CONTROL DESIGN OF HYBRID SYSTEMS VIA DEHYBRIDIZATION

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Abstract

Hybrid dynamical systems are those with interaction between continuous and discrete dynamics. For the analysis and control of such systems concepts and theories from either the continuous or the discrete domain are typically readapted. In this thesis the ideas from perturbation theory are readapted for approximating a hybrid system using a continuous one. To this purpose, hybrid systems that possess a two-time scale property, i.e. discrete states evolving in a fast time-scale and continuous states in a slow time-scale, are considered. Then, as in singular perturbation or averaging methods, the system is approximated by a slow continuous time system. Since the hybrid nature of the process is removed by averaging, such a procedure is referred to as *dehybridization* in this thesis.

It is seen that fast transitions required for dehybridization correspond to fast switching in all but one of the discrete states (modes). Here, the notion of dominant mode is defined and the maximum time interval spent in the non-dominant modes is considered as the 'small' parameter which determines the quality of approximation. It is shown that in a finite time interval, the solutions of the hybrid model and the continuous averaged one stay 'close' such that the error between them goes to zero as the 'small' parameter goes to zero.

To utilize the ideas of dehybridization for control purposes, a cascade control design scheme is proposed, where the inner-loop artificially creates the two-time scale behavior, while the outer-loop exponentially stabilizes the approximate continuous system. It is shown that if the origin is a common equilibrium point for all modes, then for sufficiently small values of the 'small' parameter, exponential stability of the hybrid model can be guaranteed. However, it is shown that if the origin is not an equilibrium point for some modes, then the trajectories of the hybrid model are ultimately

bounded, the bound being a function of the ‘small’ parameter. The analysis approach used here defines the hybrid system as a perturbation of the averaged one and works along the lines of robust stability. The key technical difference is that though the norm of the perturbation is not small, the norm of its time integral is small.

This thesis was motivated by the stick-slip drive, a friction-based micropositioning setup, which operates in two distinct modes ‘stick’ and ‘slip’. It consists of two masses which stick together when the interfacial force is less than the Coulomb frictional force, and slips otherwise. The proposed methodology is illustrated through simulation and experimental results on the stick-slip drive.

Version Abrégée

Un système dynamique est dit hybride s'il existe une étroite interaction entre des dynamiques continues et des dynamiques discrètes. Afin d'analyser et de commander ce type de systèmes, les concepts et théories les domaines discret et continu doivent être adaptées. Dans ce travail de thèse, les idées issues de la théorie des perturbations sont adaptées afin d'approximer un système hybride par un système continu. Les systèmes hybrides caractérisés par deux échelles de temps, c'est-à-dire des états discrets évoluant rapidement et des états continus évoluant lentement, sont considérés. Ensuite, comme dans les méthodes de perturbations singulières ou de moyennisation, le système est approximé par un système continu lent. Vu que la nature hybride du processus est supprimée par moyennisation, une telle procédure est appelée *déshybridation* dans cette thèse.

On montre que les transitions rapides nécessaires pour la déshybridation correspondent à la commutation rapide dans tous les états discrets (modes) sauf un. Ainsi, la notion de mode dominant est définie et l'intervalle de temps maximal passé dans les modes non dominants est considéré comme le 'petit' paramètre qui détermine la qualité de l'approximation. On prouve que, dans un intervalle de temps fini, les solutions du modèle hybride et du modèle continu moyenné restent 'proches', si bien que l'erreur entre elles tend vers zéro quand le 'petit' paramètre tend vers zéro.

Afin d'exploiterles idées de déshybridation dans un but de commande, la synthèse d'un schéma de commande en cascade est proposée. La boucle interne crée artificiellement le comportement à deux échelles de temps et la boucle externe stabilise exponentiellement le système continu approximé. On démontre que, si l'origine est un point d'équilibre commun pour tous les modes, alors pour des valeurs suffisamment petites du 'petit' paramètre, la stabilité exponentielle du système hybride peut être garantie. Si l'origine n'est pas un point d'équilibre pour certains modes, alors les trajectoires du modèle hybride sont bornées, la borne étant une fonction du 'petit' paramètre. L'approche utilisée permet de définir le système hybride comme une perturbation du système moyenné et s'inspire de la stabilité robuste. La différence technique majeure est que, malgré le fait que la norme de la perturbation n'est pas petite, la norme de son intégrale dans le temps est petite.

Cette thèse est motivée par un système d'actionneur stick-slip de laboratoire, qui réalise un micro-positionnement basé sur le frottement, et opère dans les deux modes distincts 'stick' et 'slip'. L'actionneur consiste en deux masses qui avancent ensemble quand la force interfaciale est plus faible que la force de frottement de Coulomb, et glissent dans le cas contraire. La méthodologie proposée est illustrée par des simulations et des résultats expérimentaux sur l'actionneur stick-slip.

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Chapter 1

Introduction

1.1 Motivation

The theory of automatic control deals with developing methodologies for modeling, analysis and control of dynamical systems. Two major classes of dynamical systems are commonly distinguished in the literature: i) systems dealing with continuous state variables, i.e. continuous dynamical systems, and ii) systems dealing with discrete states, i.e. discrete event systems. Yet, most practical dynamical systems consist of a mixture of both continuous and discrete variables. Such systems, referred to as *hybrid systems* are those where an interaction between continuous and discrete dynamics exists.

Most of the dynamical systems around us are hybrid in nature. Nevertheless, it is common that the hybrid nature of systems is neglected, and consequently systems are modeled by purely continuous or discrete dynamics. One of the main reasons for such a simplification is to facilitate analysis and synthesis. However, there are many systems that cannot be modeled without consideration of their hybrid nature. Furthermore, new technological developments which have led to the use of more sophisticated and complex systems have made hybrid modeling inevitable. Since the notion of the hybrid system covers many domains and application fields, people from different backgrounds, e.g. control theory, computer science and mathematics, have contributed to its development.

In control theory, dealing with both continuous and discrete variables is not new. In fact, the well-known approaches such as sliding mode control, gain scheduling, relay control, variable structure control, stabilization of nonlinear systems by switching, bang-bang control and programmablelogic controller deal with both continuous and discrete dynamics. So, what is novel in the new field of hybrid systems? The main objective in recent research activities is to find more systematic and unified ways to deal with hybrid systems rooted in a rigorous mathematical foundation. Particular emphasis has been placed on a unified representation of hybrid models, and working on fundamental properties such as existence and uniqueness of solutions and issues such as stability analysis and control synthesis. With the new approach, all techniques mentioned earlier in this paragraph can be dealt with in a unified framework. In addition, a wider class of problems can be solved.

Recent research results on hybrid systems have revealed the complexity of the problems and challenges regarding analysis and control design. The difficulties that arise are mainly due to the *interaction* of the continuous and discrete dynamics. Since the dynamics of continuous and discrete variables interact and are influenced by each other, most well-known results developed for either of the classes are inappropriate. Thus, concepts and theories in each of the domains should be readapted in the hybrid framework. The idea in this thesis is to readapt the notions of 'perturbation theory' and 'averaging' for transforming a hybrid system to a continuous one so as to be able to use standard analysis and control techniques available for continuous systems. Generally, the averaging methods deal with systems having multiple-time scale property, wherein the states are classified into 'fast' and 'slow'. The slow response of the system is then approximated by an averaged model. For a hybrid system where the transition of discrete states is faster than the evolution of the continuous states, an approximate continuous model can be defined in the slow time-scale. Since the hybrid nature of the process is removed such an averaging procedure will be referred to as *dehybridization* in this work. The idea of dehybridization opens up several questions: If a continuous model approximates the behavior of the hybrid one, then what is the quality of the approximation? Is it possible to use control techniques in the continuous framework? If a control law is designed for the approximated continuous model, is the stability of the hybrid system guaranteed? Answers to these questions in a general setting form the subject of this thesis.

This thesis was motivated by the stick-slip drive, a friction based laboratory setup which operates in two distinct modes 'stick' and 'slip'. It consists of two masses which stick together when the force acting at their interface is less that the Coulomb frictional force. The masses slip when the interfacial force is larger than the Coulomb force. Preliminary experimental and simulation results revealed a two-time scale behavior in the system's response, where the modes 'stick' and 'slip' were changing rapidly while the velocity and position of the drive evolved smoothly in a slower time-scale. The idea of dehybridization was motivated from such a system behavior. Although controllers has been designed intuitively for such systems, stability of these control laws have never been mathematically established. The motivation for using dehybridization on this system is to prove the stability of the control scheme mathematically.

1.2 State of the Art

Since this thesis deals with perturbation theory for hybrid systems the state of the art consist of two parts: (a) hybrid system and (b) perturbation and averaging methods.

1.2.1 Hybrid Systems

The term 'hybrid systems' was first used in 1966 when Witsenhausen introduced a hybrid model consisting of continuous dynamics together with some transition sets [72]. Despite their early introduction, it is only during recent years that hybrid systems have become an independent research direction in the control community. An overview of recent activities on hybrid system theory can be found in journal special issues [8], [47], [5] and proceedings of 'Hybrid Systems: Computation and Control' ([25], [6], [3], [7], [26], [68], [41], [12], [64], [42]) and in an introductory book [69]. Below, some results related to this work regarding modeling, analysis, and control design will be surveyed.

Modeling

Three different representations of hybrid systems are addressed in the literature. Each representation has been developed for a specific purpose and for specific types of problems they are intended to solve.

- Automaton definition : This formulation is inspired by the definition of *automaton* used in computer science [2]. The hybrid system is defined by an automaton where to each discrete state (mode) a continuous behavior is attributed. In such a representation, the conditions of transition from one mode to another should be 'explicitly' defined by some 'guard' and 'jump' conditions [40].

- Equation-based definition: The hybrid system is represented by differential and algebraic equations defining the continuous dynamics together with some functions describing 'implicitly' the discrete transitions (see e.g. [61], [34], [9]). These models are influenced by the Witsenhausen [72] representation. Branicky *et al.* [15] presented a general model of this category where the possibility of jumps and transitions in continuous states was introduced.

- Behavioral representation: The hybrid system is defined by its time evolution. For instance, Ye *et al.* [73] defined a hybrid system as a family of motions or trajectories on an arbitrary metric space which they called 'time space'. A similar behavioral definition of hybrid systems is presented in [69] where the time evolution of systems is specified by a set of 'time-events'.

In this thesis, an equation-based definition of hybrid systems will be mainly used because of its compact form and the fact that it suffices for presenting the results of this work.

Analysis

Study of the fundamental properties of hybrid systems such as the existence and uniqueness of solutions, i.e. well-posedness, reachability, stability and the presence of limit cycles and chaotic phenomena has attracted the attention of many researchers. Here, some issues that are related in a way to this work will be surveyed.

The existence and uniqueness of solutions (well-posedness) are studied for different classes and formulation of hybrid systems, [24], [67], [70], [38], [39]. In this thesis the key assumption of dealing with a well-posed hybrid is based on the ideas presented in [39], where the conditions of well-posedness of finite automata used in the discrete event system's terminology are linked to the classical results for the continuous systems. The details will be discussed in Chapter 2.

Stability of hybrid systems is yet another active area of research. Stability in continuous or non-hybrid systems can be concluded from the characteristics of their vector fields. However, in hybrid systems the stability properties depend also on the switching rules. For instance, in a hybrid system by switching between two stable dynamics it is possible to get instabilities while switching between two unstable ones could result in stability. Most of the stability results for hybrid systems are extensions of the Lyapunov theories developed for continuous systems, [14], [73], [39]. They require the Lyapunov function at consecutive switching times to be a decreasing sequence. Such a requirement in general is difficult to verify without calculating the solution of the hybrid dynamics, and hence losing the advantage of Lyapunov approaches. For a survey on major developments the reader is referred to [21] and [37].

Stability of the hybrid systems presented in the literature can be classified based on two different criteria:

- Structure of the continuous dynamics of the hybrid model, i.e. linear or nonlinear.
- Structure of the Lyapunov candidate functions used, i.e. single or multiple Lyapunov functions.

Some important results reported in the literature can be arranged as shown in the following table.

	Single Lyapunov function	Multiple Lyapunov functions
Linear dynamics	[48]	[50], [37], [27], [36], [30]
Nonlinear dynamics	[43], [73], [39], [22]	[51], [14]

The stability analysis technique used in this thesis can be classified in the category of nonlinear dynamics with a single Lyapunov function. The idea is to represent the hybrid model as a perturbed model of a continuous system. So, the stability of hybrid system is transformed to a robust stability analysis of a continuous system and hence, the Lyapunov function is not needed to be decreasing at consecutive switching times. Furthermore, the origin need not be the equilibrium for all modes.

Control Design

Control design for hybrid systems is in general complex and challenging. In the literature, different design approaches are presented for different classes of hybrid systems, and different control objectives. For instance, when the control objective is concerned with issues such as safety specifications, verification and reachability (see e.g. [65], [10], [11]), the ideas in discrete event control [53] and automaton framework are used for control synthesis.

One of the most important control objectives is the stabilization problem. Since a hybrid system has in general two type of inputs, i.e. the continuous and discrete inputs, the control design methods addressed in the literature can be classified in two categories:

- Utilization of discrete inputs for stabilization: This category is mainly concerned with the stabilization problem of the switched systems, i.e. the family of different continuous dynamics governed by some switching laws. The design problem is to find switching conditions or switching strategies, i.e. stabilizing discrete inputs, which make the overall system stable (mode selection). In such a case, the system is considered either with no continuous inputs [71], [35], [51], or with predesigned controllers, e.g. gain schedulers [57] and supervisory control [46], [45].
- Utilization of continuous inputs for stabilization: In this category, the goal is to stabilize the system by using an appropriate continuous input. In such a case, the system either does not have discrete inputs or these inputs do not play a direct role in stabilization. Most of the classical switched controllers for continuous systems such as variable structure [24], sliding control [67], relay control [66], PWM control [59], and discontinuous controllers for nonholonomic systems [28] can be categorized in this class. Note that the approaches cited in this category are all developed for the case when the system to be stabilized is continuous.

In this thesis, a control design based on the averaged model will be proposed. So, the proposed approach belongs to the second category where continuous control is used for the stabilization task, while discrete inputs are used to generate the desired two-time scale property. The main difference between the proposed technique and the approaches cited in the second category is that the proposed one is intended to be applied to a hybrid system.

1.2.2 Perturbation Methods and Averaging

Perturbation theory and averaging methods represent an important research area for dynamical systems. Perturbation theory is mainly concerned with the approximate solutions for dynamical systems. Approximate solutions are sought since they might be simpler to obtain and describe the main features of the system evolution to a satisfactory level. The main tool used in perturbation theory, for the purpose of approximation, is the asymptotic approach. The idea of the asymptotic method is to find an approximative solution, such that the 'quality' of the approximation, i.e. the error between the approximate and the exact solutions, depends on a certain parameter. The error goes to zero as the parameter goes to zero [13] and [54].

As the parameter goes to zero, in many cases, a multiple time-scale structure is created. Such problems are treated extensively under the directions of singular perturbation and periodic averaging. Kokotović and Khalil [33] introduced singular perturbation theory into a control engineering framework. Rigorous mathematical results on the theory of averaging and singular perturbation from a control point of view can be found in [32] and in a recent paper [63].

In the context of hybrid systems, the idea of averaging was used in (i) analysis and control of Pulse Width Modulation (PWM) in power electronic systems [58], [62], and in (ii) linear switched systems [23], [71]. The main difficulties with the methods presented above are: (i) the class of systems considered is a restrictive one and/or (ii) only the case of infinitely fast switching is addressed. The non-limiting case, i.e. when the switching is fast yet finite is not considered, though in practice the switching is never infinite. The perturbation methods and averaging in a general hybrid formulation have not yet been addressed.

In this thesis, the problem of finite switching in a general hybrid formulation will be treated. While treating the non-limiting case, the quality of the approximation and whether the approximation can affect the stability become important issues.

1.3 Objectives of this Thesis

This thesis deals with analysis and control of hybrid systems with a two-time scale characteristic feature. In this work it will be considered that the fast and slow states correspond to discrete and continuous states respectively. Thus, similar to classical perturbation approaches, e.g. singular perturbation, it is expected that the average behavior of a hybrid system in a slow time-scale be approximated by a lower order continuous model (dehybridization). Hence, the main questions that need to be answered are,

(i) how can the approximated continuous model be constructed? (ii) how can the quality of the approximation be quantified?

The next objective of this thesis is to develop a hybrid control scheme on the basis of the averaged continuous model. To this end, a cascade control scheme will be presented where the inner-loop controller artificially imposes the desired two-time scale behavior, e.g. fast switching, while the outer-loop controller is designed on the basis of the approximated continuous model. At this stage, the relevant issue is the relationship between the stability of the averaged continuous system and the hybrid one.

As mentioned earlier, the ideas of this work were inspired by the stick-slip drive. Thus, one of the objective of this work consists on applying the developed theoretical results for such a setup. For a set-point tracking problem of stick-slip drive, design of a controller based on the proposed control scheme is sought.

1.4 Organization of the Thesis

This thesis is organized as follows. Chapter 2 contains the preliminaries and introduces the concepts and definitions relevant for understanding and developing the theoretical results of this thesis. In Chapter 3, the dehybridization procedure is introduced. It will be argued that in the case of fast switching or a domination of one of the modes the hybrid system can be approximated by a continuous model. Based on the dehybridization procedure, a cascade control schema will be introduced and the details of design will be explained.

Chapter 4 gives the theoretical results concerning the averaging approach and the stability issues.The quality of the approximation obtained using dehybrdization will be derived and and the stability results for the averaged model and the hybrid model will be investigated. In Chapter 5, the ideas of dehybridization for analysis and control of a stick-slip drive will be explored. First, the hybrid modeling of the drive will be introduced and then the proposed cascade control scheme will be applied to the set-point tracking problem of the drive. Finally, simulation and experimental results will be presented.

In Appendix A some useful concepts and examples related to the systems with fast switching will be presented. Appendix B and C give proofs of lemmas and propositions concerning Chapters 4 and 5 respectively.

Chapter 2

Preliminaries

2.1 Introduction

The results presented in this thesis are related to the approximation of a hybrid system using a continuous one. Clearly, an analysis of such an interface between continuous and hybrid systems needs concepts from each of these dynamics. Hence, this chapter will have two major subsections, the first dealing with continuous systems and the other with hybrid systems.

In this chapter, it has not been attempted to present an exhaustive or complete view of these subjects. Thus, it does not reflect the state of the art in of either these directions. Contrarily, the goal is just to introduce the concepts that are necessary to understand the novel ideas of this thesis and present the definitions and already existing results required to prove the theoretical results. So, the presentation is not cohesive, in the sense that there might not be any link between the different elements presented.

In the domain of continuous systems, only two major concepts are undertaken: i) the Lyapunov stability ii) perturbation methods. The reason for presenting the stability results is that one of the goals in this thesis is to prove the stability of the proposed scheme, i.e. control of hybrid systems via dehybridization. So, it is important to ascertain when the approximate continuous system is stable and if so, when the original hybrid system is stable as well. This immediately poses the question of the quality of approximation, which is typically studied using perturbation methods. Therein, the original system is considered as a perturbation (small deviation) of the approximate system. One of the main issues in perturbation methods is whether or not the approximate system has the same order as the original one. If the order of dynamics is reduced, the perturbation will be referred to as singular perturbation, the details of which will be discussed in this chapter.

In the domain of hybrid systems, only two concepts will be stressed: i) inter-connection of hybrid systems and ii) modeling of discontinuous dynamics in a hybrid framework. The former is important since a hybrid controller will be used in the proposed scheme, and the latter is useful when addressing the application part of this thesis. Note that there are different frameworks or representations available in the literature for hybrid systems. The formulation chosen here is not the most general one. A compromise was made between simplicity and illustrating different essential features.

Except for the notations, this chapter can be skipped without loss of continuity. It is organized as follows. In Section 2.2 continuous dynamical systems will be revisited. First a formal definition of continuous systems will be given, and then solution concepts (existence and uniqueness) will be discussed (Section 2.2.2). The concepts of Lyapunov stability for continuous dynamics, i.e. direct and converse theorems, boundedness of solutions, and some useful mathematical lemmas known as comparison results will be introduced in Sections 2.2.3 and 2.2.4. The concept of singular perturbation will be discussed in Section 2.2.5.

In Section 2.3, hybrid dynamical systems will be introduced. A brief description of discrete event systems will be given, and then a formal definition of a hybrid dynamical system as the interaction of continuous dynamical systems and discrete event systems will be presented (Section 2.3.1). The conditions for a well-posed hybrid system (existence and uniqueness) and the class of hybrid system treated in this thesis are given in Section 2.3.2. The interconnection of a hybrid plant and hybrid controller is presented in Section 2.3.3. Finally, a methodology for modeling of discontinuous dynamics in the hybrid framework will be presented in Section 2.3.4 where the notion of sliding mode regularization will be discussed as well.

2.2 Continuous Dynamical Systems

2.2.1 Definitions

The concept of a *system* is one which is often left to intuition rather than to exact definition [19]. There are two main features regarding the qualitative definitions that can be found in the literature. First, a system consists of interacting 'components' and, secondly, a system is associated with a 'function' it is intended to perform. In order to analyze and to develop design and control techniques for systems, quantitative definitions are needed. Hence, a *model* or an abstraction of a system is sought. A model is a mathematical device that duplicates the behavior of a system. A mathematical model describes the relations between different variables and quantities in a system by means of mathematics. Due to close connection of a model and system, it is common to drop the distinction and use the meanings of terms interchangeably.

Dynamical systems are those in which their behavior depends on the present and the *past* values of their variables. For a dynamic system, three important concepts need to be distinguished, i) the state variables, x, ii) the in input, u and iii) the outputs, y. The state variables, correspond to the minimal information that is necessary to describe the system behavior uniquely for a given time instant. In other words, the state variables represent the 'memory' that the system has of its past. The *state* of a system refers to a specific value of state variables and is generally denoted by x. The inputs constitute the external variables that influence the system. The outputs are typically the variables that can be measured. They can also variables acting as inputs to open connected systems.

There are two alternative formulations of dynamical systems: i) the input-output formulation and ii) the state space representation. The former describes the relationship between the inputs and the outputs using differential/difference equations. While in latter, the relationship between the inputs and the states is represented using a set of first-order differential/difference equations. These relationships are also referred to as the *dynamics*. In this thesis, only the state space representation will be considered.

The continuous time dynamical system is one where the state variables take their values in a non-denumerable set, i.e. $X \subset \mathbb{R}^n$, where n is the number of state variables, also referred to as the order of the system. The set X is also called the state space. For a continuous system, the dynamics are described by differential equations and can be defined as follows:

Definition 2.1 *A continuous-time continuous state dynamical system is described by a triple* (X, U, F)*, where* $X \subset \mathbb{R}^n$ *is the set of state space,* $U \subset \mathbb{R}^m$ *is the set of external values (input, disturbances), and* $F : X \times U \to \mathbb{R}^n$ *is a vector field describing the dynamics of system:*

$$\dot{x}(t) \quad = \quad F(x(t), u(t)), \quad x(t_0) = x_0 \tag{2.1}$$

with $x \in X$ being the state variables, $u \in U$ the inputs.

The inputs $u(t)$ are either specified *a priori* in time or are computed as a function of time and states, i.e. $u = k(t, x)$. The latter is referred to as 'state feedback control'. Substituting $u = k(t, x)$ in (2.1) leads to

$$\dot{x} \quad = \quad F(t, x), \quad x(t_0) = x_0 \tag{2.2}$$

Such a system now has no more external inference. The concepts like equilibrium point or stability are analyzed, typically after giving a particular form to the input and so the analysis will only address systems of type (2.2).

A special case of (2.2) where F does not depend explicitly on t, (time-invariant or autonomous systems), will be considered in this thesis:

$$\dot{x} \quad = \quad F(x), \quad x(t_0) = x_0 \tag{2.3}$$

It has to be stressed that (2.3) is not a restriction of (2.1), rather it takes the form of (2.3) when a structure is given to the inputs.

2.2.2 State Trajectory: Existence and Uniqueness

For a mathematical model to predict the behavior of a system, the initial value problem of differential equation (2.3) must have a unique solution. Solution of (2.3) over an interval $[\tau_0, \tau_0 + \delta]$ is a continuous function $x : [\tau_0, \tau_0 + \delta] \rightarrow \mathbb{R}^n$, satisfying

$$x(t) = x(\tau_0) + \int_{\tau_0}^{\tau_0+\delta} F(x(s))ds \tag{2.4}$$

The function $x(t)$ defined by (2.4) is known as the Carathéodory solution . An interesting property of such a definition is that the solution $x(t)$ is not required to be differentiable everywhere. The existence and uniqueness of the *state trajectory* or the solution of the system depends on the characteristics of the vector fields in (2.3). It can be seen that continuity of $F(x)$ to its argument ensures the existence of a solution [44]. Note that the converse is not true; if $F(x)$ is discontinuous there are certain cases where a solution (even many) exists and certain cases where no solution exists.

The next issue is that of uniqueness of solution, for which continuity is not sufficient. A sufficient condition guaranteeing *local* existence and uniqueness of solution is that the vector field $F(x)$ satisfies the Lipschitz condition in a neighborhood of the initial condition x_0

$$||F(x) - F(y)|| \leq L||x - y||, \quad \forall x, y \in B_r = \{x \in \mathbb{R}^n |\, ||x - x_0|| \leq r\} \tag{2.5}$$

The condition implies that the function does not grow faster than some linear function. A function satisfying the above inequality is said to be Lipschitz and the positive constant L is called the Lipschitz constant. A function can be either *locally* Lipschitz or *globally* Lipschitz. A function is called locally Lipschitz on a domain $D \subset \mathbb{R}^n$, if each point of D has a neighborhood D_0 such that F satisfies (2.5) with some Lipschitz constant L_0. $F(x)$ is said to be locally Lipschitz on a set W, if it satisfies (2.5) for all points of W with the same Lipschitz constant L, and is said to be globally Lipschitz, if it is Lipschitz on $\mathbb{R}^n$.

A locally Lipschitz function only guarantees the uniqueness of solution over an interval $t \in [\tau_0,\, \tau_0 + \delta]$ where δ is small. The reason is simply the fact that the solution may leave in finite time the compact set over which the Lipschitz condition is satisfied. For example, in some cases, nonlinear systems with a locally Lipschitz vector field can have 'finite escape time', in the sense that a trajectory escapes to infinity in finite time [32]. If the vector field satisfies the global Lipschitz condition, then the time interval of existence and uniqueness of the solution can also be extended indefinitely. Note also the local Lipschitz property is just a smoothness requirement and except for systems with a discontinuous right hand side $F(x)$, most of the physical systems satisfy this condition. However, the global Lipschitz condition is very restrictive and except for systems modeled by linear dynamics, few nonlinear physical systems possess this property.

2.2.3 Lyapunov Stability

An important concept dealing with continuous systems represented by the state equations (2.3) is the concept of equilibrium point.

Definition 2.2 *A state x^{eq} is an equilibrium point of a dynamic system, if $x(t_0) = x^{eq}$ implies that $x(t) = x^{eq}$, $\forall t$. The equilibrium point for dynamics (2.3) is the solution of the equation $F(x) = 0$.*

Since any equilibrium point using a change of variables can be shifted to the origin without altering the vector field, the equilibrium point at the origin $x^{eq} = 0$ for definitions and theorems will be considered. The stability of an equilibrium point is related to whether all solutions nearby the equilibrium point will stay nearby the equilibrium. Among many notions of stability, important results concerning the stability of equilibrium point for continuous dynamics, in the sense of Lyapunov, will be presented here.

Definition 2.3 *The equilibrium point $x = 0$ of dynamics (2.3) is*

- *stable, if for each $R > 0$, there is $r = r(R) > 0$ such that*

$$||x(t_0)|| < r \Rightarrow ||x(t)|| < R, \quad \forall t \geq t_0$$

- *asymptotically stable, if it is stable and, $r > 0$ can be chosen such that*

$$||x(t_0)|| < r \Rightarrow \lim_{t \to \infty} x(t) = 0$$

- *exponentially stable, if there exist positive constants c, k, and a such that*

$$||x(t)|| \leq k||x(t_0)||e^{-a(t-t_0)}, \quad \forall ||x(t_0)|| < c$$

- *unstable, if it is not stable.*

The definition claims that a stable equilibrium point is one with the property that trajectories starting arbitrarily close to the equilibrium point can be kept, in another neighborhood, arbitrarily close to it. Asymptotical stability means that the equilibrium point is stable, and in addition all trajectories nearby will converge to it as time goes to infinity. Note that, the convergence of trajectories alone without satisfying the stability condition, does not imply asymptotical stability. An exponential stable equilibrium point has the property that all trajectories converge to the equilibrium point faster than an exponential function. The exponent a indicates how fast a trajectory can converge to the equilibrium point.

The above definitions describe the local behavior of the trajectories nearby equilibrium point. When an equilibrium point is asymptotically (exponentially) stable, one wants to know how far from the equilibrium point (what region in the state space) the trajectory can start, and still converge to it. The largest set of such initial trajectories is called domain of attraction or basin. If the domain of attraction is entire state space, the equilibrium point is said to be *globally* asymptotic (exponential) stable.

Before stating the Lyapunov stability theorems, definition of comparison functions, known as class $\mathcal{K}$ and class $\mathcal{KL}$ functions will be presented. The use of these functions will become clear when discussing Lyapunov stability results.

Definition 2.4 *A continuous function* $\gamma : [0, a) \to [0, \infty)$ *is said to belong to class* $\mathcal{K}$ *if, it is strictly increasing and* $\gamma(0) = 0$. *It is said to belong to class* $\mathcal{K}_\infty$, *if* $a = \infty$ *and* $\gamma(r) \to \infty$ *as* $r \to \infty$.

Definition 2.5 *A continuous function* $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ *is said to belong to class* $\mathcal{KL}$, *if for each fixed* s, *the mapping* $\beta(r, s)$ *belongs to class* $\mathcal{K}$ *with respect to* r *and for each fixed* r, *the mapping* $\beta(r, s)$ *is decreasing with respect to* s *and* $\beta(r, s) \to 0$ *as* $s \to \infty$.

Note that the inverse of a class $\mathcal{K}$ ($\mathcal{KL}$) function belongs to class $\mathcal{K}$ ($\mathcal{KL}$) and composition of class $\mathcal{K}$ functions also belong to class $\mathcal{K}$, i.e. $\gamma_1(\gamma_2(.)) = \gamma \in \mathcal{K}$.

Lyapunov Stability Theorems

To ascertain stability of an equilibrium point, one needs to show that the trajectories starting from every initial condition in its neighborhood have certain property. This problem cannot be addressed directly since an explicit solution of the differential equation governing a system is not always available, especially for nonlinear systems. However, in some cases by investigating the system's total energy, the stability of the equilibrium point can be verified.

Roughly speaking, the idea behind the energy function can be explained as follows: The energy function is related to the magnitude of the states of a system, and if the total energy of the system decreases (dissipates) as the solution of the system tends to zero, then the system's trajectory approaches the origin (equilibrium point). Lyapunov showed that instead of energy functions, a general class of scalar positive definite functions known as 'Lyapunov functions', can be used to determine the stability of a system. In the following theorems, such Lyapunov functions are used to investigate the stability of an equilibrium point.

Direct Lyapunov Results

Theorem 2.1 *[32] Let $x = 0$ be an equilibrium point for system (2.3) and $D \subset \mathbb{R}^n$ be the domain containing $x = 0$. Let $V : D \to \mathbb{R}$ be a continuously differentiable function such that*

$$\gamma_1(||x||) \leq V(x) \leq \gamma_2(||x||) \tag{2.6}$$

$$\dot{V} = \frac{\partial V}{\partial x} F(x) \leq 0 \tag{2.7}$$

$\forall x \in D$, where γ_1 and γ_2 are class $\mathcal{K}$ functions on D. Then, $x = 0$ is stable. If the condition (2.7) is strengthened to

$$\dot{V} = \frac{\partial V}{\partial x} F(x) \leq -\gamma_3(||x||) \tag{2.8}$$

with γ_3 a class $\mathcal{K}$ function, then $x = 0$ is asymptotically stable. If in addition, $D = \mathbb{R}^n$ the origin is globally asymptotically stable.

Note that, if the scalar function $V(x)$ is positive definite, then there exist class $\mathcal{K}$ functions γ_1 and γ_2 such that the condition (2.6) is satisfied (Lemma 4.3 [32]). Condition (2.8) guarantees that the derivative of $V(x)$ along the the trajectories of system is negative definite.

A special case of above theorem is when the class $\mathcal{K}$ functions are defined by $\gamma_i(||x||) = c_i||x||^a$, $i = 1, 2, 3$ with c_i and a some positive constants. In such a case the equilibrium point is exponentially stable:

Theorem 2.2 *[32] Let $x = 0$ be an equilibrium point for system (2.3) and $D \subset \mathbb{R}^n$ be a domain containing $x = 0$. Let $V : D \to \mathbb{R}$ be a continuously differentiable function such that*

$$c_1||x||^a \leq V(x) \leq c_2||x||^a \tag{2.9}$$

$$\frac{\partial V}{\partial x} F(x) \leq -c_3||x||^a \tag{2.10}$$

$\forall x \in D$, where c_1, c_2, c_3 and a are some positive constants. Then, $x = 0$ is exponentially stable. If $D = \mathbb{R}^n$, then $x = 0$ is globally exponentially stable.

Converse Lyapunov Results

The direct Lyapunov theorems show the stability (asymptotic or exponential) of the equilibrium point when Lyapunov functions satisfying conditions (2.6)-(2.8) can be found. Quite often, it would be interesting to know whether such functions exist at all. There are results concerning the existence of Lyapunov functions when the equilibrium point possesses some stability properties. Such results are known as *converse theorems*. The converse theorems suggest that, if an equilibrium point is asymptotically (exponentially) stable, then the existence of a Lyapunov function is guaranteed.

Theorem 2.3 *[32] Let $x = 0$ be an equilibrium point for the dynamics (2.3) where $F : D \to \mathbb{R}^n$ is continuously differentiable and $\frac{\partial F}{\partial x}$ is bounded on $D = \{x \in \mathbb{R}^n \mid ||x|| \leq r\}$. Then,*

- *If the origin is a stable equilibrium point, there exists a positive definite function $V(x)$ whose time derivative along the trajectories of the system is negative semidefinite.*
- *If the origin is an asymptotically stable equilibrium point, there exists a continuously differentiable function $V : D \to \mathbb{R}$ that satisfies, for some class $\mathcal{K}$ function γ_i, i=1,2,3,4,*

$$\gamma_1(||x||) \leq V(x) \leq \gamma_2(||x||), \qquad \dot{V} \leq -\gamma_3(||x||), \qquad \left|\left|\frac{\partial V}{\partial x}\right|\right| \leq \gamma_4(||x||).$$

- *If the origin is an exponentially stable equilibrium point, there exists a continuously differentiable function $V : D \to \mathbb{R}$ that satisfies, for some positive constants c_i, i=1,2,3,4,*

$$c_1||x||^2 \leq V(x) \leq c_2||x||^2, \qquad \dot{V} \leq -c_3||x||^2, \qquad \left|\left|\frac{\partial V}{\partial x}\right|\right| \leq c_4||x||.$$

Moreover, if $r = \infty$, and the origin is globally asymptotically (exponentially) stable, then V(x) satisfies the above inequalities on $\mathbb{R}^n$.

