu\|_\Lambda$$

(see, e.g., [QV99]).
Thanks to (2.89) we can now prove the continuity of S_f; in fact, for all $\mu, \eta \in \Lambda_0$, we have

$$|\langle S_f\mu,\eta\rangle| = |a_f(R_f^1\mu,R_f^1\eta)| \le \beta_f\|\mu\|_\Lambda\|\eta\|_\Lambda\ ,$$

where β_f is the positive continuity constant

$$\beta_f = \nu\left[\alpha^*(1+C_{\Omega_f})\left(1+\frac{1}{\beta^0}\right)\right]^2\ . \tag{2.90}$$

We now turn to the issue of continuity of S_p. Let m_K be the positive constant introduced in (2.44). Thanks to the Poincaré inequality and to (2.74) we have:

$$\begin{aligned}\|R_p\mu\|_{1,p}^2 &\le (1+C_{\Omega_p})\|\nabla R_p\mu\|_{0,p}^2\\ &\le \frac{1+C_{\Omega_p}}{m_K}a_p(R_p\mu,R_p\mu)\\ &= \frac{1+C_{\Omega_p}}{m_K}\int_\Gamma n(R_p\mu)_{|\Gamma}\,\mu\ .\end{aligned}$$

Finally, the Cauchy–Schwarz inequality and the trace inequality (2.3) allow us to deduce that

$$\|R_p\mu\|_{1,p} \le \frac{(1+C_{\Omega_p})}{m_K}nC_{tr,p}^1\|\mu\|_\Lambda \qquad \forall\mu\in\Lambda_0\ .$$

Then, $\forall\mu,\eta\in\Lambda_0$,

$$\begin{aligned}|\langle S_p\mu,\eta\rangle| &\le g\|R_p\mu_{|\Gamma}\|_{L^2(\Gamma)}\|\eta\|_{L^2(\Gamma)}\\ &\le gC_{tr,p}^1\|R_p\mu\|_{1,p}\|\eta\|_\Lambda \le \frac{ng(C_{tr,p}^1)^2(1+C_{\Omega_p})}{m_K}\|\mu\|_\Lambda\|\eta\|_\Lambda\ .\end{aligned}$$

Thus S_p is continuous, with continuity constant

$$\beta_p = \frac{ng(C_{tr,p}^1)^2(1+C_{\Omega_p})}{m_K}\,. \tag{2.91}$$

2. S_f is symmetric thanks to (2.84). Again using the Poincaré inequality and the trace inequality (2.5), for all $\mu\in\Lambda_0$ we obtain

$$\begin{aligned}\langle S_f\mu,\mu\rangle &\ge \min\left(\frac{\nu}{2},\frac{\nu}{2C_{\Omega_f}}\right)\|R_f^1\mu\|_{1,f}^2\\ &\ge \frac{1}{C_{tr,f}^3}\min\left(\frac{\nu}{2},\frac{\nu}{2C_{\Omega_f}}\right)\|(R_f^1\mu\cdot\mathbf{n})_{|\Gamma}\|_\Lambda^2\ =\alpha_f\|\mu\|_\Lambda^2\ ;\end{aligned}$$

thus S_f is coercive, with a coercivity constant given by

$$\alpha_f = \frac{1}{C_{tr,f}^3}\min\left(\frac{\nu}{2},\frac{\nu}{2C_{\Omega_f}}\right)\ . \tag{2.92}$$

3. S_p is symmetric: for all $\mu,\eta\in\Lambda$:

$$\begin{aligned}\langle S_p\mu,\eta\rangle &= \frac{g}{n}\int_\Gamma n(R_p\eta)_{|\Gamma}\mu = \frac{g}{n}a_p(R_p\mu,R_p\eta)\\ &= \frac{g}{n}a_p(R_p\eta,R_p\mu) = \frac{g}{n}\int_\Gamma n\eta(R_p\mu)_{|\Gamma} = \langle S_p\eta,\mu\rangle.\end{aligned}$$

Moreover, thanks to (2.74), $\forall\mu\in\Lambda_0$

$$\langle S_p\mu,\mu\rangle = \int_\Gamma g\,(R_p\mu)\mu = \frac{g}{n}a_p(R_p\mu,R_p\mu)\ .$$

On the other hand, we have

$$\begin{aligned}\|\mu\|_{\Lambda'} &= \sup_{\eta\in\Lambda_0}\frac{\langle n\mu,\eta\rangle}{n\|\eta\|_\Lambda} = \sup_{\eta\in\Lambda_0}\frac{\langle \mathsf{K}\dfrac{\partial R_p\mu}{\partial\mathbf{n}_p},\eta\rangle}{n\|\eta\|_\Lambda}\\ &= \sup_{\eta\in\Lambda_0}\frac{a_p(R_p\mu,\mathcal{H}_p\eta)}{n\|\eta\|_\Lambda} \le \sup_{\eta\in\Lambda_0}\frac{\alpha_* a_p(R_p\mu,\mathcal{H}_p\eta)}{n\|\mathcal{H}_p\eta\|_{1,p}}\\ &\le \sup_{\eta\in\Lambda_0}\frac{\alpha_*\max_j\|\mathsf{K}_j\|_{\infty,p}}{n}\cdot\frac{\|R_p\mu\|_{1,p}\|\mathcal{H}_p\eta\|_{1,p}}{\|\mathcal{H}_p\eta\|_{1,p}}\\ &= \frac{\alpha_*\max_j\|\mathsf{K}_j\|_{\infty,p}}{n}\|R_p\mu\|_{1,p}\ .\end{aligned}$$

We have denoted by Λ' the dual space of Λ_0, and by $\langle\cdot,\cdot\rangle$ the duality pairing between Λ' and Λ_0. Moreover, $\mathcal{H}_p\eta$ is the harmonic extension of η to $H^1(\Omega_p)$, i.e., the (weak) solution of the problem:

$$\begin{aligned}&\nabla\cdot(\mathsf{K}\nabla\mathcal{H}_p\eta) = 0 \qquad \text{in } \Omega_p\\ &\mathsf{K}\nabla(\mathcal{H}_p\eta)\cdot\mathbf{n}_p = 0 \qquad \text{on } \Gamma_p\\ &\mathcal{H}_p\eta = 0 \qquad \text{on } \Gamma_p^b\\ &\mathcal{H}_p\eta = \eta \qquad \text{on } \Gamma,\end{aligned}$$

and we have used the equivalence of the norms

$$\alpha_*\|\eta\|_\Lambda \le \|\mathcal{H}_p\eta\|_{1,p} \le \alpha^*\|\eta\|_\Lambda$$

(see, e.g., [Nečˇ67] or [QV99] chapter 4).
We conclude that $\langle S_p\mu,\mu\rangle \ge C\|\mu\|^2_{\Lambda'}$, for a suitable constant $C>0$. □

The following result is a straightforward consequence of Lemma 2.5.1.

Corollary 2.5.1. *The global Steklov-Poincaré operator S is symmetric, continuous and coercive. Moreover S and S_f are spectrally equivalent, i.e., there exist two positive constants k_1 and k_2 (independent of η) such that*

$$k_1\langle S_f\eta,\eta\rangle \ \le\ \langle S\eta,\eta\rangle \ \le\ k_2\langle S_f\eta,\eta\rangle \qquad \forall\eta\in\Lambda_0.$$

2.6 Interface Equation for the Trace of the Piezometric Head

Now, let us consider the interface variable σ corresponding to the trace of the piezometric head on Γ, as indicated in (2.57). In this case, if we recovered the value of σ on Γ, then we would have a Dirichlet boundary condition $\varphi = \sigma$ on Γ for Darcy's equation and a Neumann boundary condition $(-\nu\partial\mathbf{u}_f/\partial\mathbf{n} + p_f\mathbf{n})\cdot\mathbf{n} = g\varphi$ on Γ for Stokes problem. We assume again the interface condition (2.58) for Stokes problem as in Sect. 2.5.

We introduce the following auxiliary problems whose solutions depend on the data of the original problem:

Π1) find $\boldsymbol{\varpi}_0^* \in H_f^\tau$, $\overline{\pi}^* \in Q$ such that

$$a_f(\boldsymbol{\varpi}_0^* + E_f\mathbf{u}_{in}, \mathbf{v}) + b_f(\mathbf{v}, \overline{\pi}^*) = \int_{\Omega_f} \mathbf{f}\,\mathbf{v} \qquad \forall \mathbf{v} \in H_f^\tau \tag{2.93}$$

$$b_f(\boldsymbol{\varpi}_0^* + E_f\mathbf{u}_{in}, q) = 0 \qquad \forall q \in Q\,; \tag{2.94}$$

Π2) find $\phi_0^* \in H_p^0$ such that

$$a_p(\phi_0^* + E_p\varphi_p, \psi) = 0 \qquad \forall \psi \in H_p^0 \tag{2.95}$$

Then, we define the following extension operators:

$$\mathcal{R}_f : \Lambda_\dagger \to H_f^\tau \times Q, \quad \eta \to \mathcal{R}_f\eta = (\mathcal{R}_f^1\eta, \mathcal{R}_f^2\eta)$$

(where $\Lambda_\dagger$ is the space introduced in (2.15)) such that

$$a_f(\mathcal{R}_f^1\eta, \mathbf{v}) + b_f(\mathbf{v}, \mathcal{R}_f^2\eta) + \int_\Gamma g\eta\mathbf{v}\cdot\mathbf{n} = 0 \qquad \forall \mathbf{v} \in H_f^\tau \tag{2.96}$$

$$b_f(\mathcal{R}_f^1\eta, q) = 0 \qquad \forall q \in Q; \tag{2.97}$$

$$\mathcal{R}_p : \Lambda_\dagger \to H_p, \quad \eta \to \mathcal{R}_p\eta$$

such that $\mathcal{R}_p\eta = \eta$ on Γ and

$$a_p(\mathcal{R}_p\eta, \psi) = 0 \qquad \forall \psi \in H_p^0. \tag{2.98}$$

We define the *Steklov-Poincaré* operator $\mathcal{S}$ as follows: for all $\eta, \mu \in \Lambda_\dagger$

$$\langle \mathcal{S}\eta, \mu\rangle = a_p(\mathcal{R}_p\eta, R_2\mu) - \int_\Gamma n(\mathcal{R}_f^1\eta\cdot\mathbf{n})\mu \tag{2.99}$$

which can be split as $\mathcal{S} = \mathcal{S}_f + \mathcal{S}_p$:

$$\langle \mathcal{S}_f\eta, \mu\rangle = -\int_\Gamma n(\mathcal{R}_f^1\eta\cdot\mathbf{n})\mu \tag{2.100}$$

$$\langle \mathcal{S}_p \eta, \mu \rangle = a_p(\mathcal{R}_p \eta, R_2 \mu) \tag{2.101}$$

for all $\eta, \mu \in \Lambda_\dagger$.
Finally, we define the functional $\varsigma : \Lambda_\dagger \to \mathbb{R}$,

$$\langle \varsigma, \mu \rangle = \int_\Gamma n(\boldsymbol{\varpi}_0^* \cdot \mathbf{n})\mu - a_p(\phi_0^* + E_p \varphi_p, R_2 \mu) \qquad \forall \mu \in \Lambda_\dagger \,. \tag{2.102}$$

Now, we can rewrite the solution of the global problem in terms of σ and, in particular, we can state the following result, which is the counterpart of Theorem 2.5.1.

Theorem 2.6.1. *The solution to (2.59)-(2.63) can be characterized as follows:*

$$\mathbf{u}_f^0 = \boldsymbol{\varpi}_0^* + \mathcal{R}_f^1 \sigma, \quad p_f = \overline{\pi}^* + \mathcal{R}_f^2 \sigma, \quad \varphi_0 = \phi_0^* + \mathcal{R}_p \sigma \tag{2.103}$$

where $\sigma \in \Lambda_\dagger$ is the solution of the Steklov-Poincaré equation

$$\langle \mathcal{S}\sigma, \mu \rangle = \langle \varsigma, \mu \rangle \qquad \forall \mu \in \Lambda_\dagger \,. \tag{2.104}$$

Proof. By direct inspection and knowing that $(H_0^1(\Omega_f))^d \subset H_f^\tau$, (2.59)-(2.61) are satisfied. Then, if σ is solution to (2.104), (2.62) is satisfied for all $\mu \in \Lambda$, since it does for all $\mu \in \Lambda_\dagger \supset \Lambda$. Finally, we substitute (2.103) in (2.63) and we check if the equality is true. Recalling the definition of the extension operator $\mathcal{R}_p$ and that $\phi_0^* \in H_p^0$, then the left hand side is equal to $\int_\Gamma g\sigma\mu$. On the other hand, thanks to (2.93) and to (2.96), the right hand side can be written as $\int_\Gamma g\sigma(\mathcal{R}_1^\tau \mu \cdot \mathbf{n})$ and using the definition of R_1^τ we can conclude that the equality is satisfied. □

In the next section we shall study the properties of the Steklov-Poincaré operators (2.100), (2.101).

2.6.1 Analysis of the Steklov-Poincaré Operators $\mathcal{S}_f$ and $\mathcal{S}_p$

We can prove the following result.

Lemma 2.6.1. *The Steklov-Poincaré operators $\mathcal{S}_f$ and $\mathcal{S}_p$ enjoy the following properties:*

1. *$\mathcal{S}_f$ and $\mathcal{S}_p$ are linear continuous operators on $\Lambda_\dagger$;*
2. *$\mathcal{S}_f$ is symmetric and positive;*
3. *$\mathcal{S}_p$ is symmetric and coercive;*
4. *the global Steklov-Poincaré operator $\mathcal{S}$ is symmetric, continuous and coercive. Moreover, $\mathcal{S}$ and $\mathcal{S}_p$ are spectrally equivalent, i.e. there exist two positive constants $\tilde{k}_1$ and $\tilde{k}_2$ such that*

$$\tilde{k}_1 \langle \mathcal{S}_p \eta, \eta \rangle \;\le\; \langle \mathcal{S}\eta, \eta \rangle \;\le\; \tilde{k}_2 \langle \mathcal{S}_p \eta, \eta \rangle \,.$$

Proof. 1. $\mathcal{S}_f$ and $\mathcal{S}_p$ are obviously linear. Then, we consider $\mathbf{v} = \mathcal{R}_f^1 \eta$ in (2.96) so that we obtain

$$|a_f(\mathcal{R}_f^1 \eta, \mathcal{R}_f^1 \eta)| = \left| \int_\Gamma g\eta \mathcal{R}_f^1 \eta \cdot \mathbf{n} \right| \qquad \forall \eta \in \Lambda_\dagger.$$

Thanks to the Cauchy-Schwarz inequality and recalling that $\Lambda_\dagger$ is continuously embedded in $L^2(\Gamma)$, we have

$$\left|\int_\Gamma g\eta\mathcal{R}_f^1\eta\cdot\mathbf{n}\right| \le C_0 g\|\eta\|_{\Lambda_\dagger}\|\mathcal{R}_f^1\eta\cdot\mathbf{n}\|_{\Lambda_\dagger}\,, \quad C_0>0. \tag{2.105}$$

On the other hand, we can apply the Poincaré inequality to get

$$\begin{aligned} |a_f(\mathcal{R}_f^1\eta,\mathcal{R}_f^1\eta)| &\ge \nu\min\left(\frac{1}{2C_{\Omega_f}},\frac{1}{2}\right)\|\mathcal{R}_f^1\eta\|_{1,f}^2 \\ &\ge \frac{\nu}{C_{tr}}\min\left(\frac{1}{2C_{\Omega_f}},\frac{1}{2}\right)\|\mathcal{R}_f^1\eta\cdot\mathbf{n}\|_{\Lambda_\dagger}^2 \end{aligned} \tag{2.106}$$

where we have applied the trace inequality for vector functions (see, e.g., [LM68]). Therefore, from (2.105) and (2.106) we have

$$\|\mathcal{R}_f^1\eta\cdot\mathbf{n}\|_{\Lambda_\dagger} \le \frac{C_0C_{tr}}{\min\left(\frac{1}{2C_{\Omega_f}},\frac{1}{2}\right)}\cdot\frac{g}{\nu}\|\eta\|_{\Lambda_\dagger} \qquad \forall\eta\in\Lambda_\dagger\,. \tag{2.107}$$

Then, for all $\eta,\mu\in\Lambda_\dagger$ it holds

$$|\langle\mathcal{S}_f\eta,\mu\rangle| \le C_0 n\|\mathcal{R}_f^1\eta\cdot\mathbf{n}\|_{\Lambda_\dagger}\|\mu\|_{\Lambda_\dagger} \le \tilde{\beta}_f\|\eta\|_{\Lambda_\dagger}\|\mu\|_{\Lambda_\dagger}$$

where we have used (2.107) and we have defined

$$\tilde{\beta}_f = \frac{C_0^2C_{tr}}{\min\left(\frac{1}{2C_{\Omega_f}},\frac{1}{2}\right)}\cdot\frac{ng}{\nu}\,, \tag{2.108}$$

$C_0>0$ being a positive constant.
Now, we consider the continuity of $\mathcal{S}_p$. We observe that for every $\mu\in\Lambda_\dagger$ we can make the special choice $R_2\mu=\mathcal{R}_p\mu$. Then, thanks to well-known estimates on the solution of elliptic problems (see, e.g., [Neč67]) we have

$$|\langle\mathcal{S}_p\eta,\mu\rangle| \le \max_j\|\mathsf{K}_j\|_{\infty,p}\|\mathcal{R}_p\eta\|_{1,p}\|\mathcal{R}_p\mu\|_{1,p} \le \tilde{\beta}_p\|\eta\|_{\Lambda_\dagger}\|\mu\|_{\Lambda_\dagger}$$

where we have denoted by $\tilde{\beta}_p$ the positive continuity constant

$$\tilde{\beta}_p = C_1\max_j\|\mathsf{K}_j\|_{\infty,p}$$

$C_1>0$ being a positive constant.

2. $\mathcal{S}_f$ is symmetric by definition. On the other hand, we can write

$$\begin{aligned} \|\mu\|_{\Lambda'} &= \sup_{\eta\in\Lambda}\frac{\langle g\mu,\eta\rangle}{g\|\eta\|_\Lambda} = \sup_{\eta\in\Lambda}\frac{\left|\langle\left(-\nu\dfrac{\partial\mathcal{R}_f^1\mu}{\partial\mathbf{n}}+\mathcal{R}_f^2\mu\mathbf{n}\right)\cdot\mathbf{n},R_f^1\eta\cdot\mathbf{n}\rangle\right|}{g\|\eta\|_\Lambda} \\ &= \sup_{\eta\in\Lambda}\frac{|a_f(\mathcal{R}_f^1\mu,R_f^1\eta)|}{g\|\eta\|_\Lambda} \le \sup_{\eta\in\Lambda}\frac{\nu\|\mathcal{R}_f^1\mu\|_{1,f}\|R_f^1\eta\|_{1,f}}{gC\|R_f^1\eta\|_{1,f}}\,, \end{aligned}$$

the last inequality follows from (2.89). Then, there exists a positive constant $C_2 > 0$ such that

$$\|\mu\|_{\Lambda'_\dagger} \leq C_2 \|\mathcal{R}_f^1 \mu\|_{1,f}\,.$$

We conclude that $\langle \mathcal{S}_f \mu, \mu \rangle \geq C_2 \|\mu\|^2_{\Lambda'_\dagger}$.

3. The operator $\mathcal{S}_p$ is obviously symmetric and using the Poincaré inequality and the trace inequality (2.5) it can be easily seen that

$$\langle \mathcal{S}_p \eta, \eta \rangle = a_p(\mathcal{R}_p \eta, \mathcal{R}_p \eta) \geq \tilde{\alpha}_p \|\eta\|^2_{\Lambda_\dagger}$$

where $\tilde{\alpha}_p$ is the positive coercivity constant

$$\tilde{\alpha}_p = \frac{m_K}{2} \min\left(\frac{1}{C_{\Omega_p}}, 1\right) \cdot \frac{1}{C^3_{tr,p}} \tag{2.109}$$

m_K being defined in (2.44). □

2.7 General Framework and Possible Iterative Procedures

In this section we summarize the results obtained in Sect. 2.5.1 and 2.6.1 giving a general formal framework. Then, since we aim at solving the coupled Stokes/Darcy problem by appropriate numerical schemes based on domain decomposition methods, we will set possible iterative substructuring procedures at the differential level, that we shall replicate at the discrete stage.

We have seen that:

Weak coupled problem (2.30), (2.31)	$\longleftrightarrow$	Interface equations	find $\lambda_0 \in \Lambda_0$: $S\lambda_0 = \chi$ *or else* find $\sigma \in \Lambda_\dagger$: $\mathcal{S}\sigma = \varsigma$

where the Steklov-Poincaré operators are such that:

1. *Case of interface variable* λ (see (2.56)): we have

   1a. $S_f : \Lambda_0 \to \Lambda'_0$ that maps

   $$S_f : \{\text{normal velocities on } \Gamma\} \to \{\text{normal stresses on } \Gamma\};$$

   and there exists $S_f^{-1} : \Lambda'_0 \to \Lambda_0$;

   1b. $S_p : \Lambda_0 \to \Lambda'_0$ which maps

   $$S_p : \{\text{fluxes of } \varphi \text{ on } \Gamma\} \to \{\text{traces of } \varphi \text{ on } \Gamma\}.$$

   1c. S_f is spectrally equivalent to S.

2. *Case of interface variable* σ (see (2.57)): we have

   2a. $\mathcal{S}_f : \Lambda_\dagger \to \Lambda'_\dagger$ that maps

   $$\mathcal{S}_f : \{\text{normal stresses on } \Gamma\} \to \{\text{normal velocities on } \Gamma\};$$

   2b. $\mathcal{S}_p : \Lambda_\dagger \to \Lambda'_\dagger$ which maps

   $$\mathcal{S}_p : \{\text{traces of } \varphi \text{ on } \Gamma\} \to \{\text{fluxes of } \varphi \text{ on } \Gamma\}$$

   and it admits the inverse $\mathcal{S}_p^{-1} : \Lambda'_\dagger \to \Lambda_\dagger$;

   2c. $\mathcal{S}_p$ is spectrally equivalent to $\mathcal{S}$.

Therefore, we can devise two iterative methods to solve either one of the interface equations using either S_f or $\mathcal{S}_p$ as preconditioner. In particular, we can propose the following Richardson methods at the differential level:

1. to solve $S\lambda_0 = \chi$ we consider: let λ_0^0 be given, for $k \geq 0$ do

   $$\lambda_0^{k+1} = \lambda_0^k + \theta S_f^{-1}(\chi - (S_f + S_p)\lambda_0^k) \tag{2.110}$$

   where $\theta > 0$ is a suitable relaxation parameter;

2. to solve $\mathcal{S}\sigma = \varsigma$, we consider: let σ^0 be given, for $k \geq 0$ do

   $$\sigma^{k+1} = \sigma^k + \vartheta \mathcal{S}_p^{-1}(\varsigma - (\mathcal{S}_f + \mathcal{S}_p)\sigma^k)$$

   where $\vartheta > 0$ is a suitable acceleration parameter.

In chapter 3 we shall prove that these methods correspond to Dirichlet-Neumann type schemes to solve the coupled Stokes/Darcy problem.

Finally, let us point out which differential problems correspond to the different Steklov-Poincaré operators:

i) Operator S_f:
computing $S_f\lambda_0$ involves solving a Stokes problem in Ω_f with the boundary conditions $\mathbf{u}_f \cdot \mathbf{n} = \lambda_0$ and $\mathbf{u}_f \cdot \boldsymbol{\tau}_j = 0$ on Γ, and then to compute the normal stress $(-\nu \partial \mathbf{u}_f / \partial \mathbf{n} + p_f \mathbf{n}) \cdot \mathbf{n}$ on Γ.

ii) Operator S_f^{-1}:
the application of S_f^{-1} to a given $\mu \in \Lambda'_0$ corresponds to solve a Stokes problem with the boundary conditions $(-\nu \partial \mathbf{u}_f / \partial \mathbf{n} + p_f \mathbf{n}) \cdot \mathbf{n} = \mu$ and $\mathbf{u}_f \cdot \boldsymbol{\tau}_j = 0$ on Γ, and to compute the normal velocity $\mathbf{u}_f \cdot \mathbf{n}$ on Γ.

iii) Operator S_p:
computing $S_p\lambda_0$ corresponds to solve a Darcy problem in Ω_p with the Neumann boundary condition $-(\mathsf{K}/n)\partial\varphi/\partial\mathbf{n} = \lambda_0$ on Γ and to recover φ on Γ.

On the other hand,

iv) Operator $\mathcal{S}_f$:
the application of $\mathcal{S}_f$ to a given $\sigma \in \Lambda_\dagger$ corresponds to solve the same type of Stokes problem as in *ii)*. However, notice that here $\sigma \in \Lambda_\dagger$, so that it has a higher regularity than the datum $\mu \in \Lambda_0'$ used in *ii)*. The $\Lambda_\dagger$-regularity is more than required to guarantee the well-posedness of the Stokes problem with Neumann datum on Γ.

v) Operator $\mathcal{S}_p$:
computing $\mathcal{S}_p\sigma$ corresponds to solve a Darcy problem in Ω_p with the Dirichlet boundary condition $\varphi = \sigma$ on Γ and to recover the flux $-(\mathsf{K}/n)\partial\varphi/\partial\mathbf{n}$ on Γ.

vi) Operator $\mathcal{S}_p^{-1}$:
the application of $\mathcal{S}_p^{-1}$ to a given $\eta \in \Lambda_\dagger'$ implies the solution of a Darcy problem with Neumann boundary condition $-(\mathsf{K}/n)\partial\varphi/\partial\mathbf{n} = \eta$ on Γ to get φ on Γ, like in *iii)*. Remark that here η is less regular than the datum $\lambda_0 \in \Lambda_0$ taken in *iii)*. However, this regularity is enough to guarantee the well-posedness of the Darcy problem.

3. Substructuring Methods for the Finite Element Approximation of the Stokes/Darcy Problem

In this chapter we consider a Galerkin finite element approximation of the Stokes/Darcy problem and we propose iterative subdomain methods for its solution, inspired to domain decomposition theory. The convergence analysis that we develop is based on the properties of the discrete Steklov-Poincaré operators associated to the given coupled problem. Optimal preconditioners for Krylov methods are proposed and analyzed.
The results presented in this chapter extend those published in [DQ04].
This chapter is rather technical; figure 3.1 resumes the key points and should serve as a guide to the reader.

3.1 Introduction

We consider a triangulation $\mathcal{T}_h$ of the domain $\overline{\Omega}_f \cup \overline{\Omega}_p$, depending on a positive parameter $h > 0$, made up of triangles if $d = 2$, or tetrahedra in the 3-dimensional case. We assume that:

1. each triangle or thetrahedra, say T, is such that $int(T) \neq \emptyset$;
2. $int(T_1) \cap int(T_2) = \emptyset$ for each pair of different T_1, $T_2 \in \mathcal{T}_h$, and if $T_1 \cap T_2 = F \neq \emptyset$, then F is a common face or edge or vertex to T_1 and T_2;
3. $diam(T) \leq h$ for all $T \in \mathcal{T}_h$;
4. $\mathcal{T}_h$ is *regular* that is there exists a constant $C_{reg} \geq 1$ such that
$$\max_{T \in \mathcal{T}_h} \frac{diam(T)}{\rho_T} \leq C_{reg} \qquad \forall h > 0$$
with $\rho_T = \sup\{diam(B) | B \text{ is a ball contained in } T\}$;
5. the triangulations $\mathcal{T}_{fh}$ and $\mathcal{T}_{ph}$ induced on the subdomains Ω_f and Ω_p are compatible on Γ, that is they share the same edges (if $d = 2$) or faces (if $d = 3$) therein;
6. the triangulation $\mathcal{T}_{\Gamma h}$ induced on Γ is *quasi-uniform*, that is it is regular and there exists a constant $\tau > 0$ such that $\min_{T \in \mathcal{T}_{\Gamma h}} h_T \geq \tau h$, for all $h > 0$.

We shall denote by $\mathbb{P}_r$, with r a non negative integer, the usual space of algebraic polynomials of degree less or equal to r.

Prior to the analysis, we briefly discuss some possible choices of finite element (FE) spaces that may be adopted to compute the solution of the fluid and the porous media problems.

3.1.1 Overview on the Classical FE Spaces for Stokes and Darcy Equations

The literature on FE methods for the (Navier-)Stokes equations is quite broad. The crucial issue concerning the finite dimensional spaces, say V_h and Q_h, approximating the spaces of velocity and pressure, respectively, is that they must satisfy the discrete compatibility condition: there exists a positive constant $\beta^* > 0$, independent of h, such that

$$\forall q_h \in Q_h, \quad \exists \mathbf{v}_h \in V_h,\ \mathbf{v}_h \neq \mathbf{0} : \quad b_f(\mathbf{v}_h, q_h) \geq \beta^* \|\mathbf{v}_h\|_{1,f} \|q_h\|_{0,f} \ . \tag{3.1}$$

Several choices can be made in this direction featuring both discontinuous pressure FE (e.g. the $\mathbb{P}_2$-$\mathbb{P}_0$ elements or the Crouzeix-Raviart elements defined using cubic bubble functions) and continuous pressure FE among which we recall the Taylor-Hood (or $\mathbb{P}_2$-$\mathbb{P}_1$) elements and the $(\mathbb{P}_1 iso \mathbb{P}_2)$-$\mathbb{P}_1$ elements. See, e.g., [QV94] chapter 9, or [BF91] chapter VI.

Concerning the solution of the Darcy problem (1.10), (1.11), currently used numerical methods are based on two different approaches.

The former is based on the primal, single field formulation (1.21) for the piezometric head: it amounts to solving the Poisson problem in the unknown φ using classical FE spaces and then to recover the velocity field by numerically computing the gradient of φ. This approach may lead to a loss of accuracy, i.e. to lower-order approximations for fluxes than the primal variable; besides, mass conservation is not guaranteed.
However, post-processing techniques for the velocity field based on gradient superconvergence phenomena, like those studied by Zienkiewicz and Zhu, have been successfully used to improve accuracy and enforce mass conservation. In [LRM95] the authors show that these methods and their variants may provide higher rates of convergence if compared with the classical displacement or mixed methods.

The latter and more popular approach is however based on the mixed formulation (1.10), (1.11), since it permits to recover simultaneously both the primal unknown and its gradient with the same order of convergence. Moreover, mass is locally conserved and the continuity of fluxes is preserved.
This approach comprises the so-called mixed (MFE) and mixed-hybrid (MHFE) finite elements, among which we recall the Raviart-Thomas (RT) elements, the Brezzi-Douglas-Marini (BDM) and the Brezzi-Douglas-Fortin-Marini (BDFM) elements, only to quote the most classical ones (see [BF91]). In this context we cite also the recent work [MH02] which presents a new stabilized MFE method without mesh-dependent parameters, and the comparative study [HEM$^+$02] concerning the numerical reliability of MFE and MHFE methods applied to porous media flows under the influence of mesh parameters and medium heterogeneity.
Other approaches are based on the Discontinuous Galerkin (DG) methods (see [ABCM02, CK00]) which are attractive for porous media flow due to their high order convergence property, local conservation of mass, flexibility with respect of meshing and *hp*-adaptive refinement, and their robustness with respect to strongly discontinuous coefficients. A numerical comparison between DG and MFE for porous media can be found in [Bas03].

MFE and DG have been also adopted in the recent works [LSY03, Riv03, RY03] for the Stokes/Darcy coupling.
In particular, in [LSY03] a coupling between LBB-stable FE for Stokes and MFE for Darcy equations is realized using hanging nodes on the interface Γ. The analysis developed shows that optimal error bounds can be obtained in both the fluid and the porous region.
DG methods based on Interior Penalty are considered in [Riv03] for both the fluid and the groundwater problem, and all unknowns are approximated by totally discontinuous polynomials of different orders.
The two approaches are used together in [RY03] where the fluid velocity and pressure are obtained by MFE in the porous media region, while they are approximated by DG in the incompressible flow region. Error estimates are derived for two-dimensional problems and the authors point out that non-matching grids on the interface can be used, with the space of discrete normal velocities on Γ playing the role of a mortar space.
Finally, the issue of adopting different meshes in the two subdomains has been considered also in [BH02], where $\mathbb{P}_1$-$\mathbb{P}_0$ FE, stabilized through a generalization of the Brezzi-Pitkäranta penalization, have been used for both the fluid and the porous medium, realizing the coupling via a Nitsche method.

3.2 Galerkin FE Approximation of the Stokes/Darcy Problem

Our analysis will consider the single field formulation (1.21) for Darcy equation. This approach will allow us to treat the interface conditions as natural conditions for both the fluid and the porous media, as for the continuous case; moreover, this approach will perfectly serve our purpose to characterize iterative substructuring methods to solve the coupled problem.
As a first step, we can approximate the velocity field in Ω_p using one of the post-processing techniques recalled above. Then, we shall indicate how Darcy's mixed formulation can be accounted for in our iterative methods and in chapter 4 we shall also present numerical results based on this latter formulation.

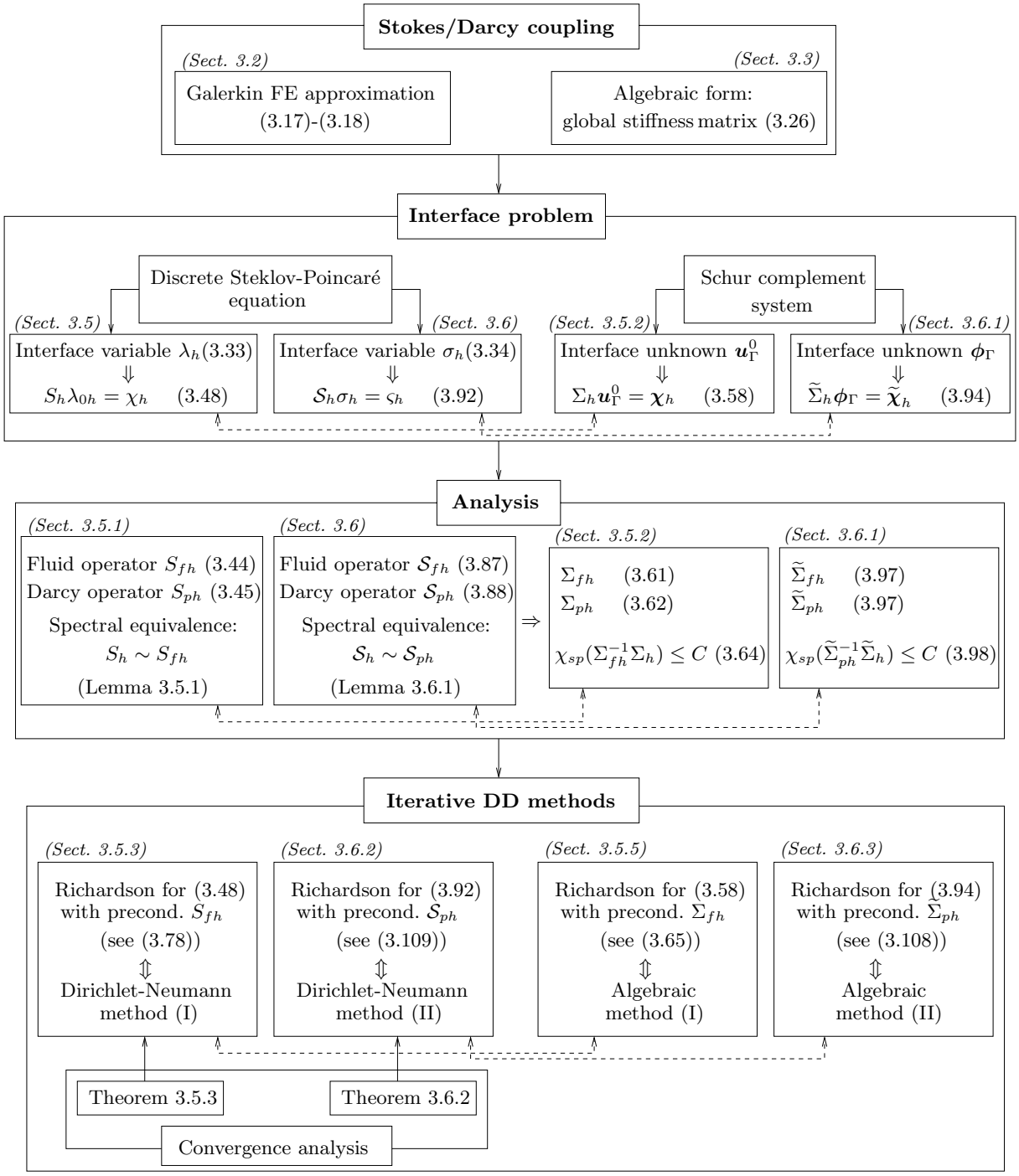


Fig. 3.1. Scheme of the content of chapter 3.

The setting of the coupled problem is the same as in Sect. 2.2.
What matters for the analysis we are going to develop, is only to guarantee that the compatibility condition (3.1) holds. Therefore, in the following, for the sake of exposition, we will consider the special choice of piecewise quadratic elements for the velocity components and piecewise linear for the pressure in the fluid domain ($\mathbb{P}_2$-$\mathbb{P}_1$ FE), while we shall consider quadratic $\mathbb{P}_2$ elements for the piezometric head in the porous media domain. For the sake of clarity let us show the degrees of freedom we are considering and how they match across the interface Γ: in Fig. 3.2 we sketch two triangles of a conforming regular mesh and we indicate the degrees of freedom corresponding to the velocity $\mathbf{u}_f$ and the pressure p_f in Ω_f, and to the piezometric head φ in Ω_p.

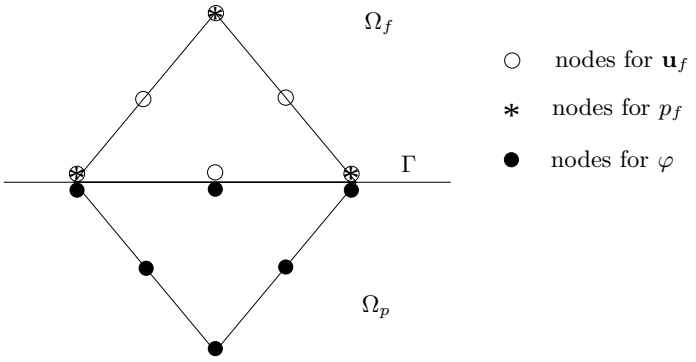


Fig. 3.2. Degrees of freedom of the FE used for approximating velocity, pressure and piezometric head.

We define the discrete spaces:

$$H_{fh} = (V_{fh})^d, \qquad d = 2, 3\,, \tag{3.2}$$

where

$$V_{fh} = \{v_h \in X_{fh} |\, v_h = 0 \text{ on } \Gamma_f^{in}\}\,, \tag{3.3}$$

$$X_{fh} = \{v_h \in C^0(\overline{\Omega}_f) |\, v_h = 0 \text{ on } \Gamma_f \text{ and } v_{h|K} \in \mathbb{P}_2(K),\, \forall K \in \mathcal{T}_{fh}\}\,, \tag{3.4}$$

and

$$\widetilde{H}_{fh} = (\widetilde{V}_{fh})^d, \qquad d = 2, 3, \tag{3.5}$$

where

$$\widetilde{V}_{fh} = \{v_h \in X_{fh} |\, v_h = 0 \text{ on } \Gamma\}\,. \tag{3.6}$$

Moreover, let

$$H^0_{fh} = \{\mathbf{v}_h \in H_{fh} |\ \mathbf{v}_h \cdot \mathbf{n} = 0 \text{ on } \Gamma\}\ ; \tag{3.7}$$

$$Q_h = \{q_h \in C^0(\overline{\Omega}_f) |\ q_{h|K} \in \mathbb{P}_1(K),\ \forall K \in \mathcal{T}_{fh}\}\ ; \tag{3.8}$$

$$X_{ph} = \{\psi_h \in C^0(\overline{\Omega}_p) |\ \psi_{h|K} \in \mathbb{P}_2(K),\ \forall K \in \mathcal{T}_{ph}\}\ ; \tag{3.9}$$

$$H_{ph} = \{\psi_h \in X_{ph} |\ \psi_h = 0 \text{ on } \Gamma^b_p\}\ ; \tag{3.10}$$

$$H^0_{ph} = \{\psi_h \in H_{ph} | \psi_h = 0 \text{ on } \Gamma\}\ ; \tag{3.11}$$

$$W_h = H_{fh} \times H_{ph}\ . \tag{3.12}$$

Finally, we consider the spaces:

$$\Lambda_h = \{v_{h|\Gamma} |\ v_h \in V_{fh}\} \qquad \text{and} \qquad \Lambda_{\dagger h} = \{\psi_{h|\Gamma} |\ \psi_h \in X_{ph}\} \tag{3.13}$$

to approximate the trace spaces Λ and $\Lambda_\dagger$ (see (2.15)) on Γ, respectively.

Now, let us consider the approximation of the boundary data. If we suppose that the Darcy datum φ_p on Γ^b_p belongs to $\varphi_p \in H^{1/2}(\Gamma^b_p) \cap C^0(\Gamma^b_p)$, we can take the quadratic interpolant φ_{ph} of its nodal values on Γ^b_p, and then the extension $E_{ph}\varphi_{ph} \in X_{ph}$, such that $E_{ph}\varphi_{ph} = \varphi_{ph}$ at the nodes lying on Γ^b_p and $E_{ph}\varphi_{ph} = 0$ at the nodes of $\overline{\Omega}_p \setminus \Gamma^b_p$.
We can proceed in the same way for the boundary datum $\mathbf{u}_{in}$ for the Stokes problem, provided it belongs to $(H^{1/2}(\Gamma^{in}_f))^d \cap (C^0(\Gamma^{in}_f))^d$. We consider again its quadratic interpolant, say $\mathbf{u}_{inh}$, and then its extension

$$E_{fh}\mathbf{u}_{inh} \in \widetilde{H}_{fh}\,. \tag{3.14}$$

Remark 3.2.1. The discrete extension operator E_{fh} is the counterpart of the continuous operator E_f defined in (2.16). Note that also in this case we could have considered a discrete divergence free extension operator, say $\widetilde{E}_{fh}$, corresponding to $\widetilde{E}_f$ that we have characterized in Remark 2.2.1. We point out that to define $\widetilde{E}_{fh}$ we should consider the discrete counterpart of problem (2.17) whose solvability is now guaranteed thanks to (3.1). □

Now, we proceed as in Sect. 2.2. Recalling the definitions (2.26), (2.27) and considering the linear functionals

$$\langle \mathcal{F}^*, \underline{w}\rangle = \int_{\Omega_f} n\mathbf{f}\,\mathbf{w} - n\,a_f(E_{fh}\mathbf{u}_{inh}, \mathbf{w}) - g\,a_p(E_{ph}\varphi_{ph}, \psi) \qquad \forall \underline{w} = (\mathbf{w}, \psi) \in W, \tag{3.15}$$

$$\langle \mathcal{G}^*, q\rangle = -nb_f(E_{fh}\mathbf{u}_{inh}, q) \qquad q \in Q, \tag{3.16}$$

the Galerkin approximation of the coupled Stokes/Darcy problem reads:

find $\underline{u}_h = (\mathbf{u}^0_{fh}, \varphi_{0h}) \in W_h$ and $p_{fh} \in Q_h$:

$$\mathcal{A}(\underline{u}_h, \underline{v}_h) + \mathcal{B}(\underline{v}_h, p_{fh}) = \langle \mathcal{F}^*, \underline{v}_h\rangle \qquad \forall \underline{v}_h \in W_h \tag{3.17}$$

$$\mathcal{B}(\underline{u}_h, q_h) = \langle \mathcal{G}^*, q_h\rangle \qquad \forall q_h \in Q_h\ . \tag{3.18}$$

Notice that considering a divergence null discrete extension of $\widetilde{E}_{fh}\mathbf{u}_{inh}$, the linear functional $\mathcal{G}^*$ would be null.