One may wonder if the stability of an equilibrium point is already known, why should the question of existence of a Lyapunov function be raised. The converse theorems, however, show their utility in different areas such as stability analysis of perturbed systems (robust analysis). For instance, when some stability information on the unperturbed system is available, then the existence of a Lyapunov function can help to derive stability conditions for the perturbed system.

Boundedness

In many cases, it is desirable to show boundedness of trajectories even if their origin in not stable. There are also cases in which the energy or Lyapunov function is not always decreasing, specially nearby the equilibrium point. For instance, for some systems subjected to perturbation, the Lyapunov function may even increase near to the origin. The Lyapunov analysis is useful to show the boundedness of trajectories.

Definition 2.6 *The trajectories of the dynamics (2.3) are*

- *bounded, if for some $a > 0$, there is a positive constant possibly dependent on a, i.e. $\beta = \beta(a) > 0$, such that*

$$||x(t_0)|| \leq a \Rightarrow ||x(t)|| < \beta(a),\ \forall t \geq t_0 \tag{2.11}$$

- *ultimately bounded with ultimate bound b, if for some $a > 0$ there exist positive constants b and $\delta = \delta(a, b) < \infty$ such that*

$$||x(t_0)|| \leq a \Rightarrow ||x(t)|| < b,\ \forall t \geq t_0 + \delta \tag{2.12}$$

- *globally ultimately bounded if, for arbitrarily large a,(2.12) holds.*

Theorem 2.4 *[32] Let $V : D \rightarrow \mathbb{R}$ be a continuously differentiable function with $D \subset \mathbb{R}^n$ a domain that contains the origin and satisfies*

$$\gamma_1(||x||) \leq V(x) \leq \gamma_2(||x||) \tag{2.13}$$

$$\frac{\partial V}{\partial x} F(x) \leq -\gamma_3(||x||), \quad \forall ||x|| \geq \mu > 0 \tag{2.14}$$

$\forall x \in D$, where γ_i, $i = 1, 2, 3$ are class $\mathcal{K}$ functions defined on D. Take r such that $B_r = \{x \in \mathbb{R}^n |\ ||x|| \leq r\} \subset D$ and is sufficiently larger than μ, then there exists a class $\mathcal{KL}$ function β, and for every initial state $x(t_0)$ satisfying $||x(t_0)|| \leq \gamma_2^{-1}(\gamma_1(r))$, there exists $\delta \geq 0$ such that the solution of (2.3) satisfies

$$||x(t)|| \leq \beta(||x(t_0)||, t - t_0), \quad \forall\, t_0 \leq t \leq t_0 + \delta \tag{2.15}$$

$$||x(t)|| \leq \gamma_2^{-1}(\gamma_1(\mu)) = b, \quad \forall\, t \geq t_0 + \delta \tag{2.16}$$

If in addition $D = \mathbb{R}^n$, then (2.15) and (2.16) hold for any initial condition.

Based on the above theorem, if the derivative of the Lyapunov function along the trajectories of system is negative definite only outside a set containing the origin, e.g. B_μ, then the inequalities (2.15) and (2.16) suggest that the solution is bounded for all $t \geq t_0$ and ultimately bounded with $b = \gamma_2^{-1}(\gamma_1(\mu))$. Note that with $\gamma_2^{-1}(\gamma_1)$ being a class $\mathcal{K}$ function, as $\mu \to 0$ the ultimate bound $b \to 0$.

2.2.4 Comparison Principles

The objective of the mathematical theories proposed by Lyapunov is to determine the stability of a system by checking the sign of the time derivative of V, without needing to solve the differential equation describing the dynamics governing the system. Yet, in certain cases it will be instructive to see how at least the Lyapunov function evolves with time. However, from (2.8) it can be seen that only a bound on the time derivative of function $V(t)$ is known. In such situations, the derivative of the Lyapunov function is expressed by inequalities of the form $\dot{V}(t) \leq g(V(t))$ (differential inequality) where $V(t)$ is a scalar positive function. Though this inequality does not admit an exact solution, it is possible to have an estimate of a bound on $V(t)$ using some useful tools given in following:

Lemma 2.1 *[44] Consider the scalar differential equation*

$$\dot{u} = g(t, u), \quad u(t_0) = u_0$$

where $g(t, u)$ is continuous in t and locally Lipschitz in u, for all $t \geq 0$ and all $u \in D \subset \mathbb{R}$. Let $[0, \delta)$, be the maximal interval of existence of the solution $u(t)$, and suppose $u(t) \in D$ for all $t \in [0, \delta)$. Let $v(t)$ be a continuous function whose derivative $\dot{v}(t)$ satisfies the differential inequality

$$\dot{v}(t) \leq g(t, v(t)), \quad v(t_0) \leq u_0$$

with $v(t) \in D$, $\forall t \in [0, \delta)$. Then, $v(t) \leq u(t)$, $\forall t \in [0, \delta)$.

The above lemma, known as the comparison principle, compares the solution of the differential inequality $\dot{v}(t) \leq g(v(t))$ with the solution of the differential equation $\dot{u} = g(t, u)$.

The following useful result, is a direct consequence of the comparison principle (Lemma 2.1):

Lemma 2.2 *Consider the differential inequality*

$$\dot{v}(t) \leq -cv + \kappa(t)v \quad v(0) = v_0 \geq 0 \tag{2.17}$$

where $v(t)$ is a positive scalar function and $c > 0$, $\kappa(t) > 0$. Then $v(t)$ satisfies

$$v(t) \leq v_0 \exp\left[-ct + \int_0^t \kappa(s)ds\right] \tag{2.18}$$

Proof: By using the comparison principle (Lemma 2.1), it can be seen that $v(t) \leq u(t)$ where $u(t)$ is the solution of $\dot{u} = -cu(t) + \kappa(t)u(t)$, with $v_0 = u_0$. $u(t)$ satisfies

$$u(t) = u_0 \exp\left[\int_0^t (-c + \kappa(s))ds\right]$$

hence

$$v(t) \leq v_0 \exp\left[\int_0^t (-c + \kappa(s))ds\right] = v_0 \exp\left[-ct + \int_0^t \kappa(s)ds\right] \tag{2.19}$$

which completes the proof.

■

Integral inequalities that give explicit bounds on unknown functions provide a useful tool in the study of many properties of solutions of nonlinear differential equations. One of the best known and widely used inequalities is stated by the Gronwall-Bellman lemma:

Lemma 2.3 *[32] Let $\lambda(t)$ and $\mu(t)$ be real and continuous functions which satisfy $\lambda(t) \geq 0$, $\mu(t) \geq 0$. If a continuous function $v(t)$ satisfies*

$$v(t) \leq \lambda(t) + \int_0^t \mu(s)v(s)ds, \quad t \in [0, \delta] \tag{2.20}$$

then

$$v(t) \leq \lambda(t) + \int_0^t \lambda(s)\mu(s)\exp\left[\int_s^t \mu(s)ds\right]ds, \quad t \in [0, \delta] \tag{2.21}$$

In particular, if $\lambda(t) \equiv \lambda$ is a constant, then

$$v(t) \leq \lambda\exp\left[\int_s^t \mu(s)ds\right], \quad t \in [0, \delta] \tag{2.22}$$

As a result of several driving forces, the Gronwall-Bellman inequality has been extended and different variations have been introduced. An useful inequality was given by Pachpatte:

Lemma 2.4 *[49] Let v, μ_1 and μ_2 be continuous functions which satisfy $\mu_1(t) \geq 0$, $\mu_2(t) \geq 0$ and $v(t) \geq 0$ and let λ be a nonnegative constant. If*

$$v^2(t) \leq \lambda^2 + 2\int_0^t (\mu_1(s)v^2(s) + \mu_2(s)v(s))ds, \quad t \in [0, \delta] \tag{2.23}$$

then,

$$v(t) \leq p(t)\exp\left[\int_0^t \mu_1(s)ds\right], \quad t \in [0, \delta] \tag{2.24}$$

where

$$p(t) = \lambda + \int_0^t \mu_2(s)ds, \quad t \in [0, \delta] \tag{2.25}$$

Ou-Iang studied a special case of the above lemma, with $\mu_1 = 0$, much before Pachpatte [49].

2.2.5 Perturbation Methods

As discussed in the introduction, the second concept that will be dealt in detail pertaining to continuous systems is the perturbation method. This is related to having some ‘small’ negligible parameters in the model such as masses, moments of inertia, resistance, inductance. Quite often, as a part of the modeling process (which might be an automated one), or in order to improve the accuracy, such small parameters are included in the model.

Let the model of such systems be represented by following differential equation

$$\dot{x}(t) = F(x, \varepsilon),\ x(t_0) = x_0(\varepsilon)$$

where ε is a small scalar parameter. Consider, for the sake of generality, that the initial condition is also a function of ε. Let $x(t, \varepsilon)$ represent the solution of the above differential equation. In many cases, an approximation of such a solution can be found, for instance, by considering $\varepsilon = 0$. In most cases, after such an approximation, the model has a simpler representation (e.g. dynamics with reduced order equations, nonlinear to linear dynamics, etc.) which makes the task of analysis and control easier. The approximate dynamics are given by

$$\dot{\bar{x}}(t) = F(\bar{x}, 0),\ \bar{x}(t_0) = \bar{x}_0$$

The goal of perturbation analysis is understand how close are $x(t, \epsilon)$ and $\bar{x}(t)$.

The main mathematical tool used in perturbation methods is the Taylor expansion with respect to the parameter ε. Analysis using Taylor expansion fails when the order of system is reduced for $\varepsilon = 0$. The failure is due to the fact that the small parameter ε multiplies the derivatives of some states in the differential equation, and hence for $\varepsilon = 0$, some part of the differential equation degenerates into an algebraic equation whose solution at initial time does not necessarily satisfy all the prescribed initial conditions. Such a case is known as a 'singular' perturbation problem. Singular perturbation problems are of high utility in the theory and application of control systems [33] especially since they give order reduction. An important characteristic of a singular perturbation problem is that it deals with the interaction of 'fast' and 'slow' states (two-time scale property). Some ideas behind the methodology will be used in this thesis. The following section will give a brief overview of the results concerning the singular perturbation method.

Singular Perturbation

A singular perturbed system can be represented by a model of the following form

$$\dot{x} = f(x, z, \varepsilon), \quad x(t_0) = x_0(\varepsilon) \tag{2.26}$$

$$\varepsilon \dot{z} = g(x, z, \varepsilon), \quad z(t_0) = z_0(\epsilon) \tag{2.27}$$

The vector fields f and g are assumed to be continuously differentiable in their arguments for $(x, z, \varepsilon) \in D_x \times D_z \times [0, \varepsilon_0]$, where $D_x \subset \mathbb{R}^n$ and $D_z \subset \mathbb{R}^m$. One of the main characteristics of singular perturbation models is their two-time scale property. For small values of ε, $\dot{z} = \dfrac{g}{\varepsilon}$ is very large. So z varies rapidly, while the variations of the state x are slow compared to z. This leads to the separation of 'fast' states, z, and 'slow' states, x.

When z varies rapidly, it quickly converges to an equilibrium point. For this, it needs to be assumed that the fast dynamics g/ε are stable. Such an equilibrium point can be computed by setting $\varepsilon = 0$ in (2.27). This leads to an algebraic equation

$$g(x, z, 0) = 0 \tag{2.28}$$

Note that since m differential equation have degenerated to algebraic equations, the order of the system is reduced from $(n + m)$ to n. The above model is said to have a standard form if the following assumption holds:

Assumption 2.1 *Equation (2.28) has $k \geq 1$ isolated roots*

$$z = \phi_i(x), \quad i = 1, 2, \ldots, k \tag{2.29}$$

The root $z = \phi(x)$ is called an isolated root in a domain D_x of a set of variables x, if there exists $r > 0$ such that the equation (2.28) has no solution other than $\phi(x)$ for $||z - \phi(x)|| < r$. The condition implies that for each root a well-defined low order dynamics on x exists. By substituting one of the roots $z = \phi_i(x)$, $i = 1, ..., k$, e.g. $\phi(x)$ in equation (2.26) a reduced order model results

$$\dot{\bar{x}}(t) = f(\bar{x}, \phi(\bar{x}), 0), \quad \bar{x}(t_0) = \bar{x}_0 \tag{2.30}$$

Assume that $x_0(\varepsilon) - \bar{x}_0 = O(\varepsilon)$. The dynamics (2.30) are known as *slow dynamics* or *quasi-steady state dynamics*.

The approximation of the fast states z, i.e. $\bar{z}(t)$ is computed from the algebraic equation $\bar{z}(t) = \phi(\bar{x}(t))$. In such cases, the initial value of the approximate solution $\bar{z}(t_0) = \phi(\bar{x}(t_0))$ may be different from the prescribed initial value of the original model (2.26)-(2.27), i.e. $\bar{z}(t_0) \neq z_0(\epsilon)$, and thus there is no guarantee that the approximation error be of the order $O(\varepsilon)$, unless some stability condition is satisfied. The stability conditions on fast dynamics guarantee that after some short time interval the original state z converges to $\bar{z}$, and then remains close to it. To analyze this, define a new time variable $\tau = \dfrac{t - \bar{t}}{\varepsilon}$ where $\bar{t} \in [t_0, t)$. The fast state z can be described in the new time variable τ, as

$$\varepsilon \frac{dz}{dt} = \frac{dz}{d\tau}$$

Since $t = \bar{t} + \varepsilon\tau$ when $\varepsilon \to 0$, the variables t and $x(t)$ will be frozen at $\bar{t}$ and $\bar{x}$. Consider a change of variable $\hat{z}(t) = z(t) - \bar{z}(t)$. It can be verified that at $\varepsilon = 0$, $\hat{z}$ satisfies

$$\frac{d\hat{z}}{d\tau} = g(\bar{x}, \hat{z} + \bar{z}, 0), \quad \hat{z}(0) = z(0) - \bar{z}(0) \tag{2.31}$$

with an equilibrium point at $\hat{z} = 0$. The above equation is known as the *boundary layer model*, since it describes the dynamics of the fast states in the fast time scale (also referred to as the boundary-layer interval), i.e. $[t_0, t_0 + \delta_b]$, $\delta_b < \delta$. During the boundary-layer interval, the solution of the original model, $z(t, \varepsilon)$, converges to the steady state value, $\bar{z}$, as the slow states, x, are frozen to their initial values x_0.

The following theory due to Tikhonov states the conditions under which the full (original) singular perturbed model tends to the reduced model.

Theorem 2.5 *[32] Consider the singular perturbation problem (2.26)-(2.27) and let $z = \phi(x)$ be an isolated root of (2.29). Assume the following conditions are satisfied:*

- *The vector fields $f : D_x \to \mathbb{R}^n$, $g : D_z \to \mathbb{R}^m$ and their first partial derivatives with respect to (x, z, ϵ) are continuous and the function $\phi(x)$ and the Jacobian $[\frac{\partial g}{\partial z}(x, z, 0)]$ have continuous first partial derivatives with respect to their arguments.*
- *The boundary-layer model (2.31) has an exponentially stable equilibrium point $\hat{z} = 0$.*

Then, there exists $\delta_b < \delta$ such that

$$x(t, \epsilon) - \bar{x}(t) = O(\epsilon), \quad \forall t \in [t_0, t_0 + \delta]$$

$$z(t, \epsilon) - \bar{z}(t) - \hat{z}(\frac{t}{\epsilon}) = O(\epsilon), \quad \forall t \in [t_0, t_0 + \delta_b]$$

$$z(t, \epsilon) - \bar{z}(t) = O(\epsilon), \quad \forall t \in [t_0 + \delta_b, t_0 + \delta]$$

Remark 2.1 *The Tikhonov theorem as well as other perturbation theorems are valid only on $O(1)$ time intervals, that is only for finite time intervals $[t_0, t_0 + \delta]$, where δ can be any finite number. To extend $\delta \to \infty$, the exponential stability of the reduced model is also needed.*

The Tikhonov theorem indicates that $\bar{x}$, a solution of the reduced model, is a good approximation of $x(t)$ in the entire time interval, while $\bar{z}$ is a good approximation after a certain time lapse δ_b.

The stability of the equilibrium point for a singular perturbed system is guaranteed if both the boundary-layer system and the reduced model are exponentially stable. This can be summarized by the following theorem:

Theorem 2.6 *[32] Consider the singularly perturbed system defined by equations (2.26)-(2.27). Assume the following assumptions are satisfied for all $(x, \varepsilon) \in B_r \times [0, \varepsilon_0]$:*

- $(x, z) = (0, 0)$ *is the equilibrium point for f (2.26) and g (2.27).*
- *the equation $g(x, z, 0) = 0$ has an isolated root $z = \phi(x)$ such that $\phi(0) = 0$.*
- *The origin of the reduced system $\dot{x} = f(x, \phi(x), 0)$ is exponentially stable.*
- *The origin of the boundary-layer system $\dfrac{d\hat{z}}{d\tau} = g(x, \hat{z} + \phi(x), 0)$ is exponentially stable.*

Then, there exists $\varepsilon^\star$ such that for all $\varepsilon < \varepsilon^\star$, the origin of (2.26)-(2.27) is exponentially stable.

The reason for demanding exponential stability, instead of stability, for the reduced dynamics and the boundary-layer system is that it suggests a margin for the reduction of V, i.e $\dot{V} \leq -c_3||x||^2$. This is an inherent robust property of the exponential stability which guarantees the stability in spite of unmodeled fast dynamics, or in other words, the additional terms that will be present in $\dot{V}$ when $\varepsilon \neq 0$.

2.3 Hybrid Dynamical Systems

2.3.1 Definitions

Hybrid dynamical systems are those with *interaction* between continuous and *discrete event* systems. The main characteristic of hybrid systems is that they consists of two different types of states variables; continuous state taking values in a non-denumerable set, usually real numbers $\mathbb{R}^n$, and discrete states taking values in a countable set. A hybrid model is one that specifies evolution of the continuous and discrete states and also the interaction of continuous and discrete dynamics. The definition of continuous dynamical systems was given in the last section. In order to give a mathematical definition of hybrid dynamical systems, first a formal definition of a discrete event system will be introduced.

Discrete Event Systems

Discrete event systems are those where the state variables take values in a discrete set, e.g. $\{1, 2, ..., \}$, and state transitions are only observed at discrete points in time. The state transitions are associated with 'events' [19]. Discrete event systems are in general represented by *finite automata*. An automaton is a device which generates a sequence of state transitions in accordance with a set of well-defined rules. The term finite automata reflects the fact that the state space is considered to be finite.

Definition 2.7 *Finite automata are described by a triple* (Q, Σ, R)*, where* Q *is a finite set whose elements,* q*, are the discrete states or locations.* Σ *is a set of input symbols or events* σ*, and* R *is a transition function or transition rule. The transition function defines the next discrete state:*

$$q^+ = R(q, \sigma) \tag{2.32}$$

where q^+ *refers to the discrete state after the transition due to the occurrence of the event* σ*.*

The set Σ is also known as an *alphabet*. Finite automata can also be represented by transition diagrams or graphs with vertices given by the elements of Q, and the edges by the transition rules or events.

Hybrid Systems

Now by combining the definition of the continuous system (Definition 2.1), and discrete event systems (Definition 2.7) hybrid dynamical systems can be defined:

Definition 2.8 *A hybrid system* $\mathcal{H}$ *is a collection* $\mathcal{H} = (Q, X, \Sigma, U, F, R)$*, where*

- Q *is a finite set, called the set of discrete states;*
- $X \subseteq \mathbb{R}^n$ *is the set of continuous states;*
- Σ *is a set of discrete input events or symbols;*
- $U \subseteq \mathbb{R}^m$ *is the set of continuous inputs;*
- $F : Q \times X \times U \rightarrow \mathbb{R}^n$ *is a vector field describing the continuous dynamics;*
- $R : Q \times X \times \Sigma \times U \rightarrow Q \times X$ *describes the discrete dynamics.*

The evolution of the system states (q, x) *can be described by the following relations:*

$$\dot{x}(t) = F(q(t), x(t), u(t)), \ \ x(t_0) = x_0 \tag{2.33}$$

$$(q^+, x^+) = R(q(t), x(t), u(t), \sigma(t)), \ \ q(t_0) = q_0 \tag{2.34}$$

where $q \in Q$ *are the discrete states,* $x \in X$ *are the continuous states,* $u \in U$ *are the continuous inputs and* $\sigma \in \Sigma$ *are the discrete inputs.* $(.)^+$ *refers to variables after a transition,* x_0 *and* q_0 *are respectively the initial conditions of the continuous and discrete states.*

The transition function R defines all the logical rules and conditions needed for a transition of states when an event occurs. The transition involves a switch in the discrete state and an eventual jump in the continuous state, i.e. $x^+ \neq x^-$, where x^- refers to a continuous state just before transition. As can be seen from the definition, the interaction of continuous and discrete states is present in a hybrid system. The vector field, F, is also a function of q, while the discrete transition function, R is also a function of continuous variables x and u.

The hybrid system can be viewed as finite automata where to each location (discrete state) a continuous dynamics is associated. Since each discrete state q defines a different dynamics of the system, it is often referred to as *mode* q. The changes of the discrete state imply sudden changes or discontinuities in the vector field F. This way, systems with hard nonlinearities such as relay or hysteresis can also be modeled in this framework. In this thesis, often the notation $F_q(x, u)$ will be used instead of $F(q, x, u)$ to stress the fact that the vector field F changes significantly when q changes.

Classification of Discrete Transitions

Since R is a function of continuous and discrete variables, two types of transitions can be distinguished [15], [69]:

- A transition is a consequence of changes of the external discrete variables σ. Such discrete transitions are known as *controlled* switching or externally induced switching. Controlled switching arises, for instance, when selection among a family of dynamical systems $\dot{x} = F_q(x)$, $q \in Q$ is of interest (*switched systems*) [37]. In such cases, the external input or event indicates which model or dynamics should be selected.

- A transition occurs when the continuous states x and inputs u satisfy certain conditions. Such discrete transitions are known as *autonomous* switching or internally induced switching. In other words, it is possible to define a hyper-surface

  $$\mathcal{S}^{i \to j} = \{(x, u) \in X \times U | R(i, x, u) = j,\ i, j \in Q\} \tag{2.35}$$

  where $\mathcal{S}^{i \to j}$ is the set of values of x and u that will cause a transition from discrete state $q = i$ to $q = j$.

2.3.2 Hybrid State Trajectories: Existence and Uniqueness

A hybrid trajectory is the evolution of the hybrid states $(x(t), q(t))$ over the time set $\mathbb{R}^+ = [0, \infty)$. The switching times are represented by τ_k with k taking its values in a positive integer set $\mathbb{N} = \{0, 1, 2, \ldots\}$, with $\tau_k \leq \tau_{k+1}$.

At each time instant, τ_k, the discrete state switches to another value, however between two consecutive switching times, i.e. $t \in [\tau_k, \tau_{k+1})$, the value of $q(t)$ remains constant. This way $q(t)$ is a piecewise constant function of time.

A hybrid evolution can briefly be described as follows. The system starts at time τ_0 with the initial state $(x(\tau_0), q(\tau_0)) = (x_0, q_0)$ and the continuous state x evolves according to the differential equation $\dot{x} = F(q_0, x, u)$, (x satisfies (2.4) with $F = F(q_0, x, u)$), while $q(t)$ remains constant at q_0. When a transition occurs at $t = \tau_1$, the discrete state switches to $q = q_1$. Moreover, the transition could also involve a jump in the continuous state, i.e $x^+(\tau_1) \neq x^-(\tau_1)$. After this instantaneous transition, the evolution continues from the new state $(x^+(\tau_1), q(\tau_1))$ according to the differential equation related to mode q_1, $\dot{x} = F(q_1, x, u)$, until the next transition occurs. Figure 2.1 illustrates a typical hybrid state trajectory.

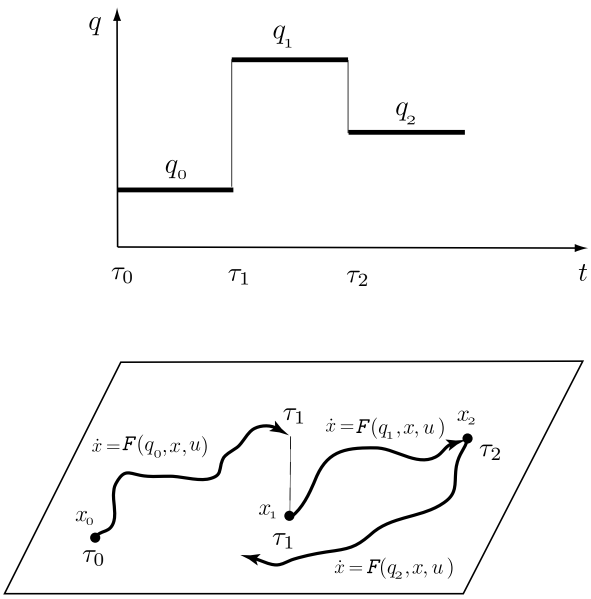

Figure 2.1: Typical hybrid trajectory

Existence and uniqueness of trajectories of hybrid systems depend on the characteristics of both continuous and discrete dynamics. When a system evolves in mode q, the existence and uniqueness of the solution depends on the properties of the continuous vector field F_q, which were discussed earlier. The conditions of existence and uniqueness for continuous evolution were also discussed earlier. In order to have a well-defined hybrid trajectory, additional conditions related to the characteristics of discrete transitions R are also needed and these will be discussed next.

Conditions on Transition Rule, R:

In order to have a well-defined and unique hybrid trajectory the discrete event dynamics or the transition rule (2.34) should satisfy certain conditions. In the theory of discrete event systems, two main problems of blocking and non-determinism could prevent a finite automaton from having a unique solution [19]. Such problems could as well occur for a hybrid dynamical system. To have a well-defined system, the discrete transition should be *deterministic* and *non-blocking*. These notions will be discussed here

- **Non-determinism**
  Non-determinism corresponds to a situation where at a given transition time, evolution in more than one mode is possible. In such a case, the transition rule no longer represents a specific new discrete state q^+, but rather a *set* of possible new states, and hence non-uniqueness of the hybrid trajectory, i.e. $R(i,x,u,\sigma) = \{j,k\}$, $j \neq k \in Q$. In the case of autonomous switching, if the hypersurfaces $\mathcal{S}^{i\to j}$ and $\mathcal{S}^{i\to k}$ intersect, i.e.

  $$\mathcal{S}^{i\to j} \cap \mathcal{S}^{i\to k} \neq \emptyset, \quad \text{for} \quad \forall i \neq j \neq k \in Q$$

  this could lead to non-determinism, and hence non-uniqueness of solutions.

- **Blocking**
  Blocking corresponds to the situation where for a given transition time, no more evolution is possible. Such situations occur if, in the model of a hybrid system the condition for continuous evolution in mode q is violated but, at the same time, the conditions for a transition to occur are not satisfied. This means no continuous evolution of x as well as transition of discrete state q is possible. This situation is usually called *deadlock* since the state is stuck or blocked, i.e. $R(i,x,u,\sigma) = \emptyset$.

  Another possibility is to block the continuous evolution on switching indefinitely among several modes without exiting the *loop*, i.e. $R(i,x,u,\sigma) = j$ and $R(j,x,u,\sigma) = i$. This condition is called *livelock*, and means that although the system is 'alive' (infinite switching), no continuous evolution is possible. In the autonomous switching case this corresponds to

$$\mathcal{S}^{i\to j} \cap \mathcal{S}^{j\to i} \neq \emptyset \quad \forall i \neq j \in Q$$

In such cases, once the switching condition for transition from mode $q = i$ is satisfied the system switches to mode $q = j$, but at the same time, because of the switching condition, the system should switch back to $q = i$. Thus, there could be infinite switches between mode $q = i$ and $q = j$, and hence livelock blocking.

From the arguments presented so far, existence and uniqueness of a hybrid trajectory can be summarized by the following theorem.

Theorem 2.7 *[39] A hybrid system has a unique solution if*

- *the continuous dynamics are Lipschitz continuous, and*
- *the discrete transition rule is deterministic and non-blocking.*

Remark 2.2 *In certain hybrid systems, infinitely many discrete transitions could occur within a finite time interval, that is* $\sum_{k=0}^{\infty}(\tau_{k+1} - \tau_k) < \infty$. *This phenomenon is called Zeno referring to the philosopher Zeno's famous paradox of Achilles and the turtle. The Zeno phenomenon is due to a high level of abstraction in modeling the hybrid system and thus real physical systems are not Zeno. However, analysis and simulation of hybrid models possessing such behavior could not be evaluated after a finite time (Zeno time), since the solution is not defined after such a time. In order to extend the solution beyond the Zeno time, regularization techniques need to be used [29]. Such systems are not considered in this thesis.*

Here, the class of hybrid dynamical systems which will be used in this thesis is presented.

Assumption 2.2 *The class of hybrid systems satisfying the following conditions is considered:*

- *The vector fields for each mode* $F_q(x, u)$ *are locally Lipschitz with no finite escape time;*
- *The transition rule* R *is non-blocking and deterministic;*

- *No jump or discontinuous transition on continuous states x;*
- *No Zeno phenomenon.*

2.3.3 Inter-connection of Hybrid Systems

Hybrid Systems with Outputs

In the case of physical systems modeled by hybrid models, it is not possible to measure all states (continuous and discrete) of the system in many cases. So, it would be reasonable to introduce the concept of hybrid dynamics with outputs:

Definition 2.9 *In addition to Definition 2.8, a hybrid system with outputs has the following elements:*

- *$Y \subseteq \mathbb{R}^p$ is the set of continuous outputs;*
- *O the set of discrete outputs;*
- *h: $Q \times X \times U \to Y \subset \mathbb{R}^p$ describes the continuous output map;*
- *r: $Q \times X \times U \times \Sigma \to O$ describes the discrete output map.*

The output relations can be described by the following relations:

$$y(t) = h(q(t), x(t), u(t)) \quad (2.36)$$

$$o^+ = r(q(t), x(t), u(t), \sigma(t)) \quad (2.37)$$

where $y \in Y$ are the continuous outputs and $o \in O$ are the discrete outputs or output events.

Once the hybrid system with input and output is defined, the inter-connections of hybrid systems are studied. Here, after defining the notions of hybrid plant and controller, a feedback inter-connection of these systems will be presented:

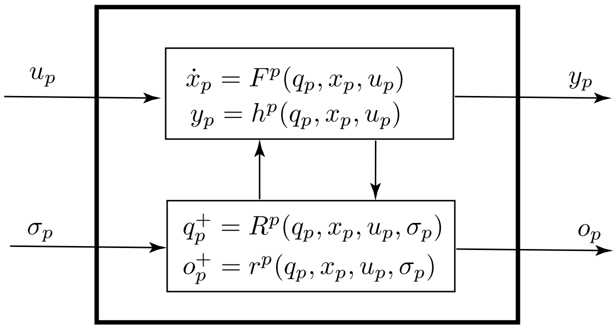

Figure 2.2: Hybrid plant $\mathcal{H}_p$

Hybrid Plant

From the control engineering point of view, the system to be controlled is referred to as the *plant*. If the plant is modeled in the hybrid framework, (Definition 2.9), it will be termed a hybrid plant.

A graphical representation of a hybrid plant is illustrated in Figure 2.2 where u_p is the continuous input, σ_p is the discrete input, x_p is the continuous state, q_p is the discrete state and y_p and o_p are continuous and discrete outputs of the hybrid plant respectively.

Hybrid Controller

The hybrid plant could be connected, via a feedback connection, to a *hybrid controller* to form a closed-loop hybrid system [51]. A hybrid controller is also a hybrid system as in Definition 2.9 and in general consists of continuous states x_c, discrete states q_c, continuous inputs u_c, discrete inputs σ_c, continuous outputs y_c and discrete outputs o_c. The special feature of the controller is that the continuous input to the hybrid controller can be partitioned in two parts: one part that comes from the hybrid plant (continuous outputs of $\mathcal{H}_p$, e.g. y_p) and the other part which does not, e.g. reference signals or new external inputs ν. The graphical representation of hybrid controller is depicted in Figure 2.3.

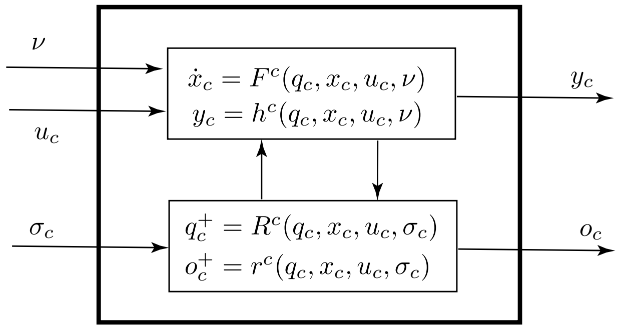

Figure 2.3: Hybrid controller $\mathcal{H}_c$

Closed-loop Hybrid System

A closed-loop hybrid system consists of the feedback configuration of the hybrid plant $\mathcal{H}_p$ and hybrid controller $\mathcal{H}_c$ [51]. The feedback configuration of plant and controller is shown in Figure 2.4.

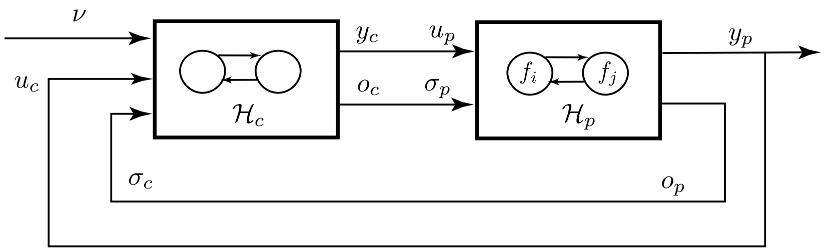

Figure 2.4: Closed-loop configuration of $\mathcal{H}_p$ and $\mathcal{H}_c$: $\mathcal{H}$

The inputs of the hybrid plant are the outputs of the hybrid controller and the inputs of the hybrid controller are outputs of the hybrid plant. The feedback connection of plant and controller is also a hybrid system, $\mathcal{H}$, which can be described by the following relations:

$$\dot{x}(t) = F(q, x, \nu) \quad (2.38)$$
$$q^{+} = R(q, x, \nu) \quad (2.39)$$
$$y(t) = h(q, x, \nu) \quad (2.40)$$

where $x = (x_c, x_p)^T$, $q = (q_c, q_p)^T$. If desired, the output y can contain the components of y_p and y_c e.g. $y = (y_c, y_p)^T$.

Assumption 2.3 *On the basis of Definition 2.9, a hybrid system could have discrete inputs σ and discrete outputs o. However, in this thesis, it will be considered that the closed-loop configuration has no discrete inputs and outputs, (2.38)-(2.40).*

2.3.4 Modeling Discontinuous Dynamics in a Hybrid Framework

Systems with discontinuous vector fields have been the subject of intense research [67, 24]. Variable structure systems, relay systems and bang-bang controllers belong to the class of such systems. Since one of the properties of hybrid formulation is to treat different dynamics (vector field switching), then it seems that discontinuous dynamics can be modeled in a hybrid modeling framework. In this section, a hybrid representation of dynamical systems with discontinuities will be presented.

Discontinuous dynamics are modeled by differential equations, $\dot{x} = F(x)$, whose vector field F is discontinuous on a smooth surface $\mathcal{S}$:

$$\mathcal{S} = \{x \in \mathbb{R}^n |\ S(x) = 0\}$$

Such a hypersurface is called the discontinuous surface. Let $\mathcal{S}^+$ and $\mathcal{S}^-$ denote the set of all points where $S(x)$ is positive and negative respectively.