The existence, uniqueness and stability of the discrete solution of (3.17), (3.18) can be proved following the same steps of the continuous case, using in addition the discrete inf-sup condition (see [Bre74]): there exists a positive constant $\beta^* > 0$, independent of h, such that

$$\forall q_h \in Q_h, \quad \exists \mathbf{v}_h \in H_{fh},\ \mathbf{v}_h \neq \mathbf{0}: \quad b_f(\mathbf{v}_h, q_h) \geq \beta^* \|\mathbf{v}_h\|_{1,f} \|q_h\|_{0,f} \ . \tag{3.19}$$

The following error estimates hold. Let $\underline{u} = (\boldsymbol{u}_f^0, \varphi_0) \in W$, $p_f \in Q$ be the solutions to (2.30), (2.31). Then, (see [Bre74])

$$\|\underline{u} - \underline{u}_h\|_W \leq \left(1 + \frac{\gamma}{\alpha}\right) \inf_{\underline{v}_h \in X_h^0} \|\underline{u} - \underline{v}_h\|_W + \frac{1}{\alpha} \inf_{q_h \in Q_h} \|p_f - q_h\|_{0,f} \tag{3.20}$$

and

$$\begin{aligned} \|p_f - p_{fh}\|_{0,f} \quad \leq \quad & \frac{\gamma}{\beta^*} \left(1 + \frac{\gamma}{\alpha}\right) \inf_{\underline{v}_h \in X_h^0} \|\underline{u} - \underline{v}_h\|_W \\ & + \left(1 + \frac{1}{\beta^*} + \frac{\gamma}{\alpha\beta^*}\right) \cdot \|p - q_h\|_{0,f} \ , \end{aligned} \tag{3.21}$$

where β^* is the positive h-independent constant in the inf-sup condition (3.19); γ and α are the h-independent continuity and coercivity constants of the bilinear form $\mathcal{A}(.,.)$ defined in (2.41) and (2.43), respectively.
Finally X_h^0 is the discrete space

$$X_h^0 = \{\underline{v}_h \in W_h |\, \mathcal{B}(\underline{v}_h, q_h) = 0 \quad \forall q_h \in Q_h\} \ .$$

We remark that since constants α, γ and β^* are all independent of the discretization parameter h, (3.20) and (3.21) give optimal convergence estimates.

Remark 3.2.2. Notice that in addition to the discrete LBB condition (3.19), no further compatibility condition is required for the discrete spaces H_{fh} and H_{ph}. In fact, the mixed coupling terms on the interface appearing in the definition of the bilinear form $\mathcal{A}(.,.)$:

$$\int_\Gamma ng\varphi_h(\mathbf{w}_h \cdot \mathbf{n}) - \int_\Gamma ng\psi_h(\mathbf{v}_h \cdot \mathbf{n}) \ , \tag{3.22}$$

give null contribution when we consider $\mathbf{w}_h = \mathbf{v}_h$ and $\psi_h = \varphi_h$. □

Finally, let us underline that in the FE approximation, the coupling condition (1.26), which imposes the continuity of normal velocity across the interface, has to be intended in the sense of the $L^2(\Gamma)$-projection on the finite element space H_{ph} on Γ. In fact, in (3.17) we are imposing

$$\int_\Gamma \left(-\frac{\mathsf{K}}{n} \nabla\varphi_h \cdot \mathbf{n} - \mathbf{u}_{fh} \cdot \mathbf{n}\right) \psi_{h|\Gamma} = 0, \tag{3.23}$$

for all $\psi_h \in H_{ph}$. This is equivalent to require that

$$\Pi(\mathbf{u}_{fh} \cdot \mathbf{n}) = -\frac{\mathsf{K}}{n} \nabla\varphi_h \cdot \mathbf{n} \,, \tag{3.24}$$

Π being the projection operator on $H_{ph|\Gamma}$ with respect to the scalar product of $L^2(\Gamma)$.

3.3 Algebraic Formulation of the Coupled Problem

We introduce the following bases for the finite dimensional spaces H_{fh}, Q_h and H_{ph}, respectively. Let $N_f = dim(H_{fh})$, $N_q = dim(Q_h)$ and $N_p = dim(H_{ph})$. Then,

a) $\{\boldsymbol{\omega}_i\}_{i=1}^{N_f}$ is a basis for H_{fh};

b) $\{\pi_j\}_{j=1}^{N_q}$ is a basis for Q_h;

c) $\{\phi_k\}_{k=1}^{N_p}$ is a basis for H_{ph}.

Finally, let N_Γ denote the number of nodes lying on the interface Γ.
We can express the unknowns $\mathbf{u}_{fh}^0$, p_{fh} and φ_{0h} as linear combinations with respect to these bases. In particular,

$$\mathbf{u}_{fh}^0 = \sum_{j=1}^{N_f} (\mathbf{u}_{fh}^0)^j \boldsymbol{\omega}_j, \quad p_{fh} = \sum_{j=1}^{N_q} (p_{fh})^j \pi_j, \quad \varphi_{0h} = \sum_{j=1}^{N_p} (\varphi_{0h})^j \phi_j, \tag{3.25}$$

where $(\mathbf{u}_{fh}^0)^j$, $(p_{fh})^j$, $(\varphi_{0h})^j$ denote the coefficients of the linear expansions.
Remark that $(p_{fh})^j$, $(\varphi_{0h})^j \in \mathbb{R}$, while, for any fixed $1 \le j \le N_f$, $(\mathbf{u}_{fh}^0)^j$ is the d-uple of $\mathbb{R}^d$: $((\mathbf{u}_{fh}^0)_1^j, \ldots, (\mathbf{u}_{fh}^0)_d^j)^T$ such that $(\mathbf{u}_{fh}^0)^j \boldsymbol{\omega}_j$ is the vector
$((\mathbf{u}_{fh}^0)_1^j (\boldsymbol{\omega}_j)_1, \ldots, (\mathbf{u}_{fh}^0)_d^j (\boldsymbol{\omega}_j)_d)^T$, $(\boldsymbol{\omega}_j)_i$ being the i-th component of $\boldsymbol{\omega}_j$.

Now, we consider equation (3.17) and choose as test functions the basis functions of H_{fh} associated with the internal nodes of Ω_f, say $\boldsymbol{\omega}_i$ for $i = 1, \ldots, N_f - N_\Gamma$. We also suppose to have reordered these basis functions in such a way that the last N_Γ are associated to the nodes on Γ, and we distinguish them with the notation $\boldsymbol{\omega}_i^\Gamma$. Therefore, thanks to (3.25), we have:

$$\begin{aligned}
&\sum_{j=1}^{N_f - N_\Gamma} na_f(\boldsymbol{\omega}_j, \boldsymbol{\omega}_i)(\mathbf{u}_{fh}^0)^j + \sum_{j=1}^{N_\Gamma} \sum_{k=1}^{d-1} na_f((\boldsymbol{\omega}_j^\Gamma \cdot \boldsymbol{\tau}_k), \boldsymbol{\omega}_i)(\mathbf{u}_{fh}^0 \cdot \boldsymbol{\tau}_k)^j \\
&+ \sum_{j=1}^{N_\Gamma} na_f((\boldsymbol{\omega}_j^\Gamma \cdot \mathbf{n}), \boldsymbol{\omega}_i)(\mathbf{u}_{fh}^0 \cdot \mathbf{n})^j + \sum_{j=1}^{N_q} nb_f(\boldsymbol{\omega}_i, \pi_j)(p_{fh})^j \\
&= \int_{\Omega_f} n\mathbf{f}\, \boldsymbol{\omega}_i \; - \; na_f(E_{fh}\mathbf{u}_{inh}, \boldsymbol{\omega}_i), \qquad i = 1, \ldots, N_f - N_\Gamma.
\end{aligned}$$

By $\boldsymbol{u}_{int}$ we indicate the vector of the values of the unknown $\mathbf{u}_{fh}^0$ at the nodes of $\Omega_f \setminus \Gamma$ plus those of $(\mathbf{u}_{fh}^0 \cdot \boldsymbol{\tau}_k)$ at the nodes lying on the interface Γ. Moreover, $\boldsymbol{u}_\Gamma$ indicate the vector of the values of $(\mathbf{u}_{fh}^0 \cdot \mathbf{n})$ at the nodes of Γ. Finally, $\boldsymbol{p}$ is the vector of the values of the unknown pressure p_{fh} at the nodes of Ω_f.
Then, we can write the following compact form (with obvious choice of notation for the matrices and the right hand side):

$$\mathrm{A}_{ff}\boldsymbol{u}_{int} + \mathrm{A}_{f\Gamma}\boldsymbol{u}_\Gamma + \mathrm{B}^T\boldsymbol{p} = \boldsymbol{f}_f \; .$$

We consider again equation (3.17), but we choose as test functions $\boldsymbol{\omega}_i^\Gamma$, $i = 1, \ldots, N_\Gamma$, associated to the nodes on Γ. Then, we obtain:

$$\sum_{j=1}^{N_f-N_\Gamma} n a_f(\boldsymbol{\omega}_j,\boldsymbol{\omega}_i^\Gamma)(\mathbf{u}_{fh}^0)^j + \sum_{j=1}^{N_\Gamma}\sum_{k=1}^{d-1} n a_f((\boldsymbol{\omega}_j^\Gamma\cdot\boldsymbol{\tau}_k),\boldsymbol{\omega}_i^\Gamma)(\mathbf{u}_{fh}^0\cdot\boldsymbol{\tau}_k)^j$$
$$+\sum_{j=1}^{N_\Gamma}\int_\Gamma n\left[\sum_{k=1}^{d-1}\frac{\nu}{\varepsilon}(\boldsymbol{\omega}_i^\Gamma\cdot\boldsymbol{\tau}_k)(\boldsymbol{\omega}_j^\Gamma\cdot\boldsymbol{\tau}_k)\right](\mathbf{u}_{fh}^0)^j$$
$$+\sum_{j=1}^{N_\Gamma} n a_f(\boldsymbol{\omega}_j^\Gamma\cdot\mathbf{n},\boldsymbol{\omega}_i^\Gamma)(\mathbf{u}_{fh}^0\cdot\mathbf{n})^j$$
$$+\sum_{j=1}^{N_q} n\, b_f(\boldsymbol{\omega}_i^\Gamma,\pi_j)(p_{fh})^j + \sum_{j=1}^{N_\Gamma}\left(\int_\Gamma n g\,\phi_j^\Gamma(\boldsymbol{\omega}_i^\Gamma\cdot\mathbf{n})\right)(\varphi_{0h})^j$$
$$=\int_{\Omega_f} n\mathbf{f}\,\boldsymbol{\omega}_i^\Gamma \;-\; n a_f(E_{fh}\mathbf{u}_{inh},\boldsymbol{\omega}_i^\Gamma)\,,$$

where ϕ_j^Γ denotes the functions of the basis of H_{ph} associated to the interface nodes.
In compact form we get:

$$\mathrm{A}_{\Gamma f}\boldsymbol{u}_{int} + \mathrm{A}_{\Gamma\Gamma}^f\boldsymbol{u}_\Gamma + \mathrm{B}_{f\Gamma}^T\boldsymbol{p} + \mathrm{M}_{\Gamma\Gamma}\boldsymbol{\phi} = \boldsymbol{f}_\Gamma\,.$$

Now, we consider for (3.17) the test functions ϕ_i, $i=1,\dots,N_p-N_\Gamma$, associated to the internal nodes of domain Ω_p. Again, we suppose the last N_Γ functions $\{\phi_i^\Gamma\}_{i=1}^{N_\Gamma}$ to correspond to the nodes on Γ. We find:

$$\sum_{j=1}^{N_p-N_\Gamma} g\,a_p(\phi_j,\phi_i)\,(\varphi_{0h})^j + \sum_{j=1}^{N_\Gamma} g\,a_p(\phi_j^\Gamma,\phi_i)\,(\varphi_{0h})^j = -g\,a_p(E_{ph}\varphi_{ph},\phi_i)\,.$$

Let $\boldsymbol{\phi}_{int}$ indicate the vector of the values of the piezometric head φ_{0h} at the nodes on $\Omega_p\setminus\Gamma$, and $\boldsymbol{\phi}_\Gamma$ those at the nodes on Γ. Therefore, we have the compact form:

$$\mathrm{A}_{pp}\boldsymbol{\phi}_{int} + \mathrm{A}_{p\Gamma}\boldsymbol{\phi}_\Gamma = \boldsymbol{f}_p\,.$$

If we consider the test functions ϕ_i^Γ, associated to the nodes on Γ, we have:

$$\sum_{j=1}^{N_p-N_\Gamma} g\,a_p(\phi_j,\phi_i^\Gamma)(\varphi_{0h})^j + \sum_{j=1}^{N_\Gamma} g\,a_p(\phi_j^\Gamma,\phi_i^\Gamma)(\varphi_{0h})^j$$
$$+\sum_{j=1}^{N_\Gamma}\left(-\int_\Gamma n g\,\phi_i^\Gamma(\boldsymbol{\omega}_j^\Gamma\cdot\mathbf{n})\right)(\mathbf{u}_{fh}^0\cdot\mathbf{n})^j = -g\,a_p(E_{ph}\varphi_{ph},\phi_i^\Gamma)\,,$$

that in compact form becomes

$$\mathrm{A}_{p\Gamma}^T\boldsymbol{\phi}_{int} + \mathrm{A}_{\Gamma\Gamma}^p\boldsymbol{\phi}_\Gamma - \mathrm{M}_{\Gamma\Gamma}^T\boldsymbol{u}_\Gamma = \boldsymbol{f}_{p\Gamma}\,.$$

Finally, we consider equation (3.18). Choosing the test functions π_i, $i=1,\dots,N_q$, we have:

$$\sum_{j=1}^{N_f-N_\Gamma} n\,b_f(\boldsymbol{\omega}_j,\pi_i)(\mathbf{u}_{fh}^0)^j + \sum_{j=1}^{N_\Gamma}\sum_{k=1}^{d-1} n\,b_f(\boldsymbol{\omega}_j^\Gamma\cdot\boldsymbol{\tau}_k,\pi_i)(\mathbf{u}_{fh}^0\cdot\boldsymbol{\tau}_k)^j$$
$$+\sum_{j=1}^{N_\Gamma} n\,b_f(\boldsymbol{\omega}_j^\Gamma\cdot\mathbf{n},\pi_i)(\mathbf{u}_{fh}^0\cdot\mathbf{n})^j = -n\,b_f(E_{fh}\mathbf{u}_{inh},\pi_i)\,,$$

or in compact form:

$$\mathrm{B}_1 \boldsymbol{u}_{int} + \mathrm{B}_{f\Gamma} \boldsymbol{u}_\Gamma = \boldsymbol{f}_{in} \, .$$

Using the notation introduced above, we can then reformulate problem (3.17), (3.18) in matrix form

$$\begin{pmatrix} \mathrm{A}_{ff} & \mathrm{B}^T & \mathrm{A}_{f\Gamma} & 0 & 0 \\ \mathrm{B}_1 & 0 & \mathrm{B}_{f\Gamma} & 0 & 0 \\ \mathrm{A}_{\Gamma f} & \mathrm{B}_{f\Gamma}^T & \mathrm{A}_{\Gamma\Gamma}^f & \mathrm{M}_{\Gamma\Gamma} & 0 \\ 0 & 0 & -\mathrm{M}_{\Gamma\Gamma}^T & \mathrm{A}_{\Gamma\Gamma}^p & \mathrm{A}_{p\Gamma}^T \\ 0 & 0 & 0 & \mathrm{A}_{p\Gamma} & \mathrm{A}_{pp} \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_{int} \\ \boldsymbol{p} \\ \boldsymbol{u}_\Gamma \\ \boldsymbol{\phi}_\Gamma \\ \boldsymbol{\phi}_{int} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}_f \\ \boldsymbol{f}_{in} \\ \boldsymbol{f}_\Gamma \\ \boldsymbol{f}_{p\Gamma} \\ \boldsymbol{f}_p \end{pmatrix} . \tag{3.26}$$

The matrix of the linear system (3.26) is positive definite, and generally it is large and sparse. To effectively solve this system using an iterative method, a preconditioning strategy is thus in order. The characterization of suitable preconditioners will be the object of the next sections.
Remark that the coupling between Stokes and Darcy equations is realized at this algebraic stage through the third and the fourth rows of the global matrix. In particular, the submatrices $\mathrm{M}_{\Gamma\Gamma}$ and $-\mathrm{M}_{\Gamma\Gamma}^T$ impose the algebraic counterpart of the coupling conditions (1.27) and (1.26), respectively.

3.4 Discrete Multidomain Formulation

The theory developed at the differential level for the Steklov-Poincaré operators associated to the Stokes/Darcy problem (see Sects. 2.5, 2.6) can be extended to the discrete operators associated with the Galerkin FE approximation (3.17), (3.18).
The characterization of these discrete operators will be crucial to set up effective iterative schemes to solve (3.26).
As already done for the continuous case, we shall consider the simplified condition

$$\mathbf{u}_{fh} \cdot \boldsymbol{\tau}_j = 0 \text{ on } \Gamma. \tag{3.27}$$

Therefore, our coupled problem (3.17), (3.18) may be rewritten in the following multidomain formulation.

Proposition 3.4.1. *Using the simplified condition (3.27), problem (3.17), (3.18) can be formulated in an equivalent way as follows:*

find $\mathbf{u}_{fh}^0 \in H_{fh}^\tau$, $p_{fh} \in Q_h$, $\varphi_{0h} \in H_{ph}$ *such that:*

$$a_f(\mathbf{u}^0_{fh} + E_{fh}\mathbf{u}_{inh}, \mathbf{w}_h) + b_f(\mathbf{w}_h, p_{fh}) = \int_{\Omega_f} \mathbf{f}\,\mathbf{w}_h \qquad \forall \mathbf{w}_h \in \widetilde{H}^0_{fh} \tag{3.28}$$

$$b_f(\mathbf{u}^0_{fh} + E_{fh}\mathbf{u}_{inh}, q_h) = 0 \qquad \forall q_h \in Q_h \tag{3.29}$$

$$a_p(\varphi_{0h} + E_{ph}\varphi_{ph}, \psi_h) = 0 \qquad \forall \psi_h \in H^0_{ph} \tag{3.30}$$

$$\int_\Gamma n(\mathbf{u}^0_{fh}\cdot\mathbf{n})\mu_h = a_p(\varphi_{0h} + E_{ph}\varphi_{ph}, R_{2h}\mu_h) \qquad \forall \mu_h \in \Lambda_h \tag{3.31}$$

$$\begin{aligned}\int_\Gamma g\varphi_{0h}\mu_h = \int_{\Omega_f} \mathbf{f}\,(R^\tau_{1h}\mu_h) - a_f(\mathbf{u}^0_{fh} + E_{fh}\mathbf{u}_{inh}, R^\tau_{1h}\mu_h)\\ -b_f(R^\tau_{1h}\mu_h, p_{fh}) \qquad \forall \mu_h \in \Lambda_h\,,\end{aligned} \tag{3.32}$$

where we have introduced the FE spaces

$$H^\tau_{fh} = \{\mathbf{v}_h \in H_{fh} |\ \mathbf{v}_h \cdot \boldsymbol{\tau}_j = 0 \ \text{on}\ \Gamma\}$$

and

$$\widetilde{H}^0_{fh} = \{\mathbf{v}_h \in H_{fh} |\ \mathbf{v}_h = \mathbf{0} \ \text{on}\ \Gamma\}\,.$$

Moreover, R^τ_{1h} is any possible continuous extension operator from Λ_h to H^τ_{fh} such that $R^\tau_{1h}\mu_h\cdot\mathbf{n} = \mu_h$ on Γ, for all $\mu_h \in \Lambda_h$, and R_{2h} is any possible continuous extension operator from $\Lambda_{\dagger h}$ to H_{ph} such that $R_{2h}\mu_h = \mu_h$ on Γ, for all $\mu_h \in \Lambda_{\dagger h}$.

Proof. The proof follows the same guidelines as in the continuous case, thus we refer the reader to Proposition 2.4.1. □

Now, let us analyze the discrete Steklov-Poincaré operators corresponding to the two possible choices of the interface variable:

1. the interface variable is the trace λ_h of the normal velocity on Γ:

$$\lambda_h = \mathbf{u}_{fh}\cdot\mathbf{n} \text{ on } \Gamma; \tag{3.33}$$

2. the interface variable is the trace σ_h of the piezometric head

$$\sigma_h = \varphi_h \text{ on } \Gamma\,. \tag{3.34}$$

3.5 Interface Problem for the Discrete Normal Velocity

We consider the interface variable λ_h on Γ as in (3.33).
From (3.23) we obtain

$$\int_\Gamma \left(-\frac{\mathsf{K}}{n}\nabla\varphi_h\cdot\mathbf{n} - \lambda_h\right)\psi_{h|\Gamma} = 0 \qquad \forall \psi_h \in H_{ph}\,,$$

that is $\Pi\lambda_h = -(\mathsf{K}/n)\nabla\varphi_h\cdot\mathbf{n}$, where Π is the projection operator introduced in (3.24).

Now, if $\int_{\Gamma_f^{in}} \mathbf{u}_{inh} \cdot \mathbf{n} \neq 0$, we introduce a function $\lambda_{*h} \in \Lambda_h$, $\lambda_{*h} = \tilde{c}_* \gamma_h$ where γ_h is a piecewise linear function on Γ such that $\gamma_h(\mathbf{x}) = 0$ if $\mathbf{x}$ is a node on $\partial\Gamma$ and $\gamma_h(\mathbf{x}) = 1$ if $\mathbf{x}$ is a node on $\Gamma \setminus \partial\Gamma$, while $\tilde{c}_* \in \mathbb{R}$ is defined as

$$\tilde{c}_* = -\frac{\displaystyle\int_{\Gamma_f^{in}} \mathbf{u}_{inh} \cdot \mathbf{n}_f}{\displaystyle\int_\Gamma \gamma_h \cdot}$$

Therefore

$$\int_\Gamma \lambda_{*h} = -\int_{\Gamma_f^{in}} \mathbf{u}_{inh} \cdot \mathbf{n}_f \ . \tag{3.35}$$

Should the normal component of the datum $\mathbf{u}_{inh}$ have zero mean value over Γ_f^{in}, the analysis we are going to develop would still be valid by setting $\lambda_{*h} = 0$ and considering the whole trace space Λ_h instead of the trace subspace Λ_{0h} defined below.
We split λ_h as the sum of two components: $\lambda_h = \lambda_{0h} + \lambda_{*h}$, where λ_{*h} is the function introduced in (3.35), and $\lambda_{0h} \in \Lambda_{0h}$ with

$$\Lambda_{0h} = \left\{ \mu_h \in \Lambda_h \,\middle|\, \int_\Gamma \mu_h = 0 \right\} \ . \tag{3.36}$$

We introduce the two auxiliary problems (counterparts of problems **P1)** and **P2)** of Sect. 2.5):

P1$_h$) find $\boldsymbol{\omega}_{0h}^* \in \widetilde{H}_{fh}^0, \pi_h^* \in Q_{0h}$ such that for all $\mathbf{v}_h \in \widetilde{H}_{fh}^0$, $q_h \in Q_{0h}$

$$a_f(\boldsymbol{\omega}_{0h}^* + E_{fh}\mathbf{u}_{inh} + E_{\Gamma h}\lambda_{*h}, \mathbf{v}_h) + b_f(\mathbf{v}_h, \pi_h^*) = \int_{\Omega_f} \mathbf{f}\,\mathbf{v}_h \tag{3.37}$$

$$b_f(\boldsymbol{\omega}_{0h}^* + E_{fh}\mathbf{u}_{inh} + E_{\Gamma h}\lambda_{*h}, q_h) = 0\,, \tag{3.38}$$

where we have set $Q_{0h} = \{q_h \in Q_h |\ \int_{\Omega_f} q_h = 0\}$ and $E_{\Gamma h}\lambda_{*h} \in H_{fh}^\tau$ denotes a suitable discrete extension of λ_{*h}, such that $E_{\Gamma h}\lambda_{*h} \cdot \mathbf{n} = \lambda_{*h}$ on Γ;

P2$_h$) find $\varphi_{0h}^* \in H_{ph}$ such that

$$a_p(\varphi_{0h}^* + E_{ph}\varphi_{ph}, \psi_h) = \int_\Gamma n\lambda_{*h}\psi_h \ , \qquad \forall \psi_h \in H_{ph} \tag{3.39}$$

Remark that **P1$_h$)** is a Galerkin FE approximation of a Stokes problem where we impose the boundary conditions (1.22), (1.23) and (3.27), while we set the normal velocity equal to λ_{*h} on Γ.
On the other hand, **P2$_h$)** is a Galerkin approximation of a Darcy problem in Ω_p with the usual boundary conditions (1.24) and (1.25), and we impose $-(\mathsf{K}/n)\nabla\varphi_{0h}^* \cdot \mathbf{n} = \lambda_{*h}$ on Γ.
These problems thus depend on the data assigned to the global coupled problem.

Moreover, let us define the following extension operators:

$$R_{fh}\colon \Lambda_{0h} \to H^{\tau}_{fh} \times Q_{0h}, \qquad \eta_h \to R_{fh}\eta_h = (R^1_{fh}\eta_h, R^2_{fh}\eta_h)$$

such that $(R^1_{fh}\eta_h)\cdot \mathbf{n} = \eta_h$ on Γ and

$$a_f(R^1_{fh}\eta_h, \mathbf{w}_h) + b_f(\mathbf{w}_h, R^2_{fh}\eta_h) = 0 \qquad \forall \mathbf{w}_h \in H^0_{fh} \tag{3.40}$$

$$b_f(R^1_{fh}\eta_h, q_h) = 0 \qquad \forall q_h \in Q_{0h} \tag{3.41}$$

$$R_{ph}\colon \Lambda_{0h} \to H_{ph}, \qquad \eta_h \to R_{ph}\eta_h$$

such that

$$a_p(R_{ph}\eta_h, R_{2h}\mu_h) = \int_\Gamma n\eta_h\mu_h \qquad \forall \mu_h \in \Lambda_{\dagger h}, \tag{3.42}$$

where R_{2h} is the extension operator introduced in Proposition 3.4.1.

Now we can define the *discrete Steklov-Poincaré* operator $S_h : \Lambda_{0h} \to \Lambda'_h$ as follows:

$$\langle S_h\eta_h, \mu_h\rangle = a_f(R^1_{fh}\eta_h, R^\tau_{1h}\mu_h) + b_f(R^\tau_{1h}\mu_h, R^2_{fh}\eta_h) + \int_\Gamma g(R_{ph}\eta_h)\mu_h \tag{3.43}$$

$\forall \eta_h \in \Lambda_{0h}$, $\forall \mu_h \in \Lambda_h$.
It can be split as sum of two suboperators $S_h = S_{fh} + S_{ph}$, associated with the Stokes and Darcy problems, respectively, and defined as

$$\langle S_{fh}\eta_h, \mu_h\rangle = a_f(R^1_{fh}\eta_h, R^\tau_{1h}\mu_h) + b_f(R^\tau_{1h}\mu_h, R^2_{fh}\eta_h)\ , \tag{3.44}$$

$$\langle S_{ph}\eta_h, \mu_h\rangle = \int_\Gamma g\,(R_{ph}\eta_h)\mu_h\ , \tag{3.45}$$

for all $\eta_h \in \Lambda_{0h}$, $\mu_h \in \Lambda_h$.
Finally, let χ_h be the linear functional:

$$\begin{aligned}\langle \chi_h, \mu_h\rangle \;=\; & \int_{\Omega_f} \mathbf{f}\,(R^\tau_{1h}\mu_h) - a_f(\boldsymbol{\omega}^*_{0h} + E_{fh}\mathbf{u}_{inh} + E_{\Gamma h}\lambda_{*h}, R^\tau_{1h}\mu_h) \\ & -b_f(R^\tau_{1h}\mu_h, \pi^*_h) - \int_\Gamma g\,\varphi^*_{0h}\mu_h\end{aligned} \tag{3.46}$$

for all $\mu_h \in \Lambda_h$.

A characterization of the solution of problem (3.28)-(3.32) in terms of the solution of a Steklov-Poincaré discrete interface problem is given by the following result, which is the discrete counterpart of Theorem 2.5.1.

Theorem 3.5.1. *The solution to (3.28)-(3.32) can be characterized as follows:*

$$\begin{gathered}\mathbf{u}^0_{fh} = \boldsymbol{\omega}^*_{0h} + R^1_{fh}\lambda_{0h} + E_{\Gamma h}\lambda_{*h}, \quad p_{fh} = \pi^*_h + R^2_{fh}\lambda_{0h} + \hat{p}_{fh}, \\ \varphi_{0h} = \varphi^*_{0h} + R_{ph}\lambda_{0h}\,,\end{gathered} \tag{3.47}$$

where $\hat{p}_{fh} = (meas(\Omega_f))^{-1} \int_{\Omega_f} p_h$, *and* $\lambda_{0h} \in \Lambda_{0h}$ *is the solution of the discrete Steklov-Poincaré interface problem:*

$$\langle S_h \lambda_{0h}, \mu_{0h} \rangle = \langle \chi_h, \mu_{0h} \rangle \quad \forall \mu_{0h} \in \Lambda_{0h} \, . \tag{3.48}$$

Moreover, $\hat{p}_{fh}$ *can be obtained from* λ_{0h} *by solving the algebraic equation*

$$\hat{p}_{fh} = \frac{1}{meas(\Gamma)} \langle S_h \lambda_{0h} - \chi_h, \zeta_h \rangle \, , \tag{3.49}$$

where $\zeta_h \in \Lambda_h$ *is a given function that satisfies*

$$\frac{1}{meas(\Gamma)} \int_\Gamma \zeta_h = 1 \, . \tag{3.50}$$

3.5.1 Analysis of the Discrete Steklov-Poincaré Operators S_{fh} and S_{ph}

Let us investigate some properties of the discrete Steklov-Poincaré operators S_{fh}, S_{ph} and S_h that will allow us to prove existence and uniqueness for problem (3.48). Since their proofs are similar to those of the continuous case, we shall only sketch them, referring to Lemma 2.5.1 for more details.

Lemma 3.5.1. *The discrete Steklov–Poincaré operators enjoy the following properties:*

1. S_{fh} *and* S_{ph} *are linear continuous operators on* Λ_{0h}, i.e. $S_{fh}\eta_h \in \Lambda_0'$, $S_{ph}\eta_h \in \Lambda_0'$, $\forall \eta_h \in \Lambda_{0h}$;
2. S_{fh} *is symmetric and coercive;*
3. S_{ph} *is symmetric and positive;*
4. S_h *and* S_{fh} *are uniformly spectrally equivalent, i.e. there exist two constants* $\hat{k}_1$ *and* $\hat{k}_2$ *independent of* h, *s.t.* $\forall \eta_h \in \Lambda_h$,

$$\hat{k}_1 \langle S_{fh}\eta_h, \eta_h \rangle \, \leq \, \langle S_h \eta_h, \eta_h \rangle \, \leq \, \hat{k}_2 \langle S_{fh}\eta_h, \eta_h \rangle.$$

Proof. 1. Making the special choice $R^\tau_{1h} = R^1_{fh}$, the operator S_{fh} can be represented as follows

$$\langle S_{fh}\eta_h, \mu_h \rangle \, = \, a_f(R^1_{fh}\eta_h, R^1_{fh}\mu_h) \, , \tag{3.51}$$

for all $\eta_h, \mu_h \in \Lambda_{0h}$.
Now, proceeding as in 1. of Lemma 2.5.1, we can define the function $\mathbf{z}_h(\mu_h) = R^1_{fh}\mu_h - \mathcal{H}_h\mu_h \in \widetilde{H}^0_{fh}$, $\mathcal{H}_h$ being the Galerkin approximation of the harmonic extension operator defined in (2.85). Using the inf-sup condition (5.3.43) of [QV94] p. 173, we have for all $\mu_h \in \Lambda_{0h}$

$$\|R^2_{fh}\mu_h\|_{0,f} \leq \frac{\nu}{\beta^*} \|R^1_{fh}\mu_h\|_{1,f} \, ,$$

and therefore

$$\|R^1_{fh}\mu_h\|_{1,f} \leq (1 + C_{\Omega_f}) \left(1 + \frac{1}{\beta^*}\right) \|\mathcal{H}_h\mu_h\|_{1,f} \, , \tag{3.52}$$

C_{Ω_f} being a positive constant due to the Poincaré inequality. Now, thanks to the Uniform Extension Theorem (see [QV94] Theorem 4.1.3; [MQ89]), there exists a positive constant $C_{|\Omega_f|} > 0$, depending on the measure of the subdomain Ω_f, but independent of the parameter h, such that

$$\|\mathcal{H}_h\mu_h\|_{1,f} \leq C_{|\Omega_f|}\|\mu_h\|_\Lambda \qquad \forall \mu_h \in \Lambda_h \ .$$

Therefore, (3.52) gives $\forall \mu_h \in \Lambda_{0h}$

$$\|R^1_{fh}\mu_h\|_{1,f} \leq C_{|\Omega_f|}(1 + C_{\Omega_f})\left(1 + \frac{1}{\beta^*}\right)\|\mu_h\|_\Lambda \, . \tag{3.53}$$

From (3.53) we deduce the continuity of S_{fh}:

$$|\langle S_{fh}\mu_h, \eta_h\rangle| \leq \hat{\beta}_f\|\mu_h\|_\Lambda\|\eta_h\|_\Lambda \ , \tag{3.54}$$

where $\hat{\beta}_f$ is the positive constant, independent of h,

$$\hat{\beta}_f = \nu\left[C_{|\Omega_f|}(1 + C_{\Omega_f})\left(1 + \frac{1}{\beta^*}\right)\right]^2 \ . \tag{3.55}$$

Proceeding as for the continuous case, we can prove that S_{ph} is continuous with constant β_p, independent of h, defined in (2.91).

2. S_{fh} is symmetric thanks to (3.51) and the proof of its coercivity follows the one in the continuous case, the coercivity constant α_f being the same (see (2.92)).

3. This property follows from point 3. of the proof of Lemma 2.5.1. □

Remark 3.5.1. Notice that the discrete operator S_{ph} is actually coercive (see, e.g., [Cia78, Ago88]); its coercivity constant, say $\alpha_{ph} > 0$ depends on h and, in particular, it vanishes for $h \to 0$. Since our aim is to characterize preconditioners optimal with respect to h for (3.48), we omit to further investigate this property. □

Remark 3.5.2. Thanks to Lax-Milgram Lemma (see, e.g., [QV94] p. 133), Lemma 3.5.1 guarantees that the discrete Steklov-Poincaré equation (3.48) has a unique solution. □

3.5.2 Algebraic Formulation of the Discrete Steklov-Poincaré Operator S_h

We consider the linear system (3.26) and we set $\boldsymbol{u}_f = (\boldsymbol{u}_{int}, \boldsymbol{p})^T$ and $\boldsymbol{\phi} = (\boldsymbol{\phi}_\Gamma, \boldsymbol{\phi}_{int})^T$. Then, with obvious choice of notation, we can rewrite (3.26) in the following block form:

$$\begin{pmatrix} \mathrm{F} & \mathrm{F}^1_\Gamma & 0 \\ \mathrm{F}^2_\Gamma & \mathrm{A}^f_{\Gamma\Gamma} & \mathrm{M}_1 \\ 0 & \mathrm{M}_2 & \mathrm{D} \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_f \\ \boldsymbol{u}_\Gamma \\ \boldsymbol{\phi} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_\Gamma \\ \boldsymbol{f}_2 \end{pmatrix} \ . \tag{3.56}$$

By writing $\boldsymbol{u}_\Gamma = \boldsymbol{u}_\Gamma^0 + \boldsymbol{\lambda}_*$, where $\boldsymbol{\lambda}_*$ is the vector whose components are the (known) values of λ_{*h} at the nodes on Γ, system (3.56) reduces to:

$$\begin{pmatrix} \mathrm{F} & \mathrm{F}_\Gamma^1 & 0 \\ \mathrm{F}_\Gamma^2 & \mathrm{A}_{\Gamma\Gamma}^f & \mathrm{M}_1 \\ 0 & \mathrm{M}_2 & \mathrm{D} \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_f \\ \boldsymbol{u}_\Gamma^0 \\ \boldsymbol{\phi} \end{pmatrix} = \begin{pmatrix} \hat{\boldsymbol{f}}_1 \\ \hat{\boldsymbol{f}}_\Gamma \\ \hat{\boldsymbol{f}}_2 \end{pmatrix} \tag{3.57}$$

where $\hat{\boldsymbol{f}}_1 = \boldsymbol{f}_1 - \mathrm{F}_\Gamma^1 \boldsymbol{\lambda}_*$, $\hat{\boldsymbol{f}}_\Gamma = \boldsymbol{f}_\Gamma - \mathrm{A}_{\Gamma\Gamma}^f \boldsymbol{\lambda}_*$ and $\hat{\boldsymbol{f}}_2 = \boldsymbol{f}_2 - \mathrm{M}_2 \boldsymbol{\lambda}_*$.
Upon eliminating the unknowns $\boldsymbol{u}_f$ and $\boldsymbol{\phi}$, we obtain the reduced Schur complement system:

$$\Sigma_h \boldsymbol{u}_\Gamma^0 = \boldsymbol{\chi}_h \tag{3.58}$$

where we have defined

$$\Sigma_h = (\mathrm{A}_{\Gamma\Gamma}^f - \mathrm{F}_\Gamma^2 \mathrm{F}^{-1} \mathrm{F}_\Gamma^1) + (-\mathrm{M}_1 \mathrm{D}^{-1} \mathrm{M}_2) \tag{3.59}$$

and

$$\boldsymbol{\chi}_h = \hat{\boldsymbol{f}}_\Gamma - \mathrm{F}_\Gamma^2 \mathrm{F}^{-1} \hat{\boldsymbol{f}}_1 - \mathrm{M}_1 \mathrm{D}^{-1} \hat{\boldsymbol{f}}_2 \, . \tag{3.60}$$

In (3.59) the first term

$$\Sigma_{fh} = \mathrm{A}_{\Gamma\Gamma}^f - \mathrm{F}_\Gamma^2 \mathrm{F}^{-1} \mathrm{F}_\Gamma^1 \tag{3.61}$$

arises from domain Ω_f, whereas

$$\Sigma_{ph} = -\mathrm{M}_1 \mathrm{D}^{-1} \mathrm{M}_2 \tag{3.62}$$

from Ω_p.
The matrices Σ_{fh} and Σ_{ph} are the algebraic counterparts of the operators S_{fh} and S_{ph}, respectively.

Remark 3.5.3. To be precise, notice that we are slightly abusing in notation, since for the algebraic system (3.26) we have considered the complete interface condition (1.28), while in order to characterize the discrete Steklov-Poincaré operators we have used its simplified form (3.27). Therefore, the exact algebraic counterpart of S_{fh} and S_{ph} should be obtained considering a basis of H_{fh}^τ instead of H_{fh}. □

Thanks to Lemma 3.5.1, the matrices Σ_{fh} and Σ_h are symmetric and positive definite and moreover

$$[\Sigma_{fh}\boldsymbol{\mu}, \boldsymbol{\mu}] \leq [\Sigma_h \boldsymbol{\mu}, \boldsymbol{\mu}] \leq \left(1 + \frac{\beta_p}{\alpha_f}\right) [\Sigma_{fh}\boldsymbol{\mu}, \boldsymbol{\mu}] \qquad \forall \boldsymbol{\mu} \in \mathbb{R}^{N_\Gamma}, \tag{3.63}$$

where $[.,.]$ is the Euclidean scalar product in $\mathbb{R}^{N_\Gamma}$ and α_f and β_p are the constants defined in (2.92) and (2.91), respectively.
Thus, the spectral condition number χ_{sp} of the matrix $\Sigma_{fh}^{-1}\Sigma_h$ is bounded independently of h:

$$\chi_{sp}(\Sigma_{fh}^{-1}\Sigma_h) \leq 1 + \frac{\beta_p}{\alpha_f} \, , \tag{3.64}$$

and Σ_{fh} is an optimal preconditioner for Σ_h. Therefore, should we use Σ_{fh} as preconditioner to solve the symmetric linear system (3.58) using the preconditioned Richardson method

$$(\boldsymbol{u}_\Gamma^0)^{k+1} = (\boldsymbol{u}_\Gamma^0)^k + \Sigma_{fh}^{-1}(\boldsymbol{\chi}_h - \Sigma_h(\boldsymbol{u}_\Gamma^0)^k), \tag{3.65}$$

we would get convergence with a rate independent of h. Same conclusion if instead of (3.65) we would use a Krylov type method (e.g., the conjugate gradient method).
In the next section, we shall interpret (3.65) as a Dirichlet-Neumann type substructuring scheme and we shall prove its convergence.

3.5.3 An Iterative Method for the Numerical Solution of the Coupled Problem

The iterative method we propose to compute the solution of the Stokes/Darcy problem (3.28)-(3.32) consists in solving first Darcy problem in Ω_p imposing the continuity of the normal velocities across Γ. Then, we solve the Stokes problem in Ω_f imposing the continuity of the normal stresses across the interface, using the value of φ_h on Γ that we have just computed in the porous media domain.
Precisely, the iterative scheme reads as follows:

given $\mathbf{u}_{inh}$, construct λ_{*h} as in (3.35);
then let $\lambda_h^0 \in \Lambda_{0h}$ be the initial guess; for $k \geq 0$:

i) find $\varphi_{0h}^{k+1} \in H_{ph}$:

$$a_p(\varphi_{0h}^{k+1}, \psi_h) - \int_\Gamma n\,\psi_h\,\lambda_{0h}^k = -a_p(E_{ph}\varphi_{ph}, \psi_h) + \int_\Gamma n\,\psi_h\lambda_{*h} \qquad \forall \psi_h \in H_{ph}\,; \tag{3.66}$$

ii) find $(\mathbf{u}_{fh}^0)^{k+1} \in H_{fh}^\tau$, $p_{fh}^{k+1} \in Q_h$:

$$\begin{aligned} a_f((\mathbf{u}_{fh}^0)^{k+1}, \mathbf{w}_h) + b_f(\mathbf{w}_h, p_{fh}^{k+1}) + \int_\Gamma g\varphi_h^{k+1}\mathbf{w}_h\cdot\mathbf{n} &= \int_{\Omega_f} \mathbf{f}\,\mathbf{w}_h \\ -a_f(E_{fh}\mathbf{u}_{inh}, \mathbf{w}_h) \qquad \forall \mathbf{w}_h \in H_{fh}^\tau, & \end{aligned} \tag{3.67}$$

$$b_f((\mathbf{u}_{fh}^0)^{k+1}, q_h) = -b_f(E_{fh}\mathbf{u}_{inh}, q_h) \qquad \forall q_h \in Q_h, \tag{3.68}$$

with $\varphi_h^{k+1} = \varphi_{0h}^{k+1} + E_{ph}\varphi_{ph}$;

iii) update λ_{0h}^k:

$$\lambda_{0h}^{k+1} = \theta(\mathbf{u}_{fh}^{k+1}\cdot\mathbf{n} - \lambda_{*h})_{|\Gamma} + (1-\theta)\lambda_{0h}^k\,, \tag{3.69}$$

θ being a positive relaxation parameter and $\mathbf{u}_{fh}^{k+1} = (\mathbf{u}_{fh}^0)^{k+1} + E_{fh}\mathbf{u}_{inh}$.

Remark 3.5.4. Note that $\lambda_{0h}^k \in \Lambda_{0h}$ for all $k \geq 0$. In fact, $\lambda_{0h} \in \Lambda_{0h}$ given, suppose $\lambda_{0h}^k \in \Lambda_0$. Then

$$\int_\Gamma \lambda_{0h}^{k+1} = \theta \int_\Gamma (\mathbf{u}_{fh}^{k+1}\cdot\mathbf{n}|_\Gamma - \lambda_{*h})\,.$$

Now, since $\int_{\Omega_f} \nabla\cdot\mathbf{u}_{fh}^{k+1} = 0$, thanks to the divergence theorem we have

$$\int_\Gamma \mathbf{u}_{fh}^{k+1}\cdot\mathbf{n} = -\int_{\Gamma_f^{in}} \mathbf{u}_{inh}\cdot\mathbf{n}$$

and recalling (3.35) the thesis follows. □

Following the general theory developed in [QV99], the above iterative method can be reinterpreted as a preconditioned Richardson method for the Steklov–Poincaré problem (3.48).

Lemma 3.5.2. *The iterative substructuring scheme (3.66)-(3.69) to compute the solution of the FE approximation of the coupled problem Stokes/Darcy (3.28)-(3.32) is equivalent to a preconditioned Richardson method for the discrete Steklov-Poincaré equation (3.48), the preconditioner being the operator S_{fh} introduced in (3.44).*

Proof. Since $E_{fh}\mathbf{u}_{inh}\cdot\mathbf{n} = 0$ on Γ, (3.69) reduces to:

$$\lambda_{0h}^{k+1} = \theta((\mathbf{u}_{fh}^0)^{k+1}\cdot\mathbf{n} - \lambda_{*h})_{|\Gamma} + (1-\theta)\lambda_{0h}^k \ . \tag{3.70}$$

Let $R_{1h}^\tau : \Lambda_h \to H_{fh}^\tau$ be the extension operator introduced in Proposition 3.4.1. For all $\mu_h \in \Lambda_h$, we can rewrite (3.67) as:

$$\begin{aligned} a_f((\mathbf{u}_{fh}^0)^{k+1}, R_{1h}^\tau\mu_h) + b_f(R_{1h}^\tau\mu_h, p_{fh}^{k+1}) + \int_\Gamma g\varphi_h^{k+1}\mu_h \\ = \int_{\Omega_f} \mathbf{f}\,(R_{1h}^\tau\mu_h) - a_f(E_{fh}\mathbf{u}_{inh}, R_{1h}^\tau\mu_h) \qquad \forall\mu_h\in\Lambda_h. \end{aligned} \tag{3.71}$$

Let us define $\hat{p}_{fh}^{k+1} = (meas(\Omega_f))^{-1}\int_{\Omega_f} p_{fh}^{k+1}$; then we set

$$p_{0h}^{k+1} = p_{fh}^{k+1} - \hat{p}_{fh}^{k+1} \ , \tag{3.72}$$

and we note that $p_{0h}^{k+1}\in Q_0$. Then (3.71) gives:

$$\begin{aligned} a_f((\mathbf{u}_{fh}^0)^{k+1}, R_{1h}^\tau\mu_h) + b_f(R_{1h}^\tau\mu_h, p_{0h}^{k+1}) + \int_\Gamma g\varphi_h^{k+1}\mu_h \\ = \int_{\Omega_f} \mathbf{f}\,(R_{1h}^\tau\mu_h) + b_f(R_{1h}^\tau\mu_h, \hat{p}_{fh}^{k+1}) - a_f(E_{fh}\mathbf{u}_{inh}, R_{1h}^\tau\mu_h) \qquad \forall\mu_h\in\Lambda_h. \end{aligned} \tag{3.73}$$

Let $\boldsymbol{\omega}_{0h}^*$, π_h^* and φ_{0h}^* be the solutions to problems (3.37), (3.38) and (3.39), respectively. Subtracting from both members in (3.73) the following terms:

$$a_f(\boldsymbol{\omega}_{0h}^* + E_{\Gamma h}\lambda_{h*}, R_{1h}^\tau\mu_h) + b_f(R_{1h}^\tau\mu_h, \pi_h^*) + \int_\Gamma g\,\varphi_{0h}^*\mu_h \ ,$$

we have

$$\begin{aligned} &a_f((\mathbf{u}_{fh}^0)^{k+1} - \boldsymbol{\omega}_{0h}^* - E_{\Gamma h}\lambda_{h*}, R_{1h}^\tau\mu_h) + b_f(R_{1h}^\tau\mu_h, p_{0h}^{k+1} - \pi_h^*) \\ &+ \int_\Gamma g(\varphi_h^{k+1} - \varphi_{0h}^*)\mu_h = \int_{\Omega_f} \mathbf{f}\,(R_{1h}^\tau\mu_h) - b_f(R_{1h}^\tau\mu_h, \pi_h^*) \\ &-a_f(\boldsymbol{\omega}_{0h}^* + E_{\Gamma h}\lambda_{h*} + E_{fh}\mathbf{u}_{inh}, R_{1h}^\tau\mu_h) - \int_\Gamma g\,\varphi_{0h}^*\mu_h + b_f(R_{1h}^\tau\mu_h, \hat{p}_{fh}^{k+1}) \end{aligned} \tag{3.74}$$

for all $\mu_h \in \Lambda_h$.
Since

$$\int_{\Omega_f} \nabla \cdot (\boldsymbol{\omega}_0^* + E_{\Gamma h}\lambda_{*h} + E_{fh}\mathbf{u}_{inh}) = 0 \quad \text{and} \quad \int_{\Omega_f} \nabla \cdot ((\mathbf{u}_{fh}^0)^{k+1} + E_{fh}\mathbf{u}_{inh}) = 0,$$

we obtain

$$\int_{\Omega} \nabla \cdot ((\mathbf{u}_{fh}^0)^{k+1} - \boldsymbol{\omega}_0^* - E_{\Gamma h}\lambda_{*h}) = 0.$$

Now, if we apply the divergence theorem and recall that $(\mathbf{u}_{fh}^0)^{k+1} \in H_{fh}^\tau$, $\boldsymbol{\omega}_{0h}^* \in \widetilde{H}_{fh}^0$ and $E_{\Gamma h}\lambda_{*h} \in H_{fh}^\tau$, we can see that $[(\mathbf{u}_{fh}^0)^{k+1} - E_{\Gamma h}\lambda_{*h}] \cdot \mathbf{n}_{|\Gamma} \in \Lambda_{0h}$. Therefore

$$a_f((\mathbf{u}_{fh}^0)^{k+1} - \boldsymbol{\omega}_{0h}^* - E_{\Gamma h}\lambda_{*h}, R_{1h}^\tau\mu_h) + b_f(R_{1h}^\tau\mu_h, p_{0h}^{k+1} - \pi_h^*) = \langle S_{fh}(((\mathbf{u}_{fh}^0)^{k+1} - E_{\Gamma h}\lambda_{*h}) \cdot \mathbf{n})_{|\Gamma}, \mu_h\rangle \quad (3.75)$$

for all $\mu_h \in \Lambda_h$.
Moreover, if we subtract (3.39) from (3.66), we obtain

$$a_p(\varphi_{0h}^{k+1} - \varphi_{0h}^*, \psi_h) = \int_\Gamma n\lambda_{0h}^k\psi_h \qquad \forall \psi_h \in H_{ph} \ ,$$

that is, thanks to (3.42), $\varphi_{0h}^{k+1} - \varphi_{0h}^* = R_{ph}\lambda_{0h}^k$. Therefore

$$\int_\Gamma g(\varphi_h^{k+1} - \varphi_{0h}^*)\mu_h = \langle S_{ph}\lambda_{0h}^k, \mu_h\rangle \quad \forall \mu_h \in \Lambda_h.$$

Finally, if we apply the divergence theorem to the last right hand side term in (3.74) and we recall the definition (3.46), we can rewrite the right hand side of (3.74) as

$$\langle \chi_h, \mu_h\rangle + \hat{p}_{fh}^{k+1}\int_\Gamma \mu_h \qquad \forall \mu_h \in \Lambda_h \ . \quad (3.76)$$

Now, for all $\mu_h \in \Lambda_{0h}$, it follows:

$$\langle S_{fh}(((\mathbf{u}_{fh}^0)^{k+1} - E_{\Gamma h}\lambda_{*h}) \cdot \mathbf{n})_{|\Gamma}, \mu_h\rangle + \langle S_{ph}\lambda_{0h}^k, \mu_h\rangle = \langle \chi_h, \mu_h\rangle \ . \quad (3.77)$$

Therefore we can conclude that (3.66)-(3.69) is equivalent to the preconditioned Richardson scheme: $\lambda_{0h}^0 \in \Lambda_{0h}$ given, for $k \geq 0$, find $\lambda_{0h}^{k+1} \in \Lambda_{0h}$ s.t.

$$\lambda_{0h}^{k+1} = \lambda_{0h}^k + \theta_h S_{fh}^{-1}(\chi_h - S_h\lambda_{0h}^k) \ , \quad (3.78)$$

□

Remark 3.5.5. The algorithm (3.66)-(3.69) does not feature the classical structure of a Dirichlet-Neumann method, which would require to solve one subproblem in the first subdomain with a Dirichlet boundary condition on the interface, and one problem in the second subdomain with a Neumann boundary condition on the interface. In fact, we are imposing natural boundary conditions for both subproblems. However, in view of (3.78), we can still refer to it as to a Dirichlet-Neumann method since the preconditioner is the Steklov-Poincaré operator associated to the second subproblem. □

The formulation (3.78) is useful to carry out the convergence analysis of scheme (3.66)-(3.69), as illustrated in the following section.

3.5.4 Convergence Analysis of the Iterative Method

Our aim is now to prove the convergence of the sequence $\{((\mathbf{u}_{fh}^0)^k, p_{fh}^k, \varphi_{0h}^k)\}_k$ generated by the iterative method (3.66)-(3.69) to the exact solution $(\mathbf{u}_{fh}^0, p_{fh}, \varphi_{0h})$ of the coupled problem (3.28)-(3.32).
To this end, we shall apply the following abstract convergence result (see [QV99] Theorem 4.2.2 and Remark 4.2.4).

Theorem 3.5.2. *Let X be a (real) Hilbert space and X' its dual. We consider a linear invertible continuous operator $\mathcal{Q}$: $X \to X'$, which can be split as $\mathcal{Q} = \mathcal{Q}_1 + \mathcal{Q}_2$, where both $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are linear operators. Taken $\mathcal{Z} \in X'$, let $x \in X$ be the unknown solution to the equation*

$$\mathcal{Q}x = \mathcal{Z} \ ,$$

and consider for its solution the preconditioned Richardson method

$$\mathcal{Q}_2(x^{k+1} - x^k) = \theta(\mathcal{Z} - \mathcal{Q}x^k), \quad k \geq 0,$$

θ being a positive relaxation parameter. Suppose that the following conditions are satisfied:

1. *$\mathcal{Q}_2$ is symmetric, continuous and coercive with constants β_2 and α_2, respectively;*
2. *$\mathcal{Q}_1$ is continuous with constant β_1;*
3. *$\mathcal{Q}$ is coercive with constant $\alpha_{\mathcal{Q}}$.*

Then, for any given $x^0 \in X$ and for any $0 < \theta < \theta_{max}$, with

$$\theta_{max} = \frac{2\alpha_{\mathcal{Q}}\alpha_2^2}{\beta_2(\beta_1 + \beta_2)^2} \ ,$$

the sequence

$$x^{k+1} = x^k + \theta\mathcal{Q}_2^{-1}(\mathcal{Z} - \mathcal{Q}x^k)$$

converges in X to the solution of problem $\mathcal{Q}x = \mathcal{Z}$.

We can now prove the main result of this section.

Theorem 3.5.3. *The iterative method (3.66)-(3.69) converges to the solution $(\mathbf{u}_{fh}^0, p_{fh}, \varphi_{0h}) \in H_{fh}^\tau \times Q_h \times H_{ph}$ of the coupled Stokes/Darcy problem (3.28)-(3.32), for any choice of the initial guess $\lambda_{0h}^0 \in \Lambda_{0h}$, and for suitable values of the relaxation parameter θ.*

Proof. Upon setting $X = \Lambda_{0h}$, $\mathcal{Q} = S_h$, $\mathcal{Q}_1 = S_{ph}$, $\mathcal{Q}_2 = S_{fh}$ and $\mathcal{Z} = \chi_h$, the proof follows from Theorem 3.5.2, whose hypotheses are satisfied thanks to Lemma 3.5.1. In fact, for an initial guess $\lambda_{0h}^0 \in \Lambda_{0h}$, and any $0 < \theta < \theta_{max}$ with

$$\theta_{max} = \frac{2\alpha_f^3}{\hat{\beta}_f(\hat{\beta}_f + \beta_p)^2} \ , \tag{3.79}$$

the sequence defined in (3.78) converges to the solution of the Steklov–Poincaré equation (3.48). Taking the limit $k \to \infty$ in the iterative procedure (3.66)-(3.69), it follows that $\{((\mathbf{u}_{fh}^0)^k, p_{fh}^k, \varphi_{0h}^k)\}_k \to (\mathbf{u}_{fh}^0, p_{fh}, \varphi_{0h})$.
The upper bound θ_{max} is independent of h as such are the constants α_f, $\hat{\beta}_f$ and β_p. □

3.5.5 Matrix Interpretation of the Substructuring Iterative Method

The iterative scheme (3.66)–(3.69) corresponds to the following steps.

Let $\boldsymbol{\lambda}_0^k \in \mathbb{R}^{N_\Gamma}$ be the vector of the values of λ_{0h}^k at the k-th step at the nodes of Γ.

The following algebraic system corresponds to (3.66):

$$\begin{pmatrix} \mathrm{A}_{\Gamma\Gamma}^p & \mathrm{A}_{p\Gamma}^T \\ \mathrm{A}_{p\Gamma} & \mathrm{A}_{pp} \end{pmatrix} \begin{pmatrix} \boldsymbol{\phi}_\Gamma^{k+1} \\ \boldsymbol{\phi}_{int}^{k+1} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}_{p\Gamma} + \mathrm{M}_\Gamma^T \boldsymbol{\lambda}_0^k + \mathrm{M}_\Gamma^T \boldsymbol{\lambda}_* \\ \boldsymbol{f}_p \end{pmatrix} . \tag{3.80}$$

By eliminating $\boldsymbol{\phi}_{int}^{k+1}$ from (3.80), we obtain

$$(\mathrm{A}_{\Gamma\Gamma}^p - A_{p\Gamma}^T \mathrm{A}_{pp}^{-1} \mathrm{A}_{p\Gamma}) \boldsymbol{\phi}_\Gamma^{k+1} = \boldsymbol{f}_{p\Gamma} - \mathrm{A}_{p\Gamma}^T \mathrm{A}_{pp}^{-1} \boldsymbol{f}_p + \mathrm{M}_\Gamma^T \boldsymbol{\lambda}^k + \mathrm{M}_\Gamma^T \boldsymbol{\lambda}_* . \tag{3.81}$$

Now use $\boldsymbol{\phi}_\Gamma^{k+1}$ to compute the unknown vector $\boldsymbol{u}_\Gamma^{k+1}$ by solving the following system, which corresponds to the Stokes problem (3.67), (3.68):

$$\begin{pmatrix} \mathrm{A}_{ff} & \mathrm{B}^T & \mathrm{A}_{f\Gamma} \\ \mathrm{B}_1 & 0 & \mathrm{B}_{f\Gamma} \\ \mathrm{A}_{f\Gamma} & \mathrm{B}_{f\Gamma}^T & \mathrm{A}_{\Gamma\Gamma}^f \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_{int}^{k+1} \\ \boldsymbol{p}^{k+1} \\ \boldsymbol{u}_\Gamma^{k+1} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}_f \\ \boldsymbol{f}_{in} \\ \boldsymbol{f}_\Gamma - \mathrm{M}_\Gamma \boldsymbol{\phi}_\Gamma^{k+1} \end{pmatrix} \tag{3.82}$$

Finally, according to (3.69), we set

$$\boldsymbol{\lambda}_0^{k+1} = \theta(\boldsymbol{u}_\Gamma^{k+1} - \boldsymbol{\lambda}_*) + (1-\theta)\boldsymbol{\lambda}_0^k , \tag{3.83}$$

and we iterate restarting from (3.80) until the convergence test

$$\frac{\|\boldsymbol{\lambda}_0^{k+1} - \boldsymbol{\lambda}_0^k\|_{\mathbb{R}^{N_\Gamma}}}{\|\boldsymbol{\lambda}_0^{k+1}\|_{\mathbb{R}^{N_\Gamma}}} \leq \epsilon$$

is satisfied for a prescribed tolerance ϵ; $\|\cdot\|_{\mathbb{R}^{N_\Gamma}}$ denotes the Euclidean norm in $\mathbb{R}^{N_\Gamma}$.

3.6 Interface Problem for the Piezometric Head

The approach based on the interface variable σ illustrated in Sect. 2.6 can be replicated at the discrete level considering the discrete variable σ_h in (3.34).
In particular, we define the discrete extension operators:

$$\mathcal{R}_{fh} : \Lambda_{\dagger_h} \to H_{fh}^\tau \times Q_h, \quad \eta_h \to \mathcal{R}_{fh}\eta_h = (\mathcal{R}_{fh}^1 \eta_h, \mathcal{R}_{fh}^2 \eta_h)$$

such that

$$a_f(\mathcal{R}_{fh}^1\eta_h, \mathbf{v}_h) + b_f(\mathbf{v}_h, \mathcal{R}_{fh}^2\eta_h) + \int_\Gamma g\eta_h \mathbf{v}_h \cdot \mathbf{n} = 0 \qquad \forall \mathbf{v}_h \in H_{fh}^\tau \tag{3.84}$$

$$b_f(\mathcal{R}_{fh}^1\eta_h, q_h) = 0 \qquad \forall q_h \in Q_h; \tag{3.85}$$

$$\mathcal{R}_{ph} : \Lambda_\dagger \to H_{ph}, \quad \eta_h \to \mathcal{R}_{ph}\eta_h$$

such that $\mathcal{R}_{ph}\eta_h = \eta_h$ on Γ and

$$a_p(\mathcal{R}_{ph}\eta_h, \psi_h) = 0 \qquad \forall \psi_h \in H^0_{ph}. \tag{3.86}$$

Then, we can characterize the local Steklov-Poincaré operators:

$$\langle \mathcal{S}_{fh}\eta_h, \mu_h\rangle = -\int_\Gamma n(\mathcal{R}^1_{fh}\eta_h \cdot \mathbf{n})\mu_h \tag{3.87}$$

$$\langle \mathcal{S}_{ph}\eta_h, \mu_h\rangle = a_p(\mathcal{R}_{ph}\eta_h, R_{2h}\mu_h) \tag{3.88}$$

for all $\eta_h, \mu_h \in \Lambda_{\dagger h}$ and the global operator $\mathcal{S}_h$:

$$\langle \mathcal{S}_h\eta_h, \mu_h\rangle = \langle \mathcal{S}_{fh}\eta_h, \mu_h\rangle + \langle \mathcal{S}_{ph}\eta_h, \mu_h\rangle \qquad \forall \eta, \mu \in \Lambda_{\dagger h}. \tag{3.89}$$

Finally, let $\varsigma_h : \Lambda_{\dagger h} \to \mathbb{R}$ be the linear functional

$$\langle \varsigma_h, \mu_h\rangle = \int_\Gamma n(\boldsymbol{\varpi}^*_{0h} \cdot \mathbf{n})\mu_h - a_p(\phi^*_{0h} + E_{ph}\varphi_{ph}, R_{2h}\mu_h) \qquad \forall \mu_h \in \Lambda_{\dagger h} \tag{3.90}$$

where $\boldsymbol{\varpi}^*_{0h} \in H^\tau_{fh}$ and $\phi^*_{0h} \in H^0_{ph}$ are the solutions to Galerkin approximations of problems (2.93), (2.94) and (2.95), respectively.

The counterpart of Theorem 2.6.1 holds:

Theorem 3.6.1. *The solution to (3.28)-(3.32) can be characterized as*

$$\mathbf{u}^0_{fh} = \boldsymbol{\varpi}^*_{0h} + \mathcal{R}^1_{fh}\sigma_h, \quad p_{fh} = \overline{\pi}^*_h + \mathcal{R}^2_{fh}\sigma_h, \quad \varphi_{0h} = \phi^*_{0h} + \mathcal{R}_{ph}\sigma_h \tag{3.91}$$

where $\sigma_h \in \Lambda_{\dagger h}$ is the solution of the Steklov-Poincaré equation

$$\langle \mathcal{S}_h\sigma_h, \mu_h\rangle = \langle \varsigma_h, \mu_h\rangle \qquad \forall \mu_h \in \Lambda_{\dagger h}\,. \tag{3.92}$$

The existence and uniqueness of the solution σ_h of (3.92) is guaranteed by the analysis of the discrete Steklov-Poincaré operators. In particular, we can state the following result, which is the discrete counterpart of Lemma 2.6.1.

Lemma 3.6.1. *The Steklov-Poincaré operators $\mathcal{S}_{fh}$ and $\mathcal{S}_{ph}$ enjoy the following properties:*

1. *$\mathcal{S}_{fh}$ and $\mathcal{S}_{ph}$ are linear continuous operators on $\Lambda_{\dagger h}$ with continuity constants $\tilde{\beta}_f$ (see (2.108)) and $\bar{\beta}_p = \hat{C}\max_j \|\mathsf{K}\|_{\infty,f}$, respectively, $\hat{C} > 0$ being a positive constant independent of h due to the Uniform Extension Theorem;*
2. *$\mathcal{S}_{fh}$ is symmetric and positive;*
3. *$\mathcal{S}_{ph}$ is symmetric and coercive with coercivity constant $\tilde{\alpha}_p$ as in (2.109);*
4. *$\mathcal{S}_h$ and $\mathcal{S}_{ph}$ are spectrally equivalent, i.e. there exist two positive constants $\bar{k}_1$ and $\bar{k}_2$, independent of h, such that*

$$\bar{k}_1\langle \mathcal{S}_{ph}\eta_h, \eta_h\rangle \le \langle \mathcal{S}_h\eta_h, \eta_h\rangle \le \bar{k}_2\langle \mathcal{S}_{ph}\eta_h, \eta_h\rangle \qquad \forall \eta_h \in \Lambda_{\dagger h}\,.$$

3.6.1 Algebraic Formulation of the Discrete Steklov-Poincaré Operator $\mathcal{S}_h$

Denoting $\boldsymbol{u} = (\boldsymbol{u}_{int}, \boldsymbol{p}, \boldsymbol{u}_\Gamma)^T$, system (3.26) can be rewritten with obvious block matrix notation as:

$$\begin{pmatrix} \widetilde{\mathrm{F}} & \widetilde{\mathrm{M}}_1 & 0 \\ \widetilde{\mathrm{M}}_2 & \mathrm{A}^p_{\Gamma\Gamma} & \mathrm{A}^T_{p\Gamma} \\ 0 & \mathrm{A}_{p\Gamma} & \mathrm{A}_{pp} \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{\phi}_\Gamma \\ \boldsymbol{\phi}_{int} \end{pmatrix} = \begin{pmatrix} \widetilde{\boldsymbol{f}}_1 \\ \boldsymbol{f}_{p\Gamma} \\ \boldsymbol{f}_p \end{pmatrix} . \tag{3.93}$$

We take the Schur complement with respect to the unknown $\boldsymbol{\phi}_\Gamma$:

$$\widetilde{\Sigma}_h \boldsymbol{\phi}_\Gamma = \widetilde{\boldsymbol{\chi}}_h \tag{3.94}$$

where

$$\widetilde{\Sigma}_h = (\mathrm{A}^p_{\Gamma\Gamma} - \mathrm{A}^T_{p\Gamma}\mathrm{A}^{-1}_{pp}\mathrm{A}_{p\Gamma}) + (-\widetilde{\mathrm{M}}_2\widetilde{\mathrm{F}}^{-1}\widetilde{\mathrm{M}}_1) \tag{3.95}$$

and

$$\widetilde{\boldsymbol{\chi}}_h = \boldsymbol{f}_{p\Gamma} - \mathrm{A}^T_{p\Gamma}\mathrm{A}^{-1}_{pp}\boldsymbol{f}_{p\Gamma} - \widetilde{\mathrm{M}}_2\widetilde{\mathrm{F}}^{-1}\widetilde{\boldsymbol{f}}_1. \tag{3.96}$$

We can split $\widetilde{\Sigma}_h = \widetilde{\Sigma}_{ph} + \widetilde{\Sigma}_{fh}$, where

$$\widetilde{\Sigma}_{ph} = \mathrm{A}^p_{\Gamma\Gamma} - \mathrm{A}^T_{p\Gamma}\mathrm{A}^{-1}_{pp}\mathrm{A}_{p\Gamma} \quad \text{and} \quad \widetilde{\Sigma}_{fh} = -\widetilde{\mathrm{M}}_2\widetilde{\mathrm{F}}^{-1}\widetilde{\mathrm{M}}_1 \tag{3.97}$$

are the algebraic counterpart of the operators $\mathcal{S}_{ph}$ and $\mathcal{S}_{fh}$, respectively. The matrices $\widetilde{\Sigma}_{ph}$ and $\widetilde{\Sigma}_h$ are symmetric and positive definite, and

$$[\widetilde{\Sigma}_{ph}\boldsymbol{\mu}, \boldsymbol{\mu}] \le [\widetilde{\Sigma}_h\boldsymbol{\mu}, \boldsymbol{\mu}] \le \left(1 + \frac{\tilde{\beta}_f}{\tilde{\alpha}_p}\right) [\widetilde{\Sigma}_{ph}\boldsymbol{\mu}, \boldsymbol{\mu}] \qquad \forall \boldsymbol{\mu} \in \mathbb{R}^{N_\Gamma},$$

so that the spectral condition number of the preconditioned matrix $\widetilde{\Sigma}^{-1}_{ph}\widetilde{\Sigma}_h$ is bounded independently of h:

$$\chi_{sp}(\widetilde{\Sigma}^{-1}_{ph}\widetilde{\Sigma}_h) \le 1 + \frac{\tilde{\beta}_f}{\tilde{\alpha}_p} . \tag{3.98}$$

3.6.2 An Iterative Method for the Solution of the Coupled Problem (II)

We propose the following iterative method which exploits σ_h as interface variable.

Let $\sigma^0_h \in \Lambda_{\dagger h}$ be given; for $k \ge 0$,

i) find $(\mathbf{u}^0_{fh})^{k+1} \in H^\tau_{fh}$, $p^{k+1}_{fh} \in Q_h$:

$$a_f((\mathbf{u}^0_{fh})^{k+1}, \mathbf{w}_h) + b_f(\mathbf{w}_h, p^{k+1}_{fh}) + \int_\Gamma g\sigma^{k+1}_h \mathbf{w}_h \cdot \mathbf{n} = \int_{\Omega_f} \mathbf{f}\, \mathbf{w}_h$$
$$-a_f(E_{fh}\mathbf{u}_{inh}, \mathbf{w}_h) \qquad \forall \mathbf{w}_h \in H^\tau_{fh}, \tag{3.99}$$

$$b_f((\mathbf{u}^0_{fh})^{k+1}, q_h) = -b_f(E_{fh}\mathbf{u}_{inh}, q_h) \qquad \forall q_h \in Q_h; \tag{3.100}$$

ii) find $\varphi_{0h}^{k+1} \in H_{ph}$:

$$a_p(\varphi_{0h}^{k+1}, \psi_h) - \int_\Gamma n\, \psi_h\, (\mathbf{u}_{fh}^{k+1} \cdot \mathbf{n}) = -a_p(E_{ph}\varphi_{ph}, \psi_h) \qquad \forall \psi \in H_{ph}\,; \tag{3.101}$$

with $\mathbf{u}_{fh}^{k+1} = (\mathbf{u}_{fh}^0)^{k+1} + E_{fh}\mathbf{u}_{inh}$;

iii) update σ_h^k:

$$\sigma_h^{k+1} = \vartheta \varphi_h^{k+1}{}_{|\Gamma} + (1-\vartheta)\sigma_h^k\,, \tag{3.102}$$

ϑ being a positive relaxation parameter and $\varphi_h^{k+1} = \varphi_{0h}^{k+1} + E_{ph}\varphi_{ph}$.

3.6.3 Matrix Formulation

The matrix formulation of scheme (3.99)-(3.102) is as follows. Let $\boldsymbol{\sigma}_h^k \in \mathbb{R}^{N_\Gamma}$ be the vector of the nodal values of σ_h^k on Γ at the k-th step. Then,

i) solve the system (corresponding to (3.99)-(3.100)):

$$\begin{pmatrix} \mathrm{A}_{ff} & \mathrm{B}^T & \mathrm{A}_{f\Gamma} \\ \mathrm{B}_1 & 0 & \mathrm{B}_{f\Gamma} \\ \mathrm{A}_{f\Gamma} & \mathrm{B}_{f\Gamma}^T & \mathrm{A}_{\Gamma\Gamma}^f \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_{int}^{k+1} \\ \boldsymbol{p}^{k+1} \\ \boldsymbol{u}_\Gamma^{k+1} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}_f \\ \boldsymbol{f}_{in} \\ \boldsymbol{f}_\Gamma - \mathrm{M}_\Gamma \boldsymbol{\sigma}_h^k \end{pmatrix} \tag{3.103}$$

and obtain $\boldsymbol{u}_\Gamma^{k+1}$;

ii) update the right hand side and solve the following system (which corresponds to (3.101)):

$$\begin{pmatrix} \mathrm{A}_{\Gamma\Gamma}^p & \mathrm{A}_{p\Gamma}^T \\ \mathrm{A}_{p\Gamma} & \mathrm{A}_{pp} \end{pmatrix} \begin{pmatrix} \boldsymbol{\phi}_\Gamma^{k+1} \\ \boldsymbol{\phi}_{int}^{k+1} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}_{p\Gamma} + \mathrm{M}_\Gamma^T \mathbf{u}_\Gamma^{k+1} \\ \boldsymbol{f}_p \end{pmatrix}\,; \tag{3.104}$$

iii) perform the relaxation:

$$\boldsymbol{\sigma}_h^{k+1} = \vartheta \boldsymbol{\phi}_\Gamma^{k+1} + (1-\vartheta)\boldsymbol{\sigma}_h^k\,. \tag{3.105}$$

The algorithm (3.103)-(3.105) (or (3.99)-(3.102)) corresponds to a preconditioned Richardson scheme to solve the interface problem (3.94) (respectively, (3.92)) with preconditioner $\widetilde{\Sigma}_{ph}$ (respectively, $\mathcal{S}_{ph}$). In fact, using the notations introduced in (3.93), system (3.103) becomes

$$\widetilde{\mathrm{F}}\boldsymbol{u}^{k+1} = \widetilde{\boldsymbol{f}}_1 - \widetilde{\mathrm{M}}_1 \boldsymbol{\sigma}_h^k \tag{3.106}$$

and

$$\mathrm{M}_\Gamma^T \boldsymbol{u}_\Gamma^{k+1} = -\widetilde{\mathrm{M}}_2 \boldsymbol{u}^{k+1}\,.$$

Eliminating $\boldsymbol{\phi}_{int}^{k+1}$ in (3.103), we find

$$\widetilde{\Sigma}_{ph}\boldsymbol{\phi}_\Gamma^{k+1} = \widetilde{\boldsymbol{\chi}}_h - \widetilde{\Sigma}_{fh}\boldsymbol{\sigma}_h^k, \tag{3.107}$$

and substituting (3.107) in (3.105) we get

$$\boldsymbol{\sigma}_h^{k+1} = \boldsymbol{\sigma}_h^k + \vartheta \widetilde{\Sigma}_{ph}^{-1}(\widetilde{\boldsymbol{\chi}}_h - \widetilde{\Sigma}_h \boldsymbol{\sigma}_h^k)\,. \tag{3.108}$$

Finally, we can prove the following convergence result.

Theorem 3.6.2. *The iterative method (3.99)-(3.102) converges to the solution of the coupled problem (3.28)-(3.32) for any choice of the initial guess $\sigma_h^0 \in \Lambda_{\dagger h}$ and for suitable values of the relaxation parameter ϑ.*

Proof. The thesis follows upon applying Theorem 3.5.2 to the counterpart of (3.108):

$$\sigma_h^{k+1} = \sigma_h^k + \vartheta \mathcal{S}_{ph}^{-1}(\widetilde{\chi}_h - \mathcal{S}_h \sigma_h^k) \tag{3.109}$$

and using Lemma 3.6.1. In particular, it must be $\vartheta \in (0, \vartheta_{max})$ with

$$\vartheta_{max} = \frac{2\tilde{\alpha}_p^3}{\tilde{\beta}_f(\tilde{\beta}_f + \overline{\beta}_p)^2}\,. \tag{3.110}$$

□

4. Algorithms and Numerical Results

In this chapter we present some numerical results obtained applying the substructuring methods introduced in chapter 3. In particular, the dependence of the convergence rate on the grid parameter h and on the physical data governing the Stokes/Darcy coupling are discussed. Some difficulties encountered when applying the algorithms are indicated together with possible improvement strategies.

4.1 Introduction

In chapter 3, we have introduced and analyzed two possible substructuring methods to solve the Stokes/Darcy problem, each one stemming from a particular choice of the governing variable on the interface Γ. These algorithms, which strongly exploit the natural decoupled structure of the problem at hand, are suited for parallel implementation and would permit to reuse existing codes specifically devised for surface and groundwater flows simulation.
The aim of this chapter is to illustrate the convergence properties of the algorithms on several test problems, with particular concern about the influence of grid and physical parameters.
Part of the results that we show have been previously published in [Dis04a].

For the sake of clarity, before presenting the numerical results, we give a schematic overview of the numerical algorithms we shall adopt, and we discuss the implementation of the preconditioned conjugate gradient (PCG) methods (see, e.g., [Saa03]) which exploit the preconditioners Σ_{fh} and $\widetilde{\Sigma}_{ph}$ that we have characterized in chapter 3.

4.1.1 Overview of Iterative Methods

The methods introduced in Sects. 3.5.3 and 3.6.2 can be written, respectively, in the following pseudo-algorithmic form.

Algorithm 4.1

0. choose an initial guess $(\mathbf{u}_f)^0 \cdot \mathbf{n}$ for the interface variable on Γ;

For $k = 0, 1, \ldots$ until convergence, Do

1. solve Darcy equation with boundary condition
   $-(\mathsf{K}/n)\nabla\varphi^{k+1} \cdot \mathbf{n} = (\mathbf{u}_f)^k \cdot \mathbf{n}$ on Γ;
2. solve Stokes problem imposing $-\mathbf{n} \cdot \mathsf{T}((\mathbf{u}_f)^{k+\frac{1}{2}}, p_f^{k+\frac{1}{2}}) \cdot \mathbf{n} = g\varphi^{k+1}$ on Γ;
3. Update: $(\mathbf{u}_f)^{k+1} \cdot \mathbf{n} = \theta\,(\mathbf{u}_f)^{k+\frac{1}{2}} \cdot \mathbf{n} + (1-\theta)\,(\mathbf{u}_f)^k \cdot \mathbf{n}$ on Γ, $\theta \in (0,1)$;

End For

Algorithm 4.2

0. choose φ as interface variable and an initial guess φ^0 on Γ;

For $k = 0, 1, \ldots$ until convergence, Do

1. solve Stokes problem imposing $-\mathbf{n} \cdot \mathsf{T}((\mathbf{u}_f)^{k+1}, p_f^{k+1}) \cdot \mathbf{n} = g\varphi^k$ on Γ;
2. solve Darcy equation with boundary condition
   $-(\mathsf{K}/n)\nabla\varphi^{k+\frac{1}{2}} \cdot \mathbf{n} = (\mathbf{u}_f)^{k+1} \cdot \mathbf{n}$ on Γ;
3. Update: $\varphi^{k+1} = \vartheta\,\varphi^{k+\frac{1}{2}} + (1-\vartheta)\,\varphi^k$ on Γ, $\vartheta \in (0,1)$;

End For

In practice, the two methods differ only in the order in which the Stokes and Darcy problems are solved; however, we have shown that they correspond to two distinct preconditioning strategies for the Schur complement systems (3.58) and (3.94), respectively.

Moreover, since these linear interface systems are symmetric and positive definite, the PCG method can be applied, using Σ_{fh} and $\widetilde{\Sigma}_{ph}$ as preconditioners for the first and the second system, respectively.
In particular, for system (3.58) the PCG method reads as follows.

Algorithm 4.3

Given an initial guess $(\boldsymbol{u}_\Gamma^0)^0$, set $\boldsymbol{r}^0 = \boldsymbol{\chi}_h - \Sigma_h(\boldsymbol{u}_\Gamma^0)^0$, $\boldsymbol{w}^0 = \boldsymbol{z}^0 = \Sigma_{fh}^{-1}\boldsymbol{r}^0$.
Then, for $k \geq 0$:

$$\boldsymbol{v}^k = \Sigma_h \boldsymbol{w}^k \tag{4.1}$$

$$\alpha_k = \frac{[\boldsymbol{w}^k, \boldsymbol{r}^k]}{[\boldsymbol{w}^k, \boldsymbol{v}^k]} \tag{4.2}$$

$$(\boldsymbol{u}_\Gamma^0)^{k+1} = (\boldsymbol{u}_\Gamma^0)^k + \alpha_k \boldsymbol{w}^k \tag{4.3}$$

$$\boldsymbol{r}^{k+1} = \boldsymbol{r}^k - \alpha_k \boldsymbol{v}^k \tag{4.4}$$

$$\text{solve} \quad \Sigma_{fh}\boldsymbol{z}^{k+1} = \boldsymbol{r}^{k+1} \tag{4.5}$$

$$\beta_k = \frac{[\boldsymbol{v}^k, \boldsymbol{z}^{k+1}]}{[\boldsymbol{w}^k, \boldsymbol{v}^k]} \tag{4.6}$$

$$\boldsymbol{w}^{k+1} = \boldsymbol{z}^{k+1} - \beta_k \boldsymbol{w}^k\,, \tag{4.7}$$

where $[\cdot,\cdot]$ denotes the Euclidean scalar product in $\mathbb{R}^{N_\Gamma}$.

The most expensive steps in terms of computational effort are (4.1) and (4.5), which require respectively:

Step (4.1):

– compute $\Sigma_{fh}\boldsymbol{w}^k$ which amounts to solving a Stokes problem in Ω_f with a Dirichlet boundary condition on Γ (see the corresponding differential operator S_f in *i)*, Sect. 2.7);
– compute $\Sigma_{ph}\boldsymbol{w}^k$ which amounts to solve a Darcy problem in Ω_p with Neumann boundary condition on Γ (see the definition of the corresponding differential operator S_p in *iii)*, Sect. 2.7).

Step (4.5):
solve the linear system $\Sigma_{fh}\boldsymbol{z}^{k+1} = \boldsymbol{r}^{k+1} \Leftrightarrow \boldsymbol{z}^{k+1} = \Sigma_{fh}^{-1}\boldsymbol{r}^{k+1}$ which amounts to solve a Stokes problem in Ω_f with Neumann boundary condition on Γ (see also the definition of the differential operator S_f^{-1} in *ii)*, Sect. 2.7).

Each step of the PCG method requires therefore to solve one Darcy problem in Ω_p and two Stokes problems in Ω_f.
Similar considerations hold when PCG is applied to the Schur complement system (3.95). In that case it is easy to see that at each step one should solve two Darcy problems in Ω_p and one fluid problem in Ω_f. Precisely, the algorithm reads as follows.

Algorithm 4.4

Given an initial guess $\boldsymbol{\phi}_\Gamma^0$, set $\boldsymbol{r}^0 = \widetilde{\boldsymbol{\chi}}_h - \widetilde{\Sigma}_h \boldsymbol{\phi}_\Gamma^0$, $\boldsymbol{w}^0 = \boldsymbol{z}^0 = \widetilde{\Sigma}_{ph}^{-1} \boldsymbol{r}^0$.
Then, for $k \geq 0$:

$$\boldsymbol{v}^k = \widetilde{\Sigma}_h \boldsymbol{w}^k \tag{4.8}$$

$$\alpha_k = \frac{[\boldsymbol{w}^k, \boldsymbol{r}^k]}{[\boldsymbol{w}^k, \boldsymbol{v}^k]} \tag{4.9}$$

$$\boldsymbol{\phi}_\Gamma^{k+1} = \boldsymbol{\phi}_\Gamma^k + \alpha_k \boldsymbol{w}^k \tag{4.10}$$

$$\boldsymbol{r}^{k+1} = \boldsymbol{r}^k - \alpha_k \boldsymbol{v}^k \tag{4.11}$$

$$\text{solve} \quad \widetilde{\Sigma}_{ph} \boldsymbol{z}^{k+1} = \boldsymbol{r}^{k+1} \tag{4.12}$$

compute β_k and update $\boldsymbol{w}^{k+1}$ as in (4.6), (4.7). (4.13)

4.2 Numerical Tests with Respect to the Grid Parameter

In this section we investigate the convergence properties of Algorithms 4.1-4.4 with respect to the grid parameter h. Throughout the whole section we shall neglect the physical parameters ν, K, g and n which shall be put all equal to 1.
We consider a test case in 2D. Let the computational domain be $\Omega \subset \mathbb{R}^2$ with $\Omega_f = (0,1) \times (1,2)$, $\Omega_p = (0,1) \times (0,1)$ and the interface $\Gamma = (0,1) \times \{1\}$. We impose Dirichlet boundary conditions on the velocity on $\partial\Omega_f \setminus \Gamma$, while we consider a Dirichlet boundary condition $\varphi = \varphi_p$ on the bottom boundary $(0,1) \times \{0\}$ and Neumann boundary conditions on the lateral boundaries $\{0,1\} \times (0,1)$ of the domain Ω_p. The boundary conditions and the forcing terms are chosen in such a way that the exact solution of the coupled Stokes/Darcy problem is

$$(\mathbf{u}_f)_1 = -\cos\left(\frac{\pi}{2}y\right)\sin\left(\frac{\pi}{2}x\right) \tag{4.14}$$

$$(\mathbf{u}_f)_2 = \sin\left(\frac{\pi}{2}y\right)\cos\left(\frac{\pi}{2}x\right) - 1 + x \tag{4.15}$$

$$p_f = 1 - x \tag{4.16}$$

$$\varphi = \frac{2}{\pi}\cos\left(\frac{\pi}{2}x\right)\cos\left(\frac{\pi}{2}y\right) - y(x-1), \tag{4.17}$$

where $(\mathbf{u}_f)_1$ and $(\mathbf{u}_f)_2$ are the components of the velocity field $\mathbf{u}_f$. Note in particular that $\mathbf{u}_f \cdot \boldsymbol{\tau} = (\mathbf{u}_f)_1 = 0$ on Γ according to (2.58). Finally, remark that in Darcy equation a non null forcing term has been considered. This implies the presence of an additional term in the definition of the functional $\mathcal{F}$ in (2.28), but it does not affect the theory we have developed.

In our computation, four different regular conforming meshes have been considered whose number of elements in Ω and of nodes on Γ are reported in table 4.1, together with the number of iterations to convergence obtained using Algorithms 4.1-4.4. The $\mathbb{P}_2$-$\mathbb{P}_1$ Taylor-Hood FE have been used for Stokes problem and $\mathbb{P}_2$ elements for Darcy equation.

A tolerance *tol* =1.e−10 has been prescribed for the convergence tests based on the relative residues. For Algorithms 4.1 and 4.2 we have chosen the relaxation parameters $\theta = \vartheta = 0.7$, respectively.

Number of mesh elements	Number of nodes on Γ	Alg. 4.1 ($\theta = 0.7$)	Alg. 4.3 (prec. Σ_{fh}^{-1})	Alg.4.2 ($\vartheta = 0.7$)	Alg. 4.4 (prec. $\widetilde{\Sigma}_{ph}^{-1}$)
172	13	18	5	20	5
688	27	18	5	20	5
2752	55	18	5	20	5
11008	111	18	5	20	5

Table 4.1. Number of iterations obtained on different grids.

Figure 4.1 shows the computed residues for the adopted iterative methods when using the finest mesh (logarithmic scale has been considered on the y-axis).