Since the vector field changes abruptly when trajectories of the system hit the discontinuous surface it is reasonable to define such a system within a hybrid framework. The hybrid modeling of discontinuous systems can be summarized by the following two main steps:

a) Modes $q = s^-$, $q = s^+$, $q = s$

Let F_{s^-}, F_{s^+} and F_s represent the vector fields defined for $x \in \mathcal{S}^-$, $x \in \mathcal{S}^+$ and $x \in \mathcal{S}$ respectively. The first step in constructing a hybrid model is to define three discrete states (modes) $q = s^-$, $q = s^+$, $q = s$ and to assign these discrete states the vector fields F_{s^-} , F_{s^+} and F_s respectively.

b) Transition Rules and Regularization

In order to define the transition rules, let x be a point of discontinuity on the surface $\mathcal{S}$ and denote $F^-(x)$ and $F^+(x)$ by the limits of $F(x)$ as the point x is approached from opposite sides of the tangent plane to $\mathcal{S}$. These vector fields are continuous in their arguments on sets $\mathcal{S}^-$ and $\mathcal{S}^+$. Let ${F^-}^N$ and ${F^+}^N$ be the projections of $F^-(x)$ and $F^+(x)$ onto the normal to the surface $\mathcal{S}$ at the point x, that is:

$${F^-}^N = \frac{\nabla S(x).F^-}{|\nabla S(x)|}, \qquad {F^+}^N = \frac{\nabla S(x).F^+}{|\nabla S(x)|}$$

where $\nabla S = \frac{\partial S}{\partial x}$. It is clear that the switching condition or transition rule between modes is defined by $S(x) = 0$. However, it is well known that for a discontinuous dynamics it is possible to have infinite switching ('sliding mode'). Such infinite switching leads to a blocking (livelock) transition function for modes $q = s^-$ and $q = s^+$. Infinite switching is related to the behavior of the vector fields F^- and F^+ or precisely to the directions of the vector fields on both sides of the discontinuous surface $S(x)$. In order to make the hybrid model well-defined, a *regularization* is needed.

If the vector fields on both sides of the discontinuous surface $\mathcal{S}$ have the same direction, i.e. ${F^-}^N(x).{F^+}^N(x) > 0$, the solution hits the discontinuous surface and passes through the discontinuity. In this case, the switching condition for transition from $q = s^-$ to $q = s^+$ is given by $\mathcal{S}^{s^- \to s^+} = \{x \in X | S(x) = 0, {F^-}^N(x) > 0, {F^+}^N(x) > 0\}$. Furthermore, $\mathcal{S}^{s^+ \to s^-} = \{x \in X | S(x) = 0, {F^-}^N(x) < 0, {F^+}^N(x)\}$.

It is possible that vector fields ${F^-}^N(x)$ and ${F^+}^N(x)$ have opposite signs pointing to the discontinuous surface, i.e. ${F^+}^N(x)$ points inside $\mathcal{S}^-$ and ${F^-}^N(x)$ points inside $\mathcal{S}^+$. In such a case, the solution after hitting the discontinuous surface has a tendency to remain on $\mathcal{S}$, since it is pushed from both sides of the switching surface. Yet, it is not clear how the vector field is defined on the discontinuous surface. To define the solution, a regularization approach is needed. Regularization means that the dynamics of the system on the discontinuous surface should be such that the evolution of the system takes place on that surface. The solution on the switching surface is known as a 'sliding motion' or 'sliding mode' solution. The term sliding mode first appeared in the context of relay and variable structure systems. The main characteristic of the sliding mode is that the new vector field defining the dynamics of the system should keep the trajectory on the surface $\mathcal{S}$, that is

$$\dot{x} = F_s(x): \quad S(x(t)) = 0 \quad \text{and} \quad \dot{S}(x(t)) = 0, \; \forall t \geq t_r > 0$$

where t_r is the time when the system's trajectory hits the surface. Different solution concepts on the switching surface are proposed in the literature, two of such will be presented next.

- The first concept was proposed by Filippov in what he called 'the simplest convex definition' [24]. According to this method, the solution is obtained by a convex combination of the vector fields $F_{s^-}(x)$ and $F_{s^+}(x)$

$$\dot{x}(t) = \alpha F_{s^+}(x) + (1-\alpha)F_{s^-}(x) \tag{2.41}$$

where $\alpha \in (0,1)$ is such that the vector field

$$F_s \triangleq \alpha F_{s^+}(x) + (1-\alpha)F_{s^-}(x)$$

is tangential to the surface $\mathcal{S}$. That is,

$$\dot{S}(x) = \frac{\partial S}{\partial x}F_s = 0 \Rightarrow \quad \alpha = \frac{\nabla S.F_{s^-}}{\nabla S.(F_{s^-} - F_{s^+})} \tag{2.42}$$

where $\nabla S = \frac{\partial S}{\partial x}$. From a geometric point of view, the end point of the vector field F_s lies on the intersection of the tangent plane to $\mathcal{S}$ with a linear segment joining the endpoints of the vectors $F_{s^-}(x)$ and $F_{s^+}(x)$. The differential equation $\dot{x} = F_s(x)$, defines the evolution of the system on $\mathcal{S}$, and is called sliding motion or sliding mode (see Figure (2.5 a)).

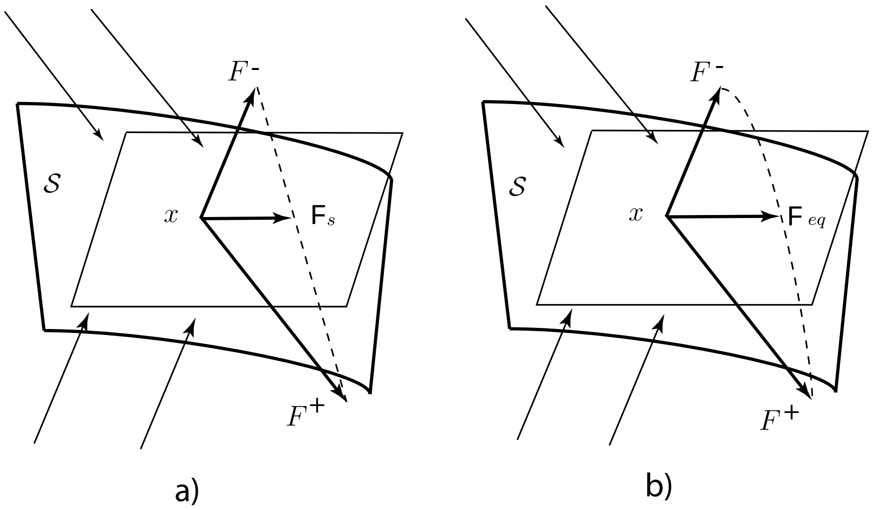

Figure 2.5: Two different interpretations of solutions on the discontinuous surface, 'sliding motion': a) simplest convex set, b) equivalent control

- The second concept is the so-called 'equivalent control' concept proposed by Utkin [67]. Let the discontinuous vector field be of the form $F(x, z(x))$ where $z(x)$ is a function which has single values $z^+(x)$ and $z^-(x)$ at the points of continuity (each side of the discontinuous surface) and indefinite at the surface of discontinuity $\mathcal{S}$, that is it can take values in the closed interval $Z(x) = [z^-(x), z^+(x)]$. Note that $z(x)$ could represent both the discontinuities on the state x and inputs u. The sliding mode occurs if the state hits the discontinuous surface and $z(x) = z_{eq}$ can be found such that it maintains the motion on the switching surface $S(x) = 0$. This can be found by calculating the vector z such that the time derivatives of the state trajectories along the switching surface are equal to zero:

  $$\dot{S}(x) = \frac{\partial S}{\partial x} F(x, z_{eq}(x)) = 0 \tag{2.43}$$

  The solution $z_{eq}(x)$ is called the equivalent control, and the sliding motions are defined by the vector field $F_{eq} \triangleq F(x, z_{eq}(x))$. In this case, the endpoint of the vector field F_{eq}, lies on the intersection of the tangent plane to $\mathcal{S}$ with an *arc* which is spanned by the endpoint of the vector field $F(x, z(x))$ when z varies over the set $Z(x)$, i.e. $z(x) \in [z^-(x), z^+(x)]$, (see Figure (2.5 b)). The vector fields F^- and F^+ shown in the figure are shorthand for $F(x, z^-(x))$ and $F(x, z^+(x))$.

Once the sliding motion is defined using one of the above mentioned vector fields, i.e. F_s or F_{eq}, a new discrete state $q = s$ can be attributed to such dynamics. The construction of the hybrid model will be completed by defining the switching conditions for mode $q = q_s$ which is given by the following sets:

$$\mathcal{S}^{s^+ \to s} = \mathcal{S}^{s^- \to s} = \{x \in X |\ S(x) = 0, F^N_{s^-} > 0,\ F^N_{s^+} < 0\}$$

Remark 2.3 *If the vector fields $F^-(x, z(x))$ and $F^+(x, z(x))$ having the opposite direction but pointing out the discontinuous surface, that is $F^N_{s^+} > 0$ and $F^N_{s^-} < 0$, then a solution which passes through a point of surface $\mathcal{S}$ at $t = \tau$ may either go off the surface into sets $\mathcal{S}^+$ or $\mathcal{S}^-$, or remain on $\mathcal{S}$. Such situations lead to non-unique solutions, and hence a non-deterministic hybrid model.*

A hybrid model of a system with discontinuity is shown in Figure 2.6.

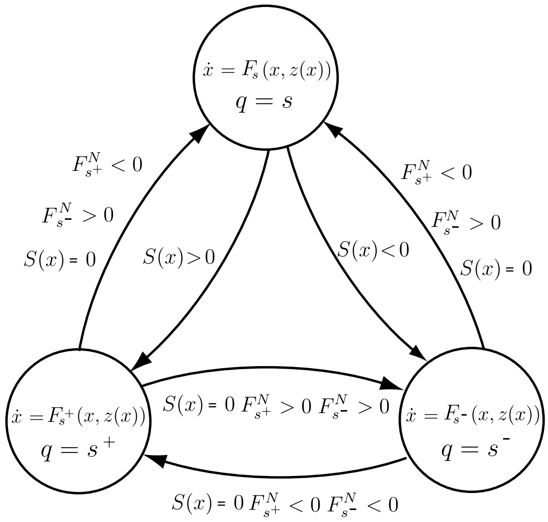

Figure 2.6: Hybrid model of discontinuous dynamics

Remark 2.4 *The vector fields F_s and F_{eq} are in general different, however, if the vector field $F(x, z(x))$ is affine in the discontinuous states $z(x)$, that is,*

$$F(x, z(x)) = F_a(x) + g(x)z(x)$$

then both definitions coincide [24].

In general, a system's vector field could have several discontinuous surfaces. In such cases, the dynamics can be represented by

$$\dot{x} = F(x, z_1(x), \ldots, z_r(x))$$

where functions $z_i(x)$ are discontinuous, respectively, on the switching surfaces $\mathcal{S}_i$, $i \in \{1, 2, \ldots, r\}$. Let us define a set $Q_i = \{s_i^+, s_i^-, s_i\}$ whose elements, q_i, as described earlier, indicate the values that $z_i(x)$ can take in each set $\mathcal{S}^+$, $\mathcal{S}^-$ and $\mathcal{S}$, that is $z_i^+(x)$, $z_i^-(x)$ and z_i^{eq} respectively. z_i^{eq} is the solution of the equivalent control equation

$$\nabla S_i(x).F(x, z_1(x), \ldots, z_r(x)) = 0$$

with $z_j(x)$, $j \neq i$ given by their appropriate values on the set $\mathcal{S}_i$.

For a discontinuous dynamics a hybrid model can be constructed where its discrete states take values in the set $Q = Q_1 \times Q_2 \times \ldots Q_r$ of all r-tuples, $q = (q_1, q_2, \ldots, q_r)$ with $q_i \in Q_i$, $i = 1, 2, \ldots, r$. The hybrid model could have a maximum of 3^r discrete states or modes. The continuous dynamics and transition rules are derived in a similar way to that was explained earlier for a system with one discontinuous surface, i.e. $r = 1$. Obviously, the construction procedure will be complicated and cumbersome as the number of the discontinuous surfaces r increases. However, the goal of this section was just to show the procedure of modeling, and to introduce the concepts such as sliding motion and regularization of the solution for discontinuous systems, without arguing about the advantages or disadvantages of such a formulation.

Example 2.1 Consider a mass, m, and a spring, k, and this system subject to an input force, u, and a friction force $\mathcal{F}$ (see Figure 2.7).

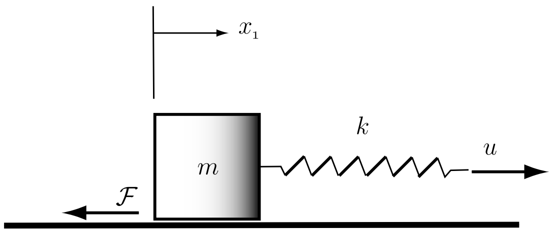

Figure 2.7: A mass spring system subjected to friction

The dynamics of the system is described by the following state space equations:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{m}(u - kx_1 - \mathcal{F}) \end{aligned} \tag{2.44}$$

where x_1 and x_2 are states representing the position and velocity of the mass. The friction is defined by the function $\mathcal{F} = F_c \text{sgn}(x_2) = z(x)$ where F_c represents the Coulomb friction level and $z(x) \in Z = [-F_c, F_c]$ is discontinuous on the line defined by $\mathcal{S} = \{x \in \mathbb{R}^2 | S(x) = x_2 = 0\}$ (Figure 2.8).

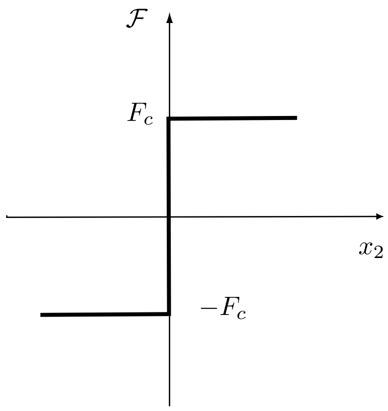

Figure 2.8: Coulomb friction

Let us define two modes $q = s^- = slip^-$ and $q = s^+ = slip^+$ with continuous dynamics given by the vector fields on both sides of the discontinuous surface $\mathcal{S}$, $x_2 < 0$ and $x_2 > 0$, that is

$$F_{s^-}(x, u, z(x)) = \begin{pmatrix} x_2 \\ \frac{u - kx_1 + F_c}{m} \end{pmatrix}, \quad F_{s^+}(x, u, z(x)) = \begin{pmatrix} x_2 \\ \frac{u - kx_1 - F_c}{m} \end{pmatrix} \tag{2.45}$$

The projection of F_{s^-} and F_{s^+} on to the normal of the surface $S(x) = x_2 = 0$ is given by

$$F_{s^-}^N = \frac{\nabla S(x).F_{s^-}}{|\nabla S(x)|} = (0 \quad 1).\begin{pmatrix} x_2 \\ \frac{u - kx_1 + F_c}{m} \end{pmatrix} = \frac{1}{m}(u - kx_1 + F_c) \tag{2.46}$$

$$F_{s^+}^N = \frac{\nabla S(x).F_{s^+}}{|\nabla S(x)|} = (0 \quad 1).\begin{pmatrix} x_2 \\ \frac{u - kx_1 - F_c}{m} \end{pmatrix} = \frac{1}{m}(u - kx_1 - F_c) \tag{2.47}$$

It can be easily seen that for $|u - kx_1| > F_c$, the vector fields on both sides of the switching surface have the same direction, i.e. $F_{s^+}^N.F_{s^-}^N > 0$, trajectories of the system after hitting the line $x_2 = 0$ will pass through it. In such a case the system switches between modes $q = s^-$ and $q = s^+$.

When $|u - kx_1| \leq F_c$, the vector fields are pointing into the switching surface $x_2 = 0$, and thus sliding motion occurs (trajectories approach from both sides of the surface and they will remain there), i.e. $F^N_{s^+}.F^N_{s^-} < 0$. In such a case, regularization is needed. Define a new mode $q = s = stick$ and by using the equivalent control solution concept, the sliding mode vector field can be calculated:

$$\nabla S(x).F(x, u, z_{eq}) = 0 \Rightarrow (0 \quad 1).\begin{pmatrix} x_2 \\ \frac{u - kx_1 - z_{eq}}{m} \end{pmatrix} = 0 \Rightarrow z_{eq} = u - kx_1 \tag{2.48}$$

where $-F_c \leq z_{eq} \leq F_c$. The dynamics of the system in mode $q = s$ then given by

$$F_s(x, u, z_{eq}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.49}$$

Note that, from the dynamics of the sliding mode (2.49), it can be seen that every continuous state belonging to the set $\{(x_1, x_2)|x_2 = 0, |x_1| \leq \frac{F_c}{k}\}$ is an equilibrium point of the system.The direction of the vector fields on a phase diagram (for $u = 0$) is illustrated in Figure 2.9.

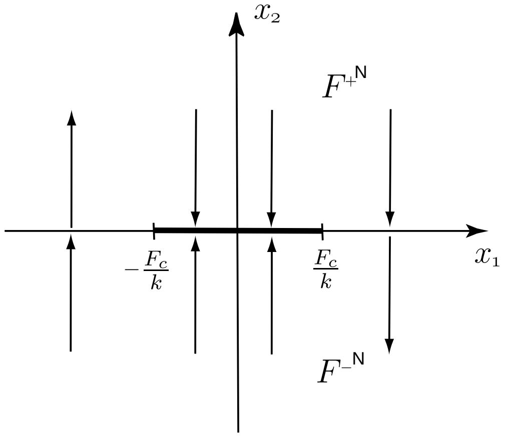

Figure 2.9: Phase plane of mass-spring system: direction of the vector fields and sliding surface

The hybrid model $\mathcal{H} = (Q, X, U, F, R)$ of the mass-spring system with friction can be defined as follows:

$q \in Q = \{s^-, s^+, s\}$, $(x_1, x_2) \in X = \mathbb{R}^2$ and $u \in U = \mathbb{R}$ with the following continuous and discrete dynamics

$$F : \begin{cases} F(s_-, x, u) = \begin{pmatrix} x_2 \\ \frac{u - kx_1 + F_c}{m} \end{pmatrix} \\ F(s_+, x, u) = \begin{pmatrix} x_2 \\ \frac{u - kx_1 - F_c}{m} \end{pmatrix} \\ F(s, x, u) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{cases}$$

$$R(q, x, u) : \quad q^+ = \begin{cases} s & \text{if} \quad (q = s^- \text{ or } q = s^+),\ x_2 = 0,\ |u - kx_1| \leq F_c \\ s^+ & \text{if} \quad (q = s^- \text{ or } q = s),\ x_2 = 0,\ u - kx_1 > F_c \\ s^- & \text{if} \quad (q = s^+ \text{ or } q = s),\ x_2 = 0,\ u - kx_1 < -F_c \end{cases}$$

Remark 2.5 *From the above example, it can be seen that the friction force has a two structure property (stick and slip) and coincides with the definition of friction in models such as that of Karnopp [31]. The main property of these models is that they capture the behavior of friction at 'stiction' (no movement). The friction is known to be both a function of velocity and the applied force. When the mass is in movement ('slip'), a Coulomb definition, i.e.* $\mathcal{F} = F_c \text{sgn}(v)$*, is used for describing the friction. For zero velocity, when the force acting on the mass, i.e.* $\mathcal{F}_u = u - kx_1$ *is smaller than the Coulomb level* F_c*, the friction is equal to that force, i.e.* $\mathcal{F} = \text{sat}(\mathcal{F}_u, F_c)$*, ('stick').*

Chapter 3

Control of Hybrid Systems via Dehybridization

3.1 Introduction

In the previous chapter, two-time scale systems were discussed with emphasis on the singular perturbation method for continuous systems. In systems exhibiting a two-time scale property, two types of state variables exist: 'fast' and 'slow' state variables. The main characteristic feature of such systems is that the dynamics of fast states can be neglected on a slower time scale and hence the system can be approximated by a lower order dynamics describing only the evolution of the slow states. The approximation properties depend on a small parameter ε which is related to the speed or time constant of fast states. The above mentioned time scale separation in dynamical systems could be either 'natural' or 'artificial'. The former is caused by the small parasitic parameters inherent to dynamical systems, e.g. masses, capacitances, while the latter arises from the control scheme, e.g high-gain parameters in feedback systems or cascade control.

In hybrid dynamical systems, it is also possible to have the two-time scale property. Consequently, fast and slow state variables can be distinguished in the system. When fast states correspond to discrete states and slow states to continuous states of the system, then similar to the arguments mentioned above, it can be expected that the slow response of a hybrid system be approximated by a reduced order model. Since the fast transitions are defined by discrete states, then the reduced order model consists *only* of the 'slow' continuous states, and hence the reduced order model will be a *purely continuous-time* one that defines the 'average' behavior of the system. Since by averaging, the hybrid nature of system is removed, the approximation procedure is more than a model order reduction and in this thesis is termed as *dehybridization*. The ideas of dehybridization was first introduced in [56].

Since the discrete states are piecewise constant in time, their evolution is only marked by their transitions. So, the discrete states evolving faster than the continuous ones corresponds to fast switching , i.e. small switching time intervals. Hence, the parameter ε that characterizes the properties of the approximation depends on the switching time intervals. Thus, when the time spent in all discrete states is small, a good continuous approximation can be obtained.

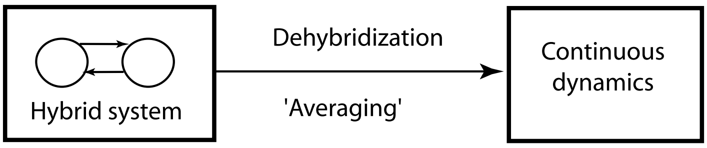

Figure 3.1: Dehybridization: Fast discrete transition and slow continuous dynamics $\rightarrow$ approximation by an averaged continuous model

It will be noted that dehybridization can also be applied when the time spent in *one* of the states is not small, while that spent in all other states is small. This class of systems is referred to as systems with a *dominant* mode. So, the real requirement for good approximation is that the switching time intervals of *all but one* modes should be small. In reference to this extended class of systems, a hybrid system with *recurrent* mode is defined, where there is at least one mode that repeats in the sequence of discrete transitions. As a generalization, the approximation parameter ε depends on the time spent on *all but one* modes. It can be noted that this class encompasses the systems with fast switching and those with a dominant mode. The link between systems with recurrent mode and dominant mode is easy to see.

As discussed earlier, the reasons exhibiting a two-time scale property, could be either natural or artificial. Not all systems have a natural two-time scale separation. So, in order to take advantage of 'dehybridization', it becomes important to propose a control scheme in which the two-time scale property can be artificially created, if necessary. In this chapter, a cascade control design scheme for hybrid systems will be presented. In the cascade scheme, the inner-loop controller imposes the two-time scale property (fast switching/mode domination), while the outer-loop takes the task of satisfying the desired control objectives. The design of the outer-loop is based on the approximated continuous dynamics and hence, it is termed control design via dehybridization.

This chapter is organized as follows. An alternative representation of hybrid dynamics is presented in the first part of Section 3.2. Hybrid systems with recurrent mode are introduced in 3.2.1. Based on such structures, the parameter ε is defined in Section 3.2.2. The averaged continuous model is presented in Section 3.2.3 where analogies with sliding mode and singular perturbation are examined. Finally, the cascade control scheme using dehybridization is introduced in Section 3.3 where the inner-loop and outer-loop designs are discussed in Sections 3.3.2 and 3.3.3 respectively.

3.2 Dehybridization

Consider the dynamics of a hybrid system defined by equations

$$\dot{x} = F(q,x) \tag{3.1}$$
$$q^{+} = R(q,x,\sigma) \tag{3.2}$$

where $F : Q \times D \to \mathbb{R}^n$ are locally Lipschitz on a domain $D \subset X = \mathbb{R}^n$, and R defines the transition rules for discrete states $q \in Q$. Here, an alternative representation of the above dynamics will be given, where the discrete state is represented as an l dimensional vector. This representation will be used later to formulate and relate the hybrid model to the continuous averaged one. Let ψ be a one-to-one map that maps each element of a discrete set, $q \in Q$, to the set E, which is the canonical basis of the real vector space $\mathbb{R}^l$, $l = \mathrm{Card}(Q)$, that is $\psi : Q \to \mathbf{W}$ where $w_1 = (1,0,\ldots,0)^T$, $w_2 = (0,1,\ldots,0)^T$, ..., $w_l = (0,0,\ldots,1)^T$ and $\mathrm{Card}(Q)$ stands for the cardinality of the set Q. The vector fields in each mode, $F(q,x)$, now can be represented by $F_{w_i}(x)$ or $F_i(x)$, $i = 1,2,...,l$ for ease of notation.

Let $\theta = \psi(q)$, then the hybrid dynamics (3.1)-(3.2) can be represented by the following equations:

$$
\begin{aligned}
\dot{x} &= F(\theta, x) = \mathbf{F}(x).\theta = \sum_{i=1}^{l} \theta_i F_i(x) && (3.3) \\
\theta^+ &= R(\theta, x, \sigma) && (3.4)
\end{aligned}
$$

where $(x, \theta) \in \mathbb{R}^n \times \mathbb{R}^l$, $\mathbf{F} = [F_1(x), F_2(x), \ldots, F_l(x)]$ and $\theta = (\theta_1, ..., \theta_l)^T$ with $\theta_i = 1$ for some i and $\theta_j = 0$ for $j \neq i$.

3.2.1 Hybrid Systems with Fast Mode Transitions: Recurrent Mode

Let the dynamics of the hybrid system (3.1)-(3.2) be such that the discrete transitions are faster than the evolution of system's continuous states, i.e. fast switching (small switching time intervals). In the case of fast switching, it can be deduced that at least one mode repeats since the number of discrete states is finite. If in addition, the switching time intervals in *all modes except one* are small, then still the two-time scale property will be preserved (mode domination). The advantage of such an assumption is that the class of systems with fast switching can be enlarged to the class of systems containing both fast switching and mode domination. This class of systems is termed a hybrid system with a recurrent mode. Let such a mode be represented by $q^\star$, then a hybrid system with a recurrent mode $q^\star$ can be defined as follows:

Definition 3.1 *A hybrid system $\mathcal{H}$ has a recurrent mode $q^\star$, if for $\forall t$ with $q(t) \neq q^\star$, $\exists \bar{t} > t$ such that $q(\bar{t}) = q^\star$.*

The definition implies that if the system switches from a particular mode $q^\star$, it will be switched back again to the same mode after some finite time. In the last chapter, the switching times were denoted by τ_k, $k = \{0, 1, 2, ...\}$. Without loss of generality let τ_0 denote the time instant at which the system starts in mode $q^\star$, and τ_N denote the time instant at which the system switches back again to $q^\star$ after transitions to other modes, i.e. $q \neq q^\star$.

A typical sequence of discrete states for a system with a recurrent mode $q^\star$ is as follows:

$$
\ldots \rightarrow \underbrace{q_0 = q^\star \rightarrow q_1 \rightarrow q_2 \rightarrow \ldots \rightarrow q_N = q^\star}_{\tau_N - \tau_0 = T} \rightarrow
$$

where q_j indicates the discrete states at the switching instants τ_j with $q_j \neq q^\star$ for $j = 1, 2, ..., N-1$ and $q_0 = q_N = q^\star$.

Definition 3.2 *The time interval between two occurrences of $q^\star$, $\tau_N - \tau_0$, is called the cycle time T. The fraction of cycle time T—that the system spends in mode q is called the duty ratio α_q.*

In other words, the duty ratio of a mode is given by the ratio of the duration of that mode to the cycle time. For example, when the time duration of a mode is 1 second while the cycle time is 4 seconds, the duty ratio is equal to 0.25. From the above definition it is clear that the duty ratios satisfy $\sum_Q \alpha_q = 1$ with $\alpha_q \in [0,1]$. On the basis of Definitions 3.1 and 3.2, $\alpha_q = 0$ indicates that mode $q \neq q^\star$ is not visited in the cycle, and $\alpha_{q^\star} = 1$ means that after some cycles, the system switches to mode $q^\star$, and from that moment no further switching occurs in the system.

Definition 3.3 *In a hybrid system with a recurrent mode, the cycle time T can vary from one cycle to another. Let T_ℓ denote the cycle time in the ℓ'th cycle. The duty ratio for mode q in the ℓ'th cycle is represented by α_q^ℓ.*

Definition 3.4 *In the special case where in each cycle, i) the sequence of discrete states, ii) cycle times T_ℓ, iii) duty ratios α_q^ℓ, and iv) states at switching instants are identical, the hybrid system is termed periodic.*

3.2.2 Parameter ε and Time-scale Properties

It was discussed earlier that fast switching is the main source for a hybrid system to exhibit a multiple-time scale behavior. Similar to standard two-time scale approximation approaches, the parameter ε, by which the approximation is characterized, is related to the rate at which the fast transition takes place. This means that parameter ε is a function of switching time intervals.

ε will be defined as the maximum time interval the system spends on *non-dominant modes*. Here, 'dominant mode' refers to the mode with the largest time interval compared to all other modes visited during a cycle.Consider a hybrid system with a recurrent mode defined by (3.1)-(3.2) and let T_ℓ be the cycle time at the ℓ'th cycle, then $\max_q\{\alpha_q^\ell T_\ell\}$, $q \in Q$, is the largest time interval during which the system stays in a particular mode in the ℓ'th cycle. Denote $\varepsilon_\ell = T_\ell - \max_q\{\alpha_q^\ell T_\ell\} = \min_q\{T_\ell(1-\alpha_q^\ell)\}$

the time interval during which the system stays in all other modes in that cycle. Define ε as the maximum of such time intervals over all cycles, that is $\varepsilon = \max_{\ell}\{\varepsilon_\ell\}$

$$\varepsilon = \max_{\ell} \min_{q} T_\ell(1 - \alpha_q^\ell) \tag{3.5}$$

Time-scale properties of the hybrid system determined by parameter ε are explained as follows. The variation of continuous states in each cycle, $\Delta x = x(\tau_0 + T_\ell) - x(\tau_0)$ is given by the *time integral* of vector fields F_q visited during that cycle. Consider the following two cases:

i) When the cycle times T_ℓ and consequently the switching time intervals are small, then the variation of continuous states in each cycle is also small. For small T_ℓ and consequently small ε, continuous states $x(t)$ vary more slowly in time compared to discrete states $q(t)$. Thus, a time-scale separation between continuous and discrete states is obvious. The average behavior of continuous states is observable in a slower time scale (after several cycle numbers). ii) When in a cycle a particular mode is dominant, i.e. $(1 - \alpha_{q^\star}) \approx 0$, the system spends most of the cycle time in that mode. Thus the variation of the continuous states in all other modes compared to that mode is small. In such a case, the time-scale separation appears since the slow and fast time scales correspond to the time intervals of dominant and non-dominant modes respectively. Also, it is possible that the cycle time be small with a particular mode being dominant, leading to a small ε. In such a case, the combination of two time-scale effects i) and ii) is observed.

This shows that to have a two-time scale, either the transition between all modes should be fast or the transition between all modes *except* one should be fast. The latter case, which in this thesis will be called mode domination, is a new concept which is combined to the well-known concept of fast switching to generalize the notion of averaging for a larger class of hybrid systems.

Similar to standard averaging approaches, it can be expected that for sufficiently small values of ε, the behavior of the slow states be described by an averaged model. The averaged model of the hybrid system with a recurrent mode will be introduced in the next section.

Remark 3.1 *Two possible cases for calculating the parameter ε can be distinguished: i) For a hybrid system with a controlled switching rule, the transitions are controlled by the external inputs. In this case, since the transition of external inputs is fixed in advance, the cycle times T_ℓ and duty ratios α_q*

are known. Hence the value of varepsilon is known and can be calculated using (3.5). ii) For a hybrid system with a autonomous switching rule, the switching times are state dependent (See Appendix A.2.1 for illustrative example). Hence the switching time intervals depend on the initial condition of the continuous states. This suggests that ε, as a function of switching time intervals, be dependent on the initial condition, i.e. $\varepsilon(x_0)$. However, if a superior bound on the switching time intervals regardless of the initial condition can be found, then ε is given by such a bound. This bound can be a function of parameters of system.

3.2.3 Averaged Continuous Model

Consider a hybrid system with a recurrent mode represented by equations (3.3)- (3.4), and let parameter ε be defined by (3.5). An 'average model' for the hybrid system can be obtained by setting $\varepsilon = 0$. Here, a quasi-steady-state value of θ, i.e. $\bar{\theta}$ will be used

$$\dot{\bar{x}} = F(\bar{\theta}, \bar{x}) = \sum_{i=1}^{l} \bar{\theta}_i F_i(\bar{x}) \tag{3.6}$$

When $\varepsilon = 0$, from (3.5) it can be deduced either that $T_\ell = 0$, or that there exists $q^\star$ such that $(1 - \alpha^\ell_{q^\star}) = 0$. This means either that there are an infinite number of switching, or that no switching occurs at all.

In the case of $(1 - \alpha^\ell_{q^\star}) = 0$, since no transition occurs, the dynamics of the hybrid system are exclusively defined by the continuous dynamics (3.1) with $\bar{\theta} = \theta_i = 1, \theta_j = 0 \; i \neq j$, for $\psi(q^\star) = w_i$. In the case $T_\ell = 0$, as the cycle time goes to zero, the discrete states θ change faster and faster such that in the extreme case, there will be *instantaneous* switching between discrete states (infinite switching). Similar to sliding mode regularization discussed in Section 2.3.4, a *regularization* is needed. Regularization means that a new value for $\bar{\theta}$ should be defined.

Let $\bar{\theta}$, $\forall t \in \mathbb{R}^+$ be the regularization solution of the transition rule $\bar{\theta} = R(\theta, x, \sigma)$, then the dynamics of the hybrid system degenerates to continuous dynamics (3.6). Depending on the type of the regularization used for R, there could be different ways to define $\bar{\theta}$. In this thesis, the regularization solution $\bar{\theta}$ is defined by the duty ratios as $\varepsilon \to 0$, that is $\bar{\theta} = \lim\limits_{\varepsilon \to 0} \alpha$, where α is a vector whose elements are the duty ratios, i.e. $\alpha = (\alpha^\ell_1, \alpha^\ell_2, ..., \alpha^\ell_l)^T$ (Definition 3.2). This is equivalent to considering $\bar{\theta}$ as the time average of discrete states θ (see Figure 3.2), that is

$$\bar{\theta} = \lim_{\varepsilon \to 0} \frac{1}{T_\ell} \int_{\tau_0}^{\tau_0 + T_\ell} \theta dt \tag{3.7}$$

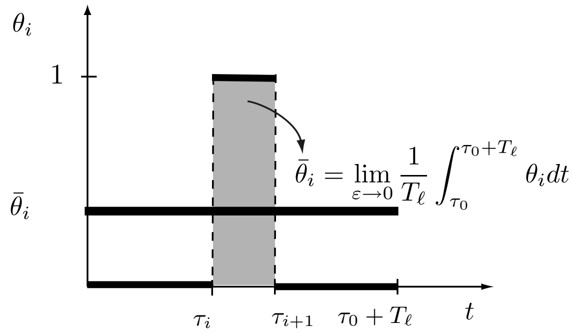

Figure 3.2: Regularization : θ and its time averaged $\bar{\theta}$

Now, $\bar{\theta}$ belongs to the interior of the convex set Θ

$$\Theta = \{\theta = (\theta_1, ..., \theta_l) | \sum_{i=1}^{l} \theta_i = 1, \ \theta_i \geq 0\} \subset \mathbb{R}^l$$

as opposed to θ used in (3.3), where they form the vertices of Θ.