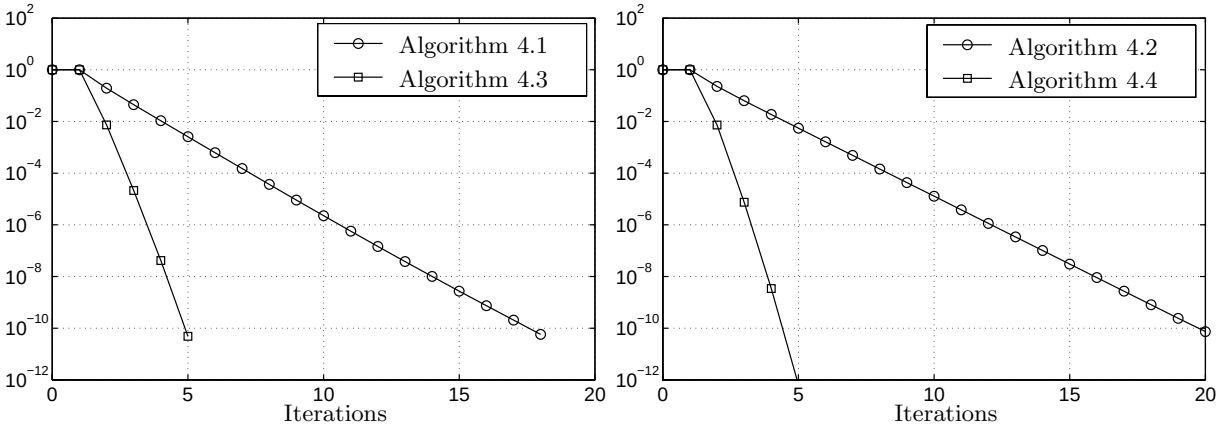


Fig. 4.1. Computed relative residues for the interface variable λ_h (left) and σ_h (right) using Richardson and PCG iterations.

Table 4.2 reports the spectral condition numbers of the preconditioned Schur complement matrices $\Sigma_{hf}^{-1}\Sigma_h$ and $\widetilde{\Sigma}_{ph}^{-1}\widetilde{\Sigma}_h$ illustrating the optimality of both preconditioners with respect to h.

$h_{\|\Gamma}$ approx.	$\chi_{sp}(\Sigma_{fh}^{-1}\Sigma_h)$	$\chi_{sp}(\widetilde{\Sigma}_{ph}^{-1}\widetilde{\Sigma}_h)$
0.1429	1.083655	1.017733
0.0714	1.083670	1.017764
0.0357	1.083658	1.017768
0.0179	1.083656	1.017769

Table 4.2. Spectral condition numbers for the preconditioned Schur complements.

Finally, figure 4.2 report the errors with respect to the exact solution for both choices of the interface variables. Precisely, we have computed the errors

$$E^h_{Stokes} = \|\nabla \mathbf{u}_f - \nabla \mathbf{u}_{fh}\|_{0,f} + \|p_f - p_{fh}\|_{0,f}$$
$$E^h_{Darcy} = \|\varphi - \varphi_h\|_{1,p}$$
$$E_{\lambda_h} = \|\lambda - \lambda_h\|_{0,\Gamma} \quad \text{and} \quad E_{\sigma_h} = \|\sigma - \sigma_h\|_{0,\Gamma}\,.$$

We recall that the following theoretical estimates hold (see, e.g., [QV94]):

$$E^h_{Darcy} \leq C_D h^{l+1} \|\varphi\|_{l,p} \qquad C_D > 0,$$

with $l = \min(2, s-1)$ if $\varphi \in H^s(\Omega_p)$ $(s \geq 2)$, and

$$E^h_{Stokes} \leq C_S h^r (\|\mathbf{u}_f\|_{r+1,f} + \|p_f\|_{r,f}) \qquad C_S > 0,$$

with $r = 1, 2$, provided the solution $(\mathbf{u}_f, p_f)$ is regular enough so that the norms at the right hand side make sense. The numerical results show that these theoretical estimates are fulfilled.

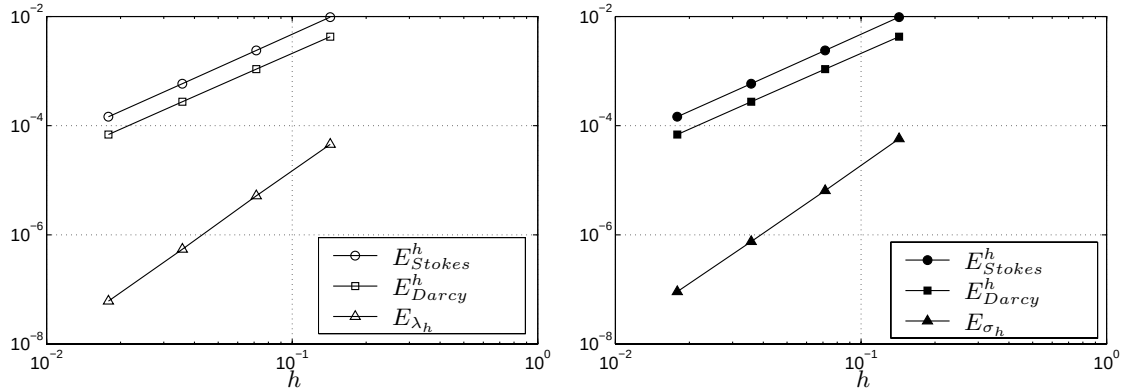


Fig. 4.2. Computed errors with respect to the exact solution versus h obtained using Algorithm 4.3 (left) and 4.4 (right).

The numerical tests we have presented show that according to the theory developed in chapter 3, the preconditioners Σ_{fh} and $\widetilde{\Sigma}_{ph}$ are equally optimal with respect to the grid parameter h since the corresponding preconditioned substructuring methods yield convergence in a number of iterations independent of h.

4.3 Numerical Tests with Respect to the Physical Parameters

We consider now the influence of the physical parameters, which govern the coupled problem, on the convergence of the given algorithms. We shall adopt the Algorithms 4.3 and 4.4 instead of

Algorithms 4.1 and 4.2, as the PCG methods embed the choice of dynamic optimal acceleration parameters. We take the same computational domain as in the test of Sect. 4.2 with the same kind of boundary conditions, but here the boundary data and the forcing terms are chosen in such a way that the exact solution of the coupled problem is

$$(\mathbf{u}_f)_1 = y^2 - 2y + 1 \tag{4.18}$$

$$(\mathbf{u}_f)_2 = x^2 - x \tag{4.19}$$

$$p_f = 2\nu(x + y - 1) + \frac{gn}{3\mathsf{K}} \tag{4.20}$$

$$\varphi = \frac{n}{\mathsf{K}}\left(x(1-x)(y-1) + \frac{y^3}{3} - y^2 + y\right) + \frac{2\nu}{g}x. \tag{4.21}$$

The most relevant physical quantities for the coupling are the fluid viscosity ν and the hydraulic conductivity K. Therefore, we test our algorithms with respect to different values of ν and K, and set $g = n = 1$. We consider a convergence test based on the relative residue with tolerance $tol = 1.\text{e}{-10}$.

In table 4.3 we report the number of iterations obtained for several choices of ν and K (the symbol # indicates that the method did not converge within $maxit = 150$ iterations), while in Fig. 4.3 we show the spectral condition numbers $\chi_{sp}(\Sigma_{fh}^{-1}\Sigma_h)$ (left) and $\chi_{sp}(\widetilde{\Sigma}_{ph}^{-1}\widetilde{\Sigma})$ (right) versus h for the considered test cases.

We can see that both algorithms encounter some difficulties to deal with values of ν and K different from 1. In particular, the convergence is troublesome when the values of ν and K decrease. In fact, in that case the methods converge in a large number of iterations which depends on h, losing their optimality properties that we have illustrated in Sect. 4.2.

The Dirichlet-Neumann type methods we have proposed are then effective only when the product $\nu\mathsf{K}$ is sufficiently large, while dealing with small values causes severe difficulties.

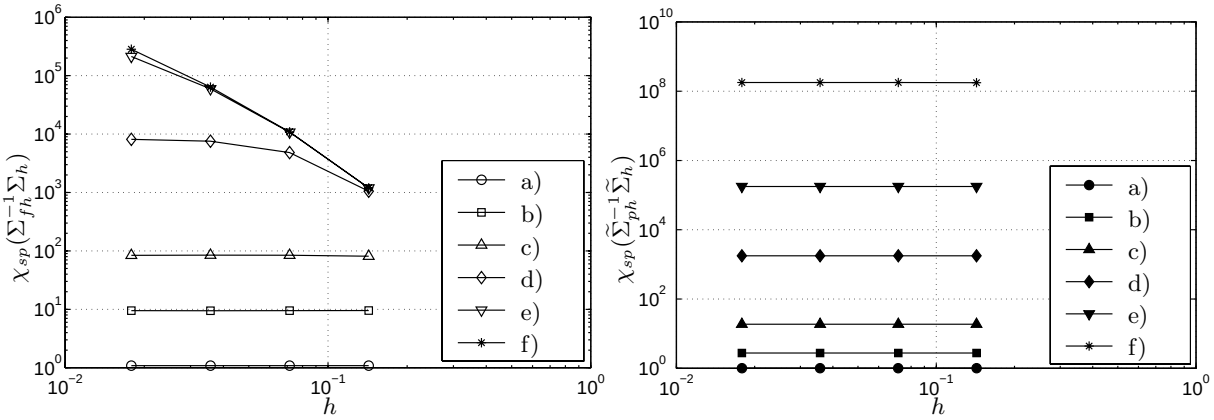


Fig. 4.3. Condition number $\chi_{sp}(\Sigma_{fh}^{-1}\Sigma_h)$ (left) and $\chi_{sp}(\widetilde{\Sigma}_{ph}^{-1}\widetilde{\Sigma}_h)$ (right) versus h for the test cases reported in table 4.3.

Remark that the latter are the very values of interest in real-life applications: see, for example, the values of K reported in table 1.1 and recall that water has a kinematic viscosity $\nu = 1.\text{e}{-06}\, m^2/s$.

			Iter. using Alg. 4.3 (prec. Σ_{fh}^{-1})			
	ν	K	$h = 0.1428$	$h = 0.0714$	$h = 0.0357$	$h = 0.0178$
a)	1.e+00	1.e+00	5	5	5	5
b)	1.e−01	1.e−01	11	11	10	10
c)	1.e−02	1.e−01	15	19	18	17
d)	1.e−03	1.e−02	20	54	73	56
e)	1.e−04	1.e−03	20	59	#	#
f)	1.e−06	1.e−04	20	59	148	#

			Iter. using Alg. 4.4 (prec. $\widetilde{\Sigma}_{ph}^{-1}$)			
	ν	K	$h = 0.1428$	$h = 0.0714$	$h = 0.0357$	$h = 0.0178$
a)	1.e+00	1.e+00	6	6	6	6
b)	1.e−01	1.e−01	10	10	9	9
c)	1.e−02	1.e−01	15	15	14	14
d)	1.e−03	1.e−02	19	46	52	43
e)	1.e−04	1.e−03	22	55	82	88
f)	1.e−06	1.e−04	41	78	102	123

Table 4.3. Iterations using Algorithms 4.3 (above) and 4.4 (below) with respect to several values of ν and K.

Remark 4.3.1. Should we adopt Algorithm 4.1 (or 4.2), when the fluid viscosity and the hydraulic conductivity decrease, small relaxation parameters θ (or ϑ) must be adopted to guarantee convergence, in accordance with the theoretical estimate of the upper bound θ_{max} (ϑ_{max}, respectively) given in (3.79) ((3.110), respectively). Unfortunately, in some cases θ should be so small that in practice it prevents the numerical scheme from converging. To quote an example, if $\nu = 1.\text{e}{-}03$ and $\mathsf{K} = 1.\text{e}{-}02$, then θ should be unreasonably small (smaller than 1.e−04 !) to prevent divergence. □

We introduce a formal argument for better understanding these results and to set up a more effective numerical scheme.
Our conjecture is that the difficulties may come from the different structure of the Stokes equation (1.29) and of the Darcy law (1.10), which become even more dissimilar when $\nu \ll 1$ and $\mathsf{K} \ll 1$. In fact, in that case, under the physically reasonable hypothesis that $\triangle \mathbf{u}_f$ and $\nabla\varphi$ are sufficiently small, (1.29) reduces to

$$C_f \mathbf{I} + \nabla p_f \cong \mathbf{f},$$

while (1.10) becomes

$$\mathbf{u}_p + C_p \mathbf{I} \cong \mathbf{0},$$

where C_f and C_p denote two positive constants $\ll 1$. We rewrite (1.10) as

$$(\mathsf{K}/n)^{-1}\mathbf{u}_p + \nabla\varphi = \mathbf{0} \quad \text{in } \Omega_p\,, \tag{4.22}$$

and formally comparing (4.22) to (1.29), we are led to modify the latter by adding a mass term like $\mathsf{K}^{-1}\mathbf{u}_p$ as follows:

$$\overline{\gamma}\mathsf{K}^{-1}\mathbf{u}_f - \nu\triangle\mathbf{u}_f + \nabla p_f = \tilde{\mathbf{f}}\,, \quad \overline{\gamma} \in \mathbb{R}^+\,, \tag{4.23}$$

possibly with a consequent modification of the right hand side (see Sect. 4.4) that we have denoted by $\tilde{\mathbf{f}}$. In this way we obtain a generalized Stokes momentum equation, and note that now (4.23) has the same behaviour of (4.22) in the cases of our interest, that is when $\nu \ll 1$ and $\mathsf{K} \ll 1$.

We expect that the mass term $\overline{\gamma}\mathsf{K}^{-1}\mathbf{u}_f$ would help improving the positivity of the discrete Steklov-Poincaré operator Σ_{fh} which acts as a preconditioner in Algorithm 4.1 (or equivalently, Algorithm 4.3), thus enhancing the rate of convergence of the substructuring method. With this aim, we have carried out some numerical tests using the PCG algorithm 4.3 to solve the modified problem Stokes/Darcy where (4.23) is considered instead of (1.29). The convergence results reported in table 4.4 and the corresponding spectral condition numbers in Fig. 4.4 show that the numerical scheme has improved substantially.

ν	K	$\overline{\gamma}$	Iterations on the mesh with			
			$h = 0.1428$	$h = 0.0714$	$h = 0.0357$	$h = 0.0178$
1.e−03	1.e−02	0.1	15	24	28	28
		1	12	14	16	14
		10	8	9	9	8
1.e−06	1.e−04	0.1	15	23	28	33
		1	13	14	17	18
		10	8	9	9	9

Table 4.4. Number of iterations to solve problem the modified Stokes/Darcy problem using (4.23) for different values of ν, K and $\overline{\gamma}$.

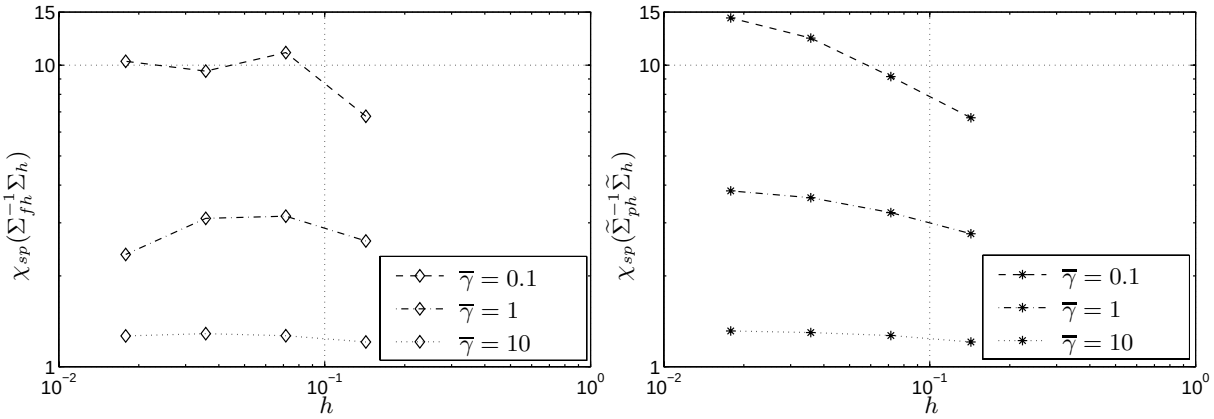


Fig. 4.4. Condition number $\chi_{sp}(\Sigma_{fh}^{-1}\Sigma_h)$ for $\nu = 1.\text{e}{-03}$, $\mathsf{K} = 1.\text{e}{-02}$ (left) and $\nu = 1.\text{e}{-06}$, $\mathsf{K} = 1.\text{e}{-04}$ (right) versus h for different values of $\overline{\gamma}$.

4.4 Dirichlet-Neumann for a Time-Dependent Problem

Equation (4.23) can be regarded as a discretization in time of the time-dependent Stokes momentum equation

$$\frac{\partial \mathbf{u}_f}{\partial t} - \nu \triangle \mathbf{u}_f + \nabla p_f = \mathbf{f} \quad \text{in } \Omega_f. \tag{4.24}$$

Precisely, if we consider

$$\overline{\gamma}\,\mathsf{K}^{-1}\mathbf{u}_{f,n+1} - \nu \triangle \mathbf{u}_{f,n+1} + \nabla p_{f,n+1} = \tilde{\mathbf{f}}_{n+1} \qquad n \geq 0$$

with

$$\tilde{\mathbf{f}}_{n+1} = \mathbf{f}(\mathbf{x}, t_{n+1}) + \overline{\gamma}\,\mathsf{K}^{-1}\mathbf{u}_{f,n},$$

we have a backward Euler discretization in time with $\overline{\gamma}\,\mathsf{K}^{-1}$ playing the role of the inverse of a time step.
From the physical viewpoint, since the fluid velocities in Ω_f are much higher than the ones through the porous medium (see the analysis in [ESP75] that we have briefly summarized in Sect. 1.2), a time-dependent model better represents the phenomena occurring during the filtration process.

4.4.1 The tDN Algorithm

Let $[0, T]$ be a characteristic time interval; using for the sake of simplicity the first-order backward Euler scheme, denoting by $\Delta t > 0$ the time step and $N = T/\Delta t$, the iterative method that we propose to solve the time-dependent coupled problem reads (the subscript n refers to the n-th time level):

Algorithm 4.5 (tDN method)

For $n = 0, \ldots, N-1$, Do

0. choose an initial guess $(\mathbf{u}_f)^0_{n+1} \cdot \mathbf{n}$ for the normal velocity on Γ at the $(n+1)$-th time level;

For $k = 0, 1, \ldots$ until convergence, Do

1. solve Darcy equation with boundary condition
   $-(\mathsf{K}/n)\nabla \varphi^{k+1}_{n+1} \cdot \mathbf{n} = (\mathbf{u}_f)^k_{n+1} \cdot \mathbf{n}$ on Γ;
2. solve the Stokes problem

$$(\Delta t)^{-1}\mathbf{u}^{k+\frac{1}{2}}_{f,n+1} - \nu \triangle \mathbf{u}^{k+\frac{1}{2}}_{f,n+1} + \nabla p^{k+\frac{1}{2}}_{f,n+1} = (\Delta t)^{-1}\mathbf{u}_{f,n} + \mathbf{f}_{n+1} \quad \text{in } \Omega_f$$

$$\nabla \cdot \mathbf{u}^{k+\frac{1}{2}}_{f,n+1} = 0 \quad \text{in } \Omega_f$$

imposing $-\mathbf{n} \cdot \mathsf{T}(\mathbf{u}^{k+\frac{1}{2}}_{f,n+1}, p^{k+\frac{1}{2}}_{f,n+1}) \cdot \mathbf{n} = g\varphi^{k+1}_{n+1}$ on Γ;

3. Update: $(\mathbf{u}_f)_{n+1}^{k+1} \cdot \mathbf{n} = \theta\,(\mathbf{u}_f)_{n+1}^{k+\frac{1}{2}} \cdot \mathbf{n} + (1-\theta)\,(\mathbf{u}_f)_{n+1}^{k} \cdot \mathbf{n}$ on Γ, $\theta \in (0,1)$;

End For

End For

4.4.2 Numerical Tests

We consider the horizontal section of a channel $12\,m$ long and $8\,m$ wide which is partially occupied by a porous medium with discontinuous conductivity, as represented in Fig. 4.5. A parabolic inflow profile is imposed on the left hand side boundary with maximal velocity equal to $0.1\,m/s$. On the right an outflow condition is imposed. The time interval is $t \in [0, 0.5]$ and the time step $\Delta t = 1.\text{e}{-}03\,s$; for space discretization three different computational meshes have been adopted.
In a first case we have considered $\nu = 1.\text{e}{-}05\,m^2/s$ and a discontinuous coefficient $\mathsf{K} = 1.\text{e}{-}03\,m/s$ in $\Omega_p^{(1)}$, $\mathsf{K} = 1.\text{e}{-}07\,m/s$ in $\Omega_p^{(2)}$.
In Fig. 4.6 we have represented the computed solution at time $t = 0.05\,s$, while in Fig. 4.7 a zoom of the velocity field through the porous medium is shown; it can be seen that the velocity is almost null in the less permeable areas of the porous medium. Finally, table 4.5 (left) reports the number of iterations obtained for three computational grids at different time levels, showing that the number of iterations is low and independent of h.

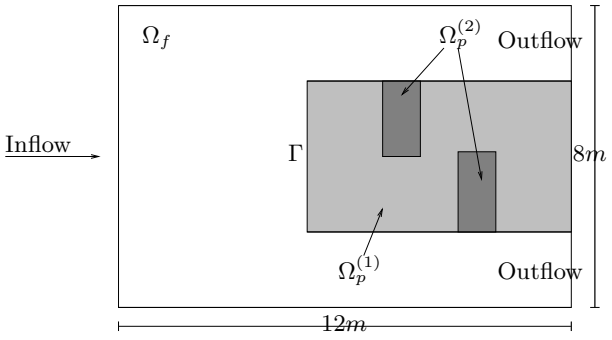


Fig. 4.5. Computational domain.

The same test has been performed considering different values of the parameters: $\nu = 1.\text{e}{-}02\,m^2/s$, $\mathsf{K} = 1.\text{e}{-}01\,m/s$ in $\Omega_p^{(1)}$ and $\mathsf{K} = 1.\text{e}{-}05\,m/s$ in the less permeable part $\Omega_p^{(2)}$ of the porous medium. The convergence results show that the number of iterations is essentially independent of these parameters, as it can be seen comparing the previous convergence results with those reported in table 4.5 (right).

Numerical results show that considering a time-dependent problem allows to set up a far more efficient iterative method for problems with parameters in a range of physical interest. However, as we have pointed out in the preliminary tests of Sect. 4.3 (see table 4.4), the value of Δt

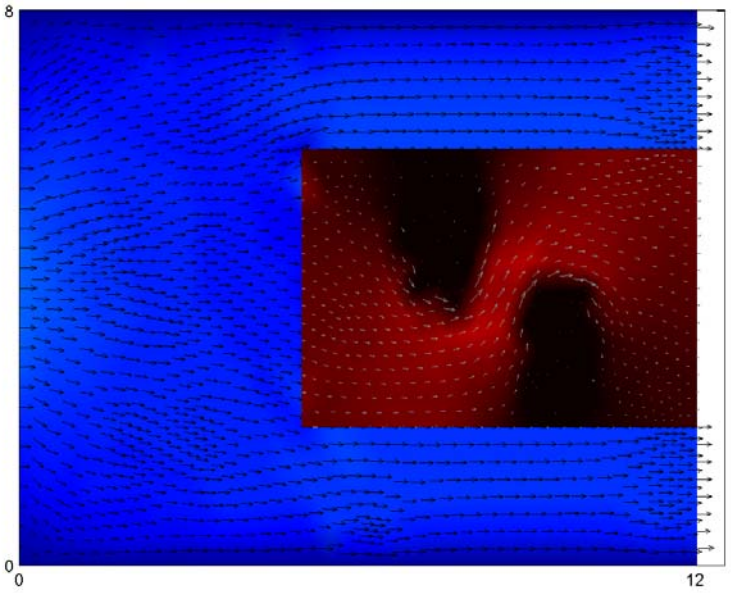


Fig. 4.6. Computed velocity field at $t = 0.05\,s$.

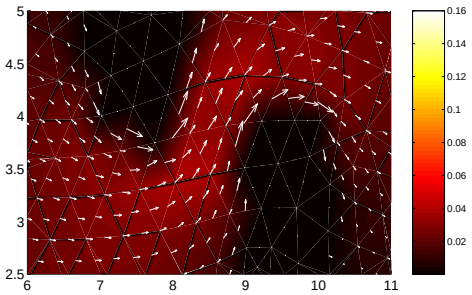


Fig. 4.7. Zoom of the velocity field through the porous medium.

Time level	Iterations on the mesh with 232 el.	928 el.	3712 el.
0.001	21	21	21
0.003	20	19	19
0.006	12	11	11
0.009	10	10	10
0.01	10	10	10

Time level	Iterations on the mesh with 232 el.	928 el.	3712 el.
0.001	22	22	22
0.003	20	20	20
0.006	15	15	15
0.009	15	15	15
0.01	15	15	15

Table 4.5. Number of iterations on different grids with $\nu = 1.\text{e}{-05}\,m^2/s$, $\mathsf{K} = 1.\text{e}{-03}\,m/s$ and $\mathsf{K} = 1.\text{e}{-07}\,m/s$ (left); with $\nu = 1.\text{e}{-02}\,m^2/s$, $\mathsf{K} = 1.\text{e}{-01}\,m/s$ and $\mathsf{K} = 1.\text{e}{-05}\,m/s$ (right).

generally depends on ν and K, and in some cases we could be forced to consider very small time steps $\Delta t \ll 1$. This could be quite annoying since one might be interested in considering long time scales, for example in modeling the filtration of pollutants in groundwater.

This limitation on Δt drives us to reconsider the steady coupled model. In fact, should we find an algorithm whose behaviour were as much as possible independent of the physical parameters, then not only we would be able to solve the steady problem itself, but we could also use it in the framework of the time-dependent model where Δt would be chosen under the sole requirements of stability and accuracy.

Remark 4.4.1. The results we have presented in this section have been obtained considering the mixed formulation of the Darcy equation (1.10), (1.11). Problem 1. in Algorithm 4.5 thus becomes (taking homogeneous boundary conditions for simplicity):

find $(\mathbf{u}_p, \varphi) \in W_p$ such that

$$\int_{\Omega_p} \mathsf{K}^{-1}(\mathbf{u}_p)_{n+1}^{k+1}\mathbf{v} - \int_{\Omega_p} \varphi_{n+1}^{k+1}\nabla\cdot\mathbf{v} = 0 \tag{4.25}$$

$$\int_{\Omega_p} \psi\nabla\cdot(\mathbf{u}_p)_{n+1}^{k+1} = 0 \tag{4.26}$$

$$(\mathbf{u}_p)_{n+1}^{k+1}\cdot\mathbf{n} = (\mathbf{u}_f)_{k+1}^{n+1}\cdot\mathbf{n} \quad \text{on } \Gamma \tag{4.27}$$

where W_p is a suitable subspace of $H(div;\Omega_p)\times L^2(\Omega_p)$.
We have adopted $\mathbb{P}_2$ and $\mathbb{P}_1$ elements for the velocity and piezometric head, respectively. The essential boundary condition (4.27) has been imposed via Lagrange multipliers. □

4.5 The Steady Case

We consider the bounds (3.64) and (3.98) for the spectral condition number of the preconditioned Schur complement matrices. In both cases these upper bounds involve the ratios of the continuity and coercivity constants: β_p/α_f and $\tilde{\beta}_f/\tilde{\alpha}_p$, respectively. Using the definitions of these constants that we have given in chapter 3, we can see that the corresponding ratios reduce essentially to the quantity

$$C\frac{ng}{m_K\nu}$$

where m_K has been defined in (2.44) and $C > 0$ is a positive constant due to Poincaré and trace inequalities. Focusing our attention on the parameters ν and K, we can therefore write the following approximate estimates

$$\chi_{sp}(\Sigma_{fh}^{-1}\Sigma_h) \simeq \chi_{sp}(\widetilde{\Sigma}_{ph}^{-1}\widetilde{\Sigma}_h) \lesssim 1 + \frac{1}{m_K\,\nu}\,, \tag{4.28}$$

which shows that when $\nu \ll 1$ and $\mathsf{K} \ll 1$ the spectral condition numbers deteriorates causing the bad convergence behaviours we have presented (see table 4.3) and it justifies why the methods were still quite effective when the product $\nu\mathsf{K}$ was not too small.

4.5.1 Parameter Dependence in Homogeneous Domain Decomposition

In order to find an effective way to improve our iterative methods, let us briefly review the strategies that are commonly adopted to overcome similar difficulties in homogeneous domain decomposition.
We consider an open bounded domain $\Omega \subset \mathbb{R}^2$ and the elliptic model problem

$$\mathcal{L}u = -\nabla \cdot (\varrho \nabla u) = f \text{ in } \Omega \tag{4.29}$$

$$u = 0 \text{ on } \partial\Omega, \tag{4.30}$$

$\varrho \in L^\infty(\Omega)$ being a positive real valued function.
We assume that Ω is partitioned into two nonoverlapping subdomains Ω_1 and Ω_2, and we denote their common interface by $\Gamma := \overline{\Omega}_1 \cap \overline{\Omega}_2$ (see Fig. 4.8).

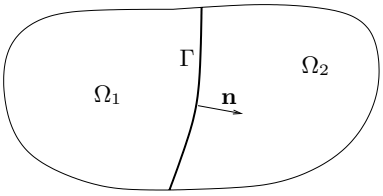


Fig. 4.8. Nonoverlapping partition of the computational domain Ω.

After introducing a suitable Galerkin approximation of (4.29), we can equivalently reformulate the associated algebraic problem in terms of the Schur complement system

$$\Sigma \boldsymbol{\lambda} = \boldsymbol{\chi} \tag{4.31}$$

with $\Sigma = \Sigma_1 + \Sigma_2$, Σ_i being the local Schur complement associated to the subdomain Ω_i (a precise characterization of system (4.31) will be introduced in chapter 5; we refer the reader also to [QV99, TW04]).
The following estimate holds: there exist two positive constants $c_i, C_i > 0$, independent of the mesh parameter h, such that

$$c_i \varrho_i \|\boldsymbol{\mu}\|^2_{\mathbb{R}^{N_\Gamma}} \leq [\Sigma_i \boldsymbol{\mu}, \boldsymbol{\mu}] \leq C_i \varrho_i \|\boldsymbol{\mu}\|^2_{\mathbb{R}^{N_\Gamma}} \qquad \forall \boldsymbol{\mu} \in \mathbb{R}^{N_\Gamma},$$

where $\varrho_i = \varrho_{|\Omega_i}$, $i = 1, 2$. Then, we can see that

$$\chi_{sp}(\Sigma_i^{-1}\Sigma) \leq 1 + \frac{C_j}{c_i} \cdot \frac{\varrho_j}{\varrho_i} \qquad i = 1, 2,\ j \neq i,$$

so that the physical parameters enter in the estimate of the spectral condition number $\chi_{sp}(\Sigma_i^{-1}\Sigma)$ as their ratio.
Notice the difference with the heterogeneous Stokes/Darcy case (4.28) where the inverse of the product of the physical parameters comes into play.

Therefore, in the homogeneous case, if ϱ_1 and ϱ_2 are small, but with a not so small ratio, then a Dirichlet-Neumann algorithm may yield anyway quite good convergence results.
We show a simple test taking Ω as the unit ball in $\mathbb{R}^2$ with $\Omega_1 = \{\mathbf{x} = (x_1, x_2) \in \Omega | x_1 > 0\}$ and $\Omega_2 = \{\mathbf{x} \in \Omega | x_1 < 0\}$ and we have set $f = 1$ in (4.29). We have considered $\mathbb{P}_2$ FE and we have applied the PCG to the Schur complement system (4.31) with preconditioners Σ_2^{-1}. Table 4.6 reports the number of iterations obtained for different values of ϱ_1, ϱ_2 and h. We have indicated also the acceleration coefficients α_k computed by the PCG method. Even for this simple example, we can notice that the number of iterations depends on the parameters and that α_k becomes small if the coefficient ϱ_2 reduces.

ϱ_1	ϱ_2	$h = 0.0833$	$h = 0.0417$	$h = 0.0208$	α_k (mean)
1	1.e+02	7	7	6	0.9901
1	1	11	11	11	0.5013
1	1.e−02	12	12	12	0.0098

Table 4.6. Number of iterations for the PCG method with preconditioner Σ_2^{-1} for several values of ϱ_i and h.

A well-known strategy to improve this behaviour is to use the Neumann-Neumann preconditioner

$$\mathrm{P}_{NN}^{-1} = \theta_1 \Sigma_1^{-1} + \theta_2 \Sigma_2^{-1}$$

with a particular choice of the weights θ_1 and θ_2 (see, e.g., [MB96, TW04]).
Precisely, if

$$\theta_i = \left(\frac{\varrho_i}{\varrho_1 + \varrho_2} \right)^2 \qquad i = 1, 2, \tag{4.32}$$

it can be easily seen that

$$\chi_{sp}(\mathrm{P}_{NN}^{-1}\Sigma) \leq \left(\frac{(C_1 C_2)^2}{c_1 c_2} \right)^2 \cdot \frac{(c_2\varrho_1 + c_1\varrho_2)^2}{(c_1 C_2^2 \varrho_1 + c_2 C_1^2 \varrho_2)} \cdot \frac{C_1\varrho_1 + C_2\varrho_2}{c_1\varrho_1 + c_2\varrho_2}$$

so that the condition number is almost uniformly bounded with respect to ϱ_1 and ϱ_2 yielding a number of iterations almost independent of the two parameters as shown in table 4.7. Notice also that now the mean value of α_k remains essentially the same with respect to ϱ_i.

ϱ_1	ϱ_2	$h = 0.0833$	$h = 0.0417$	$h = 0.0208$	α_k (mean)
1	1.e+02	5	6	5	0.9999
1	1	7	7	7	0.9988
1	1.e−02	5	6	5	0.9999

Table 4.7. Iterations for PCG with the Neumann-Neumann preconditioner and mean values of α_k.

Remark 4.5.1. The weighting coefficients θ_i might be computed dynamically according to a suitable error minimization strategy; an example is provided by the k-dependent preconditioner

$$(\mathrm{P}_{NN}^{k})^{-1} = \theta_1^k \Sigma_1^{-1} + \theta_2^k \Sigma_2^{-1}$$

where θ_i^k $(i = 1, 2)$ are computed automatically using the Aitken acceleration strategy (see, e.g., [DDQ04]), yielding a good control of the spectral condition number. □

4.5.2 Application to the Stokes/Darcy Case

The results of Sect. 4.5.1 would suggest to set up a Neumann-Neumann method also for the Stokes/Darcy coupling.
Should we consider the normal velocity as interface variable, the corresponding algorithm would read as in Fig. 4.9.
However, such a method poses some additional difficulties. In fact, we cannot guarantee that the regularity of the interface data is preserved: for example, in general we expect that $\xi^k \in H^{-1/2}(\Gamma)$ which is not regular enough to assure the solvability of the homogeneous Darcy problem in (B) and consequently that $(\mathbf{u}_f)^k \cdot \mathbf{n} \in \Lambda$ for all k. Moreover, the associated preconditioning operator would read (at the differential level)

$$P_{NN}^{-1} = \theta_1 S_f^{-1} + \theta_2 S_p^{-1}, \qquad \theta_1, \theta_2 > 0$$

(or $\mathrm{P}_{NN}^{-1} = \theta_1 \Sigma_{fh}^{-1} + \theta_2 \Sigma_{ph}^{-1}$ at the algebraic level); however, considering the analysis we have developed in chapter 2, we cannot guarantee the existence of the inverse S_p^{-1}.

On the other hand, using the trace $\sigma_{|\Gamma}$ of the piezometric head as interface variable, we would encounter similar difficulties concerning the regularity of the interface data and the issue of inverting the operator $\mathcal{S}_f^{-1}$.

A possible strategy to overcome the problem of invertibility would be to consider the Moore-Penrose pseudo-inverse of the local Schur complements Σ_{ph} and $\widetilde{\Sigma}_{fh}$ (which is very expensive to compute), or to modify them adding a positive matrix, say $\mathrm{E} \in \mathbb{R}^{N_\Gamma \times N_\Gamma}$, and then to take the inverses $(\mathrm{E} + \Sigma_{ph})^{-1}$ or $(\mathrm{E} + \widetilde{\Sigma}_{fh})^{-1}$.
In that case we could foresee a modified Neumann-Neumann preconditioner like

$$\widetilde{\mathrm{P}}_{NN}^{-1} = \theta_1(\tilde{\gamma}\mathrm{E} + \Sigma_{fh})^{-1} + \theta_2(\mathrm{E} + \Sigma_{ph})^{-1}$$

with $\tilde{\gamma}$ possibly equal to zero.
This option seems quite advantageous in terms of computational effort with respect to computing the Moore-Penrose pseudo-inverses, but it would be interesting to characterize the subdomain problem associated to the modified inverses in order to be able to compute the products

$$(\mathrm{E} + \Sigma_{ph})^{-1}\boldsymbol{\mu} \quad \text{or} \quad (\mathrm{E} + \widetilde{\Sigma}_{fh})^{-1}\boldsymbol{\mu} \qquad \forall \boldsymbol{\mu} \in \mathbb{R}^{N_\Gamma}$$

without explicitly constructing the inverse matrices.
This issue will be considered in the next chapter 5, where, considering a generic elliptic problem, we shall prove the equivalence of the so-called Robin-Robin method with a preconditioning strategy involving the modified inverses $(\gamma_1\mathrm{E} + \Sigma_1)^{-1}$ and $(\gamma_2\mathrm{E} + \Sigma_2)^{-1}$ $(\gamma_i > 0,\ i = 1, 2)$, and we shall investigate its convergence properties. Then, in chapter 6 we shall show how to apply this study to the Stokes/Darcy case.

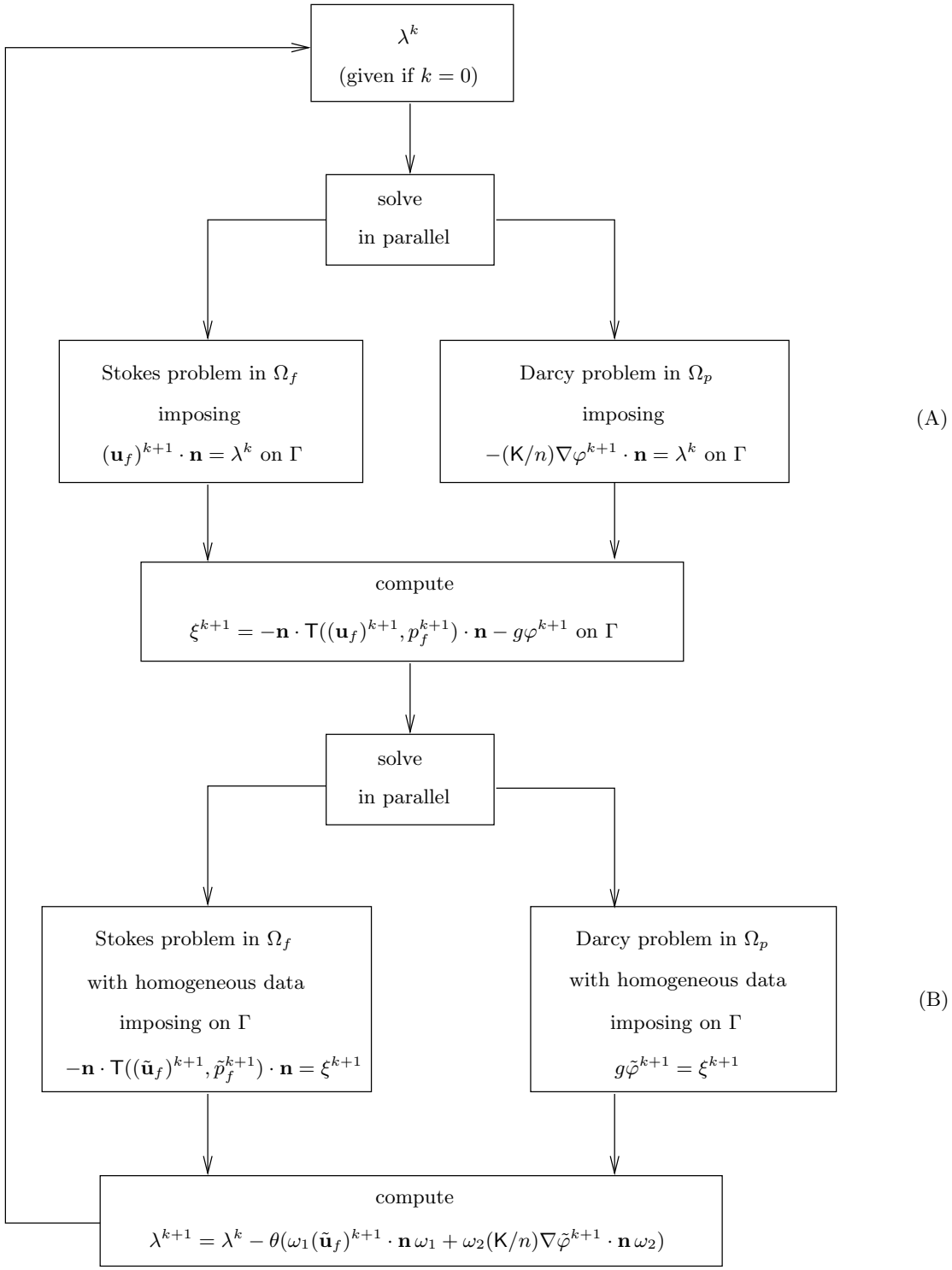


Fig. 4.9. Schematic representation of a possible Neumann-Neumann method; the problems in steps (A) and (B) may be solved in parallel.

5. An Operator-Splitting Approach to Nonoverlapping Domain Decomposition methods

The Robin-Robin method is an iterative substructuring method to solve boundary-value problems on domains partitioned into nonoverlapping subdomains. It involves mixed-type boundary conditions on the interface, which depend on suitable weighting coefficients in order to maximize the convergence rate. In this chapter we interpret this algorithm as an alternating direction iterative method to compute the solution of the Steklov-Poincaré equation associated to the given boundary value problem. This interpretation allows us to characterize new preconditioners for the interface problem and provides a strategy to compute optimal relaxation parameters. Finally, some numerical examples are presented. The results of this chapter have been published in [Dis04b].
The analysis we present will allow us to characterize robust substructuring schemes to solve the Stokes/Darcy problem also for small physical parameters, as we will see in chapter 6.

5.1 Introduction

When solving elliptic boundary value problems using nonoverlapping domain decomposition methods, the problem given on a global domain Ω can be rewritten in terms of an interface equation, say,

$$\widetilde{S}\,\widetilde{\lambda} = \widetilde{\chi} \tag{5.1}$$

solely defined on the interface Γ separating the subdomains in which Ω has been split. These methods are based on adopting suitable coupling conditions across Γ which impose the continuity of the solution and of its flux.
As we have already mentioned, the so-called Dirichlet-Neumann and Neumann-Neumann methods (see, e.g., [QV99, TW04]) exploit Dirichlet and Neumann conditions on Γ and can be interpreted as preconditioned Richardson methods to solve the interface equation (5.1), therefore characterizing optimal preconditioners which can be used in the framework of Krylov type methods.
Proper combinations of Dirichlet and Neumann conditions across Γ can be considered as well, giving rise to the so-called Robin-Robin methods. These methods, which were early introduced and analyzed in [Lio90], are currently widely used in domain decomposition, especially to

treat advection-diffusion-reaction problems (see, e.g., [NR95, ATV98, OL99, LMO00, Zun03, GGTN04]), but also, to quote two more examples, Oseen equations (see [OL98]) and Helmholtz equations (see, e.g., [BD97]).
Several convergence results for these methods have been proved, however, to our knowledge, their interpretation in terms of preconditioners for the interface equation has not been given yet. Moreover, a critical issue in the setting of these methods is the choice of suitable relaxation parameters which appear in the definition of the mixed interface conditions and which strongly influence the convergence rate.
In this chapter we apply an operator-splitting strategy (see [Yan71, Mar90] for a general setting of operator-splitting methods) to solve the interface equation and we show that Robin-Robin methods can be obtained as a particular case of an alternating direction iterative (ADI) algorithm and may be seen as generalizations of the Dirichlet-Neumann and Neumann-Neumann ones. This interpretation allows us to characterize suitable multiplicative and additive preconditioners to solve the interface problem (5.1) and to devise a purely algebraic strategy to compute optimal relaxation parameters. We present some numerical results for the Laplace operator and for advection-diffusion problems. Finally, an extension of this approach to the case of many subdomains is presented.