Depending on how the switching transition is defined, i.e. whether it is a controlled switching rule, $R(q, \sigma)$, or it is an autonomous switching rule, $R(q, x)$ (see Section 2.3), the regularization solution, $\bar{\theta}$, has different characteristics. In the case of autonomous switching, $\bar{\theta}$ is a function of states x, i.e. $\bar{\theta} = \bar{\theta}(x)$ (See Appendix A.2.1 for illustrative example), while in the case of controlled switching, $\bar{\theta}$ is not state dependent.

There are analogies between the regularization introduced above and sliding mode regularization. In sliding mode regularization, the vector fields are pointing to the switching surface $\mathcal{S}$, and hence there would be infinite switching around the surface. In order to overcome the problem of infinite switching, a regularization is needed. In such cases, a new vector field is defined that is a tangent to the surface and makes the trajectory slide along the surface. Such a vector field is defined by the convex combination of vector fields at both sides of the surface, e.g. Filippov simplest convex solution

(See Appendix A.2.2 for illustrative example).

It can be verified that for $\varepsilon = 0$, in some special cases, $\bar{\theta}$ indeed coincides with the smallest convex definition of Filippov's sliding mode regularization.

In dehybridization, similar to singular perturbation methodology, the fast states (discontinuous states θ) are eliminated, and the slow variables (continuous states x) satisfy a lower order differential equation that describes the average behavior of the system (equation 3.6). This can be regarded as a model order reduction where the hybrid states $(\theta, x) \in \mathbb{R}^l \times \mathbb{R}^n$, degenerate to continuous states $x \in \mathbb{R}^n$.

In the singular perturbed system defined by (2.26) and (2.27) when $\varepsilon = 0$, as $\dot{z} = \frac{g}{\varepsilon}$, the fast states *instantaneously* converge to the equilibrium of dynamics defined by $g(x, z, 0) = 0$, i.e. $\bar{z} = \phi(x)$. So, the dynamics of the system degenerates to dynamics of a lower order system given by equation $\dot{x} = f(x, \phi(x), 0)$. Such dynamics describes the evolution of systems on a slow time scale. Now, consider the hybrid system defined by (3.3)-(3.4) with $\varepsilon = \max_{\ell} \min_{q} T_\ell(1 - \alpha_q^\ell) = 0$. Similar to singular perturbation, as $\varepsilon = 0$, the fast states θ 'converge' to an 'equilibrium' or 'quasi-steady-state' of $\theta = R(\theta, x, \sigma)$, $\bar{\theta}$. Depending on whether there is infinite or zero switching, it is either defined by (3.7) or takes the form $\bar{\theta} = w_i$.

The approximation can also be interpreted in a different way: for small values of ε, i.e. when fast transitions occur, the evolution of the trajectories of the hybrid system can be considered as fluctuations around the trajectories of a 'slowly' varying dynamics. The continuous averaged model approximates the slow response of the hybrid model, while the difference between the averaged model and the hybrid model solutions is the fast transition of states in the hybrid dynamics.

To summarize, the evolution of continuous states of a hybrid system with a recurrent mode, for sufficiently small values of ε, can be approximated by the evolution of a continuous model defined by (3.6) where $\bar{\theta}_i = \lim_{\varepsilon \to 0} \alpha_i$. In the next chapter, it will be shown that the approximation error in finite time is $O(\varepsilon)$.

3.3 Controller Design via Dehybridization

To apply the dehybridization technique for control design, the main requirement is the two-time scale specification. This time-scale separation in a hybrid system could be natural, i.e. the continuous dynamics and switching conditions are such that the continuous states evolve more slowly than the discrete ones. However, if this is not the case, it is also possible to generate such a property artificially. So, in the control design problem, an external hybrid controller is proposed for creating a time-scale separation artificially. Here, a design methodology based on a cascade control scheme is proposed, where the inner-loop and outer-loop have different specifications. The inner-loop consists of a hybrid controller which imposes the desired two-time scale behavior. The controller on the outer-loop takes the task to fulfill the requirements of the control objectives based on the dehybridized continuous problem, e.g. stabilization.

There are analogies between ideas used in control design via dehybridization and control design via linearization for nonlinear systems. It is well known that, except for some special cases, the qualitative behavior of a nonlinear system can be locally determined via linearization of a nonlinear system trajectory. The main advantage in using linearization techniques is the fact that methods for analysis and control design are more readily available for linear systems than for their nonlinear counterparts. The control design procedure via dehybridization shares the same point of view in the way that the controller design is based on a continuous model rather than a hybrid system. Similar to linearization, some rich and more complex behaviors of the system could be neglected by hybrid to continuous approximation.

3.3.1 Control Methodology

Here, the stabilization problem for a hybrid plant $\mathcal{H}_p$ represented in Figure 2.2 is considered, that is designing a controller which makes the closed-loop combination of plant and controller stable:

Control Design Problem: Let $x^{eq} = 0$ be an equilibrium for the hybrid plant $\mathcal{H}_p$, design a control scheme such that x^{eq} becomes an asymptotically stable equilibrium for the closed-loop system, $\mathcal{H}$.

The proposed control scheme consists of the following two loops:

I. Inner-loop Design: Hybrid Controller

Since the hybrid plant to be controlled may not satisfy the two-time scale separation needed for dehybridization, the first step consists of designing a hybrid controller $\mathcal{H}_c$ which renders the feedback combination of plant and controller (see Chapter 2) with a recurrent mode satisfying a two-time scale property. The resulting hybrid system, as shown in Figure 3.3, is denoted by $\mathcal{H}$. Note that the goal of the switching logic or hybrid controller is not to stabilize $\mathcal{H}_p$, but only to impose a desired two-time scale separation on the feedback combination, $\mathcal{H}$.

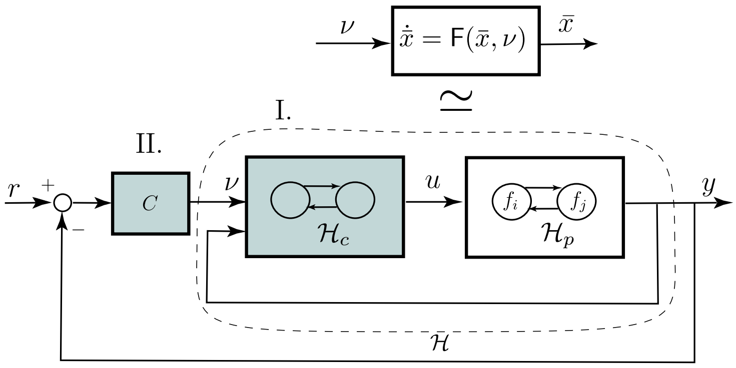

Figure 3.3: Block diagram for the controller design via dehybridization

II. Outer-loop Design: Stabilizing Control Law

Once the inner-loop (closed-loop combination of $\mathcal{H}_p$ and $\mathcal{H}_c$) controller is designed to impose the desired time-scale properties (fast switching and/or mode domination), an averaged 'slow' continuous dynamics can be constructed via dehybridization. Such a continuous model will be used as a basis for designing the outer-loop controller (denoted as C in Figure 3.3). The controller C could be designed using standard control techniques such as PID control, feedback linearization, or Lyapunov-based techniques.

Remark 3.2 *Similar to all cascade control schemes, the inner-loop response should be faster than that of the outer-loop, which is created artificially by the hybrid controller.*

3.3.2 Inner-loop: Hybrid Controller

This part addresses the problem of designing a switching law that renders the discrete transition of $\mathcal{H}$ fast. Based on the classification of discrete transitions (Section 2.3.1), the design problem can be categorized in two different cases: i) controlled or externally induced switching ii) autonomous or internally induced switching. Depending on whether the hybrid plant has a controlled or autonomous switching rule, different strategies could be realized. Here, the aim is not to identify possible classes of hybrid plants nor to define the possible switch strategy for each class. Nevertheless, in the next part, certain issues of the inner-loop design will be discussed.

- **Controlled Switching**

  For a hybrid plant with a controlled switching rule, the transitions are externally induced via the discrete inputs σ, i.e. $q^+ = R(q, \sigma)$. The hybrid controller should generate σ in such a way that the combination of the hybrid controller and plant has a recurrent mode with the desired two-time scale property. Since σ are generated by the hybrid controller, the switching time intervals can be controlled externally. Thus, the cycle time T and the duty ratios α_q and consequently parameter ε can be chosen arbitrarily and independent of the continuous states. Thus, the two-time scale behavior can be generated at will.

  For switched systems [37], that are characterized by $q^+ = R(q, \sigma) = \sigma$, a control strategy for inner-loop control design is discussed in Appendix A.1.1. Pulse Width Modulation (PWM) which is a special case of switched systems is discussed in Appendix A.1.2.

- **Autonomous Switching**

  In the autonomous switching case, the task of the hybrid controller is to i) find a sequence of modes that would take the system back to $q^\star$ and ii) find the continuous inputs, u, that can make this transition possible. The main difference between the controlled and autonomous switching cases are i) the discrete inputs σ are used in the former, while the continuous inputs are used in the latter and ii) in the autonomous case, the cycle time and duty ratios depend on the continuous states, and cannot be arbitrarily imposed. Thus, the design of the inner-loop controller in the case of autonomous switching is more involved. Without defining a special class of hybrid plants (structure of continuous dynamics and switching hypersurfaces), a general design procedure is hardly possible.

  One idea is to make the hypersurfaces separating different modes into attractive ones. As was discussed in Section 2.3.1, in the case of autonomous switching, the discrete transition or switching occurs when certain conditions on the continuous states and inputs are met, i.e. $(x, u) \in \mathcal{S}^{i \to j}$, $i, j \in Q$. Such conditions are generally defined by the equation for hypersurfaces such as $S^{i \to j}(x) = 0$, $i, j \in Q$ where $S : \mathbb{R}^n \to \mathbb{R}$. In order to render a hybrid plant with a recurrent mode $q^\star$, the hybrid controller should be designed such that hypersurfaces $S^{q \to q^\star}(x) = 0$, $\forall q \neq q^\star$ be attractive. For this, the ideas of *reachability* in sliding mode control can be used. An example illustrating these issues is discussed in Appendix A.2.3.

3.3.3 Outer-loop: Control Law for Averaged Model

Once the closed-loop combination of hybrid plant and controller $\mathcal{H}$ satisfies two-time scale behavior, then an averaged continuous model described by

$$\dot{\bar{x}} = \sum_{i=1}^{l} \bar{\theta}_i(\bar{x}) F_i(\bar{x}, \nu) = \mathsf{F}(\bar{x}, \nu) \tag{3.8}$$

can approximate the behavior of the system's slow dynamics (dehybridization).

On the basis of such an averaged dynamics, and the objectives of control, the control law $\nu(x)$ can be designed. The main problem with this approach is that the control law based on the averaged continuous model requires the description of $\bar{\theta}(x)$ to be known. Except for the classes of systems with controlled switching transition rules, where the duty ratios are known *a priori*, an analytical expression for $\bar{\theta}(x)$ is not available.

However, in control design different scenarios could be possible to cope with uncertainties in $\bar{\theta}_i(x)$:

- **Estimation of $\bar{\theta}$**

  In many cases, the variation of duty cycles, α from one cycle to another, is small. Furthermore, to calculate the duty ratios, only switching time instants need to be known. The switching instants can be memorized as the system evolves and so the duty ratios can be *measured* on line. In such cases, after one or two cycles an estimate of $\bar{\theta}$ would be available. The control strategy could incorporate these measurements in order to compensate for the variation in $\bar{\theta}(x)$.

- **Robust Approach**

Since duty cycles and thus $\bar{\theta}(x)$ are bounded, i.e $||\bar{\theta}|| \leq 1$, they can be considered as bounded *uncertain* terms in the model. Smaller polytopical regions can be obtained by studying the problem at hand. Then, depending on the locations of the these uncertain terms in the averaged system equations, a robust controller can be designed [52]. The robust control designs, which in general rely on the bounds of perturbation terms, give conservative results. In the case when the variation of $\bar{\theta}_i(x)$ is small, it turns out that the approach based on estimation discussed above can be combined with a robust design to reduce the conservatism of the control design.

3.4 Conclusions

In this chapter, a two-time scale property of a hybrid system has been investigated and based on such a property a control design methodology was proposed. It was arranged such that the fast and slow states in a hybrid system were the discrete states and continuous states respectively. Fast evolution of discrete states was related to fast switching in systems. Also, it was noted that it is sufficient to have fast switching in *all but one* of the discrete states. So, an extended class of systems, i.e. hybrid systems with recurrent mode, was considered. For such systems, an approximate model was proposed by setting the small parameter ε to zero. Here, ε was defined as the time spent in all but one of the modes. The quality of approximation for a nonzero ε will be discussed in the next chapter.

A control design scheme based on approximate continuous dynamics was proposed. The design scheme led to a cascade control structure where the inner-loop artificially imposes a two-time scale property while the outer-loop takes the stabilizing task. The design of control in the outer-loop is based on the continuous model. The main drawback of the control scheme is that the exact values of duty ratios, which are required to construct the average model, are not always known. Nevertheless, in the outer-loop control design, uncertainties could be dealt either via a robust control or by estimating the same. The control law addresses the stability of the averaged model when $\varepsilon = 0$, however there is no guarantee that the overall scheme be stable for the hybrid model for a nonzero ε. The stability relations of the averaged model and the hybrid model for nonzero values of ε will be investigated in the next chapter.

Chapter 4

Dehybridization: Theoretical Results

4.1 Introduction

In the previous chapter, it was discussed that for a hybrid system in which fast switching occurs or one mode is dominant, i.e. infinite fast switching in all but one mode, the dynamics of the hybrid system is reduced to a continuous one. It was also argued that the averaged continuous model is defined by the convex combination of the vector fields where the convex parameters correspond to the duty ratios when $\varepsilon = 0$. The important issue is the quality of the approximation for nonzero ε. This has an important practical implication, since in many physical hybrid systems either infinite switching is impossible or the objective is not to have such a property. A realistic way of looking at the problem is to take into account *sufficiently fast* switching instead of infinite switching. In this case, it is important to be aware of issues such as the quality of the approximation, i.e. how close are the trajectories of the hybrid model and the continuous averaged one, and the conditions needed for validating the approximation. In this chapter these issues will be studied in detail.

Moreover, in the previous chapter a cascade control design scheme based on the approximate continuous model was proposed. Therein the outer-loop controller is designed to fulfill the control objectives, e.g. stabilization. The stability of the averaged model does not necessary imply the stability of the hybrid model when ε is not equal to zero. In this chapter the link between the stability of the averaged and the hybrid model will be established.

The approach that will be undertaken to study approximation as well as stability, is based on defining the hybrid dynamics as a perturbation of the continuous averaged model. The important point to note is that although the sizes of the perturbation terms are not small, it will be shown that the bounds on their time integral depends linearly on a small parameter ε. This interesting property will be used to characterize both the approximation and the stability results.

This chapter is organized as follows. First, in Section 4.2.1 the hybrid system as a perturbed model of a continuous system will be presented, wherein the properties of the perturbation term will be discussed. In Section 4.2.2, the main approximation results will be presented, where the discrete states and fast continuous states are eliminated. In Section 4.3, the stability relations of the averaged model and the hybrid model for nonzero values of ε will be investigated.

4.2 Dehybridization: Approximation Results

In order to develop approximation results, an alternative representation of hybrid dynamics will be introduced. In such a representation the dynamics in each mode is presented as the perturbed model of a continuous system. The continuous model belongs to the set of all convex combinations of the vector fields F_i, $i \in \mathbf{L} = \{1, 2, ..., l\}$, $l = \text{Card}(Q)$, of the hybrid system.

4.2.1 Hybrid Model as a Perturbed Model of a Continuous System

Consider a hybrid system, $\mathcal{H}$, represented by

$$\dot{x} = F_i(x), \; x(t_0) = x_0 \tag{4.1}$$

$$\theta^+ = R(\theta, x) \tag{4.2}$$

where $(x, \theta) \in \mathbb{R}^n \times \mathbb{R}^l$ with $\theta = (\theta_1, ..., \theta_l)^T$ and $\theta_i = 1$ for some i and $\theta_j = 0$ for $j \neq i$ and $i \in \mathbf{L} = \{1, 2, ..., l\}$. $F_i(x) : \mathbf{L} \times D \to \mathbb{R}^n$ are Lipschitz and bounded in $D \subset \mathbb{R}^n$ a domain that contains the origin $x = 0$.

Define $\mathcal{C}$ as the set of all convex combinations of the vector fields $F_i(x)$, $i \in \mathbf{L}$.

$$\mathcal{C} = \{\mathsf{F} | \;\; \mathsf{F} = \sum_{i=1}^{l} \alpha_i F_i, \alpha_i \geq 0, \; \sum_{i=1}^{l} \alpha_i = 1\}$$

where α_i are convex multiplicative factors. Consider the dynamics defined by vector fields belonging to $\mathcal{C}$:

$$\dot{x} = \mathsf{F}(x) = \sum_{i=1}^{l} \alpha_i F_i(x), \; x(t_0) = x_0 \tag{4.3}$$

When the mode $\dot{x} = F_i(x)$ is active, it can be seen that the dynamics can be represented as a perturbation of a nominal model, F, that is

$$\dot{x}(t) = \mathsf{F}(x) + \Delta(x) \tag{4.4}$$

with

$$\Delta(x) = \Delta_i = F_i - \mathsf{F} \tag{4.5}$$

Δ_i can be rewritten as:

$$\begin{aligned} \Delta_i &= F_i - \mathsf{F} = (1-\alpha_i)F_i - \sum_{j \neq i} \alpha_j F_j \\ &= \left(\sum_{j \neq i} \alpha_j \right) F_i - \sum_{j \neq i} \alpha_j F_j = \sum_{j \neq i} \alpha_j (F_i - F_j) \end{aligned} \tag{4.6}$$

As the combination is convex, Δ_i's have the following property:

Lemma 4.1 *The perturbation terms Δ_i, $i \in \mathbf{L}$ defined by (4.6) satisfy*

$$\sum_{i=1}^{l} \alpha_i \Delta_i = 0 \tag{4.7}$$

Proof: From the definition of Δ_i given by equation (4.6), and the fact that $\sum_{i=1}^{l} \alpha_i = 1$, it can be seen that

$$\begin{aligned}\sum_{i=1}^{l}\alpha_i\Delta_i &= \sum_{i=1}^{l}\alpha_i\sum_{j=1}^{l}\alpha_j\left(F_i-F_j\right)\\ &= \sum_{i=1}^{l}\alpha_iF_i\left(\sum_{j=1}^{l}\alpha_j\right)-\left(\sum_{i=1}^{l}\alpha_i\right)\sum_{j=1}^{l}\alpha_jF_j\\ &= \sum_{i=1}^{l}\alpha_iF_i-\sum_{j=1}^{l}\alpha_jF_j=0 \qquad (4.8)\end{aligned}$$

■

Lemma 4.2 *Suppose the vector fields in each mode are Lipschitz in* $D\subset\mathbb{R}^n$ *with Lipschitz constant* L_i*, and* $F_i(0)=0\ \forall i\in\mathbf{L}$*. Then, perturbation terms* $\Delta_i(x)$ *satisfy*

$$||\Delta_i(x)||\leq 2(1-\alpha_i)L||x|| \qquad (4.9)$$

with $L=\max_i\{L_i\}$.

Proof: From (4.6), it can be seen that

$$\Delta_i(x) = (1-\alpha_i)F_i(x)-\sum_{j\neq i}\alpha_jF_j(x) \qquad (4.10)$$

Since F_i are Lipschitz and $F_i(0)=0$, then $||F_i(x)||\leq L_i||x||, \forall(i,x)\in\mathbf{L}\times D$. Taking the norm on Δ yields

$$\begin{aligned}||\Delta_i(x)|| &\leq (1-\alpha_i)||F_i(x)||+||\sum_{j\neq i}\alpha_jF_j(x)||\\ &\leq (1-\alpha_i)L_i||x||+\sum_{j\neq i}\alpha_jL_j||x|| \qquad (4.11)\end{aligned}$$

Let $L=\max_i\{L_i\}$, then

$$\begin{aligned}\Delta_i(x) &\leq (1-\alpha_i)L||x||+\sum_{j\neq i}\alpha_jL||x|| = (1-\alpha_i)L||x||+(1-\alpha_i)L||x||\\ &\leq 2(1-\alpha_i)L||x|| \qquad (4.12)\end{aligned}$$

■

In the previous chapter the small parameter ε was defined as

$$\varepsilon = \max_{\ell} \min_{i} T_\ell (1 - \alpha_i^\ell) \tag{4.13}$$

where, T_ℓ is the cycle time and α_i^ℓ is the duty ratio at the ℓ'th cycle. It will be shown that if the convex coefficients of (4.3) are chosen as the duty ratios, certain other properties hold:

Lemma 4.3 *For ε defined as (4.13), the following are true:*

- $T_\ell \alpha_i^\ell \alpha_j^\ell < \varepsilon$, $\forall i \neq j \in \mathbf{L}$
- $T_\ell \alpha_i^\ell (1 - \alpha_i^\ell) < \varepsilon$, $\forall i \in \mathbf{L}$

Proof: Let $i^\star = \arg\min_i (1 - \alpha_i^\ell)$, then it can be seen that

$$\alpha_i^\ell \leq \sum_{i \neq i^\star} \alpha_i^\ell = (1 - \alpha_{i^\star}^\ell), \ \forall i \neq i^\star \tag{4.14}$$

From the definition of ε given by (4.13), $T_\ell(1 - \alpha_{i\star}^\ell) < \varepsilon$, hence using (4.14) it can be seen that

$$T_\ell \alpha_i^\ell < T_\ell (1 - \alpha_{i\star}^\ell) < \varepsilon, \ \forall i \neq i^\star \tag{4.15}$$

In order to show the first inequality, two cases need to be considered:

i) if $i \neq i^\star$, then from (4.15), $T_\ell \alpha_i^\ell \leq \varepsilon$. Now, using the fact that $\alpha_j^\ell \leq 1$, it can be seen that $T_\ell \alpha_i^\ell \alpha_j^\ell < \varepsilon$.

ii) if $i = i^\star$, then from (4.15), $T_\ell \alpha_j^\ell \leq \varepsilon, \ \forall j \neq i^\star$. Using the fact that $\alpha_i < 1$ it can be seen that $T_\ell \alpha_j^\ell \alpha_i^\ell \leq \varepsilon$ and $T_\ell (1 - \alpha_i^\ell) \alpha_i^\ell < \varepsilon$.

As for the second inequality, by definition $T_\ell (1 - \alpha_i^\ell) \leq \varepsilon \ \forall i \in \mathbf{L}$, since $\alpha_i^\ell \leq 1$ then $T_\ell (1 - \alpha_i^\ell) \alpha_i^\ell \leq \varepsilon$. ■

Next, it will be shown that the time integral of the perturbation term Δ (equation 4.6) is bounded by a linearly dependent function of ε.

Proposition 4.1 *Consider a hybrid system with a recurrent mode, where the vector fields $F_i(x)$ and their Jacobian $\frac{\partial F_i(x)}{\partial x}$ are continuous and bounded in $D \subset \mathbb{R}^n$, $\forall i \in \mathbf{L}$ as $\varepsilon \to 0$, then the perturbation term $\Delta(x)$ (4.6) satisfies*

$$\left|\left|\int_{t_0}^{t} \Delta(x)ds\right|\right| < \varepsilon K_1(t - t_0) \tag{4.16}$$

with K_1 being a positive constant and ε defined as (4.13).

Proof: Without loss of generality let the sequence of discrete states in each cycle be $i = 1 \rightarrow i = 2 \rightarrow ... \rightarrow i = l$. Denote τ_{i-1}^{ℓ} and x_{i-1}^{ℓ}, $i \in \mathbf{L} = \{1, 2, ..., l\}$ respectively as the time instant and the state when the system switches to mode i in the ℓ'th cycle.

Let $t_m = \tau_0 + \sum_{\ell=1}^{m} T_\ell$ be the time at the end of the m'th cycle. From the summation property of integrals it can be verified that

$$\int_{t_0}^{t_m} \Delta(x)ds = \sum_{\ell=1}^{m}\sum_{i=1}^{l}\int_{\tau_{i-1}^{\ell}}^{\tau_i^{\ell}} \Delta_i(x)ds \tag{4.17}$$

Using the fact that $\tau_i^{\ell} - \tau_{i-1}^{\ell} = \alpha_i^{\ell} T_\ell$, the second order Taylor series of the integral can be written as

$$\begin{aligned}\int_{t_0}^{t_m} \Delta(x)ds &= \sum_{\ell=1}^{m}\left\{T_\ell \sum_{i=1}^{l}\alpha_i^{\ell}\Delta_i(x_{i-1}^{\ell}) + \frac{1}{2}T_\ell^2 \sum_{i=1}^{l}\alpha_i^{\ell\,2}\frac{\partial \Delta_i}{\partial x}F_i(\hat{x}_i^{\ell})\right\}\\ &= \sum_{\ell=1}^{m} T_\ell\left\{\sum_{i=1}^{l}\alpha_i^{\ell}\Delta_i(x_{i-1}^{\ell}) + \frac{1}{2}\sum_{i=1}^{l}T_\ell\alpha_i^{\ell\,2}\frac{\partial \Delta_i}{\partial x}F_i(\hat{x}_i^{\ell})\right\}\end{aligned} \tag{4.18}$$

where $\hat{x}_i^{\ell}$ represents the state at a time instant between τ_{i-1}^{ℓ} and τ_i^{ℓ}, that is $x(\hat{\tau}_i^{\ell})$ with $\hat{\tau}_i^{\ell} \in [\tau_{i-1}^{\ell},\ \tau_i^{\ell}]$. Using the properties of the integral and the first order Taylor series expansion, the term $\Delta_i(x_{i-1}^{\ell})$, for $i = 1, \ldots, l$, can be written as

$$\begin{aligned}\Delta_i(x_{i-1}^{\ell}) &= \Delta_i(x_0^{\ell}) + \int_{\tau_0}^{\tau_{i-1}} \frac{\partial}{\partial t}\Delta_i ds\\ &= \Delta_i(x_0^{\ell}) + T_\ell\sum_{j=1}^{i-1}\alpha_j\frac{\partial \Delta_i}{\partial x}F_j(\hat{x}_j^{\ell})\end{aligned} \tag{4.19}$$

Replacing (4.19) in (4.18) yields

$$\int_{t_0}^{t_m} \Delta(x)ds \quad = \quad \sum_{\ell=1}^{m} T_\ell \left\{ \sum_{i=1}^{l} \alpha_i \Delta_i(x_0^\ell) + \Omega(\hat{x}_i^\ell) \right\} \tag{4.20}$$

where

$$\Omega(\hat{x}_i^\ell) = \sum_{i=1}^{l} \sum_{j=1}^{i-1} T_\ell \alpha_i^\ell \alpha_j^\ell \frac{\partial \Delta_i}{\partial x} F_j(\hat{x}_j^\ell) + \frac{1}{2} \sum_{i=1}^{l} T_\ell {\alpha_i^\ell}^2 \frac{\partial \Delta_i}{\partial x} F_i(\hat{x}_i^\ell) \tag{4.21}$$

From Lemma 4.2, the term $\sum_{i=1}^{l} \alpha_i \Delta_i(x_0^\ell) = 0$, and hence (4.20) becomes

$$\int_{t_0}^{t_m} \Delta(x)ds = \sum_{\ell=1}^{m} T_\ell \Omega(\hat{x}_i^\ell) \tag{4.22}$$

From (4.6), $\dfrac{\partial \Delta_i}{\partial x} = \sum_{j \neq i}^{l} \alpha_j \dfrac{\partial (F_i - F_j)}{\partial x}$, then by replacing it in the second term of equation (4.21) and taking the norm it can be verified that

$$\begin{aligned} ||\Omega(\hat{x}_i^\ell)|| \quad \leq \quad & \left\| \sum_{i=1}^{l} \sum_{j=1}^{i-1} T_\ell \alpha_i^\ell \alpha_j^\ell \frac{\partial \Delta_i}{\partial x} F_j(\hat{x}_j^\ell) \right\| + \\ & \left\| \frac{1}{2} \sum_{i=1}^{l} \sum_{j \neq i}^{l} T_\ell \alpha_i^\ell \alpha_j^\ell \alpha_i^\ell \frac{\partial (F_i - F_j)}{\partial x} F_i(\hat{x}_i^\ell) \right\| \end{aligned} \tag{4.23}$$

From the first inequality of Lemma 4.3, and the fact that vector fields $F_i(x)$ and their jacobian $\frac{\partial F_i}{\partial x}$ are bounded, it can be seen that

$$||\Omega(\hat{x}_i)|| < \varepsilon K_1 \tag{4.24}$$

where K_1 is a positive constant related to the bounds on $F_i(x)$ and $\frac{\partial F_i(x)}{\partial x}$.

Taking the norm of (4.22) and using $\sum_{\ell=1}^{m} T_\ell = t_m - t_0$ yields

$$\left|\left|\int_{t_0}^{t_m} \Delta(x)ds\right|\right| \leq \sum_{\ell=1}^{m} T_\ell ||\Omega(\hat{x}_i^\ell)|| < \varepsilon K_1 \sum_{\ell=1}^{m} T_\ell = \varepsilon K_1 (t_m - t_0) \tag{4.25}$$

which completes the proof. ∎

4.2.2 Approximation on the Finite Time Interval: Elimination of Discrete States

Proposition 4.1 reveals an interesting property of the perturbation term Δ, i.e. even though the norm of perturbation $||\Delta(x)||$ is not *small*, the norm of its time integral is bounded by parameter ε which can be arbitrarily small. Such a property will be used to characterize the error between the trajectories of the nominal dynamics (the continuous averaged model) and the perturbed dynamics (hybrid system) using the parameter ε.

At this point, it is important to note that the continuous averaged dynamics defined in Chapter 3, does not have the duty cycle α_i^ℓ for its coefficients, but their limits $\bar{\theta}_i = \lim_{\varepsilon \to 0} \alpha_i^\ell$. The continuous averaged model is defined by

$$\dot{\bar{x}} = \sum_{i=1}^{l} \bar{\theta}_i F_i(\bar{x}) = \bar{\mathsf{F}}(\bar{x}), \ \bar{x}(t_0) = x_0 \tag{4.26}$$

Then

$$\begin{aligned} \dot{x} &= F_i(x) \\ &= \bar{\mathsf{F}}(x) + \big(\mathsf{F}(x) - \bar{\mathsf{F}}(x)\big) + (F_i(x) - \mathsf{F}(x)) \\ &= \bar{\mathsf{F}}(x) + \Delta_\varepsilon(x) + \Delta(x) = \bar{\mathsf{F}}(x) + \bar{\Delta}(x) \end{aligned} \tag{4.27}$$

where $\Delta(x)$ is as defined in (4.5), and

$$\Delta_\varepsilon(x) = \sum_{i=1}^{l} (\bar{\theta}_i - \alpha_i) F_i(x) \tag{4.28}$$

Since $\bar{\theta}_i = \lim_{\varepsilon \to 0} \alpha_i^\ell$, there exists a $k_i^\ell > 0$ such that $||\bar{\theta}_i - \alpha_i^\ell|| \leq k_i^\ell \varepsilon$, thus

$$||\Delta_\varepsilon(x)|| \leq \varepsilon \sum_{i=1}^{l} k_i^\ell ||F_i(x)|| \quad (4.29)$$

The following theorem states that the trajectories of a hybrid system with fast switching in all but one mode can be approximated in a finite time interval by continuous dynamics (4.26) with the approximation error being of the order $O(\varepsilon)$.

Theorem 4.1 *Consider the hybrid system $\mathcal{H}$ defined by (4.1)-(4.2) and the continuous dynamics defined by (4.26). Suppose the vector fields $F_i(x) : D \to \mathbb{R}^n$ and their Jacobian $\frac{\partial F_i(x)}{\partial x}$ are continuous with respect to x and bounded in $D \subset \mathbb{R}^n$, $\forall i \in \mathbf{L}$ and $\forall \varepsilon$.*

Then, the trajectories of the hybrid system and the continuous system (4.26) satisfy

$$\bar{x}(t) - x(t) = O(\varepsilon), \quad \forall t \in [t_0, \mathcal{T}),\ \mathcal{T} < \infty$$

where ε is given by (4.13).

Proof: The solutions of the continuous averaged model (4.26), $\bar{x}(t)$, and the hybrid system $x(t)$ are given by

$$\bar{x}(t) = x_0 + \int_{t_0}^{t} \bar{\mathsf{F}}(\bar{x}) ds \quad (4.30)$$

$$x(t) = x_0 + \int_{t_0}^{t} \left[\bar{\mathsf{F}}(x) + \bar{\Delta}(x)\right] ds = \int_{t_0}^{t} \left[\bar{\mathsf{F}}(x) + \Delta(x) + \Delta_\varepsilon(x)\right] ds \quad (4.31)$$

Subtracting (4.31) from (4.30) and taking the norm yields

$$||\bar{x}(t) - x(t)|| \leq \int_{t_0}^{t} ||\bar{\mathsf{F}}(\bar{x}) - \bar{\mathsf{F}}(x)|| ds + \left|\left|\int_{t_0}^{t} \Delta(x) ds\right|\right| + \int_{t_0}^{t} ||\Delta_\varepsilon(x)||\, ds \quad (4.32)$$

Since F_i are bounded, then from the definition of Δ_ε (equation 4.28), it can be seen that $||\Delta_\varepsilon(x)|| \leq \varepsilon K_2$ where $K_2 > 0$. Let L be the Lipschitz constant for $\bar{\mathsf{F}}(x)$, then from Proposition 4.1 it can be seen that

$$||\bar{x}(t) - x(t)|| \leq \int_{t_0}^{t} L||\bar{x}(s) - x(s)|| ds + \varepsilon K_1(t - t_0) + \varepsilon K_2(t - t_0) \quad (4.33)$$

Let $K = K_1 + K_2$, by application of the Gronwall-Bellman inequality (Lemma 2.3), it can be deduced that

$$\begin{aligned} ||\bar{x}(t) - x(t)|| &\leq \varepsilon K(t - t_0) + \int_{t_0}^{t} \varepsilon K L(s - t_0)\exp[L(t - s)]ds \\ &\leq \varepsilon \frac{K}{L}\{\exp[L(t - t_0)] - 1\} \end{aligned} \tag{4.34}$$

This shows that $\bar{x}(t) - x(t) = O(\varepsilon)$, $\forall t \in [t_0,\, \mathcal{T}]$.

■

Remark 4.1 *Note that since the exponential term* $\exp[L(t - t_0)]$ *grows unbounded as* $t \to \infty$*, the bound on the error* $||\bar{x}(t) - x(t)||$ *(equation (4.34)) is only valid for finite time intervals and hence the approximation is not uniform in* t*. However, for a given time interval, the parameter* ε *can be chosen to be sufficiently small such that the approximation error meets the required tolerance.*

Remark 4.2 *The approximation result of Theorem 4.1 can be extended to an infinite time interval,* $[t_0,\, \infty)$*, under certain stability conditions on the averaged model. Taking a parallel from perturbation theory, e.g. a singular perturbation, it can be postulated that if the averaged model is exponentially stable, then for sufficiently small values of* ε*, the approximation error is of the order* $O(\varepsilon)$ *in an infinite time interval.*

Theorem 4.1 suggests that if all vector fields of a hybrid system, i.e F_i, $\forall i \in \mathbf{L}$, with a recurrent mode are bounded in $D \subset \mathbb{R}^n$, then the continuous state trajectories of the hybrid system will be close to the trajectories of a continuous model defined by a convex combination of the vector fields visited during a cycle. The proof of this theorem relied on the bounds on the time integral of the perturbation term $\Delta(x)$, that is the bounds on the term $\Omega(x)$ given by equation (4.23). If $\Omega(x)$ grows as $\varepsilon \to 0$, i.e. $\Omega(x) = O(1/\varepsilon)$, then Theorem 4.1 is not applicable.

Taking into account the definition of $\Omega(x)$ (equation (4.21), there are two reasons for $\Omega(x)$ to be $O(1/\varepsilon)$: i) if some components of $F_i(x)$ are $O(1/\varepsilon)$, and ii) if $\frac{\partial F_i(x)}{\partial x}$ is $O(1/\varepsilon)$. The first case leads to dynamics of some continuous states being fast, i.e. at the same rate as the discrete states. Only this first case will be dealt with in this thesis.

In the next section, it will be shown that under certain conditions, Theorem 4.1 could still be applicable to that part of the continuous states representing the slow dynamics of the system.

Approximation on the Finite Time Interval: Removal of Fast Continuous States

Suppose in a hybrid system some components of F_i are $O(1/\varepsilon)$. These can be written as $\dfrac{g_i}{\varepsilon}$, which in turn leads to a standard singular-perturbation formulation of each of the vector fields $F_i(x)$ with $x = (\xi,\, \eta)^T$:

$$F_i(x) = \begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} f_i(\xi, \eta, \varepsilon) \\ \frac{1}{\varepsilon} g_i(\xi, \eta, \varepsilon) \end{pmatrix} \tag{4.35}$$

where ξ and η correspond to slow and fast continuous states respectively. Here, η corresponds to those continuous states that react with, or are responsible for the quick transition of the discrete states (e.g. autonomous switching events defined by switching conditions $S(\eta) = 0$). Consider continuous dynamics defined as the convex combination of f_i's that is

$$\dot{\bar{\xi}} = \sum_{i=1}^{l} \bar{\theta}_i f_i(\bar{\xi}, \bar{\eta}, \varepsilon) = \bar{\mathsf{f}}(\bar{\xi}, \bar{\eta}, \varepsilon) \tag{4.36}$$

The next corollary shows that under certain conditions, the error between the solution of the 'slow' states of the hybrid dynamics and the continuous averaged dynamics is $O(\varepsilon)$, i.e. $\bar{\xi}(t) - \xi(t) = O(\varepsilon)$.

Corollary 4.1 *Suppose the vector fields in each mode of the hybrid system $\mathcal{H}$ with a recurrent mode have the structure given by (4.35). Consider the continuous dynamics given by (4.36). Suppose the slow dynamics, f_i, in each cycle satisfy*

$$\frac{\partial f_i}{\partial \eta} = 0, \quad \forall i \in \mathbf{L} \tag{4.37}$$

Then, the 'slow' states $\bar{\xi}$ and ξ satisfy

$$\bar{\xi}(t) - \xi(t) = O(\varepsilon),\ t \in [t_0, \mathcal{T}], \quad \mathcal{T} < \infty$$

Proof: The dynamics of the 'slow' states, $\xi(t)$, of the hybrid system can be written as

$$\dot{\xi} = \bar{\mathsf{f}}(\xi) + \bar{\Delta}^s(\xi,\eta) = \bar{\mathsf{f}}(\xi) + \Delta^s(\xi,\eta) + \Delta^s_\varepsilon(\xi,\eta) \tag{4.38}$$

where f is defined by (4.36), and $\Delta^s(\xi,\eta)$ and $\Delta^s_\varepsilon(\xi,\eta)$ are given by the following equations respectively

$$\Delta^s = \sum_{j\neq i}^{l} \alpha^\ell_j (f_i - f_j) \quad \text{for } t \in [\tau_{i-1},\ \tau_i] \tag{4.39}$$

$$\Delta^s_\varepsilon = \sum_{j=1}^{l} (\bar{\theta}_i - \alpha^\ell_i) f_i,\ k_i > 0 \quad \text{with } ||\Delta^s_\varepsilon|| \leq \varepsilon K_2 \tag{4.40}$$

where $K_2 > 0$ is related to bounds on f_i. Similar to the proof of Proposition 4.1, it can be verified that Δ^s satisfies

$$\int_{t_0}^{t_m} \Delta^s(\xi,\eta)ds \quad = \quad \sum_{\ell=1}^{m} T_\ell \left\{ T_\ell \Omega_1(\hat{\xi}^\ell_i, \hat{\eta}^\ell_i) + T_\ell \Omega_2(\hat{\xi}^\ell_i, \hat{\eta}^\ell_i) \right\} \tag{4.41}$$

where Ω_1 and Ω_2 are defined by the following equations

$$\Omega_1 = \sum_{i=1}^{l} \sum_{j=0}^{i-1} T_\ell \alpha^\ell_i \alpha^\ell_j \frac{\partial \Delta^s_i}{\partial \xi} f_j(\hat{\xi}^\ell_j, \hat{\eta}^\ell_j) + \frac{1}{2} \sum_{i=1}^{l} T_\ell (\alpha^\ell_i)^2 \frac{\partial \Delta^s_i}{\partial \xi} f_i(\hat{\xi}^\ell_i, \hat{\eta}^\ell_i) \tag{4.42}$$

$$\Omega_2 = \sum_{i=1}^{l} \sum_{j=0}^{i-1} T_k \alpha^k_i \alpha^k_j \frac{\partial \Delta^s_i}{\partial \eta} g_j(\hat{\xi}^k_j, \hat{\eta}^k_j)/\varepsilon + \frac{1}{2} \sum_{i=1}^{l} T_k (\alpha^k_i)^2 \frac{\partial \Delta^s_i}{\partial \eta} g_i(\hat{\xi}^k_i, \hat{\eta}^k_i)/\varepsilon \tag{4.43}$$

If $\dfrac{\partial f_i}{\partial \eta}(\xi,\eta) = 0, \forall i \in \mathbf{L}$, then by using definition of Δ^s (4.39) it can be seen that $\dfrac{\partial \Delta^s}{\partial \eta} = \displaystyle\sum_{j\neq i}^{l} \alpha^\ell_j \frac{\partial (f_i - f_j)}{\partial \eta} = 0, \forall i,j \in \mathbf{L}$, and hence $\Omega_2(\xi,\eta) = 0$, from which (4.41) becomes

$$\int_{t_0}^{t_m} \Delta^s(\xi,\eta)ds \quad = \quad \sum_{k=1}^{m} T_k \left\{ \Omega_1(\hat{\xi}^k_i, \hat{\eta}^k_i) \right\} \tag{4.44}$$

By using similar arguments to those used in the proof of Proposition 4.1, it can be seen that

$$\left|\left|\int_{t_0}^{t_m} \Delta^s ds\right|\right| < \varepsilon K_1(t_m - t_0) \tag{4.45}$$

Now, it will be shown that $\bar{\xi}(t) - \xi(t) = O(\varepsilon)$. Consider the dynamics represented by

$$\dot{\bar{\xi}} = \sum_{i=1}^{l} \bar{\theta}_i f_i(\bar{\xi}) = \bar{\mathsf{f}}(\bar{\xi}), \quad \bar{\xi}_0 = \xi_0 \tag{4.46}$$

Subtracting the solutions of dynamics (4.46) from dynamics (4.38), and taking the norm yields

$$||\bar{\xi}(t) - \xi(t)|| \leq \int_{t_0}^{t} ||\mathsf{f}(\bar{\xi}) - \mathsf{f}(\xi)||ds + \left|\left|\int_{t_0}^{t} \Delta^s ds\right|\right| + \int_{t_0}^{t} ||\Delta^s_\varepsilon||ds \tag{4.47}$$

Since f is Lipschitz in ξ, then from (4.40) and (4.45)

$$\begin{aligned} ||\bar{\xi}(t) - \xi(t)|| &\leq \int_{t_0}^{t} L||\bar{\xi}(s) - \xi(s)||ds + \varepsilon K_1(t - t_0) + \varepsilon K_2(t - t_0) \\ &\leq \int_{t_0}^{t} L||\bar{\xi}(s) - \xi(s)||ds + \varepsilon K(t - t_0) \end{aligned} \tag{4.48}$$

with $K = K_1 + K_2$. Then application of the Gronwall-Bellman lemma yields

$$||\bar{\xi}(t) - \zeta(t)|| \leq \varepsilon\frac{K}{L}\{\exp[L(t - t_0)] - 1\}, \ \forall t \in [t_0, \mathcal{T}] \tag{4.49}$$

from which $\bar{\xi}(t) - \xi(t) = O(\varepsilon)$.

■

The above theorem states that if the fast states are only responsible for the fast switching and dynamics of the slow states are not influenced by such dynamics, then the approximation result on slow dynamics is valid.

4.3 Dehybridization: Stability Results

In the previous section, it was shown that for sufficiently small values of ε the trajectories of a hybrid system stay close to the trajectories of a continuous system dynamics. A relevant question is that if the trajectories of two dynamics are close, then is there any relationship between stability properties of the two dynamics. For instance, does the stability of the equilibrium point for the averaged model imply the same for the hybrid model. Let us remember that the outer-loop controller in the cascade control scheme is designed such that the equilibrium point of the averaged model becomes stable. So, the answer to this question has an important impact on the validity of the control design procedure presented in the previous chapter.

In this section, the theory of stability of perturbed systems will be used for stability analysis. The standard practice is to use the derivative of a Lyapunov function V along the trajectories of the perturbed model, $\dot{x} = \mathsf{F} + \Delta$, i.e.

$$\dot{V} = \frac{\partial V}{\partial x}\mathsf{F} + \frac{\partial V}{\partial x}\Delta$$

The first term, i.e. the derivative along the trajectories of the nominal model, is negative definite. Then, if the second term is small in some norm, then its 'destabilizing' effect will be compensated by the stabilizing effect of the nominal model and thus the derivative of the Lyapunov function along the trajectories of the perturbed system becomes negative.

It should be stressed that the bound on the norm of perturbation $||\Delta||$ is related to that of the vector fields, i.e. $||F_i(x)||$ (see equation (4.6)). Since the bound on perturbation is twice that of the nominal dynamics $\bar{\mathsf{F}}(x)$, the perturbation term is not a real perturbation! However, as was discussed in the last section (Proposition 4.1), the perturbation terms Δ have the property that the bound on the *integral* of the perturbation is small (see equation 4.16). So, in the case of dehybridization the integral of the perturbation is of interest and the standard stability results need to be extended.

4.3.1 From Stability of the Averaged Model to Stability of the Hybrid Model

For system (4.26), let $x = 0$ be an exponentially stable equilibrium point. Let $V(x)$ be a Lyapunov function that satisfies

$$c_1||x||^2 \leq V(x) \leq c_2||x||^2 \tag{4.50}$$
$$\frac{\partial V}{\partial x}\bar{\mathsf{F}}(x) \leq -c_3||x||^2 \tag{4.51}$$
$$\left|\left|\frac{\partial V}{\partial x}\right|\right| \leq c_4||x|| \tag{4.52}$$

for $x \in D \subset \mathbb{R}^n$ and some positive constants c_r, $r = 1, 2, 3, 4$. In order to study stability properties of the hybrid model, depending on whether the origin is a common equilibrium point for each mode of the hybrid system, two different cases will be considered:

Case 1) $x = 0$ is an equilibrium point for all modes of the hybrid system, i.e. $F_i(0) = 0$, $\forall i \in \mathbf{L}$. Case 2) $x = 0$ is not necessarily the equilibrium point for all modes, i.e. $\exists i \in \mathbf{L}$ such that $F_i(0) \neq 0$.

Case 1) Origin is an equilibrium point for all modes

When the origin, $x = 0$, is an equilibrium point for all modes of the hybrid system, i.e. $F_i(0) = 0$ then from the definition of Δ (equations (4.6) and (4.28)), it can be seen that $\Delta(0) = 0$, and hence the perturbations are vanishing at the origin. The next lemma shows that the bound on the time integral of the effect of perturbation Δ (4.6), i.e. $\dfrac{\partial V}{\partial x}\Delta$, depends linearly on ε. For the sake of generality and ease of notation, $\frac{\partial V}{\partial x}$ will be represented by $p(x)$.

Lemma 4.4 *Consider a hybrid system with recurrent mode (4.1)-(4.2) and suppose $F_i(x)$ and $[\partial F_i/\partial x](x)$ are continuous and bounded in $D \subset \mathbb{R}^n$, and $F_i(0) = 0$, $\forall i \in \mathbf{L}$. Let $p(x)$ be such that $||p(x)|| \leq c||x||$ and $||\partial p/\partial x|| < c$, with $c > 0$. Then*

$$\left|\left|\int_{t_0}^{t} \frac{p(x)\Delta(x)}{||x||^2} ds\right|\right| \leq \varepsilon\tilde{c}(t - t_0) \tag{4.53}$$

where ε is given by (4.13) and $\tilde{c}$ is a positive constant.

Proof: Similar to the proof of Proposition 4.1, it can be shown that the integral is given by

$$\int_{t_0}^{t_m} \frac{p(x)\Delta(x)}{||x||^2} ds = \sum_{\ell=1}^{m} T_\ell \Omega(\hat{x}_i^\ell) \tag{4.54}$$

where $\hat{x}_i^\ell$ represents the state at a time instant between τ_{i-1}^ℓ and τ_i^ℓ, that is $x(\hat{\tau}_i^\ell)$ with $\hat{\tau}_i^\ell \in [\tau_{i-1}^\ell,\ \tau_i^\ell]$ and

$$\Omega(\hat{x}_i^\ell) = \sum_{i=1}^{l} \sum_{j=1}^{i-1} T_\ell \alpha_i^\ell \alpha_j^\ell \nabla_i(\hat{x}_j^\ell) F_j(\hat{x}_j^\ell) + \frac{1}{2} \sum_{i=1}^{l} T_\ell (\alpha_i^\ell)^2 \nabla_i(\hat{x}_i^\ell) F_i(\hat{x}_i^\ell) \tag{4.55}$$

with ∇_i defined by

$$\nabla_i = \frac{\partial[p(x)\Delta_i(x)\frac{1}{||x||^2}]}{\partial x} = \frac{\partial p}{\partial x}\Delta_i \frac{1}{||x||^2} + p\frac{\partial \Delta_i}{\partial x}\frac{1}{||x||^2} - \frac{2}{||x||^3}\frac{\partial ||x||}{\partial x} p\Delta_i \tag{4.56}$$

Taking the norm of Ω yields

$$||\Omega(\hat{x}_i^\ell)|| \leq \sum_{i=1}^{l} \sum_{j=1}^{i-1} T_\ell \alpha_i^\ell \alpha_j^\ell ||\nabla_i(\hat{x}_j^\ell) F_j(\hat{x}_j^\ell)|| + \frac{1}{2} \sum_{i=1}^{l} T_\ell (\alpha_i^\ell)^2 ||\nabla_i(\hat{x}_i^\ell) F_i(\hat{x}_i^\ell)|| \tag{4.57}$$

Since F_i and $[\partial F_i/\partial x](x)$ are continuous and bounded in D, and $F_i(0) = 0$, $||F_i(x)|| \leq L||x||$ and $||\partial F_i/\partial x|| \leq L$, with $L = \max_i\{L_i\}$. Then, from this and Lemma 4.2 (equation (4.9)) it can be verified that $||\frac{\partial \Delta_i}{\partial x}|| \leq 2(1-\alpha_i^\ell)L$. Using $||p(x)|| \leq c||x||$, $||\partial p/\partial x|| \leq c$ and $||\frac{\partial ||x||}{\partial x}|| < 1$, it can be seen that $||\nabla_i(x)F_i(x)||$ satisfies

$$\begin{aligned} ||\nabla_i(x)F_i(x)|| &\leq \left\{ \frac{\left|\left|\frac{\partial p}{\partial x}\Delta_i\right|\right|}{||x||^2} + \frac{||p\frac{\partial \Delta_i}{\partial x}||}{||x||^2} + \frac{2||p||||\Delta_i||}{||x||^3} \right\} ||F_i|| \\ &\leq \left\{ \frac{2cL(1-\alpha_i^\ell)||x||}{||x||^2} + \frac{2cL(1-\alpha_i^\ell)||x||}{||x||^2} \right. \\ &\quad \left. + \frac{4cL(1-\alpha_i^\ell)||x||^2}{||x||^3} \right\} L||x|| \\ &\leq 8cL^2(1-\alpha_i^\ell) \end{aligned} \tag{4.58}$$

By applying the above inequality in (4.57) and using the second inequality of Lemma 4.3, i.e. $T_\ell \alpha_i^\ell (1-\alpha_i^\ell) \leq \varepsilon$, it can be seen that

$$
\begin{aligned}
||\Omega(\hat{x}_i^\ell)|| &\leq 8cL^2 \sum_{i=1}^{l} \sum_{j=1}^{i-1} T_\ell \alpha_i^\ell (1-\alpha_i^\ell)\alpha_j^\ell + 4cL^2 \sum_{i=1}^{l} T_\ell \alpha_i^\ell (1-\alpha_i^\ell)\alpha_i^\ell \\
&\leq \varepsilon \tilde{c} \quad \text{with} \quad \tilde{c} = 4cL^2(2l+1) \qquad (4.59)
\end{aligned}
$$

From the above inequality and using the fact that $\sum_{\ell=1}^{m} T_\ell \leq (t-t_0)$ it can be deduced that

$$
\begin{aligned}
\left|\left|\int_{t_0}^{t_m} \frac{p(x)\Delta(x)}{||x||^2} ds\right|\right| = ||\sum_{\ell=1}^{m} T_\ell \Omega(\hat{x}_i^\ell)|| &\leq \sum_{\ell=1}^{m} T_\ell ||\Omega(\hat{x}_i^\ell)|| \\
&\leq \varepsilon \tilde{c} \sum_{\ell=1}^{m} T_\ell \leq \varepsilon \tilde{c}(t-t_0)
\end{aligned}
\qquad (4.60)
$$

which completes the proof.

■

The next theorem characterizes the stability of the hybrid system with a recurrent mode in terms of exponential stability of the averaged model.

Theorem 4.2 *Consider a hybrid system with recurrent mode defined by (4.1)-(4.2), with $F_i(x) : \mathbf{L} \times D \to \mathbb{R}^n$ where $D \subset \mathbb{R}^n$ is a domain containing the origin. Let ε be defined as (4.13). Assume that the following conditions are satisfied:*

- *The vector fields F_i satisfy $||F_i(x)|| \leq L_i||x||$ for all $i \in \mathbf{L}$ with $L_i > 0$,*
- *The origin is an exponentially stable equilibrium point for the averaged continuous dynamics (4.26).*

Then, there exist $\varepsilon^\star > 0$ such that for all $\varepsilon \in (0, \varepsilon^\star)$, the origin is an exponentially stable equilibrium point for the hybrid system.

Proof: The dynamics of the hybrid system in each mode can be represented as

$$\dot{x} = \bar{\mathsf{F}}(x) + \bar{\Delta}(x) = \bar{\mathsf{F}}(x) + \Delta(x) + \Delta_{\varepsilon}(x), \quad (i, x) \in \mathbf{L} \times D \tag{4.61}$$

where $\bar{\mathsf{F}}(x)$ is defined by (4.26) and Δ is given by (4.6) and Δ_{ε} is defined by (4.28). From the exponential stability of the nominal dynamics F and the converse Lyapunov theorem (Theorem 2.3), it can be concluded that a Lyapunov function $V(x)$ satisfying the inequalities (4.50)-(4.52) exists.

The derivative of $V(x)$ along the trajectories of the perturbed system satisfies

$$\dot{V} \quad = \quad \frac{\partial V}{\partial x}\bar{\mathsf{F}}(x) + \frac{\partial V}{\partial x}\left[\Delta(x) + \ \Delta_{\varepsilon}(x)\right] \tag{4.62}$$

From (4.29) and $||F_i(x)|| \leq L_i||x||$ it can be seen that $||\Delta_{\varepsilon}|| \leq \varepsilon k L||x||$ with $L = \max\limits_i \{L_i\}$ and $k = \max\limits_i\{k_i\} > 0$, $i \in \mathbf{L}$. Then by using inequalities (4.51), (4.52) the following holds

$$\dot{V}(x) \leq -c_3||x||^2 + \frac{\partial V}{\partial x}\left(\Delta(x) + \Delta_{\varepsilon}\right) = -c_3||x||^2 + \frac{\frac{\partial V}{\partial x}\Delta(x)}{||x||^2}||x||^2 + \varepsilon c_4 k L||x||^2 \tag{4.63}$$

By using the left and the right parts of the inequality (4.50), it can be seen that

$$\dot{V}(x) \leq (-c + \varepsilon\bar{c})V + \kappa(t)V \tag{4.64}$$

where $c = \frac{c_3}{c_2} > 0$, $\bar{c} = \frac{c_4 k L}{c_1} > 0$ and $\kappa(t) = \dfrac{\frac{\partial V}{\partial x}\Delta(x)}{c_1||x||^2}$. Now, by using Lemma 2.2, it can be seen that

$$\begin{aligned} V(t) \quad &\leq \quad V_0 \exp\left[(\varepsilon\bar{c} - c)(t - t_0) + \int_{t_0}^{t}\kappa(s)ds\right] \\ &\leq \quad V_0 \exp\left[(\varepsilon\bar{c} - c)(t - t_0)\right]\exp\left[\int_{t_0}^{t}\kappa(s)ds\right] \end{aligned} \tag{4.65}$$

Since $||\frac{\partial V}{\partial x}|| \leq c_4||x||$, then conditions of Lemma 4.4 are met, and hence the time integral of κ satisfies

$$\left|\left|\int_{t_0}^{t} \kappa(s)ds\right|\right| = \left|\left|\int_{t_0}^{t} \frac{\frac{\partial V}{\partial x}\Delta(x)}{c_1||x||^2}ds\right|\right| \leq \varepsilon\tilde{c}(t-t_0) \tag{4.66}$$

where ε is defined as (4.13) and $\tilde{c} = 4c_4L^2(2l+1)/c_1$ given in Lemma 4.4 by equation (4.59).

By applying the above integral inequalities in (4.65), it can be seen that V satisfies

$$\begin{aligned} V(t) &\leq V_0\exp\left[(\varepsilon\bar{c}-c)(t-t_0)\right]\exp\left[\varepsilon\tilde{c}(t-t_0)\right] \\ &\leq V_0\exp\left[-a(t-t_0)\right] \end{aligned} \tag{4.67}$$

with $a = c - \varepsilon(\tilde{c}+\bar{c})$. Let ε satisfy

$$\varepsilon < \frac{c}{\tilde{c}+\bar{c}} = \varepsilon^\star$$

then $a = c-\varepsilon(\tilde{c}+\bar{c}) > 0$. From (4.50), $c_1||x(t)||^2 \leq V(t)$, and $V_0 \leq c_2||x_0||^2$. Hence, using (4.67) it can be seen that

$$c_1||x(t)||^2 \leq V(t) \leq V_0\exp\left[-a(t-t_0)\right] \leq c_2||x_0||^2\exp\left[-a(t-t_0)\right], \quad a > 0 \tag{4.68}$$

from which, it can be seen that $||x(t)||$ satisfies

$$||x(t)|| \quad \leq \quad \sqrt{\frac{c_2}{c_1}}||x_0||\exp\left[-\frac{a}{2}(t-t_0)\right], \quad a > 0 \tag{4.69}$$

which shows that the origin is an exponentially stable equilibrium point for the hybrid system. ■

Theorem 4.2 shows that when the origin is the equilibrium point for all modes of the hybrid model, i.e. $F_i(0) = 0$, and in addition if the origin is exponentially stable for the averaged continuous system, then for *sufficiently* fast switching (small cycle times) and/or domination of a particular mode, the origin is also an exponentially stable equilibrium point for the hybrid system. This means that when the averaged continuous model is stable, without admitting infinite switching or a non-switching property, it is possible to have a stable hybrid system. It should be stressed that since the upper-bound on ε, i.e. $\varepsilon^\star$, results from a worst-case analysis, then it can be expected that for values of ε greater than $\varepsilon^\star$ the hybrid system is stable.

Case 2) Origin is not necessarily the equilibrium point for all modes

In the case where the origin is not necessarily an equilibrium point for all modes in a hybrid system, even in the case of sufficiently fast switching and/or domination of one mode, the stability of the hybrid model cannot be guaranteed. Intuitively, this can be seen from the fact that in the case of finite switching, at $x = 0$ the system can switch to any mode for which the origin is not an equilibrium point and hence the system shifts away from the origin. This case is equivalent to a non-vanishing perturbation, i.e. $\Delta(0) \neq 0$. In such a case, only ultimate boundedness of solutions can be expected. Before stating the main theorem, two useful lemmas will be first introduced.

Lemma 4.5 *Consider a hybrid system with recurrent mode (4.1)-(4.2) and suppose* $F_i(x)$ *and* $[\partial F_i/\partial x](x)$ *are continuous and bounded in* $D \subset \mathbb{R}^n$*, and* $F_i(0) = \delta_i$ *with* $\sum_{i=1}^{l} \alpha_i \delta_i = 0$*,* $\forall i \in \mathbf{L}$*. Let* $\kappa_1 = \dfrac{p(x)\,[\Delta(x) - \Delta(0)]}{||x||}$ *and* $p(x)$ *be such that* $||p(x)|| \leq c||x||$ *and* $||\partial p/\partial x|| < c$*, with* $c > 0$*. Then*

$$\left|\left| \int_0^t \mathrm{e}^{-a(t-s)} \kappa_1(x) ds \right|\right| \leq \varepsilon \tilde{b}_1 \tag{4.70}$$

where $a > 0$*,* ε *is given by (4.13) and* $\tilde{b}_1$ *is a positive constant.*

Proof: See Appendix B.1. ∎

Lemma 4.6 *Consider a hybrid system with recurrent mode (4.1)-(4.2) and suppose* $F_i(x)$ *and* $[\partial F_i/\partial x](x)$ *are continuous and bounded in* $D \subset \mathbb{R}^n$*, and* $F_i(0) = \delta_i$ *with* $\sum_{i=1}^{l} \alpha_i \delta_i = 0$*, and* $\delta_{i^\star} = 0$*, where* $i^\star = \arg\min_i (1 - \alpha_i)$*. Let* $\kappa_2 = p(x)\Delta(0)$ *and* $p(x)$ *be such that* $||p(x)|| \leq c||x||$ *and* $||\partial p/\partial x|| < c$*, with* $c > 0$*. Then*

$$\left|\left| \int_0^t \mathrm{e}^{-a(t-s)} \kappa_2(x) ds \right|\right| \leq \varepsilon^2 \tilde{b}_2 \tag{4.71}$$

where $a > 0$*,* ε *is given by (4.13) and* $\tilde{b}_2$ *is a positive constant.*

Proof: See Appendix B.2. ∎

Theorem 4.3 *Consider a hybrid system with recurrent mode defined by (4.1)-(4.2), with* $F_i(x) : \mathbf{L} \times D \rightarrow \mathbb{R}^n$ *where* $D \subset \mathbb{R}^n$ *is a domain containing the origin. Let* ε *be defined as (4.13). Assume that the following conditions are satisfied*

- $F_i(x)$ *and* $[\partial F_i/\partial x](x)$ *are continuous and bounded in* $D \subset \mathbb{R}^n$, *and* $F_i(0) = \delta_i$ *with* $\sum_{i=1}^{l} \alpha_i \delta_i = 0$, $\forall i \in \mathbf{L}$ *and* $\delta_{i^\star} = 0$, *where* $i^\star = \arg\min_i(1 - \alpha_i)$.
- *the origin is an exponentially stable equilibrium point for the averaged continuous dynamics (4.26).*

Then, there exist $\varepsilon^\star > 0$ *such that for all* $\varepsilon \in (0, \varepsilon^\star)$, *the solution of hybrid dynamics is ultimately bounded by a bound which depends on* ε.

Proof: The dynamics of the hybrid system in each mode can be represented as

$$\dot{x} = \bar{\mathsf{F}}(x) + \bar{\Delta}(x) = \bar{\mathsf{F}}(x) + \Delta(x) + \Delta_\varepsilon(x), \quad (i, x) \in \mathbf{L} \times D \tag{4.72}$$

where $\bar{\mathsf{F}}(x)$ is defined by (4.26) and Δ is given by (4.6) and Δ_ε is defined by (4.28). From the exponential stability of the nominal dynamics F and the converse Lyapunov theorem (Theorem 2.3), it can be concluded that a Lyapunov function $V(x)$ satisfying the inequalities (4.50)-(4.52) exists.

The derivative of $V(x)$ along the trajectories of the perturbed system satisfies

$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial x}\bar{\mathsf{F}}(x) + \frac{\partial V}{\partial x}\Delta(x) + \frac{\partial V}{\partial x}\Delta_\varepsilon(x) \\ &= \frac{\partial V}{\partial x}\bar{\mathsf{F}}(x) + \frac{\partial V}{\partial x}\left[\Delta(x) - \Delta(0)\right] + \frac{\partial V}{\partial x}\Delta(0) + \frac{\partial V}{\partial x}\Delta_\varepsilon \end{aligned} \tag{4.73}$$

Since $||F_i(x)|| \leq L_i||x|| + ||\delta_i||$ then from (4.29) $||\Delta_\varepsilon|| \leq \varepsilon k L||x|| + \varepsilon k \delta$ with $L = \max_i\{L_i\}$, $k = \max_i\{k_i\} > 0$, and $\delta = \max_i\{||\delta_i||\} > 0$, $i \in \mathbf{L}$. Now by using inequalities (4.51), (4.52) it can be seen that

$$\begin{aligned}\dot{V}(x) &\leq -c_3||x||^2 + \frac{\frac{\partial V}{\partial x}[\Delta(x)-\Delta(0)]}{||x||}||x|| + \frac{\partial V}{\partial x}\Delta(0)\\ &+\varepsilon c_4 kL||x||^2 + \varepsilon c_4 k\delta||x||\end{aligned} \tag{4.74}$$

By using the left and the right parts of the inequality (4.50), it can be seen that

$$\dot{V}(x) \leq (-c+\varepsilon\bar{c})V + \kappa_1\sqrt{V} + \kappa_2 \tag{4.75}$$

where $c = \frac{c_3}{c_2} > 0$, $\bar{c} = \frac{c_4 kL}{c_1} > 0$, $\kappa_1 = \dfrac{\frac{\partial V}{\partial x}[\Delta(x)-\Delta(0)]}{\sqrt{c_1}||x||} + \dfrac{\varepsilon c_4 k\delta}{\sqrt{c_1}}$ and $\kappa_2 = \frac{\partial V}{\partial x}\Delta(0)$. Let ε satisfy

$$\varepsilon < \frac{c}{\bar{c}} = \varepsilon^{\star}$$

then $a = c - \varepsilon\bar{c} > 0$.

Two cases will be considered: (i) If $\exists\mathcal{T}$ such that $\sqrt{V} \leq \varepsilon b$ for all $t \geq t_0 + \mathcal{T}$, then from (4.50) it can be seen that $||x|| \leq \varepsilon(\frac{b}{\sqrt{c_1}})$ and thus the proof is completed. (ii) For $\sqrt{V} > \varepsilon b$, $\dot{V}$ in (4.75) satisfies

$$\dot{V}(x) \leq -aV + \kappa_1\sqrt{V} + \frac{\kappa_2}{\varepsilon b}\sqrt{V} \tag{4.76}$$

From the comparison principle (Lemma 2.2.4) it can be deduced that

$$V(t) \leq V_0 \mathrm{e}^{-at} + \int_0^t \mathrm{e}^{-a(t-s)}\kappa_1\sqrt{V}ds + \frac{1}{\varepsilon b}\int_0^t \mathrm{e}^{-a(t-s)}\kappa_2\sqrt{V}ds \tag{4.77}$$

Let $V(t)\mathrm{e}^{at} = W^2$, then the above inequality can be written as

$$W^2(t) \leq \lambda^2 + \int_0^t \mathrm{e}^{\frac{a}{2}s}\kappa_1 W(s)ds + \frac{1}{\varepsilon b}\int_0^t \mathrm{e}^{\frac{a}{2}s}\kappa_2 W(s)ds \tag{4.78}$$

with $\lambda^2 = W_0^2 = V_0$. Applying Pachpatte's inequality (Lemma 2.4), it can be seen that

$$\sqrt{V}\mathrm{e}^{\frac{a}{2}t} = W(t) \leq \lambda + \frac{1}{2}\int_0^t \mathrm{e}^{\frac{a}{2}s}\kappa_1 ds + \frac{1}{2\varepsilon b}\int_0^t \mathrm{e}^{\frac{a}{2}s}\kappa_2 ds \tag{4.79}$$

from which

$$\sqrt{V} \leq \sqrt{V_0}\mathrm{e}^{-\frac{a}{2}t} + \frac{1}{2}\int_0^t \mathrm{e}^{-\frac{a}{2}(t-s)}\kappa_1 ds + \frac{1}{2\varepsilon b}\int_0^t \mathrm{e}^{-\frac{a}{2}(t-s)}\kappa_2 ds \tag{4.80}$$

From the definition of κ_1 and κ_2 and since the conditions of Lemmas 4.5 and 4.6 are met, then it can be seen that $\sqrt{V}$ satisfies

$$\begin{aligned} \sqrt{V} &\leq \sqrt{V_0}\mathrm{e}^{-\frac{a}{2}t} + \frac{1}{2}\tilde{b_1}\varepsilon + \frac{\varepsilon^2}{2\varepsilon b}\tilde{b_2} \\ &\leq \sqrt{V_0}\mathrm{e}^{-\frac{a}{2}t} + \varepsilon\left(\frac{\tilde{b_1} + \tilde{b_2}/b}{2}\right) \end{aligned} \tag{4.81}$$

where $\tilde{b_1}$, $\tilde{b_2}$ are some positive constants.

From (4.50), $\sqrt{c_1}||x(t)|| \leq \sqrt{V(t)}$, and $V_0 \leq \sqrt{c_2}||x_0||$. Hence, using (4.81) it can be seen that

$$\sqrt{c_1}||x(t)|| \leq \sqrt{V(t)} \leq \sqrt{c_2}||x_0||\mathrm{e}^{-\frac{a}{2}t} + \varepsilon\left(\frac{\tilde{b_1} + \tilde{b_2}/b}{2}\right) \tag{4.82}$$

from which, it can be seen that $||x(t)||$ satisfies

$$||x(t)|| \leq \sqrt{\frac{c_2}{c_1}}||x_0||\mathrm{e}^{-\frac{a}{2}t} + \varepsilon\tilde{b}, \quad a > 0 \tag{4.83}$$

with $\tilde{b} = \left(\frac{\tilde{b_1}+\tilde{b_2}/b}{2\sqrt{c_1}}\right)$. This shows that the solution of the hybrid system is ultimately bounded by the bound $\varepsilon\tilde{b}$ which depends on ε.