5.2 Problem Setting

We consider the elliptic model problem (4.29) in the domain Ω as in Fig. 4.8. We denote by $\mathbf{n}_i$ the normal direction on $\partial\Omega_i \cap \Gamma$ oriented outward, and for simplicity of notation we set throughout this chapter $\mathbf{n} = \mathbf{n}_1$.
We are interested in computing the solution of (4.29), (4.30) using iterative substructuring methods. We consider the trace space Λ introduced in (2.15) and we define:

$$V_i = \{v_i \in H^1(\Omega_i) |\ v_{i|\partial\Omega\cap\partial\Omega_i} = 0\} \quad i = 1, 2, \tag{5.2}$$

$$V_i^0 = H_0^1(\Omega_i), \quad i = 1, 2. \tag{5.3}$$

Moreover, let us set, for all $w_i, v_i \in V_i$, $i = 1, 2$,

$$a_i(w_i, v_i) = \int_{\Omega_i} \varrho_{|\Omega_i} \nabla w_i \cdot \nabla v_i \, . \tag{5.4}$$

For $i = 1, 2$, we denote by $\widetilde{R}_i$ any continuous extension operator from Λ to V_i such that $\widetilde{R}_i \eta = \eta$, $\forall \eta \in \Lambda$. Then, problem (4.29), (4.30) can be formulated in the multidomain form ([QV99]):

$$a_1(u_1, v_1) = \int_{\Omega_1} f\, v_1 \qquad \forall v_1 \in V_1^0 \tag{5.5}$$

$$u_1 = u_2 \qquad \text{on } \Gamma \tag{5.6}$$

$$a_2(u_2, v_2) = \int_{\Omega_2} f\, v_2 \qquad \forall v_2 \in V_2^0 \tag{5.7}$$

$$\sum_{i=1}^{2} a_i(u_i, \widetilde{R}_i \eta) = \sum_{i=1}^{2} \int_{\Omega_i} f\, \widetilde{R}_i \eta \qquad \forall \eta \in \Lambda \, . \tag{5.8}$$

Moreover, upon setting $\widetilde{\lambda} = u_{1|\Gamma} = u_{2|\Gamma}$, the solution of (5.5)-(5.8) can be characterized in terms of the solution of the Steklov-Poincaré interface equation

$$\text{find } \widetilde{\lambda} \in \Lambda : \qquad \langle \widetilde{S}\,\widetilde{\lambda}, \eta \rangle = \langle \widetilde{\chi}, \eta \rangle \qquad \forall \eta \in \Lambda\,. \tag{5.9}$$

$\widetilde{S}$ is the pseudo-differential Steklov-Poincaré operator:

$$\langle \widetilde{S}\mu, \eta \rangle = \langle \widetilde{S}_1\mu, \eta \rangle + \langle \widetilde{S}_2\mu, \eta \rangle \qquad \forall \mu, \eta \in \Lambda\,, \tag{5.10}$$

with

$$\langle \widetilde{S}_i\mu, \eta \rangle = a_i(H_i\mu, H_i\eta) \quad \forall \mu, \eta \in \Lambda, \quad i = 1, 2. \tag{5.11}$$

For any $\mu \in \Lambda$, $H_i\mu$ is the harmonic extension of μ in Ω_i $(i = 1, 2)$, that is

$$H_i\mu \in V_i,\ H_i\mu_{|\Gamma} = \mu\ : \quad a_i(H_i\mu, v_i) = 0 \qquad \forall v_i \in V_i^0\,. \tag{5.12}$$

The operators $\widetilde{S}_i$ acts between the space of trace functions Λ and its dual Λ' and they are symmetric, continuous and coercive, i.e. there exist two constants $C_{0,i}, c_{0,i} > 0$ such that

$$\|\widetilde{S}_i\eta\|_{\Lambda'} \le C_{0,i}\|\eta\|_{\Lambda} \quad \text{and} \quad \langle \widetilde{S}_i\eta, \eta \rangle \ge c_{0,i}^2\|\eta\|_{\Lambda}^2\,. \tag{5.13}$$

$\langle \cdot, \cdot \rangle$ denotes the duality pairing between Λ' and Λ.

Finally,

$$\langle \widetilde{\chi}, \eta \rangle = \langle \widetilde{\chi}_1, \eta \rangle + \langle \widetilde{\chi}_2, \eta \rangle \quad \forall \eta \in \Lambda\,, \tag{5.14}$$

with

$$\langle \widetilde{\chi}_i, \eta \rangle = \int_{\Omega_i} f_i\, H_i\eta \; - a_i(w_i, H_i\eta) \quad \forall \eta \in \Lambda,\ i = 1, 2, \tag{5.15}$$

where we denote by $w_i \in V_i^0$ the solution of the following problem:

$$a_i(w_i, v_i) = \int_{\Omega_i} f\, v_i \qquad \forall v_i \in V_i^0\,. \tag{5.16}$$

In the next section we introduce a general approach for solving the interface equation (5.9) which includes and extends the classical Dirichlet-Neumann and Neumann-Neumann iterative methods.

5.3 An Operator-Splitting Approach to Solve the Interface Equation

To compute the solution $\widetilde{\lambda} \in \Lambda$ of the interface equation (5.9), we adopt an operator-splitting approach based on the splitting of the Steklov-Poincaré operator $\widetilde{S}$ as sum of the local operators $\widetilde{S}_i$ (see (5.10)).
Our aim is to set up a method generating two sequences of traces $\{\mu_1^k\}$, $\{\mu_2^k\}$ which approximate $u_{|\Gamma}$, say, from Ω_1 and Ω_2, respectively, and converge to the exact trace $u_{|\Gamma}$ for $k \to \infty$.

This idea is quite similar to the one characterizing a Schwarz method where a decomposition of Ω with overlapping is taken (see Fig. 5.1). In fact, in that case, the classical additive or multiplicative Schwarz methods generate two sequences of functions $\{\hat{u}_i^k\}$ in Ω_i' $(i = 1, 2)$ satisfying the Dirichlet condition $\hat{u}_i^k = \hat{u}_{j|\Gamma_i}^{k-1}$ or $\hat{u}_i^k = \hat{u}_{j|\Gamma_i}^k$ on Γ_i $(i = 1, 2,\ j \neq i)$.
Obviously, in the Schwarz methods the traces $\hat{u}_{1|\Gamma_1}$, $\hat{u}_{2|\Gamma_2}$ obtained at convergence do not necessarily coincide, but we could intend our approach as a particular Schwarz algorithm with zero overlap.

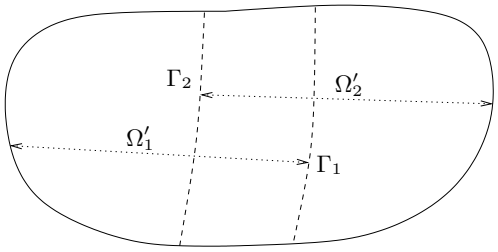


Fig. 5.1. Overlapping partition of the computational domain Ω.

We consider the *Alternating Direction Iterative* (ADI) method to generate two sequences of functions $\{\mu_1^k\}$ and $\{\mu_2^k\}$ corresponding to approximations of the traces $u_{1|\Gamma}^k$ and $u_{2|\Gamma}^k$ on Γ, respectively.

Consider an initial guess $\mu_2^0 \in \Lambda$; then, for $k \geq 0$ we look for $\mu_1^{k+1} \in \Lambda$ and then $\mu_2^{k+1} \in \Lambda$ s.t. for all $\eta \in \Lambda$

$$\langle(\gamma_1 I + \widetilde{S}_1)\mu_1^{k+1}, \eta\rangle = \langle\widetilde{\chi}_1, \eta\rangle + \langle\widetilde{\chi}_2 + (\gamma_1 I - \widetilde{S}_2)\mu_2^k, \eta\rangle \tag{5.17}$$

$$\langle(\gamma_2 I + \widetilde{S}_2)\mu_2^{k+1}, \eta\rangle = \langle\widetilde{\chi}_2, \eta\rangle + \langle\widetilde{\chi}_1 + (\gamma_2 I - \widetilde{S}_1)\mu_1^{k+1}, \eta\rangle\,. \tag{5.18}$$

We have denoted by γ_1 and γ_2 two non-negative real acceleration coefficients such that $\gamma_1 + \gamma_2 > 0$, which could be chosen dynamically according to a suitable error minimization strategy; we shall investigate this option in Sect. 5.6.1.

Should the iterative method (5.17)-(5.18) converge to two limit functions, say μ_1 and μ_2, then necessarily $\mu_1 = \mu_2 = \widetilde{\lambda}(= u_{|\Gamma})$, the solution of (5.9).

5.4 Differential Interpretation of the ADI Method

In this section we interpret the ADI method (5.17)-(5.18) in terms of a sequence of Poisson problems in Ω_1 and Ω_2, respectively, with suitable boundary conditions on the interface Γ. Precisely, we have the following result.

Proposition 5.4.1. *The ADI method (5.17)-(5.18) is equivalent to a Robin-Robin method to solve the Steklov-Poincaré equation (5.9).*

Before proving this equivalence, let us recall the definition of the classical Robin-Robin method.

5.4.1 The Robin-Robin Method

This method has been introduced and analyzed in [AL90a] and [Lio90]; in particular, Lions addressed the general case of $M \geq 2$ subdomains and proved a convergence result using a technique based on energy estimates.

The Robin-Robin algorithm for problem (5.5)-(5.8) reads: for $k \geq 0$, find $u_1^{k+1} \in V_1$, then $u_2^{k+1} \in V_2$ s.t.

$$a_1(u_1^{k+1}, \widetilde{R}_1\eta) + \int_\Gamma \gamma_1 u_1^{k+1}\eta = \int_{\Omega_1} f\,\widetilde{R}_1\eta - a_2(u_2^k, \widetilde{R}_2\eta) + \int_\Gamma \gamma_1 u_2^k\eta + \int_{\Omega_2} f\,\widetilde{R}_2\eta \qquad \forall\eta \in \Lambda\,; \quad (5.19)$$

$$a_2(u_2^{k+1}, \widetilde{R}_2\eta) + \int_\Gamma \gamma_2 u_2^{k+1}\eta = \int_{\Omega_2} f\,\widetilde{R}_2\eta - a_1(u_1^{k+1}, \widetilde{R}_1\eta) + \int_\Gamma \gamma_2 u_1^{k+1}\eta + \int_{\Omega_1} f\,\widetilde{R}_1\eta \qquad \forall\eta \in \Lambda\,, \quad (5.20)$$

where γ_1 and γ_2 are non-negative acceleration parameters satisfying $\gamma_1 + \gamma_2 > 0$.
The following convergence result holds (see [Lio90] or [QV99] p. 135):

Theorem 5.4.1. *If $\gamma_1 = \gamma_2$, then u_i^k $(i = 1, 2)$ converges weakly to $u_{|\Omega_1}$ in V_i and, in particular, $u_{i|\Gamma}^k$ converges to $u_{|\Gamma}$ weakly in $H^{1/2}(\Gamma)$ as k goes to $+\infty$. Moreover, if a finite element approximation of (5.5)-(5.8) is considered, convergence is uniform in the mesh size.*

Remark 5.4.1. Theorem 5.4.1 states that if $\gamma_1 = \gamma_2$, then the Robin-Robin method converges; note that the converse is not true as it can be seen from this simple example.
Let $\Omega = (0, 1)$ with $\Omega_1 = (0, 1/2)$, $\Omega_2 = (1/2, 1)$ and $\Gamma = \{1/2\}$. We consider the following problem: $-u'' = 0$, $0 < x < 1$, $u(0) = u(1) = 0$ (with null solution).
The Robin-Robin method with $\gamma_1 = 1$ and $\gamma_2 = 10$ generates the following sequences:

$$u_1^k(x) = (-1)^{k+1}\frac{2^k}{3^{2k-1}}\,x, \quad u_2^k(x) = (-1)^k\frac{2^{k+1}}{3^{2k}}\,(x-1),\ k \geq 1$$

and for all $x \in \Omega$, $u_i^k(x) \to 0$, $k \to \infty$, $i = 1, 2$, so that we have uniform convergence to the exact solution.
We shall see that suitably chosen different parameters γ_1 and γ_2 may increase the convergence rate. □

5.4.2 Proof of Proposition 5.4.1

If we assume for simplicity $\widetilde{\chi}_i = 0$ $(i = 1, 2)$, then the algorithm (5.17)-(5.18) corresponds to the following steps.

1a) To a given $\mu_2^k \in \Lambda$ apply the operator $\gamma_1 I - \widetilde{S}_2$, that is (see (5.11)), compute ζ^k such that

$$\langle \zeta^k, \eta \rangle = \int_\Gamma \gamma_1 \mu_2^k \, \eta - a_2(H_2\mu_2^k, H_2\eta) \qquad \forall \eta \in \Lambda \tag{5.21}$$

1b) Find $\mu_1^{k+1} \in \Lambda$ such that the linear problem

$$\langle (\gamma_1 I + \widetilde{S}_1)\mu_1^{k+1}, \eta \rangle \; = \; \langle \zeta^k, \eta \rangle \qquad \forall \eta \in \Lambda$$

is satisfied. In view of (5.11), this corresponds to solve:

$$a_1(H_1\mu_1^{k+1}, H_1\eta) + \int_\Gamma \gamma_1 \mu_1^{k+1} \, \eta = \langle \zeta^k, \eta \rangle \qquad \forall \eta \in \Lambda \, . \tag{5.22}$$

Using (5.21), it follows that (5.22) is equivalent to the Robin problem (5.19), under the hypothesis that $f_i = 0$ $(i = 1, 2)$ and provided $\mu_2^k = u_{2|\Gamma}^k$ is taken (in that case $u_2^k = H_2\mu_2^k$). Note that since the solution of (5.19) is unique it holds $\mu_1^{k+1} = u_{1|\Gamma}^{k+1}$ and $u_1^{k+1} = H_1\mu_1^{k+1}$ in Ω_1.

2a) We apply the operator $\gamma_2 I - \widetilde{S}_1$ to $\mu_1^{k+1} \in \Lambda$ and we denote by ξ^{k+1} the computed function such that:

$$\langle \xi^{k+1}, \eta \rangle \; = \; \langle (\gamma_2 I - \widetilde{S}_1)\mu_1^{k+1}, \eta \rangle \qquad \forall \eta \in \Lambda \, .$$

2b) Finally, we solve the linear problem: find $\mu_2^{k+1} \in \Lambda$ s.t.

$$\langle (\gamma_2 I + \widetilde{S}_2)\mu_2^{k+1}, \eta \rangle \; = \; \langle \xi^{k+1}, \eta \rangle \qquad \forall \eta \in \Lambda \, ,$$

that is:

$$a_2(H_2\mu_2^{k+1}, H_2\eta) + \int_\Gamma \gamma_2 \mu_2^{k+1} \, \eta = \langle \xi^{k+1}, \eta \rangle \qquad \forall \eta \in \Lambda \, , \tag{5.23}$$

which is equivalent to the Robin problem (5.20). Thanks to the uniqueness of the solution of (5.20), we obtain that $\mu_2^{k+1} = u_{2|\Gamma}^{k+1}$ (and therefore $H_2\mu_2^{k+1} = u_2^{k+1}$ in Ω_2), since the right hand sides of (5.20) and (5.23) are the same.

5.4.3 Some Remarks Concerning the Robin-Robin Method

The ADI/Robin-Robin method can be represented by the following diagram, which also shows that the algorithm preserves the regularity of the interface data. Given $\mu_2^0 \in \Lambda$, for $k \geq 0$,

$$\begin{array}{ccc} \mu_2^k \in \Lambda & \xrightarrow{\gamma_1 I - \widetilde{S}_2} & \zeta^k \in \Lambda' \\ {\scriptstyle (\gamma_2 I + \widetilde{S}_2)^{-1}} \uparrow \quad k \leftarrow k+1 & & \quad \downarrow {\scriptstyle (\gamma_1 I + \widetilde{S}_1)^{-1}} \\ \xi^{k+1} \in \Lambda' & \xleftarrow[\gamma_2 I - \widetilde{S}_1]{} & \mu_1^{k+1} \in \Lambda \end{array}$$

It is therefore a fixed-point iteration,

$$\mu_2^{k+1} = \mathcal{T}_{\gamma_1,\gamma_2}\mu_2^k, \quad k \geq 0,$$

where the fixed point map $\mathcal{T}_{\gamma_1,\gamma_2} : \Lambda \to \Lambda$ is given by

$$\mathcal{T}_{\gamma_1,\gamma_2} = (\gamma_2 I + \widetilde{S}_2)^{-1}(\gamma_2 I - \widetilde{S}_1)(\gamma_1 I + \widetilde{S}_1)^{-1}(\gamma_1 I - \widetilde{S}_2)\,. \tag{5.24}$$

Remark that the inverse operators $(\gamma_i I + \widetilde{S}_i)^{-1}$ are well defined due to the continuity and coercivity of the Steklov-Poincaré operators $\widetilde{S}_i$ (see (5.13)).
We point out that a convergence result similar to Theorem 5.4.1 can be established simply exploiting the properties of the operators $\widetilde{S}_i$. To this purpose, it will be useful to replace the identity operator $I : \Lambda \to \Lambda$ in (5.24) by the linear continuous operator $\mathcal{I} : \Lambda \to \Lambda'$ which can be defined as follows using the Riesz representation theorem: given any $\eta \in \Lambda$,

$$\mathcal{I}\eta \in \Lambda' : \quad (\mathcal{I}\eta, \xi)_{\Lambda'} = \langle \xi, \eta \rangle \quad \forall \xi \in \Lambda' \tag{5.25}$$

where $(\cdot,\cdot)_{\Lambda'}$ denotes the scalar product in Λ' (for a rigorous definition of norms and scalar products in Λ' we refer to [LM68]). Then, we can prove the following result.

Theorem 5.4.2. *If $\gamma_1 = \gamma_2 = \gamma > 0$, then the sequence $\{\mu^k\}$ generated by the operator $\mathcal{T}_\gamma : \Lambda \to \Lambda$, $\mu^{k+1} = \mathcal{T}_\gamma \mu^k$, with*

$$\mathcal{T}_\gamma = (\gamma\mathcal{I} + \widetilde{S}_2)^{-1}(\gamma\mathcal{I} - \widetilde{S}_1)(\gamma\mathcal{I} + \widetilde{S}_1)^{-1}(\gamma\mathcal{I} - \widetilde{S}_2)$$

converges in Λ.

Proof. We define the auxiliary variable $\tilde{\mu}^k = (\gamma\mathcal{I} + \widetilde{S}_2)\mu^k$ and we rewrite $\mu^{k+1} = \mathcal{T}_\gamma \mu^k$ as $\tilde{\mu}^{k+1} = \widetilde{\mathcal{T}}_\gamma \tilde{\mu}^k$, $k \geq 0$, with

$$\widetilde{\mathcal{T}}_\gamma = (\gamma\mathcal{I} - \widetilde{S}_1)(\gamma\mathcal{I} + \widetilde{S}_1)^{-1}(\gamma\mathcal{I} - \widetilde{S}_2)(\gamma\mathcal{I} + \widetilde{S}_2)^{-1}.$$

Therefore, thanks to the continuity of $(\gamma\mathcal{I} + \widetilde{S}_2)$, we need only to prove that $\tilde{\mu}^k$ converges. To this aim we show that $\widetilde{\mathcal{T}}_{i,\gamma} : \Lambda' \to \Lambda'$, $\widetilde{\mathcal{T}}_{i,\gamma} = (\gamma\mathcal{I} - \widetilde{S}_i)(\gamma\mathcal{I} + \widetilde{S}_i)^{-1}$ is a contraction.
For any $\mu \in \Lambda'$, $\mu \neq 0$, we consider the ratio:

$$\frac{\|\widetilde{\mathcal{T}}_{i,\gamma}\mu\|^2_{\Lambda'}}{\|\mu\|^2_{\Lambda'}} = \frac{\|(\gamma\mathcal{I} - \widetilde{S}_i)(\gamma\mathcal{I} + \widetilde{S}_i)^{-1}\mu\|^2_{\Lambda'}}{\|\mu\|^2_{\Lambda'}} = \frac{\|(\gamma\mathcal{I} - \widetilde{S}_i)\eta\|^2_{\Lambda'}}{\|(\gamma\mathcal{I} + \widetilde{S}_i)\eta\|^2_{\Lambda'}}$$

where we have introduced the auxiliary variable $\eta = (\gamma\mathcal{I} + \widetilde{S}_i)^{-1}\mu \in \Lambda$ ($\eta \neq 0$). Therefore, we have

$$\frac{\|\widetilde{\mathcal{T}}_{i,\gamma}\mu\|^2_{\Lambda'}}{\|\mu\|^2_{\Lambda'}} = \frac{\gamma^2(\mathcal{I}\eta,\mathcal{I}\eta)_{\Lambda'} - 2\gamma(\widetilde{S}_i\eta,\mathcal{I}\eta)_{\Lambda'} + (\widetilde{S}_i\eta,\widetilde{S}_i\eta)_{\Lambda'}}{\gamma^2(\mathcal{I}\eta,\mathcal{I}\eta)_{\Lambda'} + 2\gamma(\widetilde{S}_i\eta,\mathcal{I}\eta)_{\Lambda'} + (\widetilde{S}_i\eta,\widetilde{S}_i\eta)_{\Lambda'}}\,. \tag{5.26}$$

The Riesz representation theorem implies that $(\mathcal{I}\eta,\mathcal{I}\eta)_{\Lambda'} = \|\mathcal{I}\eta\|^2_{\Lambda'} = \|\eta\|^2_{\Lambda}$, while, thanks to (5.25), $(\widetilde{S}_i\eta,\mathcal{I}\eta)_{\Lambda'} = \langle \widetilde{S}_i\eta, \eta\rangle$, so that the right hand side in (5.26) becomes

$$\frac{\gamma^2\|\eta\|^2_{\Lambda} - 2\gamma\langle \widetilde{S}_i\eta,\eta\rangle + \|\widetilde{S}_i\eta\|^2_{\Lambda'}}{\gamma^2\|\eta\|^2_{\Lambda} + 2\gamma\langle \widetilde{S}_i\eta,\eta\rangle + \|\widetilde{S}_i\eta\|^2_{\Lambda'}}\,.$$

Using the estimates (5.13) we can write:

$$\begin{aligned}
\frac{\gamma^2\|\eta\|_\Lambda^2 - 2\gamma\langle\widetilde{S}_i\eta,\eta\rangle + \|\widetilde{S}_i\eta\|_{\Lambda'}^2}{\gamma^2\|\eta\|_\Lambda^2 + 2\gamma\langle\widetilde{S}_i\eta,\eta\rangle + \|\widetilde{S}_i\eta\|_{\Lambda'}^2} &\leq \frac{\gamma^2\|\eta\|_\Lambda^2 - 2\gamma c_{0,i}^2\|\eta\|_\Lambda^2 + \|\widetilde{S}_i\eta\|_{\Lambda'}^2}{\gamma^2\|\eta\|_\Lambda^2 + 2\gamma c_{0,i}^2\|\eta\|_\Lambda^2 + \|\widetilde{S}_i\eta\|_{\Lambda'}^2} \\
&= \frac{\gamma^2 - 2\gamma c_{0,i}^2 + \dfrac{\|\widetilde{S}_i\eta\|_{\Lambda'}^2}{\|\eta\|_\Lambda^2}}{\gamma^2 + 2\gamma c_{0,i}^2 + \dfrac{\|\widetilde{S}_i\eta\|_{\Lambda'}^2}{\|\eta\|_\Lambda^2}} \\
&= 1 - \frac{4\gamma c_{0,i}^2}{\gamma^2 + 2\gamma c_{0,i}^2 + \dfrac{\|\widetilde{S}_i\eta\|_{\Lambda'}^2}{\|\eta\|_\Lambda^2}} \\
&\leq 1 - \frac{4\gamma c_{0,i}^2}{\gamma^2 + 2\gamma c_{0,i}^2 + C_{0,i}^2}
\end{aligned}$$

where the last inequality follows from noticing that the function

$$y \to 1 - \frac{4\gamma c_{0,i}^2}{\gamma^2 + 2\gamma c_{0,i}^2 + y}$$

is increasing for $y > 0$.
Then,

$$\|\widetilde{\mathcal{T}}_{i,\gamma}\mu\| = \sup_{\mu\in\Lambda',\mu\neq 0} \frac{\|\widetilde{\mathcal{T}}_{i,\gamma}\mu\|_{\Lambda'}}{\|\mu\|_{\Lambda'}} < 1$$

which ends the proof. □

Finally, we remark that, unlike the Dirichlet-Neumann method, the Robin-Robin approach to solve the Stekov-Poincaré equation (5.9) allows a specular treatment of the subdomains, since it considers the same kind of interface conditions, i.e. Robin-type conditions, for both problems in Ω_1 and Ω_2. This resembles the classical Schwarz approach which consists in passing from one subdomain to the neighboring one some "Dirichlet data" on the interfaces; in fact, one might also pass "Neumann data" or convex combinations of both. In this sense the Robin-Robin method is nothing but the illustration of this possibility when "the overlapping goes to zero" (see [Lio90]).

5.5 Relation with the Classical Schur Approach

As we have pointed out in Sect. 5.3, the ADI approach aims at computing the unique solution $\widetilde{\lambda}$ of (5.9) by generating two sequences $\{\mu_1^k\}$, $\{\mu_2^k\}$ which, in the limit, approximate $\widetilde{\lambda}$ as the common trace of functions defined in Ω_1 and Ω_2. In principle, this allows us to have a non null jump $\mu_1^k - \mu_2^k \neq 0$ on Γ, for $k \geq 0$. This approach is more general than the usual ones for nonoverlapping domain decomposition, where a single sequence of traces $\widetilde{\lambda}^k = \mu_1^k = \mu_2^k$ on Γ is generated, such that $\widetilde{\lambda}^k \to \widetilde{\lambda}$ when $k \to \infty$.

In what follows we show that the classical iterative substructuring methods, i.e. Dirichlet-Neumann and Neumann-Neumann, can be obtained from (5.17)-(5.18) under the hypothesis

$$\mu_1^k = \mu_2^k \quad \text{on } \Gamma. \tag{5.27}$$

5.5.1 The Dirichlet-Neumann Method

Under the hypothesis (5.27), algorithm (5.17)-(5.18) seems redundant since we have two equations for only one unknown. Therefore, we keep only the equation (5.17) where we indicate $\widetilde{\lambda}^k = \mu_2^k$, $\widetilde{\lambda}^{k+1/2} = \mu_1^{k+1}$ and we set $\gamma_1 = 0$. Then we have

$$\langle \widetilde{S}_1 \widetilde{\lambda}^{k+1/2}, \eta \rangle = \langle \widetilde{\chi}, \eta \rangle - \langle \widetilde{S}_2 \widetilde{\lambda}^k, \eta \rangle \qquad \forall \eta \in \Lambda. \tag{5.28}$$

On the other hand we drop (5.18) and introduce a relaxation depending on a positive acceleration parameters $\theta > 0$ in order to guarantee convergence and improve the convergence rate. In particular, we consider

$$\langle \widetilde{\lambda}^{k+1}, \eta \rangle = \theta \langle \widetilde{\lambda}^{k+1/2}, \eta \rangle + (1-\theta) \langle \widetilde{\lambda}^k, \eta \rangle \qquad \forall \eta \in \Lambda. \tag{5.29}$$

Now, thanks to (5.28) we have

$$\langle \widetilde{\lambda}^{k+1/2}, \eta \rangle = \langle \widetilde{S}_1^{-1} (\widetilde{\chi} - \widetilde{S}_2 \widetilde{\lambda}^k), \eta \rangle \tag{5.30}$$

and if we replace (5.30) into (5.29), we obtain

$$\langle \widetilde{\lambda}^{k+1}, \eta \rangle = \langle \widetilde{\lambda}^k + \theta \widetilde{S}_1^{-1} (\widetilde{\chi} - \widetilde{S} \widetilde{\lambda}^k), \eta \rangle \qquad \forall \eta \in \Lambda \,,$$

which corresponds to a Neumann-Dirichlet method to solve (5.9).
We recall that in this case we characterize the following preconditioner for the Steklov-Poincaré equation:

$$P_{ND}^{-1} = \widetilde{S}_1^{-1} \,. \tag{5.31}$$

5.5.2 The Neumann-Neumann Method

We would like to apply a parallel strategy still considering the hypothesis (5.27). Given an initial value $\widetilde{\lambda}^k = \mu_1^k = \mu_2^k$ on Γ, we consider the following problems:
find $\mu_1^{k+1} \in \Lambda$, then $\mu_2^{k+1} \in \Lambda$ s.t.

$$\langle (\gamma_1 I + \widetilde{S}_1) \mu_1^{k+1}, \eta \rangle = \langle \widetilde{\chi}, \eta \rangle + \langle (\gamma_1 I - \widetilde{S}_2) \widetilde{\lambda}^k, \eta \rangle \qquad \forall \eta \in \Lambda \tag{5.32}$$

$$\langle (\gamma_2 I + \widetilde{S}_2) \mu_2^{k+1}, \eta \rangle = \langle \widetilde{\chi}, \eta \rangle + \langle (\gamma_2 I - \widetilde{S}_1) \widetilde{\lambda}^k, \eta \rangle \qquad \forall \eta \in \Lambda \tag{5.33}$$

Notice that in general $\mu_1^{k+1} \neq \mu_2^{k+1}$. We compute the new trace $\widetilde{\lambda}^{k+1}$ on Γ as a convex combination of μ_1^{k+1} and μ_2^{k+1} using averaging positive coefficients $\theta_1, \theta_2 > 0$ such that $\theta_1 + \theta_2 = 1$. Therefore we set

$$\widetilde{\lambda}^{k+1} = \theta(\theta_1\mu_1^{k+1} + \theta_2\mu_2^{k+1}) + (1-\theta)\widetilde{\lambda}^k \qquad 0 < \theta < 1,$$

and thanks to (5.32), (5.33) we can write

$$\begin{aligned}\langle\widetilde{\lambda}^{k+1},\eta\rangle &= (1-\theta)\langle\widetilde{\lambda}^k,\eta\rangle + \theta\langle[\theta_1(\gamma_1 I+\widetilde{S}_1)^{-1}+\theta_2(\gamma_2 I+\widetilde{S}_2)^{-1}]\widetilde{\chi},\eta\rangle \\ &\quad +\theta\langle[\theta_1(\gamma_1 I+\widetilde{S}_1)^{-1}(\gamma_1 I-\widetilde{S}_2)+\theta_2(\gamma_2 I+\widetilde{S}_2)^{-1}(\gamma_2 I-\widetilde{S}_1)]\widetilde{\lambda}^k,\eta\rangle\,. \quad (5.34)\end{aligned}$$

Observe that for $i = 1, 2$, $j \neq i$,

$$\begin{aligned}\theta_i(\gamma_i I+\widetilde{S}_i)^{-1}(\gamma_i I-\widetilde{S}_j) &= \theta_i(\gamma_i I+\widetilde{S}_i)^{-1}(\gamma_i I+\widetilde{S}_i-\widetilde{S}_i-\widetilde{S}_j) \\ &= \theta_i I-\theta_i(\gamma_i I+\widetilde{S}_i)^{-1}\widetilde{S} \quad (5.35)\end{aligned}$$

so that (5.34) becomes

$$\langle\widetilde{\lambda}^{k+1},\eta\rangle = \langle\widetilde{\lambda}^k,\eta\rangle + \theta\langle P_{RR}^{-1}(\widetilde{\chi}-\widetilde{S}\widetilde{\lambda}^k),\eta\rangle \quad \forall\eta\in\Lambda. \quad (5.36)$$

Equation (5.36) corresponds to a generalized Neumann-Neumann method to solve the Steklov-Poincaré equation (5.9), where the preconditioner for the Steklov-Poincaré equation is given by

$$P_{RR}^{-1} = \theta_1(\gamma_1 I+\widetilde{S}_1)^{-1}+\theta_2(\gamma_2 I+\widetilde{S}_2)^{-1}$$

and corresponds to solving at each iteration two Robin problems in Ω_1 and Ω_2, respectively.
The classical Neumann-Neuman preconditioner

$$P_{NN}^{-1} = \theta_1\widetilde{S}_1^{-1}+\theta_2\widetilde{S}_2^{-1} \quad (5.37)$$

is found for the particular choice $\gamma_1 = \gamma_2 = 0$.

5.6 Algebraic Aspects

In this section we focus on the algebraic counterpart of the methods based on the ADI approach that we have presented in the previous sections. First of all, we recall the general setting of the ADI method and some convergence results.

5.6.1 The ADI Method

The ADI method was first introduced in [PR55] to compute the finite difference approximation of elliptic problems in a rectangular domain by splitting the matrix A into two submatrices A_1 and A_2.
The method has been extensively studied (see [Wac62, Wac63, Wac66, Var00]) if the submatrices A_1 and A_2 commute; for elliptic partial differential equations this requirement implies that the equation is separable on a rectangle. Widlund investigated also the noncommutative case for non-separable equations in rectangular regions (see [Wid66]). However, a general complete theory

for the method is still lacking. Moreover, numerical experiments have shown that the method works efficiently in many cases in which the existing theory does not rigorously apply.
In what follows we recall the definition of the method and some convergence results.

We consider a linear system $\mathrm{A}\boldsymbol{x} = \boldsymbol{b}$, where A is a given real positive definite matrix $\mathrm{A} \in \mathbb{R}^{m\times m}$. As already mentioned, the ADI method is based on representing the matrix A as a sum

$$\mathrm{A} = \mathrm{A}_1 + \mathrm{A}_2$$

where A_1 and A_2 are non-negative definite matrices and either A_1 or A_2 is positive definite. The scheme is then defined as

$$(\gamma_1\mathrm{I} + \mathrm{A}_1)\boldsymbol{y}^{k+1} = \boldsymbol{b} + (\gamma_1\mathrm{I} - \mathrm{A}_2)\boldsymbol{x}^k \tag{5.38}$$
$$(\gamma_2\mathrm{I} + \mathrm{A}_2)\boldsymbol{x}^{k+1} = \boldsymbol{b} + (\gamma_2\mathrm{I} - \mathrm{A}_1)\boldsymbol{y}^{k+1} \tag{5.39}$$

where γ_1 and γ_2 are suitably chosen real positive relaxation parameters. A convergence analysis of this method can be found in [Kel63].
In a typical case involving a linear system arising from an elliptic partial differential equation, A_1 and A_2 might be tridiagonal matrices or at least matrices with small bandwidths. For finite difference methods on rectangular mesh subdivisions, A_1 is the matrix corresponding to horizontal differences and A_2 is the matrix corresponding to vertical differences.

Note that the identity matrix I used in (5.38) and (5.39) can be replaced by any suitable positive matrix E, so that we obtain:

$$(\gamma_1\mathrm{E} + \mathrm{A}_1)\boldsymbol{y}^{k+1} = \boldsymbol{b} + (\gamma_1\mathrm{E} - \mathrm{A}_2)\boldsymbol{x}^k \tag{5.40}$$
$$(\gamma_2\mathrm{E} + \mathrm{A}_2)\boldsymbol{x}^{k+1} = \boldsymbol{b} + (\gamma_2\mathrm{E} - \mathrm{A}_1)\boldsymbol{y}^{k+1}\,. \tag{5.41}$$

Defining $\widetilde{\boldsymbol{x}} = \mathrm{E}^{\frac{1}{2}}\boldsymbol{x}$ and $\widetilde{\boldsymbol{y}} = \mathrm{E}^{\frac{1}{2}}\boldsymbol{y}$ we can rewrite (5.40)-(5.41) as

$$(\gamma_1\mathrm{I} + \tilde{\mathrm{A}}_1)\widetilde{\boldsymbol{y}}^{k+1} = \mathrm{E}^{-\frac{1}{2}}\boldsymbol{b} + (\gamma_1\mathrm{I} - \tilde{\mathrm{A}}_2)\widetilde{\boldsymbol{x}}^k \tag{5.42}$$
$$(\gamma_2\mathrm{I} + \tilde{\mathrm{A}}_2)\widetilde{\boldsymbol{x}}^{k+1} = \mathrm{E}^{-\frac{1}{2}}\boldsymbol{b} + (\gamma_2\mathrm{I} - \tilde{\mathrm{A}}_1)\widetilde{\boldsymbol{y}}^{k+1} \tag{5.43}$$

where

$$\tilde{\mathrm{A}}_i = \mathrm{E}^{-\frac{1}{2}}\mathrm{A}_i\mathrm{E}^{-\frac{1}{2}}, \quad i = 1, 2 \tag{5.44}$$

(see [WH60]).

5.6.1.1 The Preconditioner Associated to the ADI Method. The ADI method can be written as

$$\boldsymbol{x}^{k+1} = \mathrm{P}^{-1}\mathrm{N}\boldsymbol{x}^k + \mathrm{P}^{-1}\boldsymbol{b}, \qquad k \geq 0,$$

where

$$\mathrm{P} = \frac{1}{\gamma_1 + \gamma_2}(\gamma_1\mathrm{I} + \mathrm{A}_1)(\gamma_2\mathrm{I} + \mathrm{A}_2)\,, \quad \mathrm{N} = \frac{1}{\gamma_1 + \gamma_2}(\gamma_2\mathrm{I} - \mathrm{A}_1)(\gamma_1\mathrm{I} - \mathrm{A}_2)\,, \tag{5.45}$$

so that the iteration matrix is

$$\mathrm{B} = \mathrm{P}^{-1}\mathrm{N} = (\gamma_2\mathrm{I} + \mathrm{A}_2)^{-1}(\gamma_1\mathrm{I} + \mathrm{A}_1)^{-1}(\gamma_2\mathrm{I} - \mathrm{A}_1)(\gamma_1\mathrm{I} - \mathrm{A}_2)\,. \tag{5.46}$$

Note that if a positive matrix E is considered instead of I as in (5.40) and (5.41), the preconditioner P can be characterized as follows:

$$\mathrm{P} = \frac{1}{\gamma_1 + \gamma_2}(\gamma_1\mathrm{E} + \mathrm{A}_1)\mathrm{E}^{-1}(\gamma_2\mathrm{E} + \mathrm{A}_2)\,.$$

We consider now the issue of convergence and of the choice of the relaxation parameters.

Suppose that both A_1 and A_2 are positive definite and that their eigenvalues are bounded by the same constants $\tilde{\alpha}$ and $\tilde{\beta}$:

$$0 < \tilde{\alpha} \leq \delta_i^j \leq \tilde{\beta} \qquad j = 1, \ldots, m, \quad i = 1, 2. \tag{5.47}$$

The iteration matrix B is similar to

$$(\gamma_2\mathrm{I} - \mathrm{A}_1)(\gamma_1\mathrm{I} + \mathrm{A}_1)^{-1}(\gamma_1\mathrm{I} - \mathrm{A}_2)(\gamma_2\mathrm{I} + \mathrm{A}_2)^{-1}$$

and

$$\begin{aligned} \rho(\mathrm{B}) \leq |||\mathrm{B}|||_2 &\leq |||(\gamma_2\mathrm{I} - \mathrm{A}_1)(\gamma_1\mathrm{I} + \mathrm{A}_1)^{-1}|||_2 |||(\gamma_1\mathrm{I} - \mathrm{A}_2)(\gamma_2\mathrm{I} + \mathrm{A}_2)^{-1}|||_2 \\ &\leq \max_{j=1,\ldots,m} \left|\frac{\gamma_2 - \delta_1^j}{\gamma_1 + \delta_1^j}\right| \cdot \max_{j=1,\ldots,m} \left|\frac{\gamma_1 - \delta_2^j}{\gamma_2 + \delta_2^j}\right|\,, \end{aligned} \tag{5.48}$$

where $\rho(\mathrm{B})$ denotes the spectral radius of B.
We note that if $\gamma_1 = \gamma_2 = \gamma > 0$, then $\rho(\mathrm{B}) < 1$, which implies that the method converges for any $\gamma > 0$.
Moreover (see, e.g., [Axe94] p. 297), an optimal choice of the relaxation coefficients is $\gamma_1 = \gamma_2 = \sqrt{\tilde{\alpha}\tilde{\beta}}$; correspondingly,

$$\rho(\mathrm{B}) \leq \left(\frac{\sqrt{\tilde{\beta}} - \sqrt{\tilde{\alpha}}}{\sqrt{\tilde{\beta}} + \sqrt{\tilde{\alpha}}}\right)^2\,.$$

5.6.1.2 The Commutative Case. Suppose that A_1 and A_2 commute; we recall the following result (see, e.g., [Var00] chapter 7).

Theorem 5.6.1. *Let A_1 and A_2 be two $m \times m$ matrices, each of which is similar to a diagonal matrix. Then $A_1A_2 = A_2A_1$ if and only if there exists a common basis of eigenvectors $\boldsymbol{d}_j$, $j = 1, \ldots, m$, with $A_1\boldsymbol{d}_j = \delta_1^j\boldsymbol{d}_j$ and $A_2\boldsymbol{d}_j = \delta_2^j\boldsymbol{d}_j$.*

In that case, using (5.38)-(5.39) with a set of relaxation parameters $\gamma_1^n = \gamma_2^n = \gamma^n$, $n \geq 1$, the error $\boldsymbol{e}^k = \boldsymbol{x}^k - \boldsymbol{x}$ can be represented as:

$$\boldsymbol{e}^k = \sum_{j=1}^{m} \left[\prod_{n=1}^{k} \frac{\gamma^n - \delta_1^j}{\gamma^n + \delta_1^j} \cdot \frac{\gamma^n - \delta_2^j}{\gamma^n + \delta_2^j}\right] e_j\boldsymbol{d}_j$$

where $e_j \in \mathbb{R}$ are suitable real coefficients.
Under the hypothesis (5.47), choosing optimal relaxation parameters is equivalent to solve the minimax problem:

$$\text{find } \gamma^{opt}: \quad \max_{\delta_i^j \in [\tilde{\alpha},\tilde{\beta}]} \prod_{n=1}^{k} \left| \frac{\gamma^{opt} - \delta_i^j}{\gamma^{opt} - \delta_i^j} \right| \leq \min_{\gamma^n} \max_{\delta_i^j \in [\tilde{\alpha},\tilde{\beta}]} \prod_{n=1}^{k} \left| \frac{\gamma^n - \delta_i^j}{\gamma^n + \delta_i^j} \right| .$$

By applying the Chebyshev minimax theory, Wachpress proved that this problem admits a unique solution (see [Wac62]). Moreover, he extended his analysis to the more general case where eigenvalues satisfy

$$\tilde{\alpha}_i \leq \delta_i^j \leq \tilde{\beta}_i, \qquad i = 1, 2, \quad \tilde{\alpha}_1 + \tilde{\alpha}_2 > 0,$$

and he proposed an algorithm to compute two series of optimal parameters $\{\gamma_i^k\}_{k\geq 1}$, $i = 1, 2$. For all the details concerning this algorithm we refer the reader to [Wac63].