■

The theorem states that in the case where the origin is only the equilibrium point of the dominant mode and is an exponentially stable equilibrium point for the averaged model, then for sufficiently small values of ε the trajectories of the hybrid model are ultimately bounded by a bound which shrinks to zero as ε goes to zero.

Corollary 4.2 *Suppose the assumptions of Theorem 4.3 are satisfied. If in addition* $\lim_{t\to\infty} \varepsilon = 0$*, then the origin is an asymptotically stable equilibrium point for the hybrid system with a recurrent mode.*

Proof: The proof is a direct consequence of equation (4.83), where

$$\lim_{t\to\infty} ||x(t)|| \leq \lim_{t\to\infty} \varepsilon\tilde{b} = 0 \tag{4.84}$$

which shows that the origin is asymptotically stable. ∎

Corollary 4.3 *Suppose the assumptions of Theorem 4.3 are satisfied. If in addition* $\varepsilon \leq \mu||x||$ *with* μ *a positive constant satisfying* $\mu\tilde{b} < 1$*, then the origin is an asymptotically stable equilibrium point for the hybrid system with a recurrent mode.*

Proof: Equation (4.83) suggests that as $t \to \infty$, $||x(t)|| \leq \varepsilon\tilde{b}$. Now if $\varepsilon \leq \mu||x||$ then $||x|| \leq \mu\tilde{b}||x||$ which implies that $\mu\tilde{b} > 1$. This contradicts the assumption $\varepsilon \leq \mu||x||$ and thus $||x(t)|| = 0$, which means the origin is asymptotically stable. ∎

The above corollaries suggest in the case that the origin is not the equilibrium point for all modes but the dominant mode, as the time limit of ε goes to zero, then the origin becomes an asymptotically stable equilibrium point for the hybrid model. This means that as the trajectories of the system converge to a region near the origin, when infinite switching or no switching occurs, the system is defined by the dynamics of either the dominant mode or the continuous averaged one. Since from the assumption of Theorem 4.3, the origin is an asymptotically equilibrium point for both dynamics, the asymptotically stable property of the hybrid model follows.

4.4 Conclusions

In this chapter, it was shown that the error between the trajectories of the hybrid model and the continuous averaged model in a finite time interval is of the order ε. Also, if the slow continuous states are not influenced by dynamics of fast ones, then the fast states can also be eliminated from the averaged model, and the approximation result on slow continuous states is valid.

The link between stability of the averaged continuous model and the hybrid model was established. It was shown that in the case where the origin is a common equilibrium point for all modes and is exponentially stable for the averaged model, the origin is also an exponentially stable equilibrium point for the hybrid system, for sufficiently small values of ε, i.e. $\varepsilon < \varepsilon^\star$. In addition, it was shown that if the origin is not an equilibrium point for some modes, then the trajectories of the hybrid model are ultimately bounded, the bound being a function of ε. From that it was shown that if ε goes to zero asymptotically, then asymptotical stability of the hybrid model can be guaranteed.

The stability results presented here are conservative with respect to the maximum bound on the parameter ε, i.e. $\varepsilon^\star$. This is mainly the consequence of the worst-case study adopted in Lyapunov based stability approaches. The conservatism is due to several reasons which are inherent in Lyapunov based analysis approaches for uncertain systems. The only information on perturbation is considered to be the bounds on its norm $||\frac{\partial V}{\partial x}\Delta||$. This bound is usually the major source of conservatism for two main reasons. Firstly, the sign of the derivative of the Lyapunov function along the perturbation may be helpful but only the wrong sign is considered. Secondly, the uncertainties, Δ, may not always act near their bounds, and hence the real 'destabilizing' effect of the uncertainties could be smaller than that used in the analysis.

Chapter 5

Application: Control of a Stick-Slip Drive

5.1 Introduction

Stick-slip drives are becoming popular in the domain of micro and nano technologies because of their simple and compact structure, fast response and high resolution. As can be undersood from their names, these drives work in two distinct operation modes 'stick' and 'slip'. Stick and slip behavior is due to the friction force in mechanical systems. It was discussed earlier that the friction can be modeled as a hybrid system. Consequently, stick-slip drives can also be modeled in a hybrid framework. In fact, hybrid modeling is well adapted for these kind of systems, since it clearly reflects their operation principle.

While working on the modeling of a stick-slip drive available in the Laboratoire Automatique and studying the friction phenomena, the idea of modeling it in a hybrid framework and using hybrid control techniques was considered. These were actually the main motivation for developing the theoretical results discussed in Chapters 3 and 4. The aim of this chapter is to illustrate the theoretical results presented previously on the stick-slip drive. To this end, the hybrid modeling of the drive will be presented and then the cascade control scheme of Chapter 3 will be used for set-point tracking of the drive. The stability of the closed-loop system is then investigated along the lines of Chapter 4. Experimental results using the proposed control strategy will be presented as well.

This chapter is organized as follows. First, in Section 5.2 a brief representation of such drives will be presented. Then, the description of the stick-slip setup available at the Laboratoire d'Automatique and its working principle will be explained. In Section 5.3 the hybrid modeling of a stick-slip drive will be introduced. The control design of the drive via dehybridization will be discussed in Section 5.4, and the stability results are examined in Section 5.5. Finally, simulation as well as experimental results will be presented in Section 5.6.

5.2 Impact and Stick-Slip Drives

Inertial or impact drives are well developed in the domains of microsystems and micro-manipulators. They are intensively used in precise positioning and especially for the probe positioning in scanning tunneling microscopes (STM). These drives have a simple and compact structure, provide high resolution up to some nanometers, and generate long range movements with relatively high velocity. In addition, mechanisms with multiple degrees of freedom can be easily constructed.

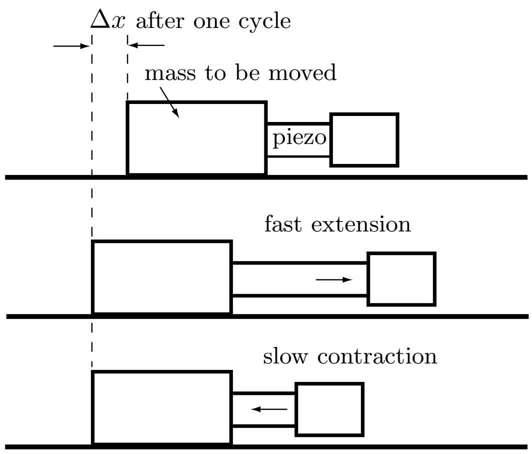

Figure 5.1: Impact Drive Working Principle

As illustrated in Figure 5.1, an impact drive consists of three basic parts: i) a main body or moving object, ii) an actuator (electromagnetic or a piezoelectric element) and iii) an inertial mass. When the actuator makes a rapid extension or contraction, a strong inertial force is generated which exceeds the friction force between the moving object and the table, and thus the main body will be displaced. When the actuator makes a slow contraction, the inertial force is smaller than the static friction so that the main body does not move. Repeating these fast and slow actuator movements carries out the motion.

Stick-slip actuators work by a similar principle and thus can be considered as a special case of impact drives. Stick-slip drives are distinguished from impact drives when one of the masses in the system is negligible compared to the other. Different applications and prototypes of such a technology can be found in the literature [16]. For instance, stick slip micromanipulators which give resolutions of some nanometers were designed and proposed in [17]. A prototype of a stick-slip drive was designed and manufactured at the Laboratoire d'Automatique for the purpose of studying friction phenomena [20]. The description of this drive and its working principle will be explained next.

5.2.1 Description of a Stick-Slip Inertial Drive (SSID)

The stick-slip inertial drive shown in Figure 5.2 is composed of the following main elements (Figure 5.3): (1) The stacked piezoactuator which has a range of 10μm and can produce a force of up to 500 N. The small mass (2) which is preloaded vertically and horizontally by means of elastic rings (4) on a load (3) (inertial mass). The force generated by the piezoactuator is transmitted to the small mass by a shaft (5) and a ball (6) which is needed to meet geometric and stress requirements. The load or inertial mass is situated on a rolling table and its position is measured by an incremental optical encoder (7) with a resolution of 0.1 μm.

The advantage of such a system is that the movement range needed by the piezoelectric actuator is much smaller than the movement range of the inertial mass. This property will explained in the next part by describing the working principle of the drive.

5.2.2 Working Principle

The name stick-slip drive comes from its working principle, where the drive operates in two different modes: 'stick' and 'slip' . Existence of these two distinct operational modes is due to the friction force between the small

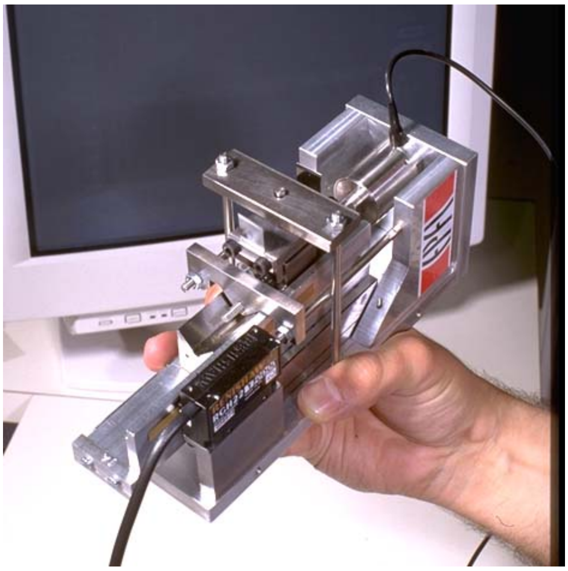

Figure 5.2: Photo of the stick-slip drive

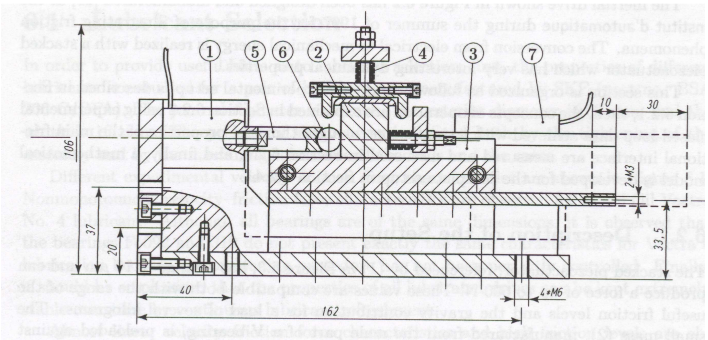

Figure 5.3: Drawing of the stick-slip drive: Piezoactuator (1), small mass (2), inertial mass (3), elastic rings (4), transmission elements shaft (5) and ball (6), optical encoder (7).

mass and the inertial mass. Two operational modes of the SSID are depicted in Figure 5.4, and described below:

- **Stick mode**: By applying a voltage input to the piezoactuator, the piezoelectric crystal deforms and generates a force on the small mass m. If the force generated is smaller than the friction force between the small mass m and inertial mass M, both masses will move together.
- **Slip mode**: By an abrupt backward movement of the piezoelectric actuator, the force acting at the frictional interface exceeds the static friction and the small mass slides on the inertial mass back to its initial position $x_m = 0$. Since the small mass is lighter than the inertial mass, during the same time interval the inertial mass has a much shorter backward movement.

During mode 'stick' both masses are attached and thus, they move together. Since the range of elongations of the piezoactuator and consequently the small mass is limited, the inertial mass cannot be displaced more than by a certain limit. Then a resetting action is needed. The act of resetting occurs in the 'slip' period during which the small mass slides on the inertial mass back to its initial position. If during this period the position of the inertial mass is not altered, then in the following step, that is in the 'stick' period, the inertial mass can be displaced further. By repeating the sequence of 'stick' and 'slip' transitions, the inertial mass can be freely displaced on the rolling table even outside the movement range of the piezoelectric actuator.

5.2.3 Modeling: A Brief History

A complete model of the drive using the bond graph methodology was proposed in [1], where a dynamic friction model (LuGre) [18] was used for defining the friction forces. LuGre is a friction model that captures most of the observed friction related phenomena such as presliding movement, Stribeck effect, and Dahl's hysteresis curve. The presliding movement is defined by introducing an additional state variable. The model of an SSID using the LuGre friction description represents the behavior of the system fairly well and is confirmed by simulation. However, because of the high complexity of the model, the analysis and control design were not apparent. In [55] a two structure friction definition was used instead of the LuGre model. The two structure friction definition was based on Karnopp's simulation model [31] that captures the stick and slip behavior of friction. A two

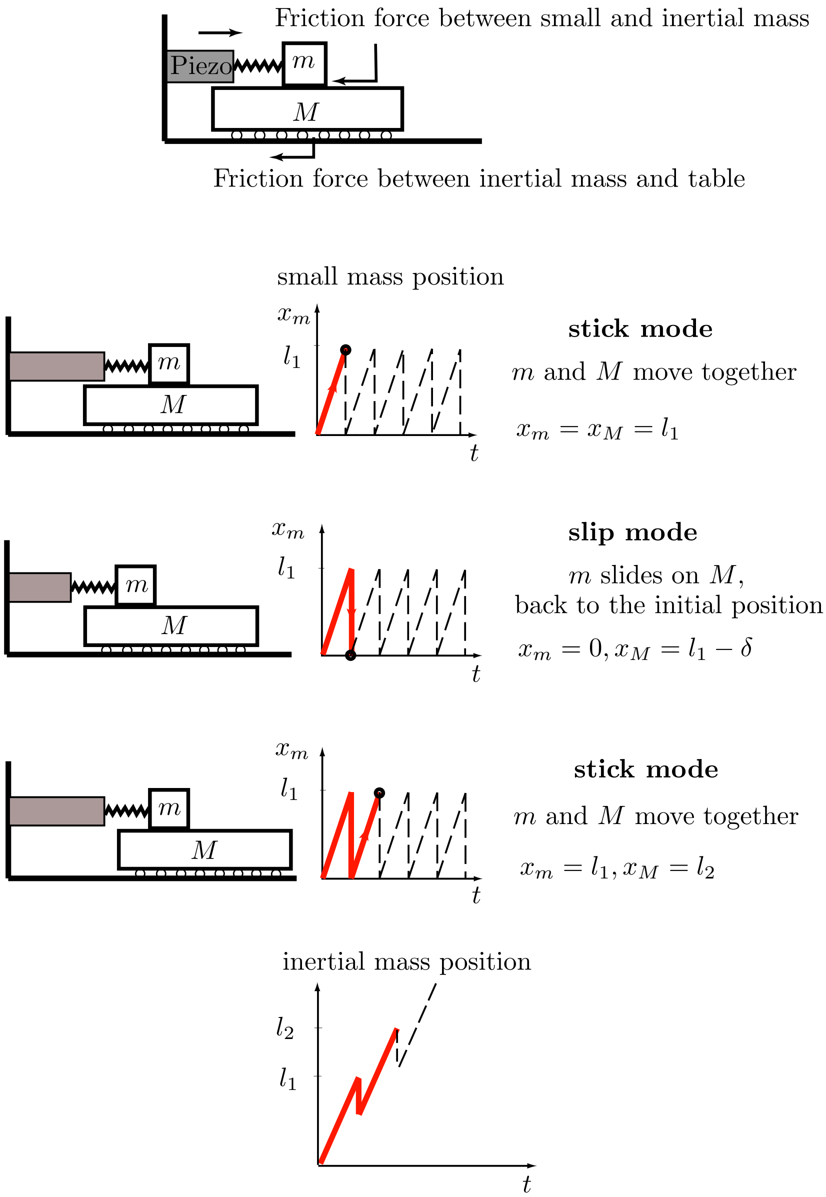

Figure 5.4: SSID Working Principle

structure model defines the friction as a function of velocity and also input force. Extensive simulation shows that Karnopp's two structure model is indeed sufficient to describe the system's behavior. The properties of the main frictional interface, as it was studied and discussed in [1], show that the presliding displacements are negligible, and thus justify the use of a simpler model. It was noted that the Karnopp model can be cast in the hybrid framework [60]. In fact, as was discussed in Section 2.3.4, by using the Coulomb friction definition, i.e. $F_c\text{sgn}(v)$, the system can also be modeled in a hybrid framework. In the next section, hybrid modeling of an SSID will be introduced. Such a model will be used throughout this chapter as a basis for analysis and control design.

5.3 Hybrid Modeling of the SSID

Consider a schematic of a stick-slip inertial drive as shown in Figure 5.5.

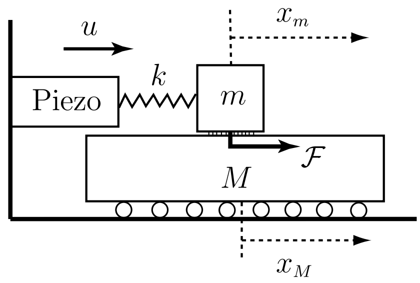

Figure 5.5: Schematic of an SSID

A simplified model of the stick–Slip inertial drive can be developed from first principles based on the following assumption:

Assumption 5.1 *The following assumptions are made prior to modeling:*

- *The friction between the inertial mass and the rolling table is neglected.*
- *The dynamics of the piezoelectric actuator are much faster than the dynamics of the rest of the system and thus the input to the piezoelectric actuator and the resulting generated force u are simply related by a gain.*

- *The force generated by the piezoactuator is related to the position of the piezo crystal,* x_{piezo}*, by the equality* $u = kx_{\text{piezo}}$*. Since the maximum elongation of the piezo crystal,* $\max\{x_{\text{piezo}}\}$*, is limited by the dimension and the load of the system, it will be assumed that the input force is bounded;* $|u| \leq u_{\max} = \max\{kx_{\text{piezo}}\}$*.*

The model is given by

$$\dot{x}_m = v_m \tag{5.1}$$

$$\dot{v}_m = \frac{1}{m}(u - kx_m - \mathcal{F}) \tag{5.2}$$

$$\dot{x}_M = v_M \tag{5.3}$$

$$\dot{v}_M = \frac{1}{M}\mathcal{F} \tag{5.4}$$

where x_m and x_M represent the positions and v_m and v_M represent the velocities of the small mass and inertial mass respectively, with m and M being their respective masses. u is the force generated by the piezoelectric actuator, k is the spring constant of the transmission elements, and $\mathcal{F}$ is the friction force between the two masses (Figure 5.6).

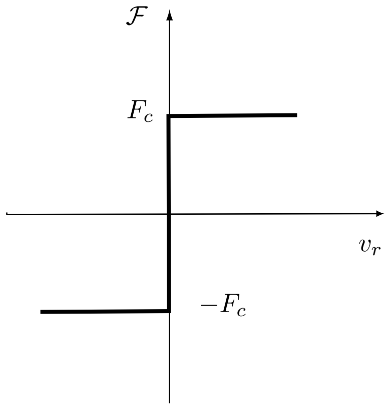

Figure 5.6: Coulomb Friction

In Section 2.3.4, it was shown that a system with friction defined by the Coulomb model (Figure 5.6), can be modeled in a hybrid framework.

Here, the hybrid modeling is achieved by using a physically motivated procedure by considering friction to be a function of both the velocity and the applied force. Depending on the external force and the relative velocity, $v_r = v_m - v_M$, friction has different definitions (structure) and three modes (discrete states) can be distinguished: $q_p = stick$, $q_p = slip_-$, and $q_p = slip_+$.

Denote ρ as the force acting at the frictional interface due to an external input. If ρ does not exceed the Coulomb friction level F_c, then the inertial and the small mass move together with $v_r = 0$. In this case mode $q_p = stick$ is active and friction $\mathcal{F}$ is defined by ρ. As soon as ρ exceeds the Coulomb friction level, one mass slips over the other with $v_r = v_m - v_M \neq 0$. The two modes $q_p = slip_+$ and $q_p = slip_-$ are respectively distinguished by positive and negative relative velocities, i.e. $v_r > 0$ or $v_r < 0$.

The expression for ρ can be derived from evolution of the relative velocity, $v_r = 0$. Setting $\dot{v}_r = \dot{v}_m - \dot{v}_M = 0$ leads to

$$\rho = \frac{M}{m+M}(u - kx_m) \tag{5.5}$$

In mode *stick* the friction force is defined by $\mathcal{F} = \rho$ and in modes $slip_-$ and $slip_+$ by $\mathcal{F} = -F_c$ and $\mathcal{F} = F_c$ respectively.

By using the definition of friction force in each mode, the vector fields $F_q(x, u)$ with $x = (x_m, v_m, x_M, v_M)^T$, $q_p \in Q = \{stick, slip_-, slip_+\}$ are defined by

$$F_{stick} = \begin{pmatrix} v_m \\ \frac{u-kx_m}{m+M} \\ v_M \\ \frac{u-kx_m}{m+M} \end{pmatrix}, \; F_{slip_-} = \begin{pmatrix} v_m \\ \frac{u-kx_m+F_c}{m} \\ v_M \\ -\frac{F_c}{M} \end{pmatrix}, \; F_{slip_+} = \begin{pmatrix} v_m \\ \frac{u-kx_m-F_c}{m} \\ v_M \\ \frac{F_c}{M} \end{pmatrix} \tag{5.6}$$

The transitions between these modes and the definition of the friction force $\mathcal{F}$ in each mode are depicted in Figure 5.7.

Remark 5.1 *Since the discrete transitions depend on the continuous states and input, i.e. (x, u), the transitions are internally induced and thus classified as autonomous switching.*

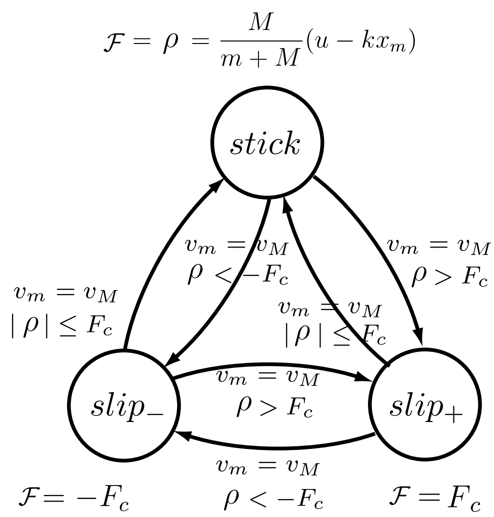

Figure 5.7: SSID Hybrid Automaton: $\mathcal{H}_p$

5.4 Control of an SSID via Dehybridization

In this section, the cascade control scheme presented in Section 3.3 will be used for the set-point tracking problem of an SSID. Using a switching rule in the inner-loop controller, a two-time scale behavior is imposed on the system. Such a property for the combination of plant and inner-loop control will be investigated. Using dehybridization the outer-loop controller will be designed for the averaged continuous model. For convenience the block diagram of the proposed control scheme is shown here (Figure 5.8).

5.4.1 Inner-loop Design: Rendering an SSID with a Recurrent Mode

In order to apply dehybridization, first, a switching logic should be designed to render the overall system with a recurrent mode. The discrete transitions being internally induced, the task of the hybrid controller is to find a discrete transition sequence that takes the system to $q^\star$, and then find an appropriate input u that makes the transition possible.

Consider a hybrid controller $\mathcal{H}_c$ defined by

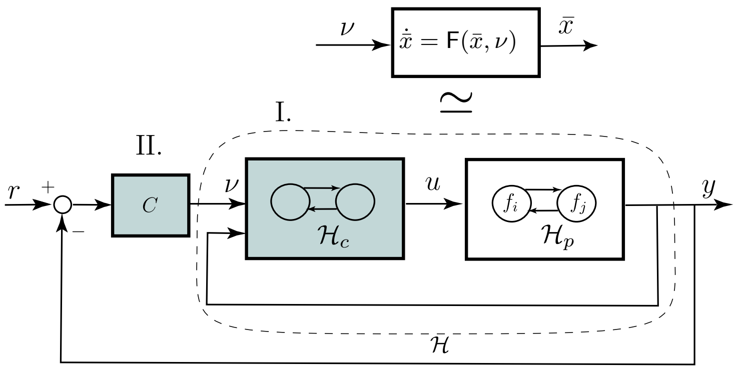

Figure 5.8: Block diagram for the controller design via dehybridization

$$
\begin{aligned}
u = y_c &= h^c(q_c, x, \nu) \\
q_c^+ &= R_c(q_c, q_p, x, u)
\end{aligned} \tag{5.7}
$$

where $q_c = \{\texttt{External}, \texttt{Reset}\}$ and R_c and h^c are given by

$$
R_c: \quad {q_c}^+ = \left\{ \begin{array}{lll} \texttt{Reset} & \text{if} & q_p = stick, \quad q_c = \texttt{External}, \quad |u| = u_{\max} \\ \texttt{External} & \text{if} & q_p = slip_-, \quad q_p = slip_-, \quad v_m = v_M \end{array} \right. \tag{5.8}
$$

$$
u = h^c(q_p, x, \nu) = \left\{ \begin{array}{lll} 0 & \text{if} & q_c = \texttt{Reset} \\ \nu + kx_m & \text{if} & q_c = \texttt{External} \end{array} \right. \tag{5.9}
$$

where the following assumptions are made

- $|\nu| \leq \bar{M} F_c$ where $\bar{M} = (\frac{M+m}{M})$,
- $u_{\max} \geq 2\bar{M}F_c$.

The first condition implies that

$$
|u - kx_m| \leq \bar{M}F_c \Rightarrow |\frac{M}{m+M}(u - kx_m)| \leq F_c \Rightarrow |\rho| \leq F_c \tag{5.10}
$$

Applying the hybrid controller $\mathcal{H}_c$ (5.7)-(5.9), the combination of the plant and controller is a hybrid system with six possible discrete states. The transition rules and discrete states are depicted in Figure 5.9.

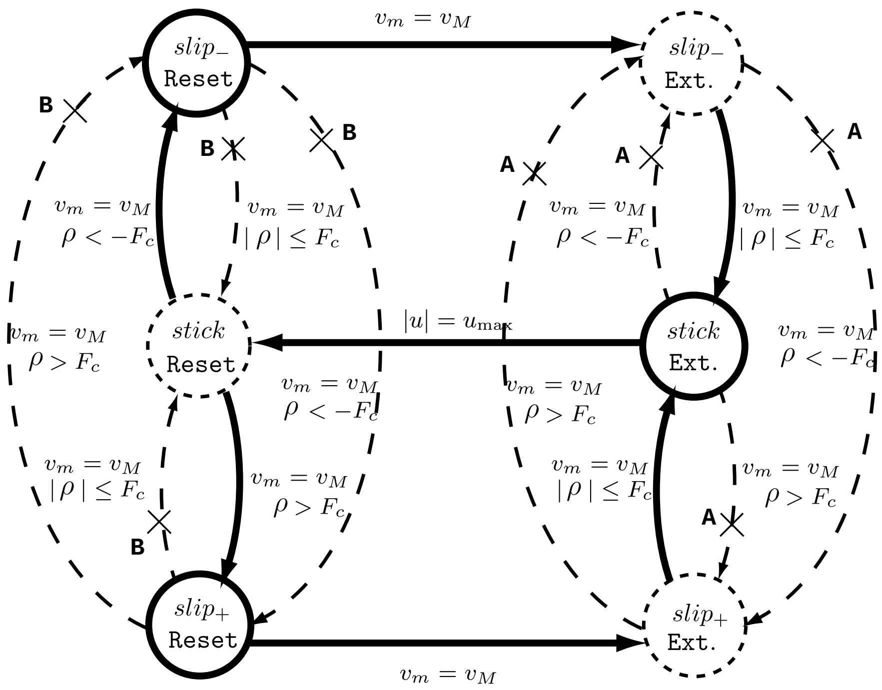

Figure 5.9: Plant and controller combination automaton $\mathcal{H}$

Among the various transitions and modes present in the overall system (Figure 5.9), quite a few transitions are not feasible and three modes are only transient, i.e. the system leaves the mode immediately.

- **Non Feasible Transitions:** Non feasible transitions can be categorized into two types and are shown as dashed lines in the figure. Transitions of type `A` are not feasible because of the choice of the external control law ν which always satisfies the condition $|\rho| \le F_c$ (see equation (5.10)). Transitions of type `B` are not possible because of the preference given to the external control, i.e. $q_c =$ `External` when $v_m = v_M$.

- **Non Feasible (transient) Modes:** There are three transient modes in Figure 5.9 that are marked by dashed lines. $q = (slip_-, \texttt{External})$ is transient since when $q_c = \texttt{External}$ and $v_m = v_M$, $|\rho| \leq F_c$ from the assumption made for $q_c = \texttt{External}$. A similar argument can be used for $q = (slip_+, \texttt{External})$. The mode $q = (stick, \texttt{Reset})$ is transient since when $u = u_{\max}$, using the assumption $u_{\max} \geq 2\bar{M}F_c$ and Proposition C.1 (see Appendix C) it has been shown that $|\rho| > F_c$.

From above argument it can be concluded that only the transitions between three modes

$$q = (q_p, q_c) \in \{(stick, \texttt{External}), (slip_-, \texttt{Reset}), (slip_+, \texttt{Reset})\}$$

are feasible. The summary of feasible transitions is depicted in Figure 5.10.

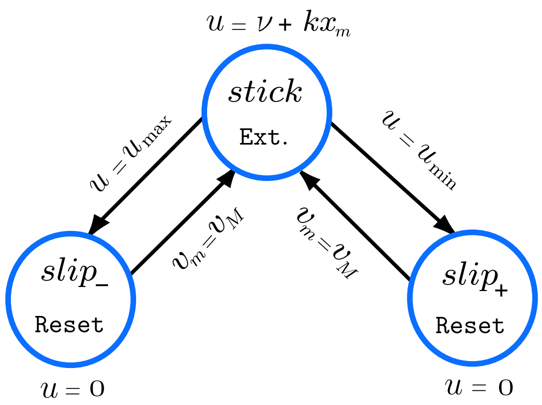

Figure 5.10: Plant and controller combination: $\mathcal{H}$

The next step is to show that the system has a recurrent mode

$$q^\star = (stick, \texttt{External})$$

To prove such a property, it suffice to show that when the system is either in mode $q = (slip_-, \texttt{Reset})$ or in mode $q = (slip_+, \texttt{Reset})$, it returns to $q^\star$ in finite time .

Proposition 5.1 *Consider the hybrid model of an SSID. The hybrid controller $\mathcal{H}_c$ defined by (5.7)-(5.9) renders the system with a recurrent mode $q^\star = (stick, \texttt{External})$. In addition, the maximum time interval during which the system can stay in modes $q = (slip_-, \texttt{Reset})$ or $q = (slip_+, \texttt{Reset})$ is $2\pi\sqrt{\frac{m}{k}}$.*

Proof: See Appendix C.2 ∎

To simplify notations let

$$q \in Q = \{(stick, \texttt{External}), (slip_-, \texttt{Reset}), (slip+-, \texttt{Reset})\} = \{1, 2, 3\}$$

The vector fields, $F_q = F_i(x)$, $i \in \mathbf{L} = \{1, 2, 3\}$, representing the dynamics of $\mathcal{H}$ are given by

$$F_1 = \begin{pmatrix} v_m \\ \frac{\nu}{m+M} \\ v_M \\ \frac{\nu}{m+M} \end{pmatrix}, \; F_2 = \begin{pmatrix} v_m \\ \frac{-kx_m+F_c}{m} \\ v_M \\ -\frac{F_c}{M} \end{pmatrix}, \; F_3 = \begin{pmatrix} v_m \\ \frac{-kx_m-F_c}{m} \\ v_M \\ \frac{F_c}{M} \end{pmatrix} \tag{5.11}$$

5.4.2 Time-scale Properties and the Averaged Continuous Model

From Proposition 5.1, it can be seen that i) $q = 1$ is dominant, ii) the maximum time interval that the system stays in non-dominant mode, i.e. modes $q = 2, 3$ is $2\pi\sqrt{\frac{m}{k}}$. Thus by using the definition of ε in Section 3 (see equation (3.5))

$$\varepsilon = \max_{\ell} T_\ell(1 - \alpha_1^\ell) \leq 2\pi\sqrt{\frac{m}{k}} \tag{5.12}$$

Remark 5.2 *From (5.12), it can be seen that for small values of m and/or large values of k, ε is small. When $m = 0$ or $k = \infty$, a slip mode transition is instantaneous. This means that the dynamics of the system is entirely defined by the dynamics of the inertial mass and in addition, during resetting periods, i.e. $q = 2, 3$, the small mass instantaneously returns to its initial position.*

Elimination of Discrete States

Once the parameter ε is defined, using dehybridization the continuous averaged model can be written as the convex combination of the vector fields $F_1(x)$ and $F_2(x)$, i.e.

$$\mathsf{F} = \begin{pmatrix} \dot{x}_m \\ \dot{v}_m \\ \dot{x}_M \\ \dot{v}_M \end{pmatrix} = \bar{\theta}_1 \begin{pmatrix} v_m \\ \frac{\nu}{m+M} \\ v_M \\ \frac{\nu}{m+M} \end{pmatrix} + \bar{\theta}_2 \begin{pmatrix} v_m \\ \frac{-kx_m+F_c}{m} \\ v_M \\ -\frac{F_c}{M} \end{pmatrix} + \bar{\theta}_3 \begin{pmatrix} v_m \\ \frac{-kx_m-F_c}{m} \\ v_M \\ \frac{F_c}{M} \end{pmatrix} \tag{5.13}$$

where $\bar{\theta}_i = \lim_{\varepsilon \to 0} \alpha_i^{\ell}$.

Elimination of the Fast Continuous States

From Theorem 4.1, the continuous model (5.13) represents the averaged model for the hybrid system, if it is bounded as $\varepsilon \to 0$. However, it can be verified that some components of F are unbounded for $\varepsilon = 0$. Recalling from Section 4.2.2 that if some components of the continuous states are evolving as fast as the discrete states, then the dehybridization procedure fails unless these fast states do not influence the dynamics of the slow continuous states. From equation (5.12), it can be deduced that the dynamics of the small mass in mode $q = 2, 3$ can be expressed in a singular perturbation formulation by the scaling $(\eta_1, \eta_2) = (\frac{x_m}{\varepsilon}, v_m)$:

$$\begin{pmatrix} \dot{x}_m \\ \dot{v}_m \end{pmatrix} = \begin{pmatrix} v_m \\ \frac{-kx_m \pm F_c}{m} \end{pmatrix} \Rightarrow \begin{pmatrix} \varepsilon\dot{\eta}_1 \\ \varepsilon\dot{\eta}_2 \end{pmatrix} = \begin{pmatrix} \eta_2 \\ \frac{1}{\sigma^2}\eta_1 \pm \varepsilon\frac{F_c}{m} \end{pmatrix} = g_i, \quad i = 2, 3 \tag{5.14}$$

where $\sigma = \frac{1}{\varepsilon}\sqrt{\frac{m}{k}}$. Hence, $\dot{\eta} = g_2(\eta, \xi)/\varepsilon$ with $\eta = (\eta_1, \eta_2)^T$ becomes unbounded as $\varepsilon \to 0$. From this it can be concluded that the small mass state variables ('fast states') cannot be approximated using the convex combination of the vector fields. However, from vector fields F_i given in (5.11) it can be observed that the inertial mass dynamics, $\dot{\xi} = f_i(\eta, \xi)$ with $\xi = (x_M, v_M)$ are not influenced by the fast states η, i.e.

$$\frac{\partial f_i}{\partial \eta} = 0, \quad i = 1, 2, 3$$

From this, and by using Corollary 4.1, it can be concluded that under the trajectories of the slow dynamics, $\xi = (x_M, v_M)$, can be approximated by the convex combination of the vector fields f_i, $i = 1, 2, 3$:

$$\begin{aligned}
\mathsf{f} &= \bar{\theta}_1 f_1 + \bar{\theta}_2 f_2 + \bar{\theta}_3 f_3 \Rightarrow \\
\begin{pmatrix} \dot{x}_M \\ \dot{v}_M \end{pmatrix} &= \bar{\theta}_1 \begin{pmatrix} v_M \\ \frac{\nu}{m+M} \end{pmatrix} + \bar{\theta}_2 \begin{pmatrix} v_M \\ -\frac{F_c}{M} \end{pmatrix} + \bar{\theta}_3 \begin{pmatrix} v_M \\ \frac{F_c}{M} \end{pmatrix} \\
&= \begin{pmatrix} v_M \\ \bar{\theta}_1 \frac{\nu}{m+M} - \bar{\theta}_2 \frac{F_c}{M} + \bar{\theta}_3 \frac{F_c}{M} \end{pmatrix} \qquad (5.15)
\end{aligned}$$

As $2\pi\sqrt{\frac{m}{k}} \to 0$, $\varepsilon \to 0$ and thus mode $q^\star = 1$ becomes dominant. In such a case $\bar{\theta}_1 = 1$, $\bar{\theta}_i = 0$, $i = 2, 3$ and thus the dynamics of the averaged model is given by

$$\begin{pmatrix} \dot{x}_M \\ \dot{v}_M \end{pmatrix} = \begin{pmatrix} v_M \\ \frac{\nu}{m+M} \end{pmatrix} \qquad (5.16)$$

which is simply a double integrator system.

5.4.3 Outer-loop Design: Set-point Tracking

In this section, the set-point tracking problem for the averaged dynamics (5.15) will be discussed. The goal is to design a control law $\nu(x)$ such that the tracking error goes to zero as t tends to infinity, that is

$$e(t) = (x_M - \mathsf{r}, v_M) \to 0 \quad \text{as} \quad t \to \infty$$

with r being the set-point for the SSID inertial mass position. Denoting $e_1 = x_M - \mathsf{r}$, $e_2 = v_M$, then the dynamics of the averaged model (5.15) can be rewritten as

$$\begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \end{pmatrix} = \begin{pmatrix} e_2 \\ \frac{\nu}{m+M} \end{pmatrix} \qquad (5.17)$$

Since from earlier assumption, $\nu(x)$ satisfies $|\nu| \leq \bar{M}F_c$, a saturated proportional-derivative control law will be proposed. Let $\nu(x)$ be defined as:

$$\nu(x) = \text{sat}(-k_p e_1 - k_v e_2,\ \bar{M}F_c) \qquad (5.18)$$

with $k_p > 0$ and $k_v > 0$. In the next section, it will be shown that such a control law (5.18) makes the origin a globally asymptotically stable equilibrium point for the averaged model.

5.5 Stability Results

In this section, the stability of the averaged model as well as the stability of the SSID subject to the proposed cascade control scheme will be investigated.

5.5.1 Stability of the Averaged Model

When the control law (5.18) is not saturated, the dynamics of the closed-loop averaged model can be written as

$$\begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \end{pmatrix} = \begin{pmatrix} e_2 \\ -\frac{k_p e_1}{m+M} - \frac{k_v e_2}{m+M} \end{pmatrix} \tag{5.19}$$

which is a second order linear system, and it can be easily verified that for $k_p > 0$ and $k_v > 0$, the origin is locally exponentially stable. However, when ν is saturated the stability proof is not trivial. Next, it will be shown that the origin for the averaged model under the control law (5.18) is globally asymptotically stable.

Proposition 5.2 *The dynamics of the averaged model (5.16) under the control law (5.18) with $k_p > 0$, $k_v > 0$, is globally asymptotically stable at the origin.*

Proof: See Appendix C.4. ∎

The proof is based on partitioning the state space into different areas where the control is saturated and non-saturated. The origin belongs to the non-saturated zone. By using a Lyapunov function whose derivative is negative definite in the non-saturated zone and is a decreasing function between entering and leaving the saturated region, the stability follows. The averaged model is globally asymptotically stable and is locally exponentially stable in the non-saturated region.

5.5.2 Stability of an SSID

In the last section, the stabilizing control law $\nu(x)$ for the averaged model was presented. By applying (5.18) in equation (5.9), the overall control for an SSID can be found. As was discussed earlier, the system is recurrent with mode $q = 1$, and the closed-loop dynamics of the SSID are given by $\dot{e} = f_1(e)$ and $\dot{e} = f_2(e)$ with $e = (e_1, e_2)^T$ and

$$f_1 = \begin{pmatrix} e_2 \\ -\frac{k_p}{m+M}e_1 - \frac{k_v}{m+M}e_2 \end{pmatrix}, \quad f_2 = \begin{pmatrix} e_2 \\ \frac{-F_c}{M} \end{pmatrix}, \quad f_3 = \begin{pmatrix} e_2 \\ \frac{F_c}{M} \end{pmatrix} \tag{5.20}$$

Now, it should be verified whether an SSID subject to the overall control (5.7)-(5.9) with (5.18) is stable. To do so, the stability results presented in Section 4.3 will be recalled.

From the above equations it can be observed that that the origin which is the equilibrium point of the averaged model is only an equilibrium point for $q = 1$, i.e. $f_1(0) = 0$, but $f_2(0) \neq 0$. In fact, since $\dot{e}_2 = v_M = -\frac{F_c}{M}$ mode $q = 2$ has no equilibrium point. Remember that from the system's working operation, mode $q = 2$ is only used for resetting the control and its not surprising that the system cannot be stabilized in such a mode. Since the origin is not an equilibrium point for all modes, only an ultimately boundedness of solutions is expected.

Proposition 5.3 *Consider the dynamics of an SSID subject to the cascade control strategy (5.7)-(5.9) and (5.18). Then there exists $k_p > 0$, $k_v > 0$ such that the trajectories of the closed-loop system are ultimately bounded by a bound which depends on ε.*

Proof: See Appendix C.5. ■

If it can be verified that the conditions of Corollary 4.2 or 4.3 are satisfied, i.e. $\varepsilon \to 0$ as $t \to \infty$, then the asymptotical stability of the origin can be guaranteed. To this end, it can be verified that there exists a ball containing the origin, i.e. $||e|| \leq \mathcal{R}$, in which the input $u = \nu + kx_m$ is never saturated, i.e. $|u| \neq u_{\max}$ and thus the system never leaves mode $q = 1$. This means that in this region the system is entirely defined by the dominant mode and hence after some finite number of cycles ℓ no further switching to the non-dominant mode occurs and hence $\varepsilon = 0$. From this fact it can be deduced that ε satisfies $\varepsilon \leq \mu||x||$ in $\mathcal{R}$ with μ a positive constant. In $\mathcal{R}$, μ can be chosen such that $\mu\tilde{b} < 1$ and thus from Corollary 4.3 asymptotic set-point tracking can be concluded.

5.6 Experimental and Simulation Results for an SSID

5.6.1 SSID: A Practical Control

Practical implementation of controllers on the system should consider certain realistic limitations. In the proposed control law (5.9), the input u is a function of the small mass position, i.e. $u = \nu + kx_m$. For implementing this control, state x_m should be measured and in addition the exact value of parameter k is needed. However, the only state measured on the system is the inertial mass position x_M. Here, an alternative controller will be presented which does not rely on the measurement of the small mass state variables. It can be observed that in mode *stick*, the difference between the small mass and the inertial mass position is constant. This can be easily seen from the fact that in mode $q = (stick, \texttt{External}) = 1$, $v_r(t) = v_m(t) - v_M(t) = 0$ and hence

$$\int_{\tau_0^\ell}^t v_m(s)ds = \int_{\tau_0^\ell}^t v_M(s)ds \Rightarrow x_m(t) = x_M(t) - \mu_\ell, \quad t \in [\tau_0^\ell, \tau_1^\ell] \tag{5.21}$$

and $\mu_\ell = x_M(\tau_0^\ell) - x_m(\tau_0^\ell)$ is constant for $t \in [\tau_0^\ell, \tau_1^\ell]$. Now, consider the control law (5.9) is replaced by the following control

$$u = h^c(q_p, x, \nu) = \begin{cases} 0 & \text{if} \quad q_c = \texttt{Reset} \\ \tilde{\nu}(x) & \text{if} \quad q_c = \texttt{External} \end{cases} \tag{5.22}$$

Now, from $x_m(t) = x_M(t) - \mu_\ell$ and the above control law, it can be verified that the averaged model can be written as

$$\begin{pmatrix} \dot{x}_M \\ \dot{v}_M \end{pmatrix} = \begin{pmatrix} v_M \\ \frac{\tilde{\nu} - kx_m}{m+M} \end{pmatrix} = \begin{pmatrix} v_M \\ \frac{\tilde{\nu} - kx_M + k\mu_\ell}{m+M} \end{pmatrix} \tag{5.23}$$

The term $k\mu_\ell$ in (5.23) can be treated as an unknown constant which can be handled by an integral term in the control law ν. Consider the control law:

$$\tilde{\nu}(x) = -k_i \int (x_M - \mathsf{r})d\tau - k_v v_M = -k_i \int e_1 d\tau - k_v e_2 \tag{5.24}$$

with $e_1 = (x_M - \mathsf{r})$, $e_2 = v_M$, $k_i > 0$ and $k_v > 0$. To introduce the integral state in the dynamics of the system, define $e_3(t) = \int_0^t e_1 d\tau$. The augmented dynamics of the closed-loop averaged model under (5.24) can be written as

$$\begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{pmatrix} = \begin{pmatrix} e_2 \\ \frac{-ke_1}{m+M} - \frac{k_v e_2}{m+M} - \frac{k_i e_3}{m+M} \\ e_1 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{k(\mu_\ell - r)}{m+M} \\ 0 \end{pmatrix} \tag{5.25}$$

The equilibrium point of the augmented system is $(e_1, e_2, e_3) = (0, 0, {e_3}^{eq})$, where ${e_3}^{eq}$ can be obtained from $\dot{e}_2 = 0$ using the above dynamics

$$\begin{aligned} \dot{e}_2 = 0 \quad &\Rightarrow \quad -\frac{k_i}{m+M}{e_3}^{eq} + \frac{k(\mu_\ell - r)}{m+M} = 0 \\ &\Rightarrow \quad {e_3}^{eq} = \frac{k}{k_i}(\mu_\ell - \mathsf{r}) \end{aligned} \tag{5.26}$$

Now, for the equilibrium point to be exponentially stable, parameters k_i and k_v should be chosen such that the following matrix be Hurwitz

$$\mathsf{A} = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{k}{m+M} & -\frac{k_v}{m+M} & -\frac{k_i}{m+M} \\ 1 & 0 & 0 \end{pmatrix} \tag{5.27}$$

Remark 5.3 *Note that $\nu_{eq} = -k_i {e_3}^{eq}$ is the steady state control value that is needed to maintain the system at the equilibrium point. Different values of $\mu_\ell = x_M(\tau_0^\ell) - x_m(\tau_0^\ell)$, change the equilibrium point ${e_3}^{eq}$, nevertheless condition $(x_M - \mathsf{r}, v_M) = (0, 0)$ is maintained.*

Note that as was discussed earlier, the control u in mode $q = (stick, \texttt{External})$ should satisfy the constraint $|\rho| \leq F_c$, i.e.

$$|\rho| = \left| \frac{M}{m+M}(u - kx_m) \right| = \left| \frac{1}{\bar{M}}(\tilde{\nu} - kx_m) \right| \leq F_c \tag{5.28}$$

It is clear that for $\tilde{\nu}(x) = -k_i \int (x_M - \mathsf{r}) - k_v v_M$ and sufficiently large values of $|x_M(t) - r|$, the above condition is violated. Hence, a saturated version of the control (5.24) should be applied, i.e.

$$\tilde{\nu} = \text{sat}\left(\tilde{\nu}(x), \bar{M}F_c + kx_m\right) \tag{5.29}$$

Since x_m is not available, the above control can not be implemented. Instead the following saturation version of $\tilde{\nu}$ is implemented for an SSID

$$\tilde{\nu}(x) = -k_i \int_{\tau_0{}^{\ell}}^{t} \text{sat}(x_M - \mathsf{r},\ \varsigma)d\tau - k_v v_M, \quad t \in [\tau_0{}^{\ell}, \tau_1{}^{\ell}] \tag{5.30}$$

with ς a design parameter selected such that condition (5.28) is satisfied. For large values of position error, $|x_M - \mathsf{r}|$, saturation occurs, and $\nu(x) = \varsigma(t - \tau_0{}^{\ell}) - k_v v_M$. This results in a second order stable system subjected to the ramp input ςt. Hence, the inertial mass M follows the ramp to reduce the position error $|x_M - \mathsf{r}|$ until $u = \pm u_{max}$ and the system switches to mode $q = (slip_{\mp}, \texttt{Reset})$. For $|x_M - \mathsf{r}| < \varsigma$, the stabilizing control law $\tilde{\nu}(x) = -k_i \int_{\tau_0{}^{\ell}}^{t} (x_M - \mathsf{r}) - k_v v_M$ is applied to the system.

Note that since the inertial mass position x_M is the only state measured in an SSID, the velocity feedback $k_v v_M$ proposed in $u = \tilde{\nu}(x)$ cannot be practically implemented. For simplicity in the model considered in this work, the effect of viscous friction, $\sigma_v v_M$ (the friction between inertial mass M and the rolling table) was neglected. Nevertheless, the viscous friction on the physical system plays the same role as the feedback velocity term $k_v v_M$ presented in the control.

Simulation Results

In this section, the simulation results for an SSID using the cascade control scheme, i.e. the control (5.7)-(5.8) together with (5.22) and (5.30), will be presented. The model parameter identification is discussed in [4]. For the model presented in Section 5.3 the identified parameters are listed in Table 5.1.

M	m	k	F_c	σ_v
1 kg	0.05 kg	1.76×10^7 Nm^{-1}	14 N	10×10^3

Table 5.1: System's model parameters

The complete model of the SSID (including the dynamics of the small mass states) using the proposed control strategy is simulated. For the control law (5.24) by choosing $k_i = 2 \times 10^8$ and k_v equal to the identified viscous friction parameter σ_v, i.e. $k_v = 10 \times 10^3$, it can be verified that the averaged model as well as the dynamics in the dominant mode $q = (stick, \texttt{External})$ are locally exponentially stable. The saturation parameter ς is set to $\varsigma = 0.5 \times 10^{-4}$. The simulation results for the set-point $\mathsf{r} = 50\,\mu$m and $u_{\max} =$ 60N are shown in Figure (5.11). Therein, the plots for the inertial and small mass position, the inertial mass velocity, the integral state and the overall input applied to the SSID are shown.

It can be observed from Figure 5.11 that set-point tracking is achieved after 0.5 seconds. Initially, when the position error is large, the control $u_1 = \nu(x)$ is saturated and the inertial mass M approaches with maximum possible velocity the reference position r. After some fast switching *stick* and *slip* transitions while the input is saturated, $|x_M - \mathsf{r}|$ is reduced and the integral action becomes active.

The two-time scale behavior of the system can be observed on the curve related to the inertial mass velocity. In a small time scale, there are very short periods of fast transition of velocity, shown by spikes (mode $q = (slip_-, \texttt{Reset})$), followed by relatively slower transitions (mode $q = (stick, \texttt{External})$). The slow response can be viewed as a smooth curve which envelopes the velocity curve from above. The two-time scale behavior of system on the inertial mass position response is not readily seen. In fact, position is the integral of the velocity, and thus the fast transitions observable on the velocity response are filtered out on the position curve.

Experimental Results

The same control strategy is applied to the setup. The real-time experiment result shows that asymptotical set-point tracking for a set-point of $\mathsf{r} = 50\mu$ m is achieved. Figure 5.12 illustrates the measured inertial mass position x_M, the reference trajectory r, and the overall applied control input u.

Note also that, since the relative velocity cannot be measured, there is no way to verify the condition $v_r = v_m - v_M$ in the control strategy. However, it can be verified that the sampling time of the control hardware is greater than the maximum time interval that the system stays in mode $q = (slip_-, \texttt{Reset})$, i.e. $t_{\text{samp}} = 1.4 \times 10^{-3} > \max\{\alpha_2 T\} = 2\pi\sqrt{\frac{m}{k}} = 9.8 \times 10^{-4}$ [sec], which guarantees that $u = 0$ for $q = (slip_-, \texttt{Reset})$.

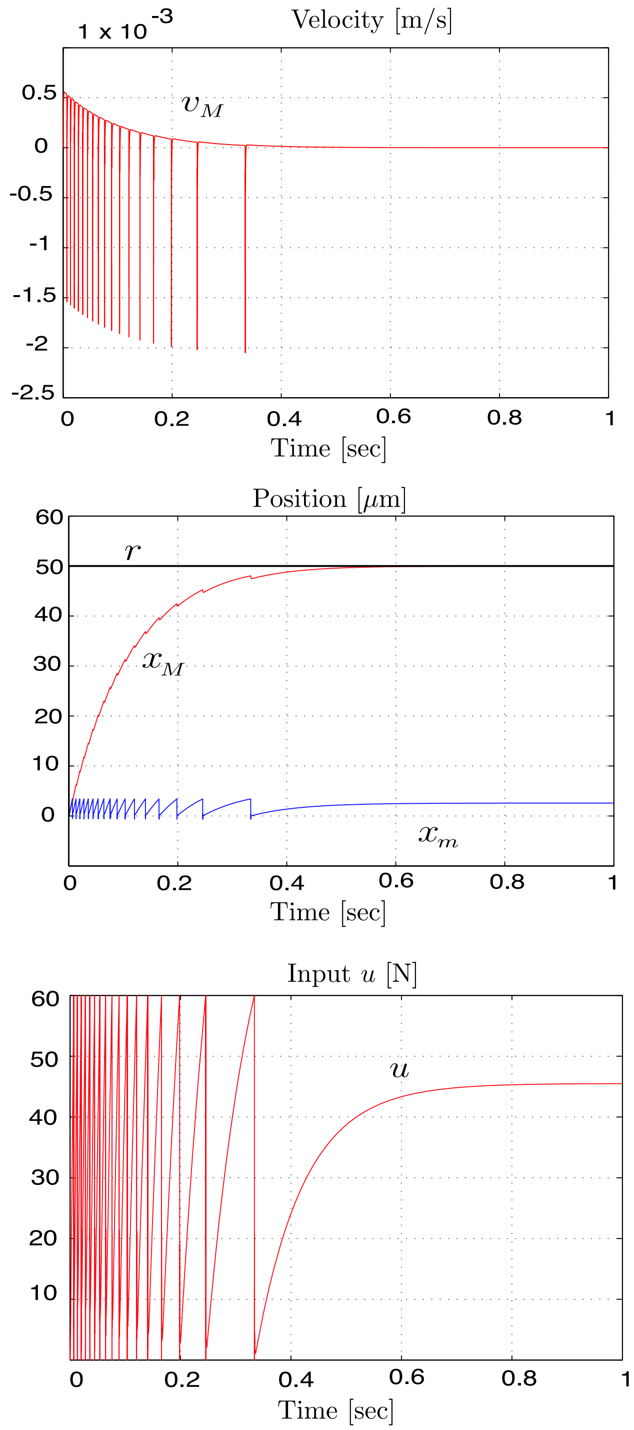

Figure 5.11: Simulation results

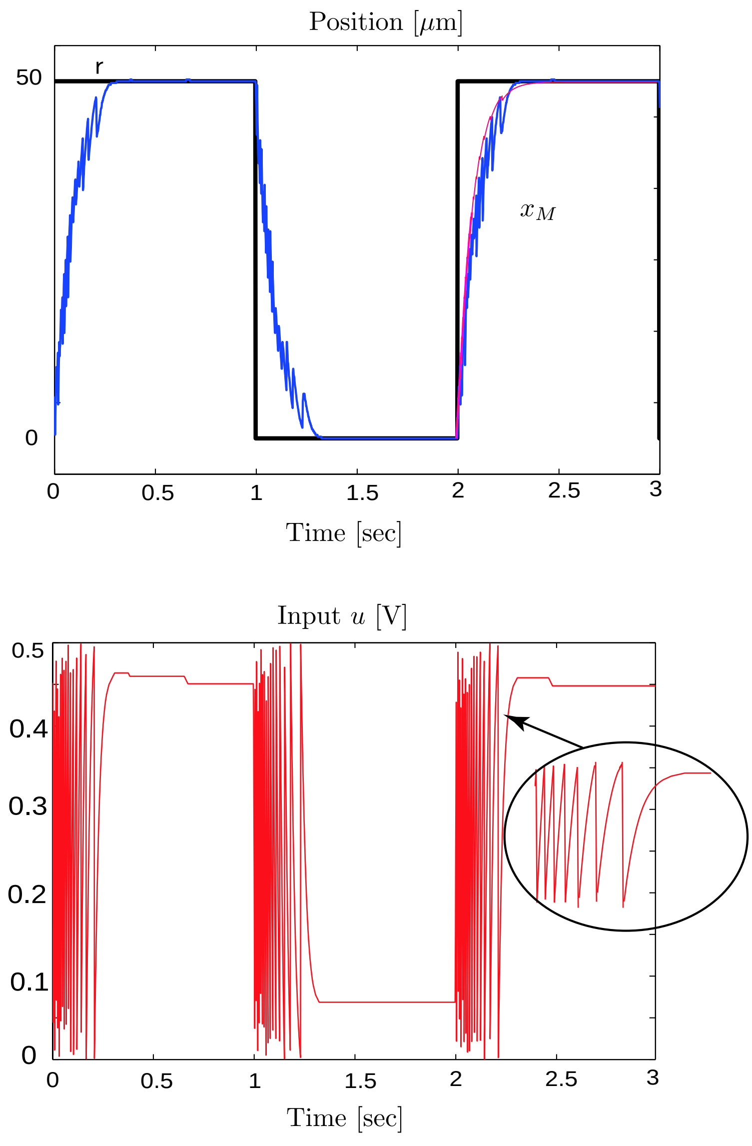

Figure 5.12: Experimental results: Inertial mass position $x_M(t)$ and input $u(t)$

5.7 Conclusions

In this chapter, the hybrid modeling and the control design for set-point tracking of a stick-slip drive were presented. It has been shown that the drive can be modeled as a hybrid system with three distinct modes, *stick*, $slip_-$ and $slip_+$. The cascade controller proposed in this thesis was used for the set-point tracking of the SSID. The cascade control scheme proposed in this thesis was used to this aim. By proposing an inner-loop hybrid controller it was shown that the combination of the hybrid plant and the controller system has a recurrent mode. By introducing the parameter ε, it was argued that for small values of the mass m and/or large values of rigidity of transmission elements k, the mode *stick* is dominant. The averaged model for the resulting hybrid system was obtained. On the basis of such a continuous model a stabilizing control law was proposed. It was shown that asymptotical set-point tracking of an SSID is possible using cascade control and an appropriate choice of controller parameters.

Practical solutions were proposed so that the control law could be implemented. Simulation and experimental results were also presented.

Chapter 6

Conclusions

6.1 Summary

The major contributions of this thesis can be traced in the following four directions:

- Dehybridization: In this work, hybrid systems with a two-time scale property caused by fast discrete state transitions and slow continuous state evolutions are studied. The idea was to approximate the behavior of the hybrid system in the slow time-scale by a 'reduced' continuous (average) model. The average model was defined by a convex combination of the vector fields, where the convex parameters were chosen to be the duty ratios. Since by using the proposed model reduction, all fast discrete states of the system are eliminated, the approximation procedure was called dehybridization. It was noted that fast discrete transitions necessary for dehybridization correspond to fast switching in *all but one* mode. So, the procedure can be utilized when one of the modes is dominant and the time spent in all other modes (defined as ε) is small.

- Quality of dehybridization: The quality of dehybridization (approximation) was investigated. It was shown that if the vector fields of each mode are bounded as $\varepsilon \to 0$, the error between the solution of the hybrid system and the averaged model in a finite time interval is of the order ε, i.e. the error goes to zero as ε goes to zero. In addition, it was shown that when some components of the vector fields grow unbounded with ε, i.e. the existence of fast continuous states, then the approximation result on slow continuous states is valid, provided the evolution of the slow ones is not influenced by the dynamics of the fast ones.

- Control design via dehybridization: A cascade control was proposed, where the inner-loop artificially imposes a two-time scale property while the outer-loop deals with the stabilizing task. With the combination of the inner-loop hybrid controller-plant having a two-time scale property, the averaged continuous model can be derived by dehybridization. Once the averaged continuous model is defined, the design of the control law for the outer-loop is carried out using the averaged model.

- Stability results: The link between stability of the averaged continuous model and the hybrid model was established. It was shown that if the origin is a common equilibrium point for all modes and is exponentially stable for the averaged model, then the origin is also an exponentially stable equilibrium point for the hybrid system, for sufficiently small values of ε. In addition, it was shown that if the origin is not an equilibrium point for some modes, then the trajectories of the hybrid model are ultimately bounded, the bound being a function of ε. From that it was shown that if ε goes to zero asymptotically—either with time or with the norm of the states—then exponential stability of the hybrid model can be guaranteed.

  The approach used for showing the stability results was based on presenting the hybrid system as a perturbed model of the continuous one. The stability results were obtained by using a robust stability approach. In such an analysis, it was shown that despite the norm of perturbation terms being large, the norm on their time integral is small, i.e. depends linearly on ε. Because of such a property, the classical robust approaches could not be directly used and the proofs were more technically involved.

In this thesis, the hybrid model of the stick-slip drive was presented. The proposed methodology was applied to the control of the stick-slip drive (set-point tracking problem) and the ideas of dehybridization were illustrated through simulation and experimental results.

Though the proposed method was successfully applied to the above mentioned setup, it is not without drawbacks. (i) It should be noted that dehybridization could lead to more complex dynamics. So, treating the continuous averaged model is not necessarily easier than treating the original hybrid one. Furthermore, in an autonomous type of switching, the convex parameters in the definition of the averaged model are not analytically defined. Thus, the outer-loop control design based on such an uncertain model is not always straightforward. (ii) No systematic method for inner-loop design was proposed. (iii) Because of the worst case analysis inherent in robust stability approaches based on Lyapunov methods, the stability results that were obtained are conservative. It was seen in simulation that the system is stable for a much larger range of ε than that predicted by the theory.

6.2 Perspectives

Several issues concerning the averaging and the proposed control design can be considered for future study:

- As mentioned in Chapter 4, the approximation results can be extended to infinite time if some stability conditions, e.g. exponential stability, are imposed on the average model. The proof of this conjecture is an important task to be accomplished in the future.

- In the case where some of the continuous states evolve as fast as the discrete states, i.e. certain modes have singularly perturbed dynamics, it would be interesting to study the behavior of these fast states. In such a case, the fast states do not necessarily converge to their steady state values. So, an important issue is to study the 'steady state' behavior of these states, e.g. stablity, boundedness, etc.

- Since a robust Lyapunov based approach was used for studying stability, the theorems give conservative bounds on ε. The main drawback is that, in many practical situations, the suggested conditions are restrictive. Thus, one can think of defining subclasses of systems where less conservative results are obtained. Also, a non Lyapunov based approach can be an alternative.

- No systematic design procedure for the inner-loop was proposed. One possibility is to restrict the class of hybrid systems and to propose a systematic procedure for inner-loop design.

- For the outer-loop controller the averaged model should be known. Yet, in the case of an autonomous type of switching rule, an analytical expression for the convex parameters needed to construct the average model does not exist. Thus, methods for determining the convex parameters could constitute possible future work.

Appendix A

Some Issues in Systems with Recurrent Mode

A.1 Controlled Switching

A.1.1 Switched Systems

Consider a hybrid plant, $\mathcal{H}_p$, consisting of a family of dynamical systems

$$\begin{aligned} \dot{x} &= F(q,x), \quad x \in D \subset \mathbb{R}^n, \quad q \in Q = \{1,2,...,l\} \\ q^+ &= R^p(q,\sigma) \end{aligned} \tag{A.1}$$

where the input to the system consists only of the discrete inputs or events σ. Since the transition rule depends only on an external discrete input σ (see equation A.1), such systems are classified as systems with a controlled switching transition rule. *Switched systems* [37] can be considered as a special case of the hybrid system defined above. Suppose $q^\star = 1$ is the desired recurrent mode with a transition rule defined as

$$R^p(q,\sigma): \quad q^+ = \sigma \tag{A.2}$$

Consider the hybrid controller, $\mathcal{H}_c$, where the time variable, t, is considered as its continuous state variable, that is $x_c = t$ with trivial dynamics $\dot{t} = \dot{x}_c = 1$:

$$\begin{aligned} \dot{x}_c &= 1, \quad x_c(0) = 0 \\ q_c^+ &= R^c(q,q_c) \end{aligned} \tag{A.3}$$

The input to the hybrid controller is the system's discrete states, q, and the output of the controller is the discrete state q_c which is the discrete input to the plant, $q_c = \sigma$. Let the transition rule be defined as follows

$$R^c(q, q_c, x_c): \quad \sigma^+ = q_c^+ = \begin{cases} q+1 & \text{if} \quad \mathrm{mod}(x_c, T) = \sum_{r=1}^{q} \alpha_r T \\ q^\star = 1 & \text{if} \quad \mathrm{mod}(x_c, T) = 0 \end{cases} \tag{A.4}$$

where $T > 0$ and $\alpha_q \in [0, 1]$, $q \in Q$ are known values. $\mathrm{mod}(x, y)$ is the modulus function and it returns the integer remainder after dividing y into x. It is easy to verify that the combination of plant and controller has a recurrent mode $q^\star = 1$:

At $x_c = t = 0$, $\mathrm{mod}(x_c, T) = 0$ and hence the system starts in mode $q = 1$ (see equations (A.2) and (A.4)). During the time interval $t \in [0, \alpha_1 T)$ the system stays in mode $q^\star = 1$. At $t = \alpha_1 T$, $\mathrm{mod}(x_c, T) = \alpha_1 T$ and then the system switches to mode $q = 1 + 1 = 2$, and stays in that mode until $x_c = t = \alpha_1 T + \alpha_2 T$. At $t = \sum_{r=1}^{\ell} \alpha_r T = T$, $\mathrm{mod}(T, T) = 0$, and the system switches back to mode $q = 1$. In fact, for values of $x_c = t = \ell T$, with $\ell \in \mathbb{Z}_+$, $\mathrm{mod}(\ell T, T) = 0$, and the system switches back to mode $q = 1$. This shows that for time instants $t = \ell T$, mode $q = 1$ is repeated, and hence the system has a recurrent mode $q^\star = 1$. By definition, T is the cycle time and α_q, $q \in Q$ are the duty ratios.

A.1.2 PWM Control

Consider a continuous system defined by

$$\dot{x} = F(x, u) \tag{A.5}$$

where $x \in X \subset \mathbb{R}^n$ and $u \in U = \{0, 1\}$ that is the input u can take values $u = 0$ and $u = 1$. Control of power converters can be modeled with such a formulation, where $u = 0$ and $u = 1$ corresponds to power 'off' and 'on' respectively. In power electronics the use of switches is popular since by performing fast on and off switching, the power can be regulated without loss of energy. Pulse Width Modulation (PWM) control which originated from control of power converters is a technique that can be described as follows:

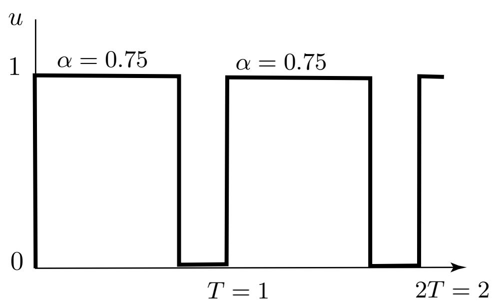

Figure A.1: Illustration of PWM control and duty cycle

On every fixed and small time interval T, the input switches once from $u = 1$ to $u = 0$. The fraction of T on which the input holds the value $u = 0$ is known as the duty cycle α. Then, the time interval for which $u = 0$ is given by αT with $\alpha \in [0,\ 1]$, and consequently $(1-\alpha)T$ describes the time interval with $u = 1$. The main idea used in PWM is that for a sufficiently small cycle time, $T \to 0$, the switching actuator can be modeled by its average behavior.

The PWM control can be formulated as a hybrid system with a controlled switching strategy:

The input $u = 0$ or $u = 1$ in $\dot{x} = F(x, u)$ results into two distinct dynamics represented by F_1 and F_2, i.e. $F(x, 0) = F_1(x)$ and $F(x, 1) = F_2(x)$. Denote $\alpha = \alpha_1$ and $(1 - \alpha) = \alpha_2$. The system with PWM control, now can be regarded as a closed loop combination of a hybrid plant (A.1) and hybrid controller (A.3), where the continuous dynamics of the plant are given by F_q, $q \in Q = \{1, 2\}$ and the hybrid controller transition rule is given by (A.4).

A.2 Autonomous Switching

A.2.1 Duty Ratios: State Dependency

Generally, in the case of autonomous switching the cycle time T_ℓ and duty ratios are a function of the continuous states, that is $\alpha_q(x)$. So, their exact values except for some special cases, cannot be known without knowing the explicit solution of system's dynamics. The following example reveals such dependency.

Example

Consider a hybrid dynamics composed of two modes $q \in \{1, 2\}$ and defined by dynamics

$$\dot{x} = A_q x \tag{A.6}$$

Suppose the system has a recurrent mode $q^\star = 1$, and the transition rules are defined by switching conditions

$$\mathcal{S}^{1\to 2} = \{x|\ C_1 x = \delta_1\}, \quad \mathcal{S}^{2\to 1} = \{x|\ C_2 x = \delta_2\} \tag{A.7}$$

Define T_ℓ as the cycle time and let $x(\tau_0^\ell) = x_0^\ell$ and $x(\tau_1^\ell) = x_1^\ell$ be the respective initial conditions for mode $q = 1$ and $q = 2$ in the ℓ'th cycle. The solution for mode $q = 1$ and $q = 2$ is given by the following equations

$$x(t) = x_0^\ell e^{A_1(t-\tau_0^\ell)}, \quad t \in [\tau_0^\ell,\ \tau_1^\ell); \qquad x(t) = x_1^\ell e^{A_2(t-\tau_1^\ell)}, \quad t \in [\tau_1^\ell,\ \tau_0^{\ell+1}) \tag{A.8}$$

From switching conditions (A.7) and (A.8), it can be seen that

$$C_1 x(\tau_1^\ell) = C_1 x_0^\ell e^{A_1 \Delta t_1} = \delta_1 \tag{A.9}$$

$$C_2 x(\tau_0^{\ell+1}) = C_2 x_1^\ell e^{A_2 \Delta t_2} = \delta_2 \tag{A.10}$$

where $\Delta t_1 = \tau_1^\ell - \tau_0^\ell$ and $\Delta t_2 = \tau_0^{\ell+1} - \tau_1^\ell$ with $T_\ell = \Delta t_1 + \Delta t_2$.