5.6.2 Multiplicative and Additive ADI Methods for the Schur Complement System

Let us consider a discrete Galerkin approximation of the Steklov-Poincaré problem (5.9). In particular, we consider a triangulation on Ω compatible on Γ and suitable finite element spaces. We denote by Σ_i the algebraic counterpart of the local Steklov-Poincaré operators $\widetilde{S}_i$ $(i = 1, 2)$. Then, (5.9) is approximated by a linear system

$$\text{find } \boldsymbol{\lambda} \in \mathbb{R}^{N_\Gamma} \ : \qquad \Sigma \boldsymbol{\lambda} = \boldsymbol{\chi} \tag{5.49}$$

where $\Sigma = \Sigma_1 + \Sigma_2$ and $\boldsymbol{\lambda}$ is the vector of the nodal values of $\widetilde{\lambda}$ on Γ, N_Γ being the number of nodes lying on the interface.
The algebraic ADI method corresponding to (5.40)-(5.41) to solve (5.49) reads: for $k \geq 0$,

$$(\gamma_1 \mathrm{E} + \Sigma_1)\boldsymbol{\mu}_1^{k+1} = \boldsymbol{\chi} + (\gamma_1 \mathrm{E} - \Sigma_2)\boldsymbol{\mu}_2^k \tag{5.50}$$

$$(\gamma_2 \mathrm{E} + \Sigma_2)\boldsymbol{\mu}_2^{k+1} = \boldsymbol{\chi} + (\gamma_2 \mathrm{E} - \Sigma_1)\boldsymbol{\mu}_1^{k+1}, \tag{5.51}$$

where $\boldsymbol{\mu}_i^k$ is the vector of the nodal values of μ_i^k on Γ at the k-th iteration $(i = 1, 2)$ and E is a suitable positive matrix.

Remark 5.6.1. The method (5.50)-(5.51) is the algebraic counterpart of the differential Robin-Robin method (5.17)-(5.18) if the particular choice $\mathrm{E} = \widetilde{\mathrm{M}}_\Gamma$ is made, where $\widetilde{\mathrm{M}}_\Gamma$ is the mass matrix

$$(\widetilde{\mathrm{M}}_\Gamma)_{ij} = \int_\Gamma \varphi_j^\Gamma \varphi_i^\Gamma \tag{5.52}$$

φ_j^Γ $(j = 1, \ldots, N_\Gamma)$ being the finite element basis functions associated to the nodes on Γ. □

The method (5.50)-(5.51) requires at each step to solve two "generalized Robin problems", one in each subdomain Ω_1 and Ω_2, as illustrated in the following Algorithm 5.1.

Algorithm 5.1 (Multiplicative ADI method)

Given an initial vector $\boldsymbol{\mu}_2^0$, compute $\boldsymbol{w}_2^0 = \Sigma_2 \boldsymbol{\mu}_2^0$. Then, For $k \geq 0$ Do:

$$\begin{aligned}
&\boldsymbol{r}^k = \boldsymbol{\chi} - (\boldsymbol{w}_2^k - \gamma_1 \mathrm{E}\boldsymbol{\mu}_2^k) \\
&(\gamma_1 \mathrm{E} + \Sigma_1)\boldsymbol{\mu}_1^{k+1} = \boldsymbol{r}^k \\
&\boldsymbol{w}^{k+\frac{1}{2}} = \boldsymbol{r}^k - \gamma_1 \mathrm{E}\boldsymbol{\mu}_1^{k+1} \\
&\boldsymbol{r}^{k+1} = \boldsymbol{\chi} - (\boldsymbol{w}^{k+\frac{1}{2}} - \gamma_2 \mathrm{E}\boldsymbol{\mu}_1^{k+1}) \\
&(\gamma_2 \mathrm{E} + \Sigma_2)\boldsymbol{\mu}_2^{k+1} = \boldsymbol{r}^{k+1} \\
&\boldsymbol{w}_2^{k+1} = \boldsymbol{r}^{k+1} - \gamma_2 \mathrm{E}\boldsymbol{\mu}_2^{k+1}
\end{aligned}$$

Applying $(\gamma_i \mathrm{E} + \Sigma_i)^{-1}$ to any vector $\boldsymbol{q} \in \mathbb{R}^{N_\Gamma}$ corresponds to

$$(\gamma_i \mathrm{E} + \Sigma_i)^{-1}\boldsymbol{q} = (0, \mathrm{I})\widetilde{\mathrm{A}}_i^{-1}(0, \mathrm{I})^T \boldsymbol{q} \tag{5.53}$$

where $\widetilde{\mathrm{A}}_i$ is the matrix associated to the Laplace operator on the local domain Ω_i defined as:

$$\widetilde{\mathrm{A}}_i = \begin{pmatrix} \mathrm{A}_{II}^{(i)} & \mathrm{A}_{I\Gamma}^{(i)} \\ \mathrm{A}_{\Gamma I}^{(i)} & \mathrm{A}_{\Gamma\Gamma}^{(i)} + \gamma_i \mathrm{E} \end{pmatrix}; \tag{5.54}$$

the subscripts I and Γ denote, respectively, the nodes internal to Ω_i and those of the interior of Γ.

With the help of a little algebra, we can rewrite (5.50)-(5.51) as:

$$\begin{pmatrix} \boldsymbol{\mu}_1^{k+1} \\ \boldsymbol{\mu}_2^{k+1} \end{pmatrix} = \begin{pmatrix} \mathrm{I} & 0 \\ \mathrm{C}_{\gamma_2} & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & \mathrm{C}_{\gamma_1} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\mu}_1^k \\ \boldsymbol{\mu}_2^k \end{pmatrix} + \begin{pmatrix} \widetilde{\boldsymbol{\chi}}_{\gamma_1} \\ \mathrm{C}_{\gamma_2}\widetilde{\boldsymbol{\chi}}_{\gamma_1} + \widetilde{\boldsymbol{\chi}}_{\gamma_2} \end{pmatrix}, \tag{5.55}$$

where we have denoted

$$\begin{aligned}
&\mathrm{C}_{\gamma_i} := (\gamma_i \mathrm{E} + \Sigma_i)^{-1}(\gamma_i \mathrm{E} - \Sigma_j) \qquad i, j = 1, 2, \quad j \neq i \\
&\widetilde{\boldsymbol{\chi}}_{\gamma_i} := (\gamma_i \mathrm{E} + \Sigma_i)^{-1}\boldsymbol{\chi} \qquad i = 1, 2.
\end{aligned}$$

The iterative method (5.55), or equivalently (5.50)-(5.51), corresponds to a block Gauss-Seidel method to solve the linear system:

$$(\mathrm{I} - \mathrm{J}_{\gamma_1, \gamma_2})\boldsymbol{\mu} = \widetilde{\boldsymbol{\chi}}, \tag{5.56}$$

being

$$\mathrm{J}_{\gamma_1, \gamma_2} = \begin{pmatrix} 0 & \mathrm{C}_{\gamma_1} \\ \mathrm{C}_{\gamma_2} & 0 \end{pmatrix}, \quad \widetilde{\boldsymbol{\chi}} = \begin{pmatrix} \widetilde{\boldsymbol{\chi}}_{\gamma_1} \\ \widetilde{\boldsymbol{\chi}}_{\gamma_2} \end{pmatrix},$$

and $\boldsymbol{\mu} = (\boldsymbol{\mu}_1, \boldsymbol{\mu}_2)^T \in \mathbb{R}^{2N_\Gamma}$.
Note that thanks to (5.55) we can characterize the iteration matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & \mathrm{T}_{\gamma_1,\gamma_2} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \mathrm{C}_{\gamma_2} & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & \mathrm{C}_{\gamma_1} \\ 0 & 0 \end{pmatrix},$$

such that $\mathrm{T}_{\gamma_1,\gamma_2}\boldsymbol{\mu}_2^k \rightarrow \boldsymbol{\mu}_2^{k+1}$, which corresponds to the discrete counterpart of (5.24).
Due to the multiplicative structure of $\mathrm{T}_{\gamma_1,\gamma_2}$ we indicate (5.55) as the *multiplicative ADI (MADI) method*.
Finally, we notice that the MADI method can be cast in the classical formulation:

$$\boldsymbol{\mu}_2^{k+1} = \boldsymbol{\mu}_2^k + \mathrm{P}_{MADI}^{-1}(\boldsymbol{\chi} - \Sigma\boldsymbol{\mu}_2^k), \qquad k \geq 0\,,$$

where we define the multiplicative preconditioner P_{MADI} for the Schur complement system $\Sigma\boldsymbol{\lambda} = \boldsymbol{\chi}$ as:

$$\mathrm{P}_{MADI} = \frac{1}{\gamma_1+\gamma_2}(\gamma_1\mathrm{E} + \Sigma_1)\mathrm{E}^{-1}(\gamma_2\mathrm{E} + \Sigma_2)\,. \tag{5.57}$$

Now, we consider again system (5.56) and we apply a block Jacobi iterative method; then we have:

$$\boldsymbol{\mu}^{k+1} = \left(\begin{pmatrix} 0 & 0 \\ \mathrm{C}_{\gamma_2} & 0 \end{pmatrix} + \begin{pmatrix} 0 & \mathrm{C}_{\gamma_1} \\ 0 & 0 \end{pmatrix}\right)\boldsymbol{\mu}^k + \widetilde{\boldsymbol{\chi}}\,. \tag{5.58}$$

It can be easily seen that this corresponds to the modified ADI method:

$$(\gamma_1\mathrm{E} + \Sigma_1)\boldsymbol{\mu}_1^{k+1} = \boldsymbol{\chi} + (\gamma_1\mathrm{E} - \Sigma_2)\boldsymbol{\mu}_2^k \tag{5.59}$$

$$(\gamma_2\mathrm{E} + \Sigma_2)\boldsymbol{\mu}_2^{k+1} = \boldsymbol{\chi} + (\gamma_2\mathrm{E} - \Sigma_1)\boldsymbol{\mu}_1^k\,, \tag{5.60}$$

where the two linear systems (5.59) and (5.60) are independent and can be solved in parallel. We indicate (5.58), or equivalently (5.59)-(5.60), as *additive* or *parallel ADI (PADI) method*.

5.7 Numerical Results

We consider the model problem (4.29), (4.30) with $f = 1$ and we take Ω as the ball centered in the origin with radius equal to 1/2. Moreover, let $\Omega_1 = \{\boldsymbol{x} = (x_1, x_2) \in \Omega | x_1 > 0\}$ and $\Omega_2 = \{\boldsymbol{x} \in \Omega | x_1 < 0\}$. We adopt lagrangian $\mathbb{P}_2$ elements.
We test the commutativity of the matrices $\widetilde{\Sigma}_1$ and $\widetilde{\Sigma}_2$, where $\widetilde{\Sigma}_i = \widetilde{\mathrm{M}}_\Gamma^{-\frac{1}{2}}\Sigma_i\widetilde{\mathrm{M}}_\Gamma^{-\frac{1}{2}}$. In particular, for some fixed values of $h_{|\Gamma}$, we compute the error

$$E^h_{\widetilde{\mathrm{M}}_\Gamma} = \max_{j=1,\dots,N_\Gamma} \|\widetilde{\Sigma}_1\widetilde{\Sigma}_2\boldsymbol{w}_j - \widetilde{\Sigma}_2\widetilde{\Sigma}_1\boldsymbol{w}_j\|_{\mathbb{R}^{N_\Gamma}}\,,$$

where $\boldsymbol{w}_j$ are the vectors of the canonical basis in $\mathbb{R}^{N_\Gamma}$. As we could expect, these matrices do not commute (see table 5.1, left) and actually $E^h_{\widetilde{\mathrm{M}}_\Gamma} \sim h^{-1}$.
However, if we consider the commutativity of Σ_1 and Σ_2, we can see that the error E^h_{I} is small and bounded by a constant independent of h (see table 5.1, right). Therefore, we apply Wachspress'

$h_{\|\Gamma}$	Error $E^h_{\widetilde{\mathrm{M}}_\Gamma}$
0.083	2.622080
0.0417	4.843364
0.0208	9.692528

$h_{\|\Gamma}$	Error E^h_{I}
0.083	0.065414
0.0417	0.057908
0.0208	0.057765

Table 5.1. Computed errors $E^h_{\widetilde{\mathrm{M}}_\Gamma}$ for different values of $h_{|\Gamma}$ on the left; commutativity error for Σ_1 and Σ_2 on the right.

algorithm considering Σ_1 and Σ_2 to obtain some estimates of the coefficients γ_1 and γ_2, and we use them to define the preconditioner P_{MADI}.
Note that we need upper and lower bounds of the eigenvalues of Σ_i in order to apply Wachspress' method. They can be obtained using, for example, the power method (see, e.g., [QSS00]) or recalling that the following estimate holds (see [QV99] Proposition 2.2.1).

Proposition 5.7.1. *Let $\widetilde{S}_{i,h}$ be internal finite element Galerkin approximation of the operator $\widetilde{S}_i$ defined in (5.11). The discrete Steklov-Poincaré operator $\widetilde{S}_{i,h}$ has real positive definite eigenvalues $\delta^j_{i,h}$, $j = 1, \ldots, N_\Gamma$, which satisfy the following bounds*

$$\underline{\varrho}_i c_i \ \leq \delta^j_{i,h} \ \leq \frac{\overline{\varrho}_i C_i}{h} \qquad j = 1, \ldots, N_\Gamma, \quad i = 1, 2, \tag{5.61}$$

where $\underline{\varrho}_i := \inf_{\boldsymbol{x}\in\Omega_i} \varrho_i(\boldsymbol{x})$, $\overline{\varrho}_i := \sup_{\boldsymbol{x}\in\Omega_i} \varrho_i(\boldsymbol{x})$, while c_i and C_i are two positive constants independent of h, but depending on the geometry of Ω_i.

Let us compare these two approaches to compute the solution of the model problem with $\varrho = 1$ on Ω; we have adopted different computational meshes imposing a tolerance $tol = 1.\mathrm{e}{-06}$ on the increment $\|\boldsymbol{\mu}_2^{k+1} - \boldsymbol{\mu}_2^k\|_{\mathbb{R}^{N_\Gamma}}$. The results reported in table 5.2 show that the coefficients computed thanks to the estimates (5.61) result in few additional iterations with respect to the case where the power method has been adopted. From now on we shall always use the estimates (5.61) instead of those that could be obtained using the more expensive power method. In fact, the latter is an iterative method which requires a matrix-vector product $\Sigma_i \boldsymbol{x}$ at each iteration.

			Iterations using		
	γ_1	γ_2	256 el.	1024 el.	4096 el.
Using the estimates (5.61) →	11.91	11.91	12	12	12
Using the power method →	3.39	3.39	8	8	8

Table 5.2. Iterations with respect to the parameters computed using the power method or (5.61).

We test the algorithm for small and discontinuous parameters ϱ_1 and ϱ_2; the results reported in table 5.3 are for a tolerance $tol = 1.\mathrm{e}{-10}$.
Our results show that if $\varrho_1 = \varrho_2$, then Wachspress' algorithm gives $\gamma_1 = \gamma_2$ as one might expect. Otherwise, it is able to account for the jump in the physical coefficients; in fact, if $\varrho_1 < \varrho_2$ (respectively, $\varrho_1 > \varrho_2$) it gives $\gamma_1 > \gamma_2$ (respectively, $\gamma_1 < \gamma_2$) so that a bigger contribution

ϱ_1	ϱ_2	γ_1	γ_2	Iterations using		
				256 el.	1024 el.	4096 el.
1.e−02	1	2.393e+01	2.376e−01	9	9	12
1	1.e−04	2.312e−03	2.392e+01	5	5	6
1.e−06	1.e−02	2.392e−01	2.312e−05	5	5	5
1.e−04	1.e−07	2.377e−06	2.384e−03	9	9	11

Table 5.3. Number of iterations on different meshes for several values of ϱ_1 and ϱ_2.

$\gamma_i\widetilde{\mathrm{M}}_\Gamma$ is added to the term in P_{MADI} involving the local Schur complement Σ_i which presents the smallest positivity.
Finally, in table 5.4 we report the number of iterations to solve the same problems with discontinuous viscosities but applying the PCG method with the Neumann-Neumann preconditioner (5.37). We have set θ_i as in (4.32) recalling that this choice of the weighting coefficients assures a convergence rate independent of the values of ϱ_i; we consider again a stopping test on the increment with $tol = 1.\mathrm{e}{-10}$. We can see that the number of iterations for the two methods are comparable.

ϱ_1	ϱ_2	θ_1	θ_2	Iterations using		
				256 el.	1024 el.	4096 el.
1.e−02	1	9.901e−03	9.901e−01	8	7	7
1	1.e−04	9.999e−01	9.999e−05	8	7	7
1.e−06	1.e−02	9.999e−05	9.999e−01	8	8	8
1.e−04	1.e−07	9.990e−01	9.990e−04	9	9	8

Table 5.4. Number of iterations on different meshes for several values of ϱ_1 and ϱ_2 using PCG with P_{NN}.

5.8 Extension to a General Diffusion-Advection-Reaction Elliptic Operator

We consider now the more general second order elliptic operator

$$\mathcal{L}_\varrho := -\nabla\cdot(\varrho\nabla u) + \nabla\cdot(\mathbf{b}u) + a_0 u$$

and the boundary value problem

$$\mathcal{L}_\varrho u = f \qquad \text{in } \Omega \tag{5.62}$$

$$u = 0 \qquad \text{on } \partial\Omega, \tag{5.63}$$

where Ω is a bounded domain in $\mathbb{R}^2$, $\varrho \in L^\infty(\Omega)$ is a positive diffusion real valued function, $\mathbf{b} \in (L^\infty(\Omega))^2$ denotes the given flow velocity, $\nabla\cdot\mathbf{b} \in L^\infty(\Omega)$, and $a_0 \in L^\infty(\Omega)$ is an absorption (or reaction) term. Finally, $f \in L^2(\Omega)$ represents a given force.
To guarantee existence and uniqueness of the solution of (5.62), (5.63), we assume that

$$\frac{1}{2}\nabla\cdot\mathbf{b}(x)+a_0(x)\geq 0 \quad \text{for almost every } x\in\Omega$$

(see, e.g., [QV94] chapter 6).
We consider again the splitting $\overline{\Omega}=\overline{\Omega}_1\cup\overline{\Omega}_2$ with nonoverlapping Ω_1 and Ω_2, and we introduce the following local bilinear forms:

$$\begin{aligned} a_i^\flat(w_i,v_i) := a_i(w_i,v_i)+\frac{1}{2}\int_{\Omega_i}[v_i(\mathbf{b}\cdot\nabla w_i)-w_i(\mathbf{b}\cdot\nabla v_i)] \\ +\frac{1}{2}\int_{\Omega_i}\nabla\cdot\mathbf{b}\,w_iv_i+\int_{\Omega_i}a_0w_iv_i \qquad \forall w_i,v_i\in H^1(\Omega_i),\ i=1,2. \end{aligned} \tag{5.64}$$

Following [ATV98] and [LMO00], we consider the following substructuring method based on Robin-type interface conditions:

$$\mathcal{L}_{\varrho_1}u_1^{k+1}=f \quad \text{in } \Omega_1 \tag{5.65}$$

$$u_1^{k+1}=0 \quad \text{on } \partial\Omega_1\cap\partial\Omega \tag{5.66}$$

$$\varrho_1\frac{\partial u_1^{k+1}}{\partial\mathbf{n}}+\left(-\frac{1}{2}\mathbf{b}\cdot\mathbf{n}+\gamma_1\right)u_1^{k+1}=\varrho_2\frac{\partial u_2^{k}}{\partial\mathbf{n}}+\left(-\frac{1}{2}\mathbf{b}\cdot\mathbf{n}+\gamma_1\right)u_2^{k} \quad \text{on } \Gamma \tag{5.67}$$

$$\mathcal{L}_{\varrho_2}u_2^{k+1}=f \quad \text{in } \Omega_2 \tag{5.68}$$

$$u_2^{k+1}=0 \quad \text{on } \partial\Omega_2\cap\partial\Omega \tag{5.69}$$

$$\varrho_2\frac{\partial u_2^{k+1}}{\partial\mathbf{n}}-\left(\frac{1}{2}\mathbf{b}\cdot\mathbf{n}+\gamma_2\right)u_2^{k+1}=\varrho_1\frac{\partial u_1^{k+1}}{\partial\mathbf{n}}-\left(\frac{1}{2}\mathbf{b}\cdot\mathbf{n}+\gamma_2\right)u_1^{k+1} \quad \text{on } \Gamma\,, \tag{5.70}$$

whose weak form is analogous to (5.19)-(5.20) with $a_i^\flat(\cdot,\cdot)$ instead of $a_i(\cdot,\cdot)$.
We apply the ADI approach presented in Sect. 5.3 after introducing the continuous and coercive local Steklov-Poincaré operators

$$\langle S_i^\flat\mu,\eta\rangle \;=\; a_i^\flat(H_i\mu,H_i\eta) \qquad \forall\mu,\eta\in\Lambda,$$

H_i being the harmonic extension operators defined in (5.12). Then, the iterative method (5.62)-(5.63) can be rewritten as: given an initial guess $\mu_2^0\in\Lambda$, for $k\geq 0$ find $\mu_1^{k+1}\in\Lambda$ and then $\mu_2^{k+1}\in\Lambda$ s.t. for all $\eta\in\Lambda$

$$\langle(\gamma_1 I+S_1^\flat)\mu_1^{k+1},\eta\rangle \;=\; \langle\chi_1^\flat,\eta\rangle+\langle\chi_2^\flat+(\gamma_1 I-S_2^\flat)\mu_2^k,\eta\rangle \tag{5.71}$$

$$\langle(\gamma_2 I+S_2^\flat)\mu_2^{k+1},\eta\rangle \;=\; \langle\chi_2^\flat,\eta\rangle+\langle\chi_1^\flat+(\gamma_2 I-S_1^\flat)\mu_1^{k+1},\eta\rangle \tag{5.72}$$

where we have indicated by $\chi_i^\flat$ the local right hand sides of the Steklov-Poincaré equation associated to the advection-diffusion problem (5.62), (5.63). The related algebraic form reads, with obvious choice of notation: for $k\geq 0$,

$$(\gamma_1\widetilde{\mathrm{M}}_\Gamma+\Sigma_1^\flat)\boldsymbol{\mu}_1^{k+1} \;=\; \boldsymbol{\chi}^\flat+(\gamma_1\widetilde{\mathrm{M}}_\Gamma-\Sigma_2^\flat)\boldsymbol{\mu}_2^k \tag{5.73}$$

$$(\gamma_2\widetilde{\mathrm{M}}_\Gamma+\Sigma_2^\flat)\boldsymbol{\mu}_2^{k+1} \;=\; \boldsymbol{\chi}^\flat+(\gamma_2\widetilde{\mathrm{M}}_\Gamma-\Sigma_1^\flat)\boldsymbol{\mu}_1^{k+1}\,. \tag{5.74}$$

5.8.1 Numerical Tests

We consider the computational domain $\Omega = (0,1)^2$ with $\Omega_1 = (0,1/2) \times (0,1)$ and $\Omega_2 = (1/2,1) \times (0,1)$, and the advection-diffusion problem (5.62), (5.63) with $a_0 = 0$, $\mathbf{b} = (-1,1)^T$ and diffusion coefficient $\varrho = 1.e{-}02$ or $\varrho = 1.e{-}05$ in Ω. We take three different structured computational grids with $h = 0.1, 0.05, 0.025$, and we consider stabilized $\mathbb{P}_1$ finite elements with the SUPG (Streamline Upwind/Petrov-Galerkin) method (see [BH82]). First of all, we test the commutativity of the Schur complements $\Sigma_i^\flat$ as done for the Laplace case; when $\mathrm{E} = \mathrm{I}$, we obtain that for both values of ϱ the error $E_{\mathrm{I}}^{h,\flat}$ is uniformly bounded with respect to h: $E_{\mathrm{I}}^{h,\flat} \leq C$ with $C \sim 1.e{-}04$. Then, we apply Wachspress' algorithm to compute the relaxation parameters γ_1 and γ_2. They are reported in table 5.5, together with the number of iterations obtained on the three computational meshes and for different values of ϱ; a tolerance $tol = 1.e{-}10$ on the relative increment has been fixed. The numerical results show that the acceleration parameters yield a number of iterations almost independent of both h and ϱ.
Finally, figure 5.2 represents two computed solutions.

ϱ_1	ϱ_2	γ_1	γ_2	Iterations using		
				200 el.	800 el.	3200 el.
1.e−02	1.e−02	2.336699	2.019213	23	24	28
1.e−05	1.e−05	2.034835	1.728585	21	21	21
1.e−02	1.e−05	2.035217	2.018824	21	22	22

Table 5.5. Number of iterations on different meshes for different values of ϱ.

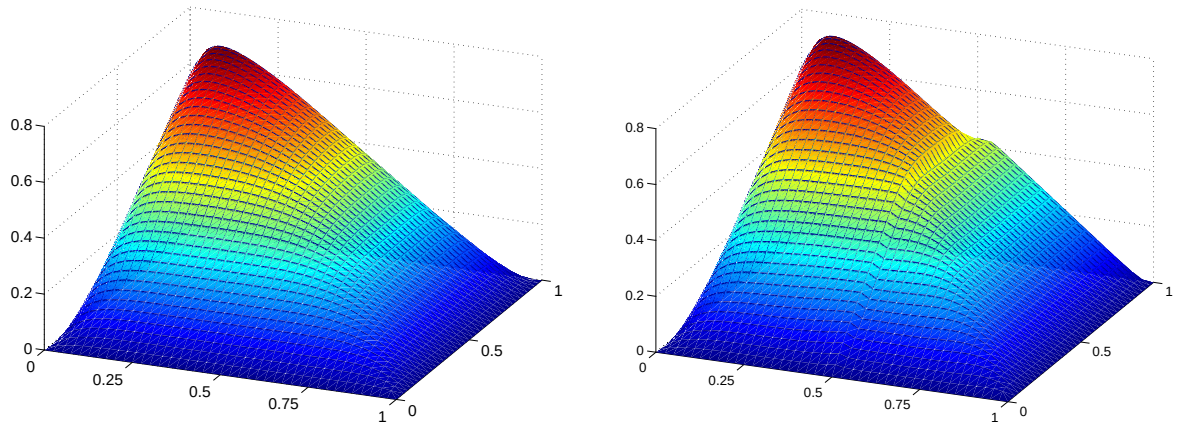


Fig. 5.2. Computed solution for the advection-diffusion problem in Ω with $\varrho = 1.e{-}02$ (left); $\varrho_1 = 1.e{-}02$, $\varrho_2 = 1.e{-}05$ (right).

5.9 Generalization to the Case of Many Subdomains

In this section we present the guidelines to generalize the approach we have presented so far when $M \geq 2$ subdomains are taken.

We consider problem (4.29), (4.30) (problem (5.62), (5.63) could be considered as well) and we suppose that Ω is partitioned into a family of nonoverlapping subdomains Ω_i, $i = 1, \ldots, M$, with $\overline{\Omega} = \bigcup_{i=1}^{M} \overline{\Omega}_i$, $\Omega_i \cap \Omega_j = \emptyset$ if $i \neq j$. We denote by $\Gamma_i = \partial\Omega_i \setminus \partial\Omega$, $\Gamma_{ij} = \overline{\Omega}_i \cap \overline{\Omega}_j$ for $i \neq j$, and the global interface Γ is defined as $\Gamma = \bigcup_{i=1}^{M} \Gamma_i$.

The Robin-Robin algorithm consists of solving the following local problems in Ω_i ($i = 1, \ldots, M$); u_i^0 given, for $k \geq 0$

$$-\nabla \cdot (\varrho_i \nabla u_i^{k+1}) = f \quad \text{in } \Omega_i \tag{5.75}$$

$$u_i^{k+1} = 0 \quad \text{on } \partial\Omega_i \cap \partial\Omega \tag{5.76}$$

$$\varrho_i \frac{\partial u_i^{k+1}}{\partial n_{ij}} + \gamma_{ij} u_i^{k+1} = \varrho_j \frac{\partial u_j^k}{\partial n_{ij}} + \gamma_{ij} u_j^k \quad \text{on } \Gamma_{ij} \quad \forall 1 \leq j \leq M,\ j \neq i \tag{5.77}$$

where $\mathbf{n}_{ij}$ is the unit outward normal to $\partial\Omega_i$ directed from Ω_i to Ω_j, and $\gamma_{ij} > 0$ are suitable relaxation parameters.

For a convergence proof of this method we refer to [Lio90] (or [ATV98, LMO00] for the advection-diffusion case).

Again, we can express the solution of a finite element Galerkin approximation of the global problem (4.29), (4.30) in terms of the solution $\boldsymbol{\lambda} \in \mathbb{R}^{N_\Gamma}$ of the linear Schur complement system $\Sigma\boldsymbol{\lambda} = \boldsymbol{\chi}$ (see [QV99]).

Now, we assume that there is a black-white partition on the subdomains Ω_i into two sets $\mathcal{B}$ and $\mathcal{W}$ such that the intersection between the boundaries of two subdomains in the same group is either empty or a vertex that we shall indicate as cross-point (see Fig. 5.3).

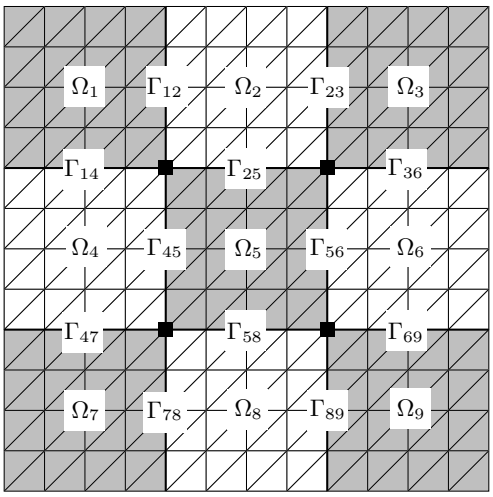


Fig. 5.3. Black and white partition of the subdomains Ω_i.

We define a family of restriction operators: given a vector of degrees of freedom $\boldsymbol{x}_\Gamma$ on Γ, we denote by $\mathrm{R}_i\boldsymbol{x}_\Gamma$ the vector of degrees of freedom of $\boldsymbol{x}_\Gamma$ on Γ_i, $i = 1, \ldots, M$; R_i is a rectangular matrix of zeros and ones and its transpose R_i^T is the extension operator by zero from $\mathbb{R}^{N_{\Gamma_i}}$ to $\mathbb{R}^{N_\Gamma}$. Then, we can decompose the Schur complement as

$$\Sigma = \sum_{i\in\mathcal{B}} \mathrm{R}_i^T \Sigma_i \mathrm{R}_i + \sum_{i\in\mathcal{W}} \mathrm{R}_i^T \Sigma_i \mathrm{R}_i \, .$$

The multidomain formulation of the multiplicative Robin-Robin algorithm (5.50)-(5.51) reads: given $\boldsymbol{x}^0 \in \mathbb{R}^{N_\Gamma}$, for $k \geq 0$,

$$\left(\sum_{i\in\mathcal{B}} \mathrm{R}_i^T(\mathrm{M}_i + \Sigma_i)\mathrm{R}_i\right)\boldsymbol{y}^{k+1} = \boldsymbol{\chi} + \left(\sum_{i\in\mathcal{W}} \mathrm{R}_i^T(\mathrm{M}_i - \Sigma_i)\mathrm{R}_i\right)\boldsymbol{x}^k \tag{5.78}$$

$$\left(\sum_{i\in\mathcal{W}} \mathrm{R}_i^T(\mathrm{M}_i + \Sigma_i)\mathrm{R}_i\right)\boldsymbol{x}^{k+1} = \boldsymbol{\chi} + \left(\sum_{i\in\mathcal{B}} \mathrm{R}_i^T(\mathrm{M}_i - \Sigma_i)\mathrm{R}_i\right)\boldsymbol{y}^{k+1} \, . \tag{5.79}$$

$\mathrm{M}_i \in \mathbb{R}^{N_{\Gamma_i}\times N_{\Gamma_i}}$ is the mass matrix obtained by assembling the local mass matrices $\mathrm{M}_{\Gamma_{ij}} \in \mathbb{R}^{N_{\Gamma_{ij}}\times N_{\Gamma_{ij}}}$ $(j \neq i)$ associated to the interfaces Γ_{ij} related to the subdomain Ω_i, each of them weighted with the corresponding coefficient γ_{ij}. For example, with respect to Fig. 5.3, the mass matrix M_1 would be defined as represented in Fig. 5.4. Note that the additive version of the algorithm would be obtained considering $\boldsymbol{y}^k$ instead of $\boldsymbol{y}^{k+1}$ in (5.79).

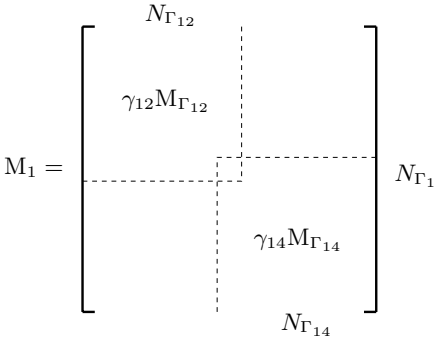


Fig. 5.4. The submatrices $\mathrm{M}_{\Gamma_{12}}$ and $\mathrm{M}_{\Gamma_{14}}$ of the stiffness matrix M_1.

Solving (5.78) and (5.79) corresponds to compute the solution of two Robin problems defined on the union of the black and white subdomains, respectively. In the case where there are no cross-points, the corresponding matrices would be block diagonal so that the Robin problems could be solved independently.

The coefficients γ_{ij} and γ_{ji} relative to the nodes on the interface Γ_{ij} $(i = 1, \ldots, M, j \neq i)$ can be computed using Wachspress' algorithm starting from the eigenvalues bounds of the local Schur complements $\widetilde{\Sigma}_k^* = \mathrm{M}_{\Gamma_{ij}}^{-\frac{1}{2}} \Sigma_k^* \mathrm{M}_{\Gamma_{ij}}^{\frac{1}{2}}$, $k = i, j$, where Σ_k^* are obtained from Σ_k eliminating the nodes which do not belong to the interface Γ_{ij} considered.

5.10 Conclusions

The interpretation of the Robin-Robin method in terms of an ADI algorithm to compute the solution of the Schur complement system allows on one hand to characterize Robin-type preconditioners for the interface problem itself, and, on the other hand, it indicates a possible strategy to automatically compute the relaxation parameters, at least in the case of "nearly commuting" submatrices.
Moreover, thanks to the presence of the positive matrices $\gamma_i \mathrm{E}$, the ADI approach permits to deal with non-negative Schur complements Σ_i without introducing suitable pseudo-inverses as it would be required by Dirichlet-Neumann or Neumann-Neumann methods.
Finally, the characterization of an iteration operator $\mathrm{T}_{\gamma_1,\gamma_2}$ and the study of its spectral radius shows how the relaxation parameters influence the convergence rate of the algorithm; this may lead to choose ad-hoc coefficients for problems where the commutativity constraint is not satisfied at all and Wachspress' approach would not work. This might be the case when dealing with the coupling of heterogeneous models; in particular, in chapter 6 we shall illustrate how this approach can help in setting up a robust iterative method to solve the linear coupled Stokes/Darcy problem thus improving the Dirichlet-Neumann type methods discussed in chapters 3 and 4.

6. An Operator-Splitting Based Method for the Stokes/Darcy Problem

We present an application of the ADI method to the Stokes/Darcy case. Possible iterative methods are illustrated at the algebraic stage together with their differential interpretation. The choice of the relaxation parameters is addressed for this particular noncommutative case. Some numerical tests are presented.

6.1 Introduction

In this chapter we apply the operator splitting approach investigated in chapter 5 to the Stokes/Darcy coupling.
More precisely, the ADI method is applied at the algebraic Schur complement system associated to the coupled problem. Both choices of the interface variables, the normal velocity on Γ and the piezometric head on Γ, will be analyzed.
We show that at the differential level the ADI-based methods can be regarded as substructuring scheme involving Robin type conditions at the interface. Most of the results we have presented for the ADI method cannot be applied in this case since that the commutativity requirement (see Sect. 5.6.1.2) is not fulfilled and Wachspress' method cannot be applied. However, exploiting the analysis carried out in chapter 5 and the characterization of the spectra of the Steklov-Poincaré operators (see chapters 2 and 3), we can set up a possible strategy to choose the acceleration parameters which allows us to obtain good convergence results.

6.2 Operator-Splitting Methods: Setting and Differential Interpretation

Case of the Schur Complement System for the Normal Velocity on Γ. We consider the Schur complement system (3.58). Using the natural splitting

$$\Sigma_h = \Sigma_{fh} + \Sigma_{ph}$$

we can write the following ADI method.

Algorithm 6.1

Given $\boldsymbol{\mu}_2^0 \in \mathbb{R}^{N_\Gamma}$,
For $k \geq 0$, until convergence Do

$$(\gamma_p \widetilde{\mathrm{M}}_\Gamma + \Sigma_{ph})\boldsymbol{\mu}_1^{k+1} = \boldsymbol{\chi}_h + (\gamma_p \widetilde{\mathrm{M}}_\Gamma - \Sigma_{fh})\boldsymbol{\mu}_2^k \tag{6.1}$$

$$(\gamma_f \widetilde{\mathrm{M}}_\Gamma + \Sigma_{fh})\boldsymbol{\mu}_2^{k+1} = \boldsymbol{\chi}_h + (\gamma_f \widetilde{\mathrm{M}}_\Gamma - \Sigma_{ph})\boldsymbol{\mu}_1^{k+1} \tag{6.2}$$

where $\widetilde{\mathrm{M}}_\Gamma$ is the mass matrix on Γ defined in (5.52).
γ_f and γ_p are two positive acceleration parameters that will be chosen in order to maximize the rate of convergence.

Algorithm 6.1 requires to solve two linear systems with matrices $(\gamma_p \widetilde{\mathrm{M}}_\Gamma + \Sigma_{ph})$ and $(\gamma_f \widetilde{\mathrm{M}}_\Gamma + \Sigma_{fh})$, respectively.

Recalling the definition (3.61) of Σ_{fh} and following (5.53) and (5.54) we can see that applying $(\gamma_f \widetilde{\mathrm{M}}_\Gamma + \Sigma_{fh})^{-1}$ to any vector $\boldsymbol{q} \in \mathbb{R}^{N_\Gamma}$ yields

$$(\gamma_f \widetilde{\mathrm{M}}_\Gamma + \Sigma_{fh})^{-1}\boldsymbol{q} = (0, \mathrm{I})\widetilde{\mathrm{F}}_*^{-1}(0, \mathrm{I})^T\boldsymbol{q}$$

where $\widetilde{\mathrm{F}}_*$ is the matrix

$$\begin{pmatrix} \mathrm{A}_{ff} & \mathrm{B}^T & \mathrm{A}_{f\Gamma} \\ \mathrm{B}_1 & 0 & \mathrm{B}_{f\Gamma} \\ \mathrm{A}_{\Gamma f} & \mathrm{B}_{f\Gamma}^T & \mathrm{A}_{\Gamma\Gamma}^f + \gamma_f \widetilde{\mathrm{M}}_\Gamma \end{pmatrix}.$$

At the differential level, $\widetilde{\mathrm{F}}_*^{-1}(0, \mathrm{I})^T\boldsymbol{q}$ corresponds to solve a Stokes problem in Ω_f supplemented with the mixed boundary condition:

$$\mathbf{n} \cdot \mathsf{T}(\mathbf{u}_f, p_f) \cdot \mathbf{n} + \gamma_f \mathbf{u}_f \cdot \mathbf{n} = q \text{ on } \Gamma,$$

where q is an assigned data function.

On the other hand, applying $(\gamma_p \widetilde{\mathrm{M}}_\Gamma + \Sigma_{ph})^{-1}$ to any vector $\boldsymbol{q} \in \mathbb{R}^{N_\Gamma}$ corresponds to

$$(\gamma_p \widetilde{\mathrm{M}}_\Gamma + \Sigma_{ph})^{-1}\boldsymbol{q} = (0, \mathrm{I})\mathrm{D}_*^{-1}(0, \mathrm{I})^T\boldsymbol{q} \tag{6.3}$$

where D_* is the matrix

$$\begin{pmatrix} \mathrm{A}_{pp} & \mathrm{A}_{p\Gamma} & 0 \\ \mathrm{A}_{p\Gamma}^T & \mathrm{A}_{\Gamma\Gamma}^p & -\mathrm{M}_{\Gamma\Gamma}^T \\ 0 & \mathrm{M}_{\Gamma\Gamma} & \gamma_p \widetilde{\mathrm{M}}_\Gamma \end{pmatrix}. \tag{6.4}$$

The differential interpretation of system (6.4) is not straightforward. Recalling the definition of the matrix $\mathrm{M}_{\Gamma\Gamma}$ (see Sect. 3.5.2), we can interpret this system as a Darcy problem in Ω_p, where we impose via Lagrange multipliers the Robin condition:

$$g\varphi - \gamma_p \frac{\mathsf{K}}{n} \frac{\partial \varphi}{\partial \mathbf{n}} = q \text{ on } \Gamma \tag{6.5}$$

for a given function q. Precisely, we consider the problem (using homogeneous boundary conditions for the sake of simplicity):

find $\varphi \in H^1(\Omega_p)$, $\mu \in H^{1/2}(\Gamma)$:

$$a_p(\varphi,\psi) - \int_\Gamma \mu\psi = 0 \qquad \forall \psi \in H^1(\Omega_p)$$
$$\langle \zeta, \left(g\varphi - \gamma_p \frac{\mathsf{K}}{n}\frac{\partial \varphi}{\partial \mathbf{n}} - q \right)\rangle = 0 \qquad \forall \zeta \in H^{1/2}(\Gamma)$$

(see [QV94] p. 226, 227), the Lagrange multiplier μ playing the role of the conormal derivative of φ on Γ.

The use of Lagrange multipliers to impose the mixed conditions (6.5) allows us to compute directly at each step k in (6.1) the vector $\boldsymbol{\mu}_1^k$ of the nodal values of the conormal derivative of the piezometric head φ, which is required to update the right hand side in (6.2).

Therefore, each step k of Algorithm 6.1 requires to solve one Robin problem in each subdomain Ω_f and Ω_p.

Case of the Schur Complement for the Trace of the Piezometric Head $\boldsymbol{\varphi_{|\Gamma}}$. We consider now the Schur complement system (3.94) associated to the choice of the interface variable φ on Γ.
In this case, our operator-splitting approach yields the following iterative method.

Algorithm 6.2

`Given` $\boldsymbol{\eta}_2^0 \in \mathbb{R}^{N_\Gamma}$`,`
`For` $k \geq 0$`, until convergence Do`

$$(\gamma_p \widetilde{\mathrm{M}}_\Gamma + \widetilde{\Sigma}_{ph})\boldsymbol{\eta}_1^{k+1} = \widetilde{\boldsymbol{\chi}}_h + (\gamma_p \widetilde{\mathrm{M}}_\Gamma - \widetilde{\Sigma}_{fh})\boldsymbol{\eta}_2^k \tag{6.6}$$
$$(\gamma_f \widetilde{\mathrm{M}}_\Gamma + \widetilde{\Sigma}_{fh})\boldsymbol{\eta}_2^{k+1} = \widetilde{\boldsymbol{\chi}}_h + (\gamma_f \widetilde{\mathrm{M}}_\Gamma - \widetilde{\Sigma}_{ph})\boldsymbol{\eta}_1^{k+1}. \tag{6.7}$$

Here, we have to solve two linear systems with matrices $(\gamma_p \widetilde{\mathrm{M}}_\Gamma + \widetilde{\Sigma}_{ph})$ and $(\gamma_f \widetilde{\mathrm{M}}_\Gamma + \widetilde{\Sigma}_{fh})$.

Concerning the first one, we can see that applying $(\gamma_p \widetilde{\mathrm{M}}_\Gamma + \widetilde{\Sigma}_{ph})^{-1}$ to any vector $\boldsymbol{q} \in \mathbb{R}^{N_\Gamma}$ corresponds to

$$(\gamma_p \widetilde{\mathrm{M}}_\Gamma + \widetilde{\Sigma}_{ph})^{-1}\boldsymbol{q} = (0, \mathrm{I})\widetilde{\mathrm{D}}^{-1}(0, \mathrm{I})^T\boldsymbol{q}$$

where $\widetilde{\mathrm{D}}$ is the matrix

$$\begin{pmatrix} \mathrm{A}_{pp} & \mathrm{A}_{p\Gamma} \\ \mathrm{A}_{p\Gamma}^T & \mathrm{A}_{\Gamma\Gamma}^p + \gamma_p \widetilde{\mathrm{M}}_\Gamma \end{pmatrix}$$

associated to the Darcy problem in Ω_p with boundary condition on Γ:

$$\gamma_p g\varphi - \frac{\mathsf{K}}{n}\frac{\partial \varphi}{\partial \mathbf{n}} = q \tag{6.8}$$

for a given function q. Notice that, unlike (6.4), in this case the mixed boundary condition (6.8) has been imposed in the usual natural way (see, e.g., [QV94] p. 162).

Finally, applying $(\gamma_f \widetilde{\mathrm{M}}_\Gamma + \widetilde{\Sigma}_{fh})^{-1}$ to any vector $\boldsymbol{q} \in \mathbb{R}^{N_\Gamma}$ corresponds to

$$(\gamma_f \widetilde{\mathrm{M}}_\Gamma + \widetilde{\Sigma}_{fh})^{-1}\boldsymbol{q} = (0, \mathrm{I})\bar{\mathrm{F}}^{-1}(0, \mathrm{I})^T\boldsymbol{q}$$

where $\bar{\mathrm{F}}$ is the matrix

$$\begin{pmatrix} \mathrm{A}_{ff} & \mathrm{B}^T & \mathrm{A}_{f\Gamma} & 0 \\ \mathrm{B}_1 & 0 & \mathrm{B}_{f\Gamma} & 0 \\ \mathrm{A}_{\Gamma f} & \mathrm{B}_{f\Gamma}^T & \mathrm{A}_{\Gamma\Gamma}^f & \mathrm{M}_{\Gamma\Gamma} \\ 0 & 0 & -\mathrm{M}_{\Gamma\Gamma}^T & \gamma_f \widetilde{\mathrm{M}}_\Gamma \end{pmatrix}. \tag{6.9}$$

Similarly to what we have done for the Darcy problem (6.4), here we can recognize that (6.9) is the matrix associated to a Stokes problem in Ω_f supplemented with the mixed boundary condition

$$-\gamma_f \mathbf{n} \cdot \mathsf{T}(\mathbf{u}_f, p_f) \cdot \mathbf{n} - \mathbf{u}_f \cdot \mathbf{n} = q \quad \text{on } \Gamma \tag{6.10}$$

for an assigned function q, where the latter condition (6.10) is imposed via Lagrange multipliers. In this case the Lagrange multiplier plays the role of the normal stress $\mathbf{n} \cdot \mathsf{T}(\mathbf{u}_f, p_f) \cdot \mathbf{n}$ on Γ.

Again, we can conclude that each step k of Algorithm 6.2 requires to solve two Robin problems in Ω_f and Ω_p.

6.2.1 Differential Interpretation

Based on the above considerations and using the theory of chapter 5, we can associate to Algorithms 6.1 and 6.2 the following differential substructuring Robin-Robin schemes.

For simplicity of notation we indicate by

$$\mathit{Darcy}\ (\varphi)$$

the Darcy problem in Ω_p with boundary conditions (1.24), (1.25) and by

$$\mathit{Stokes}\ (\mathbf{u}_f, p_f)$$

the Stokes problem in Ω_f with boundary conditions (1.22), (1.23) and (2.58) on Γ.
Remark that both problems $\mathit{Darcy}\ (\varphi)$ and $\mathit{Stokes}\ (\mathbf{u}_f, p_f)$ must be supplemented with a boundary condition on the interface Γ.

Using this notation, the ADI scheme (6.1)-(6.2) corresponds to:

Algorithm 6.3

For $k \geq 0$ until convergence,

$$\begin{cases} \texttt{solve } \textit{Darcy } (\varphi^{k+1}) \texttt{ in } \Omega_p \texttt{ with boundary condition} \\ -g\varphi^{k+1} - \gamma_p \left(-\dfrac{\mathsf{K}}{n} \dfrac{\partial \varphi^{k+1}}{\partial \mathbf{n}} \right) = \mathbf{n} \cdot \mathsf{T}(\mathbf{u}_f^k, p_f^k) \cdot \mathbf{n} - \gamma_p \mathbf{u}_f^k \cdot \mathbf{n} \texttt{ on } \Gamma \end{cases} \tag{6.11}$$

$$\begin{cases} \texttt{solve } \textit{Stokes}(\mathbf{u}_f^{k+1}, p_f^{k+1}) \quad \texttt{in } \Omega_f \texttt{ with boundary condition} \\ \mathbf{n} \cdot \mathsf{T}(\mathbf{u}_f^{k+1}, p_f^{k+1}) \cdot \mathbf{n} + \gamma_f \mathbf{u}_f^{k+1} \cdot \mathbf{n} = -g\varphi^{k+1} + \gamma_f \left(-\dfrac{\mathsf{K}}{n} \dfrac{\partial \varphi^{k+1}}{\partial \mathbf{n}} \right) \texttt{ on } \Gamma. \end{cases} \tag{6.12}$$

On the other hand scheme (6.6)-(6.7) corresponds to the following iterations.

Algorithm 6.4

For $k \geq 0$ until convergence,

$$\begin{cases} \texttt{solve } \textit{Darcy}(\varphi^{k+1}) \texttt{ in } \Omega_p \texttt{ with boundary condition} \\ -\gamma_p g\varphi^{k+1} - \left(-\dfrac{\mathsf{K}}{n} \dfrac{\partial \varphi^{k+1}}{\partial \mathbf{n}} \right) = \gamma_p \mathbf{n} \cdot \mathsf{T}(\mathbf{u}_f^k, p_f^k) \cdot \mathbf{n} - \mathbf{u}_f^k \cdot \mathbf{n} \texttt{ on } \Gamma \end{cases} \tag{6.13}$$

$$\begin{cases} \texttt{solve } \textit{Stokes}(\mathbf{u}_f^{k+1}, p_f^{k+1}) \quad \texttt{in } \Omega_f \texttt{ with boundary condition} \\ \gamma_f \mathbf{n} \cdot \mathsf{T}(\mathbf{u}_f^{k+1}, p_f^{k+1}) \cdot \mathbf{n} + \mathbf{u}_f^{k+1} \cdot \mathbf{n} = -\gamma_f g\varphi^{k+1} + \left(-\dfrac{\mathsf{K}}{n} \dfrac{\partial \varphi^{k+1}}{\partial \mathbf{n}} \right) \texttt{ on } \Gamma. \end{cases} \tag{6.14}$$

Remark 6.2.1. Note that at the differential stage, these two schemes differ only in the role of the relaxation parameters γ_f and γ_p for the Robin conditions on Γ: in fact, γ_p multiplies $\mathsf{K}\nabla\varphi \cdot \mathbf{n}$ on Γ in (6.11) rather than $\varphi_{|\Gamma}$ in (6.13), while γ_f multiplies $\mathbf{u}_f \cdot \mathbf{n}$ in (6.12) rather than $\mathbf{n} \cdot \mathsf{T}(\mathbf{u}_f, p_f) \cdot \mathbf{n}$ in (6.14). □

The Robin-Robin methods (6.11)-(6.12) and (6.13)-(6.14) are both well-posed and they preserve the regularity of the interface data at each step k. In fact, supposing that the boundary data $\mathbf{u}_{in}$, φ_p and the normal $\mathbf{n}$ are sufficiently regular, if we take at the k-th step

$$\xi^k = \varepsilon_2^p \mathbf{n} \cdot \mathsf{T}(\mathbf{u}_f^k, p_f^k) \cdot \mathbf{n} - \varepsilon_1^p \mathbf{u}_f^k \cdot \mathbf{n} \in H^{-1/2}(\Gamma)$$

(where ε_1^p and ε_2^p are positive parameters that can be put equal to 1 or γ_p to recover the right hand sides in the interface conditions (6.11) or (6.13)), then Darcy problem (6.11) (or (6.13)) with the interface condition

$$-\varepsilon_2^p g\varphi^{k+1} - \varepsilon_1^p(-(\mathsf{K}/n)\nabla\varphi^{k+1}\cdot\mathbf{n}) = \xi^k \quad \text{on } \Gamma,$$

is well-posed (see, e.g., [QV94, Gri85]). In particular, we can compute

$$\eta^{k+1} = -\varepsilon_2^f g\varphi^{k+1} + \varepsilon_1^f(-(\mathsf{K}/n)\nabla\varphi^{k+1}\cdot\mathbf{n}) \quad \text{on } \Gamma\,,$$

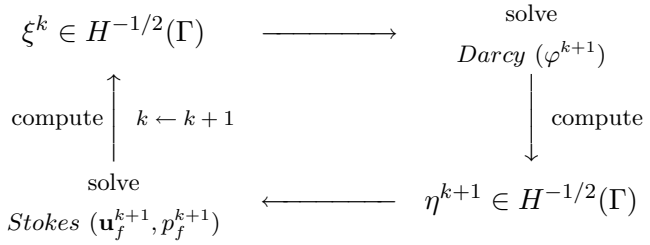
(with ε_1^f and ε_2^f positive parameters that set equal to 1 or γ_f allows us to recover conditions (6.12) or (6.14)) and we remark that $\eta^{k+1} \in H^{-1/2}(\Gamma)$.
Then, we impose the interface condition

$$\varepsilon_2^f\mathbf{n}\cdot\mathsf{T}(\mathbf{u}_f^{k+1}, p_f^{k+1})\cdot\mathbf{n} + \varepsilon_1^f\mathbf{u}_f^{k+1}\cdot\mathbf{n} = \eta^{k+1} \quad \text{on } \Gamma$$

to the Stokes problem (6.12) (or (6.14)), which is also well-posed and yields

$$\xi^{k+1} = \varepsilon_2^p\mathbf{n}\cdot\mathsf{T}(\mathbf{u}_f^{k+1}, p_f^{k+1})\cdot\mathbf{n} - \varepsilon_1^p{\mathbf{u}_f}^{k+1}\cdot\mathbf{n} \quad \in H^{-1/2}(\Gamma).$$

Therefore, we can conclude that the Robin-Robin algorithms 6.3 and 6.4 are well-posed and that at each step $k \geq 0$ the regularity of the interface data is preserved:



6.3 Numerical Results

In this section we present some numerical results obtained using Algorithms 6.3 and 6.4. In particular, we will show that we can improve the convergence results obtained in chapter 4 when applying the Dirichlet-Neumann methods to problems with small physical coefficients ν and K. We consider first of all the issue of finding suitable acceleration parameters γ_f and γ_p.

As already pointed out, the strategy proposed by Wachspress to compute the relaxation parameters for the ADI method is applicable if the local Schur-complement matrices, say Σ_1 and Σ_2, commute, and numerical tests have shown that it is successful also when the error in applying $\Sigma_1\Sigma_2 - \Sigma_2\Sigma_1$ to any vector in $\mathbb{R}^{N_\Gamma}$ is not too large (see Sects. 5.7 and 5.8).
However, this is not the case for the Stokes/Darcy coupling and an alternative strategy must be investigated.
We would like to exploit the characterization (5.46) of the iteration matrix associated to the ADI method and, in particular, we would like to study its spectral radius (5.48) at least for the case of our interest, that is for $\nu, \mathsf{K} \ll 1$.
We need to study the behaviour of the eigenvalues of the local Steklov-Poincaré operators. Thanks to the analysis developed in chapter 3, we can prove the following result.

Proposition 6.3.1. *1. Let us denote by δ_f^j, δ_p^j ($j = 1, \ldots, N_\Gamma$) the eigenvalues of the discrete operators S_{fh} and S_{ph}, respectively. Then, there exist positive constants $c_1, C_1 > 0$, $c_2, C_2 > 0$, independent of h, such that*

$$c_1 \nu \leq \delta_f^j \leq C_1 \frac{\nu}{h} \tag{6.15}$$

$$\frac{c_2}{\max_j \|\mathsf{K}_j\|_{\infty,p}} \leq \delta_p^j \leq \frac{C_2}{h m_K} \tag{6.16}$$

(with m_K defined in (2.44)).

2. Let us denote by $\tilde{\delta}_f^j$, $\tilde{\delta}_p^j$ ($j = 1, \ldots, N_\Gamma$) the eigenvalues of the discrete operators $\mathcal{S}_{fh}$ and $\mathcal{S}_{ph}$, respectively. Then, there exist positive constants $\tilde{c}_1, \tilde{C}_1 > 0$, $\tilde{c}_2, \tilde{C}_2 > 0$, independent of h, such that

$$\frac{\tilde{c}_1}{\nu} \leq \tilde{\delta}_f^j \leq \frac{\tilde{C}_1}{h\nu} \tag{6.17}$$

$$\tilde{c}_2 m_K \leq \tilde{\delta}_p^j \leq \tilde{C}_2 \frac{\max_j \|\mathsf{K}_j\|_{\infty,p}}{h}\,. \tag{6.18}$$

Proof. It follows from Lemmas 3.5.1 and 3.6.1 and from Proposition 2.2.1 in [QV99] p. 48. □

Using the power method (see [QSS00]) we have computed the extreme eigenvalues of the discrete Schur complement systems Σ_{fh}, Σ_{ph}, $\widetilde{\Sigma}_{fh}$ and $\widetilde{\Sigma}_{ph}$ for the values of the parameters ν and K adopted for the tests of table 4.3. As we can see from the results reported in Fig. 6.1, their trend correspond to the theoretical bounds of Proposition 6.3.1.

We consider now the case of method (6.1)-(6.2). We can write the following bound for the spectral radius of the iteration matrix, say B_h, associated to this algorithm:

$$\rho(\mathrm{B}_h) \leq \max_{j=1,\ldots,N_\Gamma} \left| \frac{\gamma_p - \delta_f^j}{\gamma_f + \delta_f^j} \right| \cdot \max_{j=1,\ldots,N_\Gamma} \left| \frac{\gamma_f - \delta_p^j}{\gamma_p + \delta_p^j} \right| . \tag{6.19}$$

Then, if we consider the limit $\nu \to 0$ and $\mathsf{K} \to 0$ (for fixed h), thanks to the estimate (6.15) we can see that

$$\text{if} \quad \nu \to 0, \quad \text{then} \quad \delta_f^j \to 0\,, \tag{6.20}$$

while from (6.16) it follows that

$$\text{if} \quad \mathsf{K} \to 0, \quad \text{then} \quad \delta_p^j \to \infty\,. \tag{6.21}$$

Therefore,

$$\max_{j=1,\ldots,N_\Gamma} \lim_{\nu \to 0} \left| \frac{\gamma_p - \delta_f^j}{\gamma_f + \delta_f^j} \right| \sim \frac{\gamma_p}{\gamma_f} \tag{6.22}$$

and, on the other hand,

$$\max_{j=1,\ldots,N_\Gamma} \lim_{\mathsf{K} \to 0} \left| \frac{\gamma_f - \delta_p^j}{\gamma_p + \delta_p^j} \right| \sim 1\,. \tag{6.23}$$

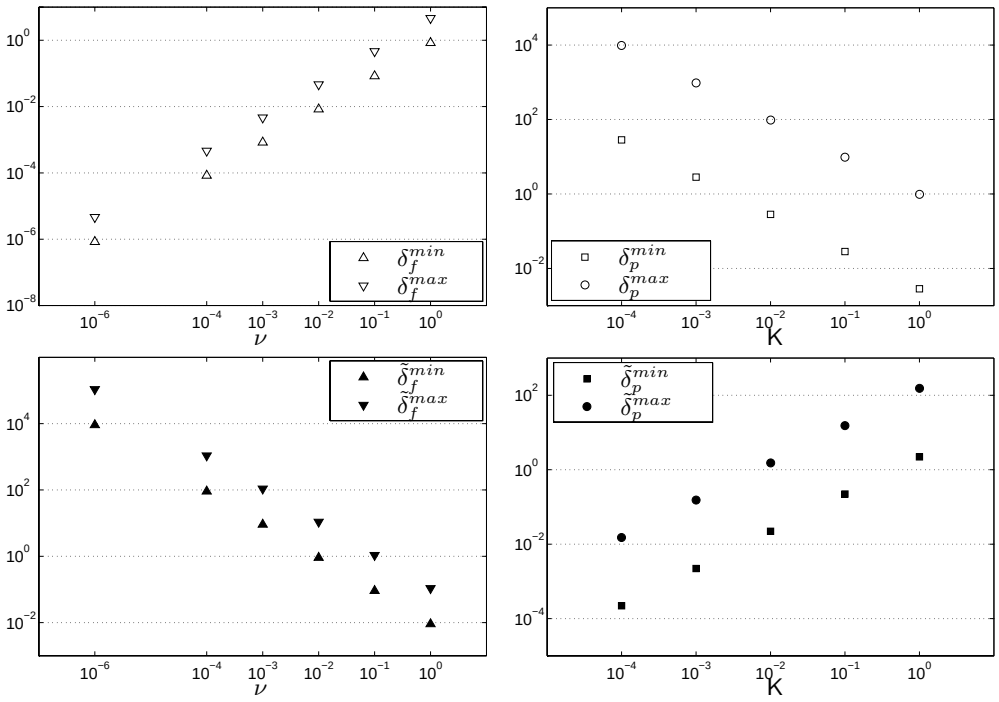


Fig. 6.1. Extreme eigenvalues for Σ_{fh} (top left), Σ_{ph} (top right), $\widetilde{\Sigma}_{fh}$ (bottom left) and $\widetilde{\Sigma}_{ph}$ (bottom right) with respect to several values of ν and K.

Then, for small values of ν and K the spectral radius of the iteration matrix B_h behaves like

$$\rho(\mathrm{B}_h) \sim \frac{\gamma_p}{\gamma_f} \,. \tag{6.24}$$

A first indication for the choice of the relaxation parameters is thus $\gamma_f > \gamma_p$. Moreover, these parameters cannot be equal to zero, or we would reduce the ADI method to one of the Dirichlet-Neumann type algorithm illustrated in chapter 4.

Finally, if we implement the differential form (6.11)-(6.12) or the algorithm (instead of (6.1)-(6.2)), to impose the Robin conditions on the interface we need to add to the stiffness matrices $\widetilde{\mathrm{F}}$ (see (3.93)) and D (see (3.56)) of Stokes and Darcy problems, respectively, the mass matrix $\widetilde{\mathrm{M}}_{\Gamma\Gamma}$ with weights γ_f and γ_p^{-1}, respectively. Due to the different scaling of these matrices, this would probably result in an increased condition number for the modified matrices $\widetilde{\mathrm{F}}$ and D. Thus, to choose γ_f and γ_p we consider the constraint of keeping the condition numbers as low as possible.

We consider again the exact solution (4.18)-(4.21) and we take the worst cases e) (i.e. $\nu =$ 1.e−04, $\mathsf{K} =$ 1.e−03) and f) (i.e. $\nu =$ 1.e−06, $\mathsf{K} =$ 1.e−04) of table 4.3, and the additional case $\nu =$ 1.e−06, $\mathsf{K} =$ 1.e−07.
We compute for these three cases the condition numbers of the modified matrices $\widetilde{\mathrm{F}}$ and D with respect to values of γ_i ranging from 1.e−03 to 10. As we can see from Fig. 6.2, for all these cases the choice $\gamma_i \simeq 0.1$ seems to be a reasonable trade-off.

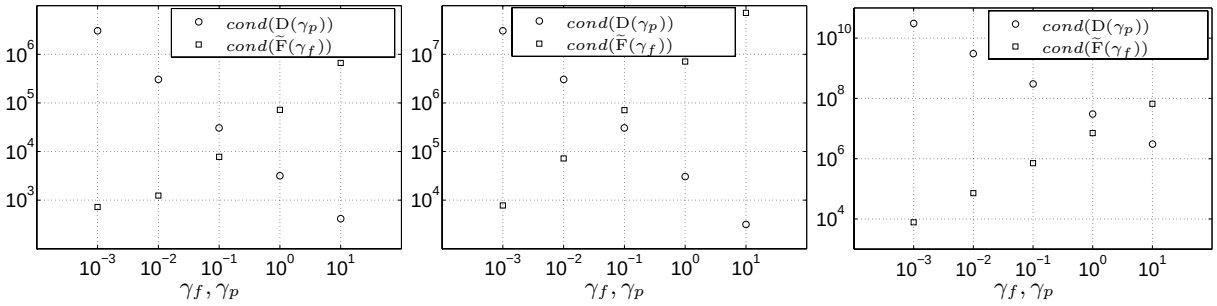


Fig. 6.2. Condition numbers of the matrices $\mathrm{D} + \gamma_p^{-1}\widetilde{\mathrm{M}}_\Gamma$ and $\widetilde{\mathrm{F}} + \gamma_f\widetilde{\mathrm{M}}_\Gamma$ as functions of γ_f and γ_p.

Thus, on the basis of these considerations we set $\gamma_f = 0.3$ and $\gamma_p = 0.1$. We fix tol = 1.e−09 and in table 6.1 we report the number of iterations obtained for the three test cases. We can see that the convergence results have sensibly improved with respect to those obtained through the Algorithm 4.3.

Parameters		Iterations for			
ν	K	$h = 0.1428$	$h = 0.0714$	$h = 0.0357$	$h = 0.0178$
1.e−04	1.e−03	19	19	19	19
1.e−06	1.e−04	20	20	20	20
1.e−06	1.e−07	20	20	20	20

Table 6.1. Number of iterations using Algorithm 6.3 with $\gamma_f = 0.3$, $\gamma_p = 0.1$, for several values of ν and K.

We can proceed in an analogous way considering now Algorithm 6.2 (or 6.4). In particular, in this case the iteration matrix, say $\widetilde{\mathrm{B}}_h$, is similar to

$$\widetilde{\mathrm{B}}_h \simeq (\gamma_f\widetilde{\mathrm{M}}_\Gamma - \widetilde{\Sigma}_{ph})(\gamma_p\widetilde{\mathrm{M}}_\Gamma + \widetilde{\Sigma}_{ph})^{-1}(\gamma_p\widetilde{\mathrm{M}}_\Gamma - \widetilde{\Sigma}_{fh})(\gamma_f\widetilde{\mathrm{M}}_\Gamma + \widetilde{\Sigma}_{fh})^{-1}$$

so that we can write the following bound for its spectral radius:

$$\rho(\widetilde{\mathrm{B}}_h) \leq \max_{j=1,\ldots,N_\Gamma}\left|\frac{\gamma_f - \tilde{\delta}_p^j}{\gamma_p + \tilde{\delta}_p^j}\right| \cdot \max_{j=1,\ldots,N_\Gamma}\left|\frac{\gamma_p - \tilde{\delta}_f^j}{\gamma_p + \tilde{\delta}_f^j}\right| . \tag{6.25}$$

Now, for h fixed, we can study the limit $\nu, \mathsf{K} \to 0$ using the eigenvalues estimates (6.17) and (6.18). We find

$$\text{if} \quad \nu \to 0, \quad \text{then} \quad \tilde{\delta}_f^j \to \infty\,, \tag{6.26}$$

and

$$\text{if} \quad \mathsf{K} \to 0, \quad \text{then} \quad \tilde{\delta}_p^j \to 0\,. \tag{6.27}$$

Therefore,

$$\max_{j=1,\ldots,N_\Gamma}\lim_{\nu\to 0}\left|\frac{\gamma_p - \tilde{\delta}_f^j}{\gamma_f + \tilde{\delta}_f^j}\right| \sim 1 \tag{6.28}$$

while

$$\max_{j=1,\dots,N_\Gamma} \lim_{\mathsf{K}\to 0} \left| \frac{\gamma_f - \tilde{\delta}_p^j}{\gamma_p + \tilde{\delta}_p^j} \right| \sim \frac{\gamma_f}{\gamma_p}\,, \tag{6.29}$$

so that for ν, K small,

$$\rho(\widetilde{\mathrm{B}}_h) \sim \frac{\gamma_f}{\gamma_p}\,, \tag{6.30}$$

as we might have expected by comparing the different role played by the parameters in (6.13)-(6.14) and in (6.11)-(6.12) (see Remark 6.2.1). Thus, in this case we have to set $\gamma_p > \gamma_f$.

Moreover, we apply the same considerations as above concerning the conditioning of the mass-added matrices $\widetilde{\mathrm{F}} + \gamma_f^{-1}\widetilde{\mathrm{M}}_\Gamma$ and $\mathrm{D} + \gamma_p\widetilde{\mathrm{M}}_\Gamma$ taking the test cases of table 6.1; in this case we obtain the rough indication $\gamma_i \simeq 10$.
Then, we set $\gamma_f = 10$ and $\gamma_p = 30$ and we report in table 6.2 the number of iterations obtained for the three test cases. The convergence results are comparable to those of table 6.1.

Parameters		Iterations for			
ν	K	$h = 0.1428$	$h = 0.0714$	$h = 0.0357$	$h = 0.0178$
1.e−04	1.e−03	18	18	18	19
1.e−06	1.e−04	20	20	20	20
1.e−06	1.e−07	20	20	20	20

Table 6.2. Iterations using Algorithm 6.4 with $\gamma_f = 10$, $\gamma_p = 30$, with respect to several values of ν and K.

Finally, we have considered the longitudinal section of a water channel $10\,m$ long with a water depth of $1\,m$. At the inlet of the channel (see Fig. 6.3) a parabolic inflow profile with maximal velocity $0.1\,m/s$ is imposed, while on the other boundaries we impose $\mathbf{u}_f = \mathbf{0}$. The fluid is thus forced to filtrate through an homogeneous porous medium $10\,m$ deep characterized by an hydraulic conductivity $\mathsf{K} = 1.\mathrm{e}{-03}\,m/s$. The fluid has a density $\nu = 1.\mathrm{e}{-06}\,m^2/s$. On the bottom of the porous media domain we impose $\varphi = 0$ while on the lateral boundaries the impermeability condition $\mathsf{K}\nabla\varphi \cdot \mathbf{n}_p = 0$ is assumed.
To compute the solution of the global problem we have considered the Algorithm 6.1 setting $\gamma_f = 0.3$ and $\gamma_p = 0.1$. The tolerance on the relative increment has been set $tol = 1.\mathrm{e}{-05}$. We have used three different computational meshes. The convergence results are reported in table 6.3, while Figs. 6.4, 6.5 represents the computed velocity field and piezometric head.

The numerical results we have presented show that the ADI method sensibly improves the convergence behaviour of the more classical Dirichlet-Neumann methods, specifically in presence of physically interesting parameters.

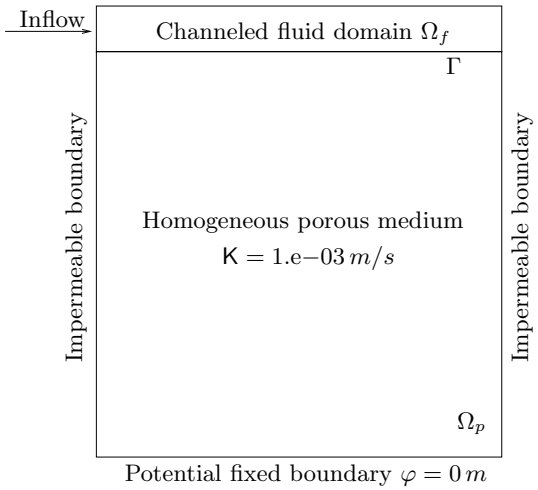


Fig. 6.3. Computational domain for the channeled fluid-porous media test case

Mesh elements	Iterations
1272	6
5088	6
20352	6

Table 6.3. Number of iterations obtained for three different computational meshes.

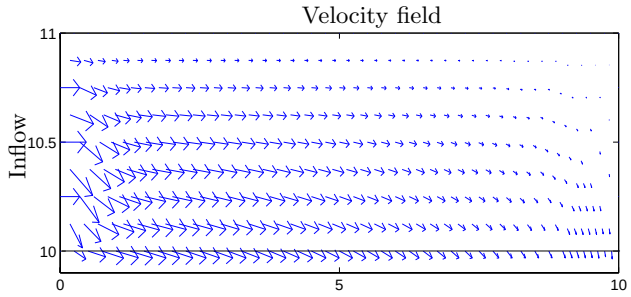


Fig. 6.4. Computed velocity field.

However, this method may still be improved introducing for example a dynamic strategy to choose the acceleration parameters. Morevoer, it would be interesting to apply the preconditioners issued by the ADI approach in the framework of the GMRES method.

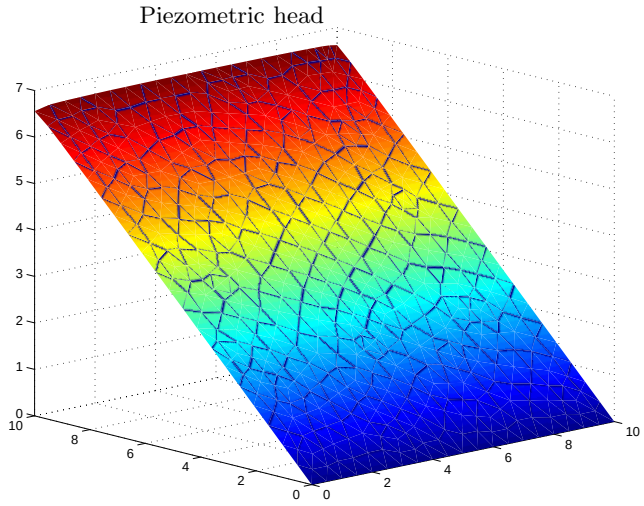


Fig. 6.5. Computed piezometric head.

7. Mathematical Analysis of a Nonlinear Coupled Problem

In this chapter we present some results concerning the analysis of the coupled Navier-Stokes/Darcy problem and the setting up of substructuring methods for its numerical approximation. In particular, we present a well-posedness result and set up a framework for the investigation of possible iterative solution strategies for this problem.

7.1 Introduction

In the previous chapters we have focused on the mathematical and numerical investigation of the coupled Stokes/Darcy model. Now, we would like to improve this basic model to account for more general fluid flows and, possibly, more accurate porous media models. For example, we could consider a nonlinear Navier-Stokes/Forchheimer model which would allow us to treat the case of high Reynolds number flows in both the fluid and the porous media domain.
However, because of the wide practical applicability of Darcy's law, we begin by considering the Navier-Stokes/Darcy model (1.19)-(1.25).
The setting of the problem will be as in chapter 2; in particular, the coupling conditions will be again (1.26), (1.27) and throughout the whole chapter we shall assume that $\mathbf{u}_f \cdot \boldsymbol{\tau}_j = 0$ on Γ instead of (1.28), as we have already done in chapters 2 and 3.

This chapter is composed of two parts.
In the first one we express the coupled model as a nonlinear interface problem and we guarantee its well-posedness.
The second part is devoted to the set up of a general framework to characterize possible iterative methods to solve the interface equation.

7.2 The Interface Problem Associated to the Navier-Stokes/Darcy Coupling

We consider the functional spaces (2.7)-(2.15) and the extension operators E_f, E_p and E_Γ defined in (2.16), (2.19) and (2.65), respectively. Moreover, we define the trilinear form:

$$c_f(\mathbf{w};\mathbf{z},\mathbf{v}) = \int_{\Omega_f} [(\mathbf{w}\cdot\nabla)\mathbf{z}]\cdot\mathbf{v} = \sum_{i,j=1}^{d} \int_{\Omega_f} w_j \frac{\partial z_i}{\partial x_j} v_i \qquad \forall \mathbf{v},\mathbf{w},\mathbf{z} \in (H^1(\Omega_f))^d. \tag{7.1}$$

The coupled Navier-Stokes/Darcy problem can be formulated in the following multidomain weak form:

find $\mathbf{u}_f^0 \in H_f^\tau$, $p_f \in Q$, $\varphi_0 \in H_p$ such that

$$\begin{aligned}
&a_f(\mathbf{u}_f^0 + E_f\mathbf{u}_{in}, \mathbf{w}) + c_f(\mathbf{u}_f^0 + E_f\mathbf{u}_{in}; \mathbf{u}_f^0 + E_f\mathbf{u}_{in}, \mathbf{w}) + b_f(\mathbf{w}, p_f) \\
&\qquad = \int_{\Omega_f} \mathbf{f}\,\mathbf{w} \qquad \forall \mathbf{w} \in H_0^1(\Omega_f) && (7.2)\\
&b_f(\mathbf{u}_f^0 + E_f\mathbf{u}_{in}, q) = 0 \qquad \forall q \in Q && (7.3)\\
&a_p(\varphi_0 + E_p\varphi_p, \psi) = 0 \qquad \forall \psi \in H_p^0 && (7.4)\\
&\int_\Gamma n(\mathbf{u}_f^0\cdot\mathbf{n})\mu = a_p(\varphi_0 + E_p\varphi_p, R_2\mu) \qquad \forall \mu \in \Lambda && (7.5)\\
&\int_\Gamma g\,\varphi_0\mu = \int_{\Omega_f} \mathbf{f}\ (R_1^\tau\mu) - a_f(\mathbf{u}_f^0 + E_f\mathbf{u}_{in}, R_1^\tau\mu) \\
&\qquad -c_f(\mathbf{u}_f^0 + E_f\mathbf{u}_{in}; \mathbf{u}_f^0 + E_f\mathbf{u}_{in}, R_1^\tau\mu) - b_f(R_1^\tau\mu, p_f) \qquad \forall \mu \in \Lambda\,, && (7.6)
\end{aligned}$$

where R_1^τ and R_2 are the continuous extension operators defined in Remark 2.5.1 and Proposition 2.4.1, respectively.

Now, we consider the interface variable $\lambda = \mathbf{u}_f\cdot\mathbf{n}$ on Γ and we split it as $\lambda = \lambda_0 + \lambda_*$ as in Sect. 2.5.

(In this section we bound ourselves to this choice of the interface variable. The case of $\sigma = \varphi_{|\Gamma}$ as interface variable will be treated in a future work [BDQ04]).

Let $(\boldsymbol{\omega}_0^*, \pi^*) \in (H_0^1(\Omega_f))^d \times Q_0$ and $\varphi_0^* \in H_p$ be the solutions to problems **P1)** and **P2)** of Sect. 2.5, and consider the *linear* extension operators R_f and R_p defined in (2.72)-(2.73) and (2.74), respectively.

Finally, let us introduce the following *nonlinear* extension operator:

$$R_{f,nl}\colon \Lambda_0 \to H_f^\tau \times Q_0, \quad \eta \to R_{f,nl}\eta := (R_{f,nl}^1\eta, R_{f,nl}^2\eta)$$

such that $(R_{f,nl}^1\eta)\cdot\mathbf{n} = \eta$ on Γ and

$$\begin{aligned}
a_f(R_{f,nl}^1\eta, \mathbf{v}) + c_f(\mathbf{u}_* + R_{f,nl}^1\eta; \mathbf{u}_* + R_{f,nl}^1\eta, \mathbf{v}) + b_f(\mathbf{v}, R_{f,nl}^2\eta) = 0 && (7.7)\\
b_f(R_{f,nl}^1\eta, q) = 0 && (7.8)
\end{aligned}$$

$\forall \mathbf{v} \in (H_0^1(\Omega_f))^d$, $\forall q \in Q_0$, where we have denoted

$$\mathbf{u}_* \in H^1(\Omega_f), \quad \mathbf{u}_* = \boldsymbol{\omega}_0^* + E_f\mathbf{u}_{in} + E_\Gamma\lambda_* \tag{7.9}$$

$\boldsymbol{\omega}_0^*$ being the solution to **P1)**.

Remark that problem (7.7)-(7.8) corresponds to a Navier-Stokes problem in Ω_f where we impose the usual boundary conditions (1.22), (1.23) and (2.58), and we require the normal velocity on Γ to be equal to $\eta + \lambda_*$. Notice that, unlike the linear problem (2.72)-(2.73), here we cannot split the dependence on the data (i.e. boundary conditions and forcing terms) from that on the interface data η and λ_* because of the nonlinearity of the problem.

In order to prove the existence and uniqueness of the operator $R_{f,nl}$ we need some preliminary results.

7.2.1 General Existence and Uniqueness Results

In this section we recall some existence and uniqueness results that we shall use in the following. For a rigorous study of the solution of nonlinear equations in Banach spaces we refer, e.g., to [BRR80, BRR81a, BRR81b, CR97] and also [GR86].

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two real normed Hilbert spaces and let us consider a bilinear continuous form:

$$b(\cdot,\cdot) : X \times Y \to \mathbb{R}, \quad (v,q) \to b(v,q)$$

and a trilinear form

$$a(\cdot;\cdot,\cdot) : X \times X \times X \to \mathbb{R}, \quad (w,u,v) \to a(w;u,v)$$

where, for $w \in X$ the mapping $(u,v) \to a(w;u,v)$ is a bilinear continuous form on $X \times X$. Then we consider the following problem:

Given $l \in X'$, find a pair $(u,p) \in X \times Y$ satisfying

$$\begin{array}{rcll} a(u;u,v) + b(v,p) & = & \langle l, v\rangle & \forall v \in X \\ b(u,q) & = & 0 & \forall q \in Y. \end{array} \tag{7.10}$$

Let us introduce the linear operators $A(w) \in \mathcal{L}(X;X')$ for $w \in X$, and $B \in \mathcal{L}(X;Y')$ defined by:

$$\langle A(w)u, v\rangle = a(w;u,v) \quad \forall u, v \in X,$$

$$\langle Bv, q\rangle = b(v,q) \quad \forall v \in X,\ \forall q \in Y\,.$$

With these notations problem (7.10) becomes:

find $(u,p) \in X \times Y$ such that

$$\begin{array}{rcll} A(u)u + B^T p & = & l & \text{in } X' \\ Bu & = & 0 & \text{in } Y'. \end{array} \tag{7.11}$$

We set $V = \mathrm{Ker}(B)$ and we associate to problem (7.10) the following one:

find $u \in V$ such that

$$a(u;u,v) = \langle l, v\rangle \qquad \forall v \in V\,. \tag{7.12}$$

Equivalently, we can write (7.12) as

$$\Pi A(u)u = \Pi l \quad \text{in } V',$$

where the linear operator $\Pi \in \mathcal{L}(X';V')$ is defined by $\langle \Pi l, v\rangle = \langle l, v\rangle$, $\forall v \in V$.
Obviously, if (u,p) is a solution of problem (7.10), then u is a solution of (7.12). The converse property may be easily established provided the inf-sup condition holds. Therefore, the real difficulty lies in solving the nonlinear problem (7.12).
The following existence result holds for (7.12).

Theorem 7.2.1 (Existence). *Assume that the following hypotheses hold:*

1. *there exists a constant $\alpha > 0$ such that*
$$a(v;v,v) \geq \alpha \|v\|_X^2 \qquad \forall v \in V; \tag{7.13}$$
2. *the space V is separable and, for all $v \in V$, the mapping $u \to a(u;u,v)$ is sequentially weakly continuous on V, i.e.*
$$u_m \rightharpoonup u \quad \text{in } V \quad \Rightarrow \quad \lim_{m\to\infty} a(u_m;u_m,v) = a(u;u,v) \qquad \forall v \in V\,.$$

Then problem (7.12) has at least one solution $u \in V$.

Concerning the uniqueness of the solution we have the following result.

Theorem 7.2.2 (Uniqueness). *Suppose that*

1. *the bilinear form $a(w;\cdot,\cdot)$ is uniformly V-elliptic with respect to w, i.e. there exists a constant $\alpha > 0$ such that*
$$a(w;v,v) \geq \alpha \|v\|_X^2 \qquad \forall v, w \in V\,;$$
2. *the mapping $w \to \Pi A(w)$ is locally Lipschitz-continuous in V, i.e. there exists a continuous and monotonically increasing function $L : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $m > 0$*
$$|a(w_1;u,v) - a(w_2;u,v)| \leq L(m)\|u\|_X \|v\|_X \|w_1 - w_2\|_X \tag{7.14}$$
$\forall u, v \in V$, $\forall w_1, w_2 \in S_m$ with $S_m := \{w \in V \,|\, \|w\|_X \leq m\}$;
3. *it holds*
$$\frac{\|\Pi l\|_{V'}}{\alpha^2} L\left(\frac{\|\Pi l\|_{V'}}{\alpha}\right) < 1\,. \tag{7.15}$$

Then (7.12) has a unique solution $u \in V$.

We end this section by addressing problem (7.10):

Theorem 7.2.3. *Assume that the bilinear form $b(\cdot,\cdot)$ satisfies the inf-sup condition*

$$\inf_{q\in Y} \sup_{v\in X} \frac{b(v,q)}{\|v\|_X \|q\|_Y} \geq \beta > 0\,. \tag{7.16}$$

Then for each solution u of (7.12) there exists a unique $p \in Y$ such that the pair (u,p) is a solution of problem (7.10).

7.2.2 Existence of the Extension Operator $R_{f,nl}$

We face now the issue of the existence and uniqueness of the extension operator $R_{f,nl}$. With this purpose, we define the auxiliary (homogeneous) nonlinear operator

$$\begin{aligned} &R : \Lambda_0 \to (H_0^1(\Omega_f))^d \times Q_0, \qquad R\eta = (R^1\eta, R^2\eta), \\ &\text{with } R^i\eta = R^i_{f,nl}\eta - R^i_f\eta, \quad i = 1, 2, \end{aligned} \tag{7.17}$$

such that $R^1\eta \cdot \mathbf{n} = 0$ on Γ and which satisfies the following problem:

$$a_f(R^1\eta, \mathbf{v}) + c_f(\mathbf{u}_* + R^1_f\eta + R^1\eta; \mathbf{u}_* + R^1_f\eta + R^1\eta, \mathbf{v}) + b_f(\mathbf{v}, R^2\eta) = 0 \tag{7.18}$$

$$b_f(R^1\eta, q) = 0 \tag{7.19}$$

for all $\mathbf{v} \in (H_0^1(\Omega_f))^d$, $q \in Q_0$.
Remark that problem (7.18)-(7.19) is analogous to (7.7)-(7.8), but here $R^1\eta \in (H_0^1(\Omega_f))^d$ while we had $R^1_{f,nl}\eta \in H^\tau_f$.

We consider the functional space

$$V_f^0 := \{\mathbf{v} \in (H_0^1(\Omega_f))^d | \nabla \cdot \mathbf{v} = 0 \quad \text{in } \Omega_f\} \tag{7.20}$$

and, given $\eta \in \Lambda_0$, we define the form:

$$\begin{aligned} \widetilde{a}(\mathbf{w}; \mathbf{z}, \mathbf{v}) := a_f(\mathbf{z}, \mathbf{v}) + c_f(\mathbf{w}; \mathbf{z}, \mathbf{v}) \\ + c_f(\mathbf{u}_* + R^1_f\eta; \mathbf{z}, \mathbf{v}) + c_f(\mathbf{z}; \mathbf{u}_* + R^1_f\eta, \mathbf{v}) \qquad \forall \mathbf{w}, \mathbf{z}, \mathbf{v} \in (H^1(\Omega_f))^d, \end{aligned} \tag{7.21}$$

and the functional

$$\langle \ell, \mathbf{v} \rangle := -c_f(\mathbf{u}_* + R^1_f\eta; \mathbf{u}_* + R^1_f\eta, \mathbf{v}) \qquad \forall \mathbf{v} \in (H^1(\Omega_f))^d\,. \tag{7.22}$$

Therefore we can rewrite problem (7.18), (7.19) as:

given $\eta \in \Lambda_0$, find $R^1\eta \in V_f^0$ such that

$$\widetilde{a}(R^1\eta; R^1\eta, \mathbf{v}) = \langle \ell, \mathbf{v} \rangle \qquad \forall \mathbf{v} \in V_f^0\,. \tag{7.23}$$

We state the following result.

Proposition 7.2.1. *There exists a positive constant $C > 0$, independent of η, such that if*

$$\eta \in \{\zeta \in \Lambda_0 |\, |R^1_f\zeta|_{1,f} < \nu/C - |\mathbf{u}_*|_{1,f}\} \subset \Lambda_0, \tag{7.24}$$

then there exists a unique nonlinear extension $R_{f,nl}\eta = (R^1_{f,nl}\eta, R^2_{f,nl}\eta) \in H^\tau_f \times Q_0$.

Remark 7.2.1. Notice that (7.24) imposes a constraint on η. In particular, recalling that the norms $|R_f^1\eta|_{1,f}$ and $\|\eta\|_\Lambda$ are equivalent (see the proof of Lemma 2.5.1), this condition implies that a unique extension $R_{f,nl}\eta$ exists provided the norm of η is small enough.
In our specific case, this means that we would be able to consider an extension $R_{f,nl}\lambda_0$ only if the normal velocity λ_0 across the interface Γ is sufficiently small. □

Proof. The proof is composed of several steps.
1. Let $\mathbf{w} \in V_f^0$. Then, we have

$$\widetilde{a}(\mathbf{w};\mathbf{v},\mathbf{v}) = a_f(\mathbf{v},\mathbf{v}) + c_f(\mathbf{w};\mathbf{v},\mathbf{v}) + c_f(\mathbf{u}_* + R_f^1\eta;\mathbf{v},\mathbf{v}) + c_f(\mathbf{v};\mathbf{u}_* + R_f^1\eta,\mathbf{v}). \tag{7.25}$$

Integrating by parts and recalling that $\mathbf{w} \in V_f^0$, then

$$c_f(\mathbf{w};\mathbf{v},\mathbf{v}) = \frac{1}{2}\int_{\partial\Omega_f} \mathbf{w}\cdot\mathbf{n}|\mathbf{v}|_{1,f}^2 - \frac{1}{2}\int_{\Omega_f} \nabla\cdot\mathbf{w}|\mathbf{v}|_{1,f}^2 = 0\,.$$

Moreover, denoting by n_j the components of the unit outward normal vector $\mathbf{n}_f$ to $\partial\Omega_f$, we have

$$\begin{aligned}
c_f(\mathbf{v};\mathbf{u}_* + R_f^1\eta,\mathbf{v}) &= \int_{\Omega_f} \sum_{i,j=1}^{d} v_j \frac{\partial(\mathbf{u}_* + R_f^1\eta)_i}{\partial x_j}\, v_i \\
&= -\sum_{i,j=1}^{d}\int_{\Omega_f} \frac{\partial}{\partial x_j}(v_i v_j)(\mathbf{u}_* + R_f^1\eta)_i + \sum_{i,j=1}^{d}\int_{\partial\Omega_f} (v_i v_j)(\mathbf{u}_* + R_f^1\eta)_i \mathbf{n}_j \\
&= -\sum_{i,j=1}^{d}\int_{\Omega_f} \frac{\partial v_i}{\partial x_j} v_j(\mathbf{u}_* + R_f^1\eta)_i - \sum_{i,j=1}^{d}\int_{\Omega_f} \frac{\partial v_j}{\partial x_j} v_i(\mathbf{u}_* + R_f^1\eta)_i \\
&\quad + \sum_{i,j=1}^{d}\int_{\partial\Omega_f} v_j\, n_j(\mathbf{u}_* + R_f^1\eta)_i v_i \\
&= -c_f(\mathbf{v};\mathbf{v},\mathbf{u}_* + R_f^1\eta).
\end{aligned}$$

The last equality follows because $\mathbf{v} \in V_f^0$.
Finally, by construction $\nabla\cdot(\mathbf{u}_* + R_f^1\eta) = 0$ and therefore $c_f(\mathbf{u}_* + R_f^1\eta;\mathbf{v},\mathbf{v}) = 0$. Then (7.25) becomes:

$$\widetilde{a}(\mathbf{w};\mathbf{v},\mathbf{v}) = a_f(\mathbf{v},\mathbf{v}) - c_f(\mathbf{v};\mathbf{v},\mathbf{u}_* + R_f^1\eta)\,, \tag{7.26}$$

and using the Poincaré inequality we obtain:

$$\begin{aligned}
\widetilde{a}(\mathbf{w};\mathbf{v},\mathbf{v}) &\geq \nu|\mathbf{v}|_{1,f}^2 - C_{\Omega_f}|\mathbf{v}|_{1,f}^2|\mathbf{u}_* + R_f^1\eta|_{1,f} \\
&\geq |\mathbf{v}|_{1,f}^2\left(\nu - 2C_{\Omega_f}(|\mathbf{u}_*|_{1,f} + |R_f^1\eta|_{1,f})\right).
\end{aligned}$$

Setting $C = 2C_{\Omega_f}$ in (7.24), it follows that the bilinear form $\widetilde{a}(\mathbf{w};\cdot,\cdot)$ is uniformly elliptic with respect to $\mathbf{w}$, with the constant $\alpha_{\widetilde{a}}$ (independent of $\mathbf{w}$):

$$\alpha_{\widetilde{a}} = \nu - 2C_{\Omega_f}(|\mathbf{u}_*|_{1,f} + |R_f^1\eta|_{1,f}).$$

2. Still using the Poincaré inequality we easily obtain:

$$|\widetilde{a}(\mathbf{w}_1;\mathbf{z},\mathbf{v}) - \widetilde{a}(\mathbf{w}_2;\mathbf{z},\mathbf{v})| = |c_f(\mathbf{w}_1 - \mathbf{w}_2;\mathbf{z},\mathbf{v})| \leq C_{\Omega_f}|\mathbf{w}_1 - \mathbf{w}_2|_{1,f}|\mathbf{v}|_{1,f}|\mathbf{z}|_{1,f}.$$

3. We have

$$\begin{aligned}
|||\Pi\ell|||_{(V_f^0)'} &= \sup_{\mathbf{v}\in V_f^0,\mathbf{v}\neq 0} \frac{|-c_f(\mathbf{u}_* + R_f^1\eta;\mathbf{u}_* + R_f^1\eta,\mathbf{v})|}{|\mathbf{v}|_{1,f}} \\
&\leq \sup_{\mathbf{v}\in V_f^0,\mathbf{v}\neq 0} \frac{C_{\Omega_f}|\mathbf{u}_* + R_f^1\eta|_{1,f}^2|\mathbf{v}|_{1,f}}{|\mathbf{v}|_{1,f}} \\
&\leq C_{\Omega_f}(|\mathbf{u}_*|_{1,f} + |R_f^1\eta|_{1,f})^2,
\end{aligned}$$

so that

$$C_{\Omega_f}\frac{|||\Pi\ell|||_{(V_f^0)'}}{\alpha_{\widetilde{a}}^2} < 1$$

owing to (7.24).

4. Thanks to (7.24) and 1.-3., $\widetilde{a}(\cdot;\cdot,\cdot)$ and ℓ satisfy the hypotheses of Theorem 7.2.2, which allows us to conclude that there exists a unique solution $R^1\eta \in V_f^0$ to (7.23).

5. Since the inf-sup condition is satisfied, from Theorem 7.2.3 there exists a unique solution $(R^1\eta, R^2\eta)$ to (7.18), (7.19). Then, the thesis follows from the definition (7.17). □

7.2.3 The Interface Equation: an Existence and Uniqueness Result

In this section we want to reformulate the global coupled problem (7.2)-(7.6) as an interface equation depending solely on λ_0.

We formally define the nonlinear pseudo-differential operator:

$$S_{nl} : \Lambda_0 \to \Lambda_0',$$

$$\begin{aligned}
\langle S_{nl}\eta,\mu\rangle &= a_f(R_{f,nl}^1\eta + \mathbf{u}_*, R_1\mu) + c_f(R_{f,nl}^1\eta + \mathbf{u}_*; R_{f,nl}^1\eta + \mathbf{u}_*, R_1\mu) \\
&\quad + b_f(R_1\mu, R_{f,nl}^2\eta + \pi^*) - \int_{\Omega_f}\mathbf{f}\,(R_1\mu) \\
&\quad + \int_\Gamma g(R_p\eta + \varphi_0^*)\mu \qquad \forall\eta\in\Lambda_0, \forall\mu\in\Lambda\,.
\end{aligned} \tag{7.27}$$

Then we have the following equivalence result.

Theorem 7.2.4. *The solution to (7.2)-(7.6) can be characterized as follows:*

$$\mathbf{u}_f^0 + E_f\mathbf{u}_{in} = R_{f,nl}^1\lambda_0 + \mathbf{u}_*, \quad p_f = R_{f,nl}^2\lambda_0 + \pi^* + \hat{p}_f, \quad \varphi_0 = R_p\lambda_0 + \varphi_0^*, \tag{7.28}$$

where $\hat{p}_f = (meas(\Omega_f))^{-1} \int_{\Omega_f} p_f$, *and* $\lambda_0 \in \Lambda_0$ *is the solution of the nonlinear interface problem:*

$$\langle S_{nl}\lambda_0, \mu_0\rangle = 0 \qquad \forall \mu_0 \in \Lambda_0\,. \tag{7.29}$$

Once λ_0 *is known,* $\hat{p}_f$ *can be obtained solving the algebraic equation*

$$\hat{p}_f = (meas(\Gamma))^{-1}\langle S_{nl}\lambda_0, \varepsilon\rangle,$$

where $\varepsilon \in \Lambda$ *is any function such that* $\int_\Gamma \varepsilon = meas(\Gamma)$.

Proof. It can be obtained by following the same guidelines of Theorem 2.5.1. □

Remark 7.2.2. With the special choice $R_1 = R_f^1$ in (7.27), thanks to the definition (2.73) of R_f, we obtain

$$b_f(R_f^1\mu, R_{f,nl}^2\eta + \pi^*) = 0 \qquad \forall \eta, \mu \in \Lambda_0\,.$$

Moreover, owing to (7.17), we have

$$\begin{aligned}\langle S_{nl}\eta, \mu\rangle \;=\; & a_f(R^1\eta + R_f^1\eta + \mathbf{u}_*, R_f^1\mu)\\ & + c_f(R^1\eta + R_f^1\eta + \mathbf{u}_*; R^1\eta + R_f^1\eta + \mathbf{u}_*, R_f^1\mu)\\ & - \int_{\Omega_f} \mathbf{f}\,(R_f^1\mu) + \int_\Gamma g(R_p\eta + \varphi_0^*)\mu.\end{aligned}$$

By taking $R^1\eta$ $(\in (H_0^1(\Omega_f))^d)$ as test function in the definition (2.72) of R_f we obtain:

$$a_f(R_f^1\mu, R^1\eta) + b_f(R^1\eta, R_f^2\mu) = 0\,.$$

Finally, since $R_f^2\mu \in Q_0$, owing to (7.19) it follows that $a_f(R_f^1\mu, R^1\eta) = 0$, so that, for all $\eta, \mu \in \Lambda_0$, the operator S_{nl} can be characterized as

$$\begin{aligned}\langle S_{nl}\eta, \mu\rangle \;=\; & a_f(R_f^1\eta + \mathbf{u}_*, R_f^1\mu)\\ & + c_f(R^1\eta + R_f^1\eta + \mathbf{u}_*; R^1\eta + R_f^1\eta + \mathbf{u}_*, R_f^1\mu)\\ & + \int_\Gamma g(R_p\eta + \varphi_0^*)\mu - \int_{\Omega_f} \mathbf{f}\,(R_f^1\mu).\end{aligned} \tag{7.30}$$

□

Now, we want to study the existence and uniqueness of the solution of the nonlinear interface problem (7.29) where S_{nl} is characterized as in (7.30).

We consider the nonlinear interface equation

$$\text{find } \lambda_0 \in \Lambda_0: \quad \langle S_{nl}\lambda_0, \mu\rangle = 0 \qquad \forall \mu \in \Lambda_0\,. \tag{7.31}$$

Note that in view of (7.30), $S_{nl}\lambda_0$ is defined in terms of the operator $R^1\lambda_0$ which in its turn satisfies the following problem:

$$a_f(R^1\lambda_0, R^1\mu) + c_f(R^1\lambda_0 + R_f^1\lambda_0 + \mathbf{u}_*; R^1\lambda_0 + R_f^1\lambda_0 + \mathbf{u}_*, R^1\mu) = 0 \quad \forall \mu \in \Lambda_0. \tag{7.32}$$

We want to apply Theorem 7.2.2 to study the existence and uniqueness of the solution λ_0. To this aim, we indicate by V the product space $V = \Lambda_0 \times V_f^0$ endowed with the following norm:

$$\|\underline{v}\|_V := (|R_f^1\mu|_{1,f}^2 + |\mathbf{v}|_{1,f}^2)^{1/2} \qquad \forall \underline{v} = (\mu, \mathbf{v}) \in V. \tag{7.33}$$

Then, for any fixed $\underline{w} = (\eta, \mathbf{w}) \in V$, we define the following operator depending on $\underline{w}$:

$$\begin{aligned} &A(\eta, \mathbf{w}) : V \to \Lambda_0' \times (V_f^0)', \\ &A(\eta, \mathbf{w}) : (\lambda, \mathbf{u}) \to (A_0(\eta, \mathbf{w})(\lambda, \mathbf{u}), A_f(\eta, \mathbf{w})(\lambda, \mathbf{u})) \end{aligned}$$

where for every test function $\mu \in \Lambda_0$:

$$\begin{aligned} \langle A_0(\eta, \mathbf{w})(\lambda, \mathbf{u}), \mu\rangle \;=\; & a_f(R_f^1\lambda, R_f^1\mu) + c_f(\mathbf{w} + R_f^1\eta; \mathbf{u} + R_f^1\lambda, R_f^1\mu) \\ & + c_f(\mathbf{u} + R_f^1\lambda; \mathbf{u}_*, R_f^1\mu) \\ & + c_f(\mathbf{u}_*; \mathbf{u} + R_f^1\lambda, R_f^1\mu) + \int_\Gamma g(R_p\lambda)\mu \,, \end{aligned}$$

whereas for any test function $\mathbf{v} \in V_f^0$:

$$\begin{aligned} \langle A_f(\eta, \mathbf{w})(\lambda, \mathbf{u}), \mathbf{v}\rangle \;=\; & a_f(\mathbf{u}, \mathbf{v}) + c_f(\mathbf{w} + R_f^1\eta; \mathbf{u} + R_f^1\lambda, \mathbf{v}) \\ & + c_f(\mathbf{u}_*; \mathbf{u} + R_f^1\lambda, \mathbf{v}) + c_f(\mathbf{u} + R_f^1\lambda; \mathbf{u}_*, \mathbf{v}) \,. \end{aligned}$$

We indicate by $\tilde{a}$ the form associated to the operator A that is

$$\tilde{a}(\underline{w}; \underline{u}, \underline{v}) = \langle A_0(\eta, \mathbf{w})(\lambda, \mathbf{u}), \mu\rangle + \langle A_f(\eta, \mathbf{w})(\lambda, \mathbf{u}), \mathbf{v}\rangle$$

for all $\underline{w} = (\eta, \mathbf{w}), \underline{u} = (\lambda, \mathbf{u}), \underline{v} = (\mu, \mathbf{v}) \in V$.
Moreover, we define the functionals $\ell_0 : \Lambda_0 \to \mathbb{R}$:

$$\langle \ell_0, \mu\rangle = \int_{\Omega_f} \mathbf{f}\,(R_f^1\mu) - a_f(\mathbf{u}_*, R_f^1\mu) - c_f(\mathbf{u}_*; \mathbf{u}_*, R_f^1\mu) - \int_\Gamma g\varphi_0^*\mu \qquad \forall \mu \in \Lambda_0, \tag{7.34}$$

and $\ell_f : V_f^0 \to \mathbb{R}$:

$$\langle \ell_f, \mathbf{v}\rangle = -c_f(\mathbf{u}_*; \mathbf{u}_*, \mathbf{v}) \qquad \forall \mathbf{v} \in V_f^0 \,, \tag{7.35}$$

and denote

$$\langle \widetilde{L}, \underline{v}\rangle = \langle \ell_0, \mu\rangle + \langle \ell_f, \mathbf{v}\rangle \qquad \forall \underline{v} = (\mu, \mathbf{v}) \in V \,.$$

Our problem (7.31)can be reformulated as:

find $\underline{u} = (\lambda_0, R^1\lambda_0) \in V$ such that

$$\tilde{a}(\underline{u}; \underline{u}, \underline{v}) = \langle \widetilde{L}, \underline{v}\rangle \qquad \forall \underline{v} = (\mu, \mathbf{v}) \in V \,. \tag{7.36}$$

We shall prove the existence and uniqueness of the solution only in a closed convex subset of V.

Let C_{Ω_f} be the Poincaré constant relative to domain Ω_f, $\mathbf{u}_* \in H^1(\Omega_f)$ the function defined in (7.9) and $\varphi_0^* \in H_p$ the solution to problem **P2)** (see Sect. 2.5). We introduce the constants

$$C_1 = C_1(\nu, \mathbf{u}_*) = \frac{\nu}{2C_{\Omega_f}} - 2|\mathbf{u}_*|_{1,f}\,, \tag{7.37}$$

$$C_2 = C_2(\mathbf{f}, \mathbf{u}_*, \varphi_0^*) = \left(\frac{C_{\Omega_f}^{1/2}\|\mathbf{f}\|_{0,f} + \nu|\mathbf{u}_*|_{1,f}^2 + C_{\Omega_f}|\mathbf{u}_*|_{1,f}^2 + C_{tr}^1 C_{\Omega_f}^{1/2} g\|\varphi_0^*\|_{0,p}}{C_{\Omega_f}} \right)^{1/2}, \tag{7.38}$$

$C_{tr}^1 > 0$ being a constant from the trace inequality.

We can prove the main result of this section.

Theorem 7.2.5. *Assume that*

$$C_1 > \frac{2+\sqrt{2}}{2} C_2\,, \tag{7.39}$$

and let $r > 0$ be such that

$$\frac{C_1 - \sqrt{C_1^2 - 2\sqrt{2}C_2^2}}{2} \le r < C_1 - C_2\,. \tag{7.40}$$

If

$$B_r = \{\underline{w} = (\eta, \mathbf{w}) \in V |\ |R_f^1 \eta|_{1,f} \le r\}\,, \tag{7.41}$$

then, there exists a unique solution $\underline{u} = (\lambda_0, R^1\lambda_0) \in B_r$ to (7.36). In particular, it follows that problem (7.31) has a unique solution λ_0 in the ball $\mathcal{B}_r \subset \Lambda_0$:

$$\mathcal{B}_r = \{\eta \in \Lambda_0 |\ |R_f^1 \eta|_{1,f} \le r\}.$$

Proof. The proof is composed of several parts.

1. *For each $\underline{w} = (\eta, \mathbf{w}) \in B_r$ the bilinear form $\tilde{a}(\underline{w}; \cdot, \cdot)$ is uniformly coercive.*
By definition, for all $\underline{v} = (\mu, \mathbf{v})$ we have

$$\begin{aligned} \tilde{a}(\underline{w}; \underline{v}, \underline{v}) &= \nu|R_f^1\mu|_{1,f}^2 + \nu|\mathbf{v}|_{1,f}^2 + \int_\Gamma g(R_p\mu)\mu \\ &\quad + c_f(\mathbf{w} + R_f^1\eta; \mathbf{v} + R_f^1\mu, \mathbf{v} + R_f^1\mu) && (7.42)\\ &\quad + c_f(\mathbf{v} + R_f^1\mu; \mathbf{u}_*, \mathbf{v} + R_f^1\mu) && (7.43)\\ &\quad + c_f(\mathbf{u}_*; \mathbf{v} + R_f^1\mu, \mathbf{v} + R_f^1\mu) && (7.44) \end{aligned}$$

Thanks to the definition (2.74) of R_p, we can see that $\int_\Gamma g(R_p\mu)\mu \ge 0$, and applying Poincaré inequality to (7.42)-(7.44) we obtain

$$\begin{aligned} \tilde{a}(\underline{w}; \underline{v}, \underline{v}) &\ge \nu(|R_f^1\mu|_{1,f}^2 + |\mathbf{v}|_{1,f}^2) - 2C_{\Omega_f}|R_f^1\eta|_{1,f}(|R_f^1\mu|_{1,f}^2 + |\mathbf{v}|_{1,f}^2) \\ &\quad -4C_{\Omega_f}|\mathbf{u}_*|_{1,f}(|R_f^1\mu|_{1,f}^2 + |\mathbf{v}|_{1,f}^2) \\ &= \alpha_{\tilde{a}}(|R_f^1\mu|_{1,f}^2 + |\mathbf{v}|_{1,f}^2) \end{aligned}$$

having set

$$\alpha_{\tilde{a}} = \nu - 2C_{\Omega_f}|R_f^1\eta|_{1,f} - 4C_{\Omega_f}|\mathbf{u}_*|_{1,f} = \nu - 2C_{\Omega_f}(|R_f^1\eta|_{1,f} + 2|\mathbf{u}_*|_{1,f})\,. \tag{7.45}$$

Condition (7.40) implies in particular that $C_1 > r$, and, since $\underline{w} \in B_r$, $C_1 > |R_f^1\eta|_{1,f}$ so that $\alpha_{\tilde{a}} > 0$. Thus, the bilinear form $\tilde{a}(\underline{w};\cdot,\cdot)$ is uniformly coercive with respect to any $\underline{w} \in B_r$.

Thanks to Lax-Milgram lemma the operator $A(\underline{w}) \in \mathcal{L}(V;V')$ is invertible for each $\underline{w} \in B_r$. Moreover, $T(\underline{w}) = (A(\underline{w}))^{-1}$ belong to $\mathcal{L}(V';V)$ and satisfies

$$\|T(\underline{w})\|_{\mathcal{L}(V';V)} \leq \frac{1}{\alpha_{\tilde{a}}}\,.$$

With this notation, we shall prove that there exists a unique $\underline{u} \in B_r$ s.t.

$$\underline{u} = T(\underline{u})\widetilde{L},$$

i.e. problem (7.36) has a unique solution in B_r.

2. *$\underline{v} \to T(\underline{v})\widetilde{L}$ maps B_r into B_r and is a strict contraction in B_r.*

For all $\underline{v} \in B_r$ we have

$$\|T(\underline{v})\widetilde{L}\|_V \leq \|T(\underline{v})\|_{\mathcal{L}(V';V)}|||\widetilde{L}|||_{V'} \leq \frac{|||\widetilde{L}|||_{V'}}{\alpha_{\tilde{a}}}\,. \tag{7.46}$$

Moreover, using Poincaré and trace inequalities we obtain:

$$\begin{aligned}
|||\widetilde{L}|||_{V'} &= \sup_{\underline{v}\in V,\underline{v}\neq\underline{0}} \frac{\left|\displaystyle\int_{\Omega_f} \mathbf{f}\,(R_f^1\mu) - a_f(\mathbf{u}_*, R_f^1\mu) - c_f(\mathbf{u}_*;\mathbf{u}_*,\mathbf{v}+R_f^1\mu) - \int_\Gamma g\varphi_0^*\mu\right|}{\|\underline{v}\|_V} \\
&\leq \sup_{\underline{v}\in V,\underline{v}\neq\underline{0}} \frac{C_{\Omega_f}^{1/2}\|\mathbf{f}\|_{0,f}|R_f^1\mu|_{1,f} + \nu|\mathbf{u}_*|_{1,f}|R_f^1\mu|_{1,f} + C_{\Omega_f}|\mathbf{u}_*|_{1,f}^2|\mathbf{v}+R_f^1\mu|_{1,f} + gC_{tr}^1C_{\Omega_f}^{1/2}\|\varphi_0^*\|_0|R_f^1\mu|_{1,f}}{\|\underline{v}\|_V} \\
&\leq \sup_{\underline{v}\in V,\underline{v}\neq\underline{0}} \frac{\left(C_{\Omega_f}^{1/2}\|\mathbf{f}\|_{0,f} + \nu|\mathbf{u}_*|_{1,f}^2 + C_{\Omega_f}|\mathbf{u}_*|_{1,f}^2 + gC_{tr}^1C_{\Omega_f}^{1/2}\|\varphi_0^*\|_{0,p}\right)(|R_f^1\mu|_{1,f} + |\mathbf{v}|_{1,f})}{\|\underline{v}\|_V}\,.
\end{aligned}$$

Finally, since $|R_f^1\mu|_{1,f} + |\mathbf{v}|_{1,f} \leq \sqrt{2}(|R_f^1\mu|_{1,f}^2 + |\mathbf{v}|_{1,f}^2)^{1/2} = \sqrt{2}\|\underline{v}\|_V$ we conclude that

$$|||\widetilde{L}|||_{V'} \leq \sqrt{2}C_{\Omega_f}C_2^2\,.$$

Now, thanks to (7.39) and (7.40) we can see that

$$\frac{|||\widetilde{L}|||_{V'}}{\alpha_{\tilde{a}}} \leq r$$

so that, owing to (7.46), $T(\underline{v})\widetilde{L}$ belongs to B_r.

Finally, to prove that the map $\underline{v} \to T(\underline{v})\widetilde{L}$ is a strict contraction in B_r, we should guarantee (see [GR86] p. 282) that $\forall \underline{w}_1, \underline{w}_2 \in B_r$

$$\|T(\underline{w}_1) - T(\underline{w}_2)\widetilde{L}\|_V \leq \frac{|||\widetilde{L}|||_{V'}}{\alpha_{\tilde{a}}^2} L(r)\|\underline{w}_1 - \underline{w}_2\|_V < \|\underline{w}_1 - \underline{w}_2\|_V, \tag{7.47}$$

$L(r)$ being the Lipschitz continuity constant associated to $\tilde{a}$. But

$$\begin{aligned}
&|\tilde{a}(\underline{w}_1; \underline{u}, \underline{v}) - \tilde{a}(\underline{w}_2; \underline{u}, \underline{v})| \\
&\quad = |c_f(\mathbf{w}_1 + R_f^1\eta_1 - (\mathbf{w}_2 + R_f^1\eta_2); \mathbf{u} + R_f^1\lambda, \mathbf{v} + R_f^1\mu)| \\
&\quad \leq C_{\Omega_f}|\mathbf{w}_1 + R_f^1\eta_1 - \mathbf{w}_2 - R_f^1\eta_2|_{1,f}|\mathbf{u} + R_f^1\lambda|_{1,f}|\mathbf{v} + R_f^1\mu|_{1,f} \\
&\quad \leq 2\sqrt{2}C_{\Omega_f}\|\underline{w}_1 - \underline{w}_2\|_V\|\underline{u}\|_V\|\underline{v}\|_V
\end{aligned}$$

so that $L(r) = 2\sqrt{2}C_{\Omega_f}$.
With the help of some algebra, thanks to (7.39) and (7.40), we can see that

$$\frac{|||\widetilde{L}|||_{V'}}{\alpha_{\tilde{a}}^2} L(r) < 1.$$

Thus, (7.47) is satisfied.

3. The existence and uniqueness of the solution $\underline{u} = (\lambda_0, R^1\lambda_0) \in B_r$ to (7.36) is now a simple consequence of the Banach contraction theorem. □

7.3 Iterative Methods for the Interface Problem

In this section we provide a general framework for devising iterative methods to solve the nonlinear interface problem. This approach has been applied also in the context of a nonlinear fluid-structure interaction problem (see [DDQ04]). An analysis of convergence of our algorithms will be the matter of a forthcoming work [BDQ04]. We shall always suppose that the hypotheses of Theorem 7.2.5 are fulfilled so that the existence of a unique solution is guaranteed.

The interface problem may be formally written as

$$\text{find } \lambda_0 \text{ s.t.} \quad S_{nl}(\lambda_0) = 0.$$

If we highlight the dependence on the fluid and the porous media problems, we can write

$$\text{find } \lambda_0 \text{ s.t.} \quad S_{f,nl}(\lambda_0) + S_p\lambda_0 = \chi_p \tag{7.48}$$

where $S_{f,nl}$ is the nonlinear fluid operator and S_p is the linear operator (2.77) associated to the groundwater problem (see also *iii)* Sect. 2.7).
Remark that computing $S_{f,nl}(\lambda_0)$ corresponds to solve a Navier-Stokes problem in Ω_f with the boundary conditions $\mathbf{u}_f \cdot \mathbf{n} = \lambda_0$ and $\mathbf{u}_f \cdot \boldsymbol{\tau}_j = 0$ on Γ, and then to compute the normal stress $(-\nu\partial\mathbf{u}_f/\partial\mathbf{n} + p_f\mathbf{n}) \cdot \mathbf{n}$ on Γ.
Finally,

$$\langle \chi_p, \mu \rangle = -\int_\Gamma g\varphi_0^*\mu \qquad \forall \mu \in \Lambda.$$

Note that the dependence on the Navier-Stokes problem data is hidden in the definition of the operators $S_{f,nl}$; this is necessary since the problem at hand is nonlinear.

We consider two approaches to solve (7.48): the first is based on the Newton method, the second one on a domain decomposition preconditioning strategy.

7.3.1 Newton Method

We denote by $J(\bar{\lambda})$ the Jacobian of $S_{f,nl}(\bar{\lambda}) + S_p\bar{\lambda} - \chi_p$ in $\bar{\lambda}$:

$$J(\bar{\lambda}) = S'_{f,nl}(\bar{\lambda}) + S_p\bar{\lambda}$$

where $S'_{f,nl}$ is the fluid tangent operator, i.e. the Fréchet derivative of $S_{f,nl}$:

$$\begin{aligned}\langle S'_{f,nl}(\delta\lambda)_{|\bar{\lambda}}, \mu\rangle = a_f(R^1_{f,nl}(\delta\lambda), R_1\mu) + c_f(R^1_{f,nl}(\delta\lambda); R^1_{f,nl}(\bar{\lambda}) + \mathbf{u}_*, R_1\mu) \\ + c_f(R^1_{f,nl}(\bar{\lambda}) + \mathbf{u}_*; R^1_{f,nl}(\delta\lambda), R_1\mu) + b_f(R_1\mu, R^2_{f,nl}(\delta\lambda)).\end{aligned}$$

The Newton algorithm reads:

Algorithm 7.1 (Newton)

Given λ_0^0, For $k \geq 0$, Do

1. compute $\sigma_p^k = S_p\lambda_0^k$;
2. compute $\sigma_f^k = S_{f,nl}\lambda_0^k$;
3. $r^k = \chi_p - (\sigma_f^k + \sigma_p^k)$;
4. solve $J(\lambda_0^k)\delta\lambda_0^k = r^k$;
5. set $\lambda_0^{k+1} = \lambda_0^k + \alpha^k\,\delta\lambda_0^k$.

End For

At each step k this algorithm requires to solve separately the fluid and the groundwater problems (steps 1 and 2, respectively) and then to solve the linear system with matrix $J(\lambda_0^k)$ (step 4). The parameter α^k is always set equal to 1.

Alternatively, we could adopt a suitable inexact Jacobian $J_{in}(\lambda_0)$ instead of $J(\lambda_0)$ and to compute the acceleration parameter α^k using a line search technique (see, e.g., [QSS00]).

7.3.2 Domain Decomposition Approach

We consider a preconditioned (nonlinear) Richardson method which, being the interface problem (7.48) nonlinear, must be interpreted in a slightly different way than what is usually done in the literature for the linear case. Precisely, the k-th step of the algorithm reads:

Algorithm 7.2 (Richardson)

Given λ_0^0, For $k \geq 0$, Do

1. compute $\sigma_p^k = S_p \lambda_0^k$;
2. compute $\sigma_f^k = S_{f,nl} \lambda_0^k$;
3. $r^k = \chi_p - (\sigma_f^k + \sigma_p^k)$;
4. solve $\mu^k = P^{-1} r^k$;
5. set $\lambda_0^{k+1} = \lambda_0^k + \omega^k \mu^k$ (with an appropriate choice of the scalar ω^k).

End For

The preconditioner P maps the interface variable onto the space of normal stresses. It is also possible to choose a preconditioner which depends on the iterate λ_0^k or more generally on the iteration step k. In these cases we will denote it by P_k.
At each step, this algorithm requires to solve independently the fluid and the porous media problems (like the Newton method) and to apply a preconditioner.

Remark 7.3.1. If no preconditioner is used, then at the differential level P should be intended as being the projection operator $\mathcal{I}$ from the space of the normal velocities Λ_0 to the space of stresses, so that

$$\lambda_0^{k+1} = \lambda_0^k + \omega^k \mathcal{I}^{-1} r^k \in \Lambda_0.$$

At the algebraic level, this remark can be omitted since in that case we are always dealing with vectors of $\mathbb{R}^{N_\Gamma}$. □

Remark 7.3.2. At the algebraic stage, a general strategy to compute the relaxation parameter ω^k is given by :

$$\omega^k = -\frac{\left(\boldsymbol{\mu}^k - \boldsymbol{\mu}^{k-1}\right) \cdot \left(\boldsymbol{\lambda}^k - \boldsymbol{\lambda}^{k-1}\right)}{\|\boldsymbol{\mu}^k - \boldsymbol{\mu}^{k-1}\|^2_{\mathbb{R}^{N_\Gamma}}}. \tag{7.49}$$

This value of ω^k is the one that minimizes the norm

$$\|(\boldsymbol{\lambda}^k - \boldsymbol{\lambda}^{k-1}) + \omega(\boldsymbol{\mu}^k - \boldsymbol{\mu}^{k-1})\|_{\mathbb{R}^{N_\Gamma}}$$

over all possible values of ω. This criterium generalizes to the vector case the Aitken extrapolation technique (see [QSS00, Dep04, DDQ04]). □

The crucial issue is how to set up a preconditioner (more precisely, a scaling operator) in order for the iterative method to converge as quickly as possible.
In the following we discuss some classical choices of the preconditioner for the Richardson method and we compare them to the Newton strategies that we have illustrated in Sect. 7.3.1.

We define a generic linear preconditioner (more precisely, its inverse):

$$P_k^{-1} = \alpha_f^k \left(S'_{f,nl}(\lambda^k)\right)^{-1} + \alpha_p^k \, S_p^\dagger \lambda^k, \tag{7.50}$$

for two scalars α_f^k and α_p^k, where $S_p^\dagger$ indicates a pseudo-inverse of S_p, since we cannot consider the inverse S_p^{-1} on Λ_0 as already discussed in chapters 2 and 4.
Instead of $S'_{f,nl}$ we could take the homogeneous operator $\bar{S}_{f,nl}^{\bar{\lambda}}$:

$$\bar{S}_{f,nl}^{\bar{\lambda}}(\delta\bar{\lambda}) = S_{f,nl}(\bar{\lambda} + \delta\bar{\lambda}) - S_{f,nl}(\bar{\lambda})\,.$$

In that case the preconditioner becomes:

$$P_k^{-1} = \alpha_f^k\,(\bar{S}_{f,nl}^{\lambda^k})^{-1} + \alpha_p^k\,S_p^\dagger\lambda^k \tag{7.51}$$

and the nonlinear operator (7.51) can be considered as an approximation of (7.50) to be used in order to avoid the solution of the linearized problem.

Remark 7.3.3. The nonlinear operator $(S'_{f,nl})^{-1}$ in (7.50) could also be replaced by its linear counterpart S_f^{-1}. However, we might expect that this would lead to the same difficulties concerning the bad behaviour of this preconditioner in presence of small physical parameters as in the full linear case. Therefore, although matematically acceptable, we do not advise considering this strategy. □

From (7.51) we retrieve the following special cases:

1. If $\alpha_f^k = 1$ and $\alpha_p^k = 0$, then
$$P_k^{-1} = P_{k,DN}^{-1} = (\bar{S}_{f,nl}^{\lambda^k})^{-1}. \tag{7.52}$$
$P_{k,DN}$ is called a *Dirichlet-Neumann preconditioner* and
$$P_{k,DN}^{-1}(r^k) = (\bar{S}_{f,nl}^{\lambda^k})^{-1}\left(\chi_p - S_{f,nl}(\lambda^k) - S_p\lambda^k\right);$$
2. If $\alpha_f^k + \alpha_p^k = 1$, then
$$P_k^{-1} = P_{k,NN}^{-1} = \alpha_f^k(\bar{S}_{f,nl}^{\lambda^k})^{-1} + \alpha_p^k S_p^\dagger$$
that we call here a *generalized Neumann-Neumann preconditioner.*

Remark 7.3.4. We neglect the case $\alpha_f^k = 0$ and $\alpha_p^k = 1$ which would give $P_k^{-1} = S_p^\dagger$, since we have already seen in the linear case (see chapter 2) that $S_p^\dagger$ is not an optimal preconditioner for the interface problem associated to the interface variable $\lambda = \mathbf{u}_f \cdot \mathbf{n}$ on Γ. □

In the Dirichlet-Neumann case the computational effort of a Richardson step may be reduced to the solution of only one Dirichlet problem in one subdomain and one Neumann problem in the other.

Remark 7.3.5. For both cases (7.50), (7.51), it is possible to choose the parameters α_f^k, α_p^k and ω^k dynamically in the following way. We define $\omega_f^k = \omega^k\alpha_f^k$ and $\omega_p^k = \omega^k\alpha_p^k$ and we look for ω_f^k and ω_p^k that minimize

$$\|(\boldsymbol{\lambda}^k - \boldsymbol{\lambda}^{k-1}) + \omega_f(\boldsymbol{\mu}_f^k - \boldsymbol{\mu}_f^{k-1}) + \omega_p(\boldsymbol{\mu}_p^k - \boldsymbol{\mu}_p^{k-1})\|_{\mathbb{R}^{N_\Gamma}},$$

over all possible values of ω_f and ω_p. This corresponds to solving the linear system

$$\mathrm{A}^T\mathrm{A}\begin{pmatrix}\omega_f^k\\ \omega_p^k\end{pmatrix} = -\mathrm{A}^T(\boldsymbol{\lambda}^k - \boldsymbol{\lambda}^{k-1}), \tag{7.53}$$

where A is the two column matrix

$$\mathrm{A} = \left((\boldsymbol{\mu}_f^k - \boldsymbol{\mu}_f^{k-1}); (\boldsymbol{\mu}_p^k - \boldsymbol{\mu}_p^{k-1})\right).$$

Again, this can be regarded as a generalized Aitken criterium; in fact, this automatic choice generalizes the one outlined in (7.49). □

A further possibility is offered by the following preconditioner

$$P_{k,RR} = \frac{1}{\gamma_f + \gamma_p}\left(\gamma_f\mathcal{I} + S'_{f,nl}(\lambda^k)\right)\mathcal{I}\left(\gamma_p\mathcal{I} + S_p\lambda^k\right), \tag{7.54}$$

where $\mathcal{I}$ is the projection operator as in Remark 7.3.1, while γ_f and γ_p are positive parameters which can be chosen according to a suitable error minimization strategy as illustrated in chapters 5 and 6. We call $P_{k,RR}$ a *Robin-Robin preconditioner*.

Thanks to the theory of chapter 5, this preconditioning strategy may be associated to the splitting scheme:

Algorithm 7.3 (Operator splitting)

Given μ_2^0, For $k \geq 0$, Do

1. solve $\gamma_f\mathcal{I}\mu_1^{k+1} + S'_{f,nl}(\mu_1^{k+1}) = \chi_p + (\gamma_f\mathcal{I}\mu_2^k - S_p\mu_2^k)$;
2. solve $\gamma_p\mathcal{I}\mu_2^{k+1} + S_p\mu_2^{k+1} = \chi_p + (\gamma_p\mathcal{I}\mu_1^{k+1} - S_{f,nl}(\mu_1^{k+1}))$.

End For

In this context one may think of replacing the tangent operator $S'_{f,nl}$ in 1 of Algorithm 7.3, (or (7.54)) by the linear operator S_f (see (2.76)), thus considering the full linear preconditioner used for the Stokes/Darcy coupling.

7.3.3 Comparison between the Newton and the DD Approaches

The Richardson algorithm 7.2 for the Steklov-Poincaré formulation (7.48) with preconditioner given by (7.50) (with $\alpha_f^k = \alpha_p^k = 1$) is not equivalent to the Newton algorithm 7.1. In fact, the latter could be regarded as a Richardson method, choosing however the nonlinear preconditioner

$$P_k(\mu) = S'_{f,nl}(\mu) + S_p\mu. \tag{7.55}$$

Note that to invert P_k one must use a (preconditioned) iterative method and may approximate the tangent problem to accelerate the computations.

Moreover, we would like to remark that the domain decomposition approach allows us to set up a completely parallel solver. In fact, the fluid and the porous media subproblems can be computed simultaneously (and independently) for both the computation of σ_f^k and σ_p^k (operators $S_{f,nl}$ and S_p) and the application of the preconditioner (operators $S'_{f,nl}$ and, eventually, $S_p^\dagger$).

The operator splitting approach (Algorithm 7.3) presents a different structure with respect to both the Newton and the Richardson ones and, in general, it is more expensive in terms of computational cost than the Richardson method with the Dirichlet-Neumann preconditioner $P_{k,DN}$ (7.52). In fact, the operator splitting method requires at each step to solve two fluid problems and two porous media problems.

A schematic representation and comparison of the three methods we have illustrated is given in table 7.1.

Interface problem $S_{f,nl}(\lambda_0) + S_p\lambda_0 = \chi_p$		
Newton iter.	Prec. Richardson iter.	Operator splitting
$\sigma_p^k = S_p\lambda_0^k$ $\sigma_f^k = S_{f,nl}(\lambda_0^k)$ $r^k = \chi_p - (\sigma_f^k + \sigma_p^k)$ $J(\lambda_0^k)\delta\lambda_0^k = r^k$ $\lambda_0^{k+1} = \lambda_0^k + \alpha^k\delta\lambda_0^k$	$\sigma_p^k = S_p\lambda_0^k$ $\sigma_f^k = S_{f,nl}(\lambda_0^k)$ $r^k = \chi_p - (\sigma_f^k + \sigma_p^k)$ $\mu^k = P^{-1}r^k$ $\lambda_0^{k+1} = \lambda_0^k + \omega^k\mu^k$	$\sigma_p^k = \chi_p + (\gamma_f\mathcal{I}\mu_2^k - S_p\mu_2^k)$ $\gamma_f\mathcal{I}\mu_1^{k+1} + S'_{f,nl}(\mu_1^{k+1}) = \sigma_p^k$ $\sigma_f^{k+1} = \chi_p + (\gamma_p\mathcal{I}\mu_1^{k+1} - S_{f,nl}(\mu_1^{k+1}))$ $\gamma_p\mathcal{I}\mu_2^{k+1} + S_p\mu_2^{k+1} = \sigma_f^{k+1}$
1 groundwater solve 1 flow solve 1 Jacobian solve	1 groundwater solve 1 flow solve 1 precond. solve	1 groundwater solve 1 (tangent) flow solve 1 flow solve 1 groundwater solve

Table 7.1. Comparison among Newton, Richardson and operator splitting approaches.

Conclusions

In this thesis we have investigated the mathematical and numerical analysis of coupled surface-groundwater flow problems.

First, a linear Stokes/Darcy model was considered. Its well-posedness has been proved at both the differential and the discrete level.

Then, domain decomposition methods have been applied to set up substructuring algorithms to compute the finite element solution of the global problem. Precisely, the original problem has been reformulated in terms of interface equations associated to the choice of the inteface variable as the trace of the fluid normal velocity or of the piezometric head on Γ.
The analysis of the pseudo-differential Steklov-Poincaré operators has allowed us to characterize optimal preconditioners, that have been replicated for the Schur complement system and used in the framework of preconditioned conjugate gradient iterations. In particular, we have first considered Dirichlet-Neumann type preconditioners which perform well with respect to grid parameters yielding a convergence rate independent of h. However, they did not scale optimally with respect to the physical parameters, which severely affect their convergence properties and make them of little interest when dealing with real life applications.
This difficulty was overcome by setting up a more sophisticated algorithm based on alternating direction iterations. In particular, a new class of multiplicative and additive preconditioners for the Steklov-Poincaré equation (or the Schur complement system) involving local Robin problems has been characterized. The convergence properties of these methods have been investigated, at least in the case of two subdomains, for a generic advection-diffusion-reaction elliptic operator, thus obtaining a general purpose algorithm which can be effectively applied also in contexts others than the surface and groundwater flows.

Finally, the nonlinear Navier-Stokes/Darcy case was addressed within the framework of domain decomposition allowing us to set up a preconditioned nonlinear Richardson method which extends the classical Newton approach. This setting may also be applied to other nonlinear couplings, for example in the case of fluid-structure interaction problems.

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1. M. Discacciati, E. Miglio and A. Quarteroni. Mathematical and numerical models for coupling surface and groundwater flows. *Appl. Numer. Math.*, 43:57−74, 2002.
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