From equation (A.9), Δt_1 can be calculated, and it can be seen that Δt_1 is a function of x_0, i.e. $\Delta t_1 = h_1(x_0^\ell)$. With Δt_1 known, Δt_2 can be calculated using equation (A.11):

$$C_2 C_1 x_0^\ell e^{A_1 \Delta t_1} e^{A_2 \Delta t_2} = \delta_2 \tag{A.11}$$

which shows that Δt_2 is also a function of x_0^ℓ, i.e. $\Delta t_2 = h_2(x_0^\ell)$. From that, it can be observed that the duty ratio for mode $q = 1$ is given by

$$\alpha_1(x_0^\ell) = \frac{\Delta t_1}{\Delta t_1 + \Delta t_2} = \frac{h_1(x_0^\ell)}{h_1(x_0^\ell) + h_2(x_0^\ell)} \tag{A.12}$$

which is a function of x.

Remark A.1 *As can be seen from this example, equations (A.9) and (A.11) giving Δt_1 and Δt_2 are not easy to solve and except for some special cases, an analytical expression cannot be found. This shows that in general the exact values of duty ratios in the case of state-controlled switching cannot be found. Yet, a numerical solution would be a possible choice for estimating duty ratios.*

A.2.2 Dehybridization and Sliding Mode Regularization

Consider that the hybrid system consists of two discrete states, $q \in Q = \{1, 2\}$ and has a repetitive mode $q = 1$. Let the transition rule defined by the internal events be

$$\mathcal{S}^{2\to1} = \{x|\, S_1(x) = 0\}, \qquad \mathcal{S}^{1\to2} = \{x|\, S_2(x) = 0\}$$

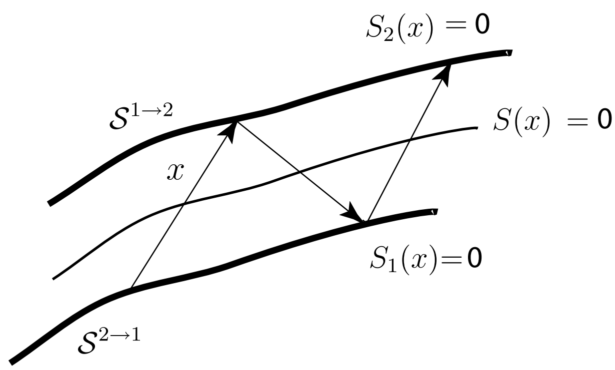

Figure A.2: Fast switches around a sliding surface $S(x) = 0$

where $S_1(x) : \mathbb{R}^n \to \mathbb{R}$ and $S_2(x) : \mathbb{R}^n \to \mathbb{R}$ are $(n-1)$-dimensional hypersurface. Let $S(x) = 0$ be an intermediate hypersurfaces such that $S(x) = S_1(x) + \delta_1 = S_2(x) + \delta_2$. The switching conditions $S_1(x) = 0$ and $S_2(x) = 0$ can be defined as $S(x) = \delta_1$ and $S(x) = \delta_2$ respectively (Figure A.2). Now consider that the system at $t = \tau_0$ is switched to mode $q = 1$, i.e. $S(x_0) = \delta_1$. The state $x(t)$ evolves by dynamics $\dot{x} = F_1(x)$ until $S(x(\tau_1^\ell)) = \delta_2$ and system switches to mode $q = 2$. If $\Delta t_1 = \tau_1^\ell - \tau_0^\ell$ is sufficiently small then the rate of change of $S(x)$ is given by

$$\dot{S}(x) = \nabla S.F_1(x) = \lim_{\Delta t_1 \to 0} \frac{\Delta S}{\Delta t_1} = \frac{S(x(\tau_1^\ell)) - S(x(\tau_0^\ell))}{\Delta t_1} = \frac{\delta_2 - \delta_1}{\Delta t_1} \quad \text{(A.13)}$$

Since the system has a recurrent mode $q = 1$, from $t = \tau_1^\ell$ the state $x(t)$ evolves by dynamics $\dot{x} = F_2(x)$ until $S(x(\tau_0^{\ell+1})) = \delta_1$ and system switches back to mode $q = 1$. With a similar argument, the rate of change of $S(x)$ during $\Delta t_2 = \tau_0^{\ell+1} - \tau_1^\ell$ can be written as

$$\dot{S}(x) = \nabla S.F_2(x) = \lim_{\Delta t_2 \to 0} \frac{\Delta S}{\Delta t_2} = \frac{-(\delta_2 - \delta_1)}{\Delta t_2} \quad \text{(A.14)}$$

Using (A.13) and (A.14), and when the cycle time $T_\ell = \Delta t_1 + \Delta t_2 \to 0$, the regularization solution $\bar{\theta}_1$ which is the time limit of the fraction of cycle time during which the system stays in mode $q = 1$, i.e. α_1^ℓ, can be written as

$$\bar{\theta}_1(x) = \lim_{\varepsilon \to 0} \alpha_1^\ell = \lim_{T_\ell \to 0} \frac{\Delta t_1}{\Delta t_1 + \Delta t_2} = \frac{\nabla S(x).F_1(x)}{\nabla S(x)(F_2(x) - F_1(x))} \quad \text{(A.15)}$$

which coincides with parametrization of the solution on the sliding surface $S(x) = 0$ as described by the simplest convex definition given by Filippov (equation 2.42).

Remark A.2 *Since the vector fields in each mode are bounded as $\varepsilon \to 0$, i.e. $\lim_{\varepsilon \to 0} F_i(x) < \infty$, then the instantaneous changes in the states x are not possible and thus when $\epsilon = 0$ the switching sets $\mathcal{S}^{1 \to 2}$ and $\mathcal{S}^{2 \to 1}$ should coincide on the sliding surface $S(x) = 0$.*

A.2.3 Control to Render the System with a Recurrent Mode

Consider a hybrid plant $\mathcal{H}_p$ defined by

$$\begin{aligned} \dot{x} &= F_q(x) + b_q(x)u \\ q^+ &= R^p(q, x) \end{aligned} \quad \text{(A.16)}$$

where $x \in X \subset \mathbb{R}^n$ and $q \in Q = \{1, 2\}$. The continuous and discrete outputs of the plant are given by $y_p = x$ and $o_p = q$ respectively. The input to the plant consists of only the continuous input $u \in U \subset \mathbb{R}$. The discrete transition rule is defined by hypersurfaces given by $S^{1\to2}(x) = 0$ and $S^{2\to1}(x) = 0$, that is

$$R^p: \quad q^+ = \begin{cases} 2 & \text{if} \quad q = 1 \quad \text{and} \quad S^{1\to2}(x) = 0 \\ 1 & \text{if} \quad q = 2 \quad \text{and} \quad S^{2\to1}(x) = 0 \end{cases} \tag{A.17}$$

Suppose a hybrid controller is needed to make an overall system with a recurrent mode $q^\star = 1$. Consider the hybrid controller, $\mathcal{H}_c$ defined as follows:

$$\begin{aligned} y_c &= h^c(q_c, x, \nu) \\ q_c^+ &= R_c(q_c, q, x) \end{aligned} \tag{A.18}$$

where $q_c \in Q_c = \{\texttt{external control}, \texttt{reaching control}\}$. The inputs of the controller are the continuous and discrete states of the hybrid plant, x and q (plant's continuous and discrete outputs), together with the external input ν. The controller output consists only of continuous variables u, that is $y_c = u$.

Let h^c and R^c be defined as:

$$R^c: \quad q_c^+ = \begin{cases} \texttt{external control} & \text{if} \quad q = 1 \\ \texttt{reaching control} & \text{if} \quad q = 2 \end{cases} \tag{A.19}$$

$$h^c(\texttt{external control}, x, \nu) = \nu(x) \tag{A.20}$$

$$h^c(\texttt{reaching control}, x, \nu) = -k(x)\text{sgn}(\hat{S}) \tag{A.21}$$

where $k(x) : \mathbb{R}^n \to \mathbb{R}^+$ and $\hat{S}(x) = S^{2\to1}\dfrac{\partial S^{2\to1}}{\partial x}b_2$. It can be seen that there exists a positive function $k(x)$ such that the closed-loop combination of $\mathcal{H}_p$ and $\mathcal{H}_c$ has a recurrent mode $q^\star = (1, \text{‘}\texttt{external control}')$:

By definition, the system has a recurrent mode $q^\star$, if the system leaves mode $q^\star$ and switches back again to $q^\star$ in finite time. The controller transition rule R_c (A.19) suggests that whenever the system is not in the mode $q^\star$, then the reaching control (A.21) is applied to the system. It will be shown that such a control forces the trajectories of the system to reach $S^{2\to1}(x) = 0$ in finite time. Denote $S^{2\to1}(x)$ by $S(x)$ and $\dfrac{\partial S}{\partial x}$ by ∇S to ease the notation.

Suppose $\nabla S b_2$ is nonsingular, and let $\kappa(x)$ be a known positive function that satisfies

$$\left|\frac{\nabla S F_2(x)}{\nabla S b_2(x)}\right| \leq \kappa(x), \quad \forall x \in X \tag{A.22}$$

Consider the Lyapunov function candidate $V(x) = \frac{1}{2}S^2(x)$. The derivative of $V(x)$ along the trajectories of the system in each mode can be written as

$$\dot{V} = S\dot{S} = S\nabla S\left(F_2(x) + b_2(x)u\right)$$

where $u = -k(x)\text{sgn}(\hat{S})$ with $\hat{S}(x) = S(x)\nabla S(x) b_2(x)$. Then

$$\dot{V} = S\nabla S F_2(x) - S\nabla S b_2(x) k(x)\text{sgn}(\hat{S})$$

From (A.22) and $\hat{S} = S\nabla S b_2$ it can be seen that

$$\begin{aligned} \dot{V} &\leq |S\nabla S b_2(x)|\kappa(x) - S\nabla S b_2(x) k(x)\text{sgn}(\hat{S}) \\ &\leq |\hat{S}|\kappa(x) - \hat{S}k(x)\text{sgn}(\hat{S}) \\ &\leq |\hat{S}|\left(\kappa(x) - k(x)\right) \end{aligned} \tag{A.23}$$

Set $k(x) \geq \kappa(x) + \kappa_0$, where $\kappa_0 > 0$, then it can be seen that

$$\begin{aligned} \dot{V} &\leq |\hat{S}|\left(\kappa(x) - \kappa(x) - \kappa_0\right) \leq -\kappa_0|\hat{S}| \\ &\leq \kappa_0|S||\nabla S b_q| \end{aligned} \tag{A.24}$$

Since $\nabla S b_2$ is nonsingular, then let $d = \min|\{\nabla S b_q\}|$. Now define $W = \sqrt{V} = |S|$, then

$$\dot{W} = \frac{\dot{V}}{2\sqrt{V}} \leq -\frac{\kappa_0|S||\nabla S b_q|}{2|S|} \leq -\frac{\kappa_0 d}{2}$$

From the comparison principle (Lemma 2.1), it can be seen that

$$W(t) \leq W(t_0) - \frac{\kappa_0 d}{2}t$$

Finally, from the above equation it can be seen that $\exists\ t_r > 0$ such that $W(t_r) = |S(x(t_r))| = 0$. This shows that surface $S^{2\to 1}(x) = 0$ can be reached in finite time and hence, the system has a recurrent mode $q^\star = (1, \text{`}\texttt{external control}\text{'})$.

Appendix B

Proof of Lemmas Concerning Ultimately Boundedness

B.1 Proof of Lemma 4.5

Proof: Similar to proof of Proposition 4.1, for $\kappa_1(x) = \frac{p(x)[\Delta(x)-\Delta(0)]}{||x||}$, it can be shown that the integral is given by

$$\int_0^{t_m} \mathrm{e}^{-a(t-s)}\kappa_1(x)ds = \sum_{\ell=1}^{m} T_\ell \Omega(\hat{x}_i^\ell) \tag{B.1}$$

where $\hat{x}_i^\ell$ represents the state at a time instant between τ_{i-1}^ℓ and τ_i^ℓ, that is $x(\hat{\tau}_i^\ell)$ with $\hat{\tau}_i^\ell \in [\tau_{i-1}^\ell,\ \tau_i^\ell]$ and

$$\begin{aligned} \Omega(\hat{x}_i^\ell) \quad = \quad & \sum_{i=1}^{l}\sum_{j=1}^{i-1} T_\ell \alpha_i^\ell \alpha_j^\ell \mathrm{e}^{-a(t-\hat{\tau}_i^\ell)} \left(\frac{\partial \kappa_1}{\partial x} F_j(\hat{x}_j^\ell) - a\kappa_1(\hat{x}_i^\ell)\right) \\ & + \frac{1}{2}\sum_{i=1}^{l} T_\ell (\alpha_i^\ell)^2 \mathrm{e}^{-a(t-\hat{\tau}_i^\ell)} \left(\frac{\partial \kappa_1}{\partial x} F_i(\hat{x}_i^\ell) - a\kappa_1(\hat{x}_i^\ell)\right) \end{aligned} \tag{B.2}$$

with $\frac{\partial \kappa_1}{\partial x}$ given by

$$\begin{aligned}
\frac{\partial \kappa_1}{\partial x} &= \frac{\partial[p(x)\,(\Delta_i(x)-\Delta_i(0))\,\frac{1}{||x||}]}{\partial x} \\
&= \frac{\partial p}{\partial x}[\Delta_i - \Delta_i(0)]\frac{1}{||x||} + p\frac{\partial \Delta_i}{\partial x}\frac{1}{||x||} - \frac{1}{||x||^2}\frac{\partial ||x||}{\partial x}p[\Delta_i - \Delta_i(0)]
\end{aligned} \tag{B.3}$$

Taking the norm of Ω yields

$$\begin{aligned}
||\Omega(\hat{x}_i^\ell)|| \leq \quad & \sum_{i=1}^{l}\sum_{j=1}^{i-1} T_\ell \alpha_i^\ell \alpha_j^\ell \mathrm{e}^{-a(t-\hat{\tau}_i^\ell)}\left(\left|\left|\frac{\partial \kappa_1}{\partial x}F_j(\hat{x}_j^\ell)\right|\right| + \left|\left|a\kappa_1(\hat{x}_i^\ell)\right|\right|\right) \\
& + \frac{1}{2}\sum_{i=1}^{l} T_\ell (\alpha_i^\ell)^2 \mathrm{e}^{-a(t-\hat{\tau}_i^\ell)}\left(\left|\left|\frac{\partial \kappa_1}{\partial x}F_i(\hat{x}_j^\ell)\right|\right| + \left|\left|a\kappa_1(\hat{x}_i^\ell)\right|\right|\right)
\end{aligned} \tag{B.4}$$

Since F_i and $[\partial F_i/\partial x](x)$ are continuous and bounded in D, then $||F_i(x) - F_i(0)|| \leq L||x||$ and $||\partial F_i/\partial x|| \leq L$, with $L = \max_i\{L_i\}$. So, from which and Lemma 4.2 (equation (4.9)), it can be verified that $||\Delta_i(x) - \Delta_i(0)|| \leq 2(1-\alpha_i^\ell)L||x||$, $||\frac{\partial \Delta_i}{\partial x}|| \leq 2(1-\alpha_i^\ell)L$. Let $||F_i(x)|| \leq K_1$, $\forall i \in \mathbf{L}$, using $||p(x)|| \leq c||x||$, $||\partial p/\partial x|| \leq c$ and $||\frac{\partial ||x||}{\partial x}|| < 1$, it can be seen that $\left|\left|\frac{\partial \kappa_1}{\partial x}F_i(x)\right|\right|$ and $||\kappa(x)||$ satisfy

$$\begin{aligned}
\left|\left|\frac{\partial \kappa_1}{\partial x}F_i(x)\right|\right| &\leq \left\{\frac{\left|\left|\frac{\partial p}{\partial x}(\Delta_i(x)-\Delta_i(0))\right|\right|}{||x||} + \frac{||p\frac{\partial \Delta_i}{\partial x}||}{||x||}\right. \\
&\quad \left. + \frac{||p||.||(\Delta_i(x)-\Delta_i(0))||}{||x||^2}\right\}||F_i|| \\
&\leq \left\{\frac{2cL(1-\alpha_i^\ell)||x||}{||x||} + \frac{2c||x||L(1-\alpha_i^\ell)}{||x||}\right. \\
&\quad \left. + \frac{2cL(1-\alpha_i^\ell)||x||^2}{||x||^2}\right\}K_1 \\
&\leq 6cLK_1(1-\alpha_i^\ell)
\end{aligned} \tag{B.5}$$

$$||\kappa_1(x)|| \quad \leq \quad \frac{||p(x)||.||\Delta_i(x) - \Delta_i(0)||}{||x||} \leq 2cK_2(1 - \alpha_i^\ell) \qquad \text{(B.6)}$$

By applying the above inequalities in (B.4) it can be seen that

$$\begin{aligned} ||\Omega(\hat{x}_i^\ell)|| \quad &\leq \quad 2c(3LK_1 + aK_2)\sum_{i=1}^{l}\sum_{j=1}^{i-1} T_\ell \alpha_i^\ell (1-\alpha_i^\ell)\alpha_j^\ell \mathrm{e}^{-a(t-\hat{\tau}_i^\ell)} \\ &+c(3LK_1 + aK_2)\sum_{i=1}^{l} T_\ell \alpha_i^\ell (1-\alpha_i^\ell)\alpha_i^\ell \mathrm{e}^{-a(t-\hat{\tau}_i^\ell)} \qquad \text{(B.7)} \end{aligned}$$

Since from the second inequality of Lemma 4.3, i.e. $T_\ell \alpha_i^\ell (1-\alpha_i^\ell) \leq \varepsilon$ and $\mathrm{e}^{-a(t-\hat{\tau}_i^\ell)} < \mathrm{e}^{-a(t-\tau_0^\ell)}$, $\forall i$, then

$$||\Omega(\hat{x}_i^\ell)|| \quad \leq \quad \varepsilon \tilde{c} \mathrm{e}^{-a(t-\tau_0^\ell)} \qquad \text{(B.8)}$$

where $\tilde{c} = c(3LK_1 + aK_2)(2l+1)$. From above inequality and using the fact that the sum $\sum_{\ell=1}^{m} T_\ell \mathrm{e}^{-a(t-\tau_0^\ell)}$ is convergent, it can be deduced that the integral

$$\begin{aligned} \left|\left|\int_{t_0}^{t_m} \mathrm{e}^{-a(t-s)}\kappa_1(x)ds\right|\right| \quad &= \quad ||\sum_{\ell=1}^{m} T_\ell \Omega(\hat{x}_i^\ell)|| \\ &\leq \quad \sum_{\ell=1}^{m} T_\ell ||\Omega(\hat{x}_i^\ell)|| \leq \varepsilon\tilde{c}\sum_{\ell=1}^{m} T_\ell \mathrm{e}^{-a(t_m-\tau_0^\ell)} \leq \varepsilon \tilde{b}_1 \end{aligned} \qquad \text{(B.9)}$$

with $\tilde{b}_1$ a positive constant.

■

B.2 Proof of Lemma 4.6

Proof: Similar to proof of Lemma 4.5 and for $\kappa_2 = p(x)\Delta(0)$ it can be shown that the integral is given by

$$\int_0^t \mathrm{e}^{-a(t-s)}\kappa_2(x)ds = \sum_{\ell=1}^{m} T_\ell \Omega(\hat{x}_i^\ell) \tag{B.10}$$

where $\hat{x}_i^\ell$ represents the state at a time instant between τ_{i-1}^ℓ and τ_i^ℓ, that is $x(\hat{\tau}_i^\ell)$ with $\hat{\tau}_i^\ell \in [\tau_{i-1}^\ell,\ \tau_i^\ell]$ and

$$\begin{aligned}\Omega(\hat{x}_i^\ell) \quad = \quad & \sum_{i=1}^{l}\sum_{j=1}^{i-1} T_\ell \alpha_i^\ell \alpha_j^\ell \mathrm{e}^{-a(t-\hat{\tau}_i^\ell)} \left(\frac{\partial \kappa_2}{\partial x} F_j(\hat{x}_j^\ell) - a\kappa_2(\hat{x}_i^\ell)\right) \\ & + \frac{1}{2}\sum_{i=1}^{l} T_\ell (\alpha_i^\ell)^2 \mathrm{e}^{-a(t-\hat{\tau}_i^\ell)} \left(\frac{\partial \kappa_2}{\partial x} F_i(\hat{x}_i^\ell) - a\kappa_2(\hat{x}_i^\ell)\right)\end{aligned} \tag{B.11}$$

with $\dfrac{\partial \kappa_2}{\partial x} = \dfrac{\partial p(x)}{\partial x}\Delta_i(0)$. Taking the norm of Ω yields

$$\begin{aligned}||\Omega(\hat{x}_i^\ell)|| \leq \quad & \sum_{i=1}^{l}\sum_{j=1}^{i-1} T_\ell \alpha_i^\ell \alpha_j^\ell \mathrm{e}^{-a(t-\hat{\tau}_i^\ell)} \left(\left|\left|\frac{\partial \kappa_2}{\partial x} F_j(\hat{x}_j^\ell)\right|\right| + \left|\left|a\kappa_2(\hat{x}_i^\ell)\right|\right|\right) \\ & + \frac{1}{2}\sum_{i=1}^{l} T_\ell (\alpha_i^\ell)^2 \mathrm{e}^{-a(t-\hat{\tau}_i^\ell)} \left(\left|\left|\frac{\partial \kappa_2}{\partial x} F_i(\hat{x}_j^\ell)\right|\right| + \left|\left|a\kappa_2(\hat{x}_i^\ell)\right|\right|\right)\end{aligned} \tag{B.12}$$

Since F_i are continuous and bounded in D, then $||F_i(x)|| \leq K_1$, $K_1 > 0$, $\forall i \in \mathbf{L}$. Also, from definition of Δ_i, i.e. equation (4.10)

$$\Delta_i(0) = (1-\alpha_i)F_i(0) - \sum_{j\neq i} \alpha_i F_j(0) \tag{B.13}$$

Since $F_i(0) = \delta_i$, by taking norm it can be shown that

$$||\Delta_i(0)|| \leq (1-\alpha_i)\delta_i + \sum_{j\neq i} \alpha_i ||F_j(0)|| \tag{B.14}$$

Let $\delta = \max_i\{||\delta_i||\}$, since $\delta_{i^\star} = 0$, where $i^\star = \arg\min_i(1-\alpha_i)$ then

$$||\Delta_{i^\star}(0)|| = 0$$

and

$$||\Delta_i(0)|| \leq (1-\alpha_i)\delta + \delta \sum_{j\neq i} \alpha_i = 2(1-\alpha_i)\delta \quad \forall i \neq i^\star \tag{B.15}$$

Since $||\frac{\partial p(x)}{\partial x}|| \leq c$, it can be seen that $\left|\left|\frac{\partial \kappa_2}{\partial x} F_i(x)\right|\right|$ and $||\kappa_2||$ satisfy

$$\left|\left|\frac{\partial \kappa_2}{\partial x} F_i(x)\right|\right| \leq ||\frac{\partial p(x)}{\partial x}||.||\Delta_i(0)||.||F_i(x)|| \leq 2c(1-\alpha_i)\delta K_1, \quad \forall i \neq i^\star \tag{B.16}$$

$$||\kappa_2(x)|| \leq 2(1-\alpha_i)\delta \bar{K}_2, \quad \forall i \neq i^\star \tag{B.17}$$

with $\bar{K}_2 > 0$ a constant related to the bounds on $p(x)$. By applying the above inequalities in (B.12) and using the fact that $\left|\left|\frac{\partial \kappa_2}{\partial x} F_i(x)\right|\right| = 0$ and $||\kappa_2|| = 0$ for $i = i^\star$ and $\mathrm{e}^{-a(t-\hat{\tau}_i^\ell)} < \mathrm{e}^{-a(t-\tau_0^\ell)}$ $\forall i$, it can be seen that

$$\begin{aligned} ||\Omega(\hat{x}_i^\ell)|| \quad \leq \quad & 2(\delta c K_1 + a\bar{K}_2) \sum_{i\neq i^\star}^{l} \sum_{j=1}^{i-1} T_\ell \alpha_i^\ell (1-\alpha_i^\ell)\alpha_j^\ell \mathrm{e}^{-a(t-\tau_0^\ell)} \\ & + (\delta c K_1 + a\bar{K}_2) \sum_{i\neq i^\star} T_\ell (\alpha_i^\ell)^2 (1-\alpha_i^\ell) \mathrm{e}^{-a(t-\tau_0^\ell)} \end{aligned} \tag{B.18}$$

and thus

$$\begin{aligned} T_\ell ||\Omega(\hat{x}_i^\ell)|| \leq \quad & 2\tilde{c}\mathrm{e}^{-a(t-\tau_0^\ell)} \left(\sum_{i\neq i^\star}^{l} \sum_{j=1}^{i-1} T_\ell^2 \alpha_i^\ell (1-\alpha_i^\ell)\alpha_j^\ell \right. \\ & \left. + \frac{1}{2} \sum_{i\neq i^\star} T_\ell^2 (\alpha_i^\ell)^2 (1-\alpha_i^\ell) \right) \end{aligned} \tag{B.19}$$

with $\tilde{c} = (\delta c K_1 + a\bar{K}_2)$. Since from the second inequality of Lemma 4.3, $T_\ell \alpha_i^\ell (1-\alpha_i^\ell) \leq \varepsilon$ and from definition of ε, $T_\ell \alpha_i^\ell < \varepsilon$, for $i \neq i^\star$, then $\sum_{i\neq i^\star}^{l} \sum_{j=1}^{i-1} T_\ell^2 \alpha_i^\ell (1-\alpha_i^\ell)\alpha_j^\ell$ and $\sum_{i\neq i^\star} T_\ell^2 (\alpha_i^\ell)^2 (1-\alpha_i^\ell)$ are $O(\varepsilon^2)$.

Now the norm on the integral reads

$$\left|\left|\int_0^t \mathrm{e}^{-a(t-s)}\kappa_2(x)ds\right|\right| = ||\sum_{\ell=1}^{m} T_\ell \Omega(\hat{x}_i^\ell)|| \leq \sum_{\ell=1}^{m} T_\ell ||\Omega(\hat{x}_i^\ell)|| \leq 2\varepsilon^2 \tilde{c} \sum_{\ell=1}^{m} \mathrm{e}^{-a(t-\tau_0^\ell)} \tag{B.20}$$

Since $\sum_{\ell=1}^{m} \mathrm{e}^{-a(t-\tau_0^\ell)}$ is a convergent sum, hence

$$\left|\left|\int_0^t \mathrm{e}^{-a(t-s)}\kappa_2(x)ds\right|\right| \leq \varepsilon^2 \tilde{b}_2 \tag{B.21}$$

with $\tilde{b}_2$ a positive constant. ∎

Appendix C

Proofs Concerning SSID

C.1 Bounds on the Small Mass Position

Proposition C.1 *In mode 'stick', if* $|u| = u_{\max}$ *with* $u_{\max} \geq 2\bar{M}F_c$ *and* u *is switched to zero, then* $|\rho| > F_c$.

Proof: In mode *stick*, the following condition always holds

$$-F_c \leq \mathcal{F} = \rho = \frac{M}{m+M}(u - kx_m) \leq F_c \tag{C.1}$$

From the right inequality of (C.1) and by replacing u with $u_{\max}$, it can be seen that

$$-kx_m \leq -u_{\max} + \left(\frac{M+m}{M}\right)F_c \Rightarrow kx_m \geq u_{\max} - \left(\frac{M+m}{M}\right)F_c$$

When $u = 0$, from definition of ρ (5.5) and by using the above inequality, it can be seen that

$$\rho = \frac{M}{m+M}(0 - kx_m) = \frac{1}{\bar{M}}(-kx_m) \leq -\frac{1}{\bar{M}}u_{\max} + F_c$$

Then using the assumption $u_{\max} \geq 2\bar{M}F_c$ leads to $\rho < -F_c$.

Similarly, from the left inequality of (C.1) and by replacing u with $-u_{\max}$, it can be seen that

$$-kx_m \geq u_{\max} - \left(\frac{M+m}{M}\right) F_c \Rightarrow kx_m \leq -u_{\max} + \left(\frac{M+m}{M}\right) F_c$$

The proof for $\rho < F_c$ follows using a similar argument to that used in the first part.

■

C.2 Proof of Proposition 5.1

Proposition 5.1 states that the combination of hybrid plant (SSID) and the hybrid controller is recurrent with mode $q^\star = (stick, \texttt{External})$. In addition this gives the maximum time interval that the system could stay in modes $q = (slip_\pm, \texttt{Reset})$.

Proof: The condition is equivalent to showing that when the relative velocity is other than zero, i.e. $v_r \neq 0$, then $\exists t$ such that $v_r = v_m(t) - v_M(t) = 0$. This will be shown for mode $slip_-$, while the proof for mode $slip_+$ follows from symmetry.

Suppose the system is in mode $(stick, \text{`}\texttt{External}')$ and at $t = 0$, $u = u_{\max}$. At this moment, the system switches to mode $q = (slip_-, \text{`}\texttt{Reset}')$ with $u = 0$. Noting that in mode $(stick, \text{`}\texttt{External}')$ friction is defined by $\rho = \mathcal{F} = \frac{M}{m+M}(u_{\max} - kx_m(t))$. Then, when the system switches from mode $(stick, \text{`}\texttt{External}')$ to $q = (slip_-, \text{`}\texttt{Reset}')$, i.e. at $t = 0$, friction ρ satisfies

$$\rho = \frac{M}{m+M}(u_{\max} - kx_m(0) \leq F_c \tag{C.2}$$

from which it can be seen that

$$u_{\max} - kx_m(0) \leq \left(\frac{M+m}{M}\right) F_c \Rightarrow -kx_m(0) \leq -u_{\max} + \bar{M}F_c \tag{C.3}$$

where $\bar{M} = \frac{m+M}{M}$. From the dynamics of mode '$slip_-$' given by (5.6) and $u = 0$, the small mass and inertial mass velocities, i.e. $v_m(t)$ and $v_M(t)$, are defined by

$$v_m(t) = a\cos(\omega t) + b\sin(\omega t) \tag{C.4}$$
$$v_M(t) = a - ct, \quad t \geq 0 \tag{C.5}$$

with $\omega = \sqrt{\frac{k}{m}}$, $a = v_m(0)$, $b = \dfrac{[F_c - kx_m(0)]}{\sqrt{km}}$ and $c = \frac{F_c}{M} > 0$. The condition $v_r = v_m - v_M = 0$ then gives

$$S(t) = v_r = a[\cos(\omega t) - 1] + b\sin(\omega t) + ct = 0 \tag{C.6}$$

with $S(0) = 0$ and S being continuous and bounded in t. The above equation does not have an analytical solution in t, however, it can be shown that when $S = v_r < 0$, then there exists $t = \tau$ such that $S(\tau) = v_r(\tau) = 0$.

From (C.6) and definitions of b, ω and c it can be verified that

$$\frac{\partial S}{\partial \tau}(0) = b\omega + c = \frac{F_c - kx_m(0)}{m} + \frac{F_c}{M} = \frac{1}{m}[\bar{M}F_c - kx_m(0)] \tag{C.7}$$

by using inequality (C.3), it can be seen that

$$\frac{\partial S}{\partial \tau}(0) \leq \frac{1}{m}[\bar{M}F_c - u_{\max} + \bar{M}F_c] \tag{C.8}$$

It was assumed that $u_{\max} > 2\bar{M}F_c$, then $\frac{\partial S}{\partial \tau}(0) < 0$. This, suggests that system at $t = 0$ system switches to mode $slip_-$ and stays there for a while. Let $\tau = \frac{2\pi}{\omega}$, then $S(\frac{2\pi}{\omega}) = c\frac{2\pi}{\omega} > 0$, from this and the continuity of S, it can be seen that there exists $t \in [0, \frac{2\pi}{\omega}]$ such that $S(t) = v_r = 0$. This means that mode $q^\star$ is reached in finite time, i.e. $t \leq \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$ and hence, the system has recurrent mode $q^\star$. ■

C.3 Stability Theorem for a Class of Hybrid Systems

Consider a hybrid system where the continuous state space is partitioned into different zones and the dynamics of the system differs from one zone to another. Hence, each zone represents a discrete state or mode of system. If at least one zone, e.g. Z_0, contains the equilibrium point, $x = 0$, and the trajectory of system be such that whenever it leaves Z_0, it returns back in finite time, then such a system can be considered as a hybrid system with a recurrent mode represented by the dynamics defined in Z_0. The following Theorem gives the sufficient conditions for global asymptotic stability of the equilibrium $x = 0$.

Theorem C.1 *Let the state space, R^n, be partitioned into different zones Z^0, Z^1,..., and let the equilibrium point, $x = 0$, be in the interior of Z^0. Suppose the system satisfies the following conditions:*

I. There exists a radially unbounded function $V: R^n \rightarrow R$ such that

$$\gamma_1(||x||) \leq V(x) \leq \gamma_2(||x||) \tag{C.9}$$

II. If the system is not in Z^0, it reaches Z^0 in finite time.

III. Whenever the system leaves Z^0, it returns with $V^{en} \leq V^{ex}$, where V^{ex} and V^{en} are the values of V when the system leaves and reenters Z^0.

IV. $\dot{V}(x) \leq -\gamma_3(||x||)$ for $x \in Z^0$.

where γ_1, γ_2 and γ_3 are class $\mathcal{K}$ functions, then $x = 0$, is globally asymptotically stable.

Proof: Since $x = 0$ is in the interior of Z^0, there exists a ball $B_r = \{x \in Z^0 | \, ||x|| < r\}$, such that $B_r \subset Z^0$ and the system does not leave Z^0. For initial conditions starting in B_r, i.e. $||x_0|| \leq r$, from condition 4 the local asymptotic stability of $x = 0$ follows. If $x_0 \notin B_r$, two cases are possible:

(i) $x_0 \in Z^0$ and system never leaves Z^0. From condition 4, the asymptotic stability of $x = 0$ can be concluded.

(ii) $x_0 \notin Z^0$ or system trajectory leaves Z^0. In this case it will be shown that after some transitions, all trajectories will converge to B_r. Condition 2, suggests that if x leaves Z^0 it returns in finite time. Let t_i and t'_i denote time instants at which the system enters and exits Z^0 in the i'th transition, with $t'_i > t_i$. The value of V when the system leaves Z^0 is given by

$$V^{ex}(x(t'_i)) = V^{en}(x(t_i)) + \int_{t_i}^{t'_i} \dot{V}(x(\tau))d\tau$$

Since $\dot{V}(x) \leq -\gamma_3(||x||)$ and $V^{en}(x(t_{i+1})) \leq V^{ex}(x(t'_i))$ (condition 3), then

$$V^{en}(x(t_{i+1})) \leq V^{en}(x(t_i)) - \int_{t_i}^{t_i'} \gamma_3(||x||)d\tau$$

By induction it can be seen that

$$V^{en}(x(t_{i+1})) \leq V^{en}(x(t_1)) - \sum_{j=1}^{i} \int_{t_j}^{t_j'} \gamma_3(||x||)d\tau$$

By contradiction, it can be shown that the trajectories of the system converge to B_r. For this, let $||x|| > r, \forall t > t_{i+1}$, then from (C.9) and the above inequality

$$0 < \gamma_1(r) < \gamma_1(||x(t_{i+1})||) < V^{en}(x(t_{i+1})) \leq V^{en}(x(t_1)) - \sum_{j=1}^{i} \int_{t_j}^{t_j'} \gamma_3(r)d\tau$$

From which

$$0 < V^{en}(x(t_1)) - \sum_{j=1}^{i} \int_{t_j}^{t_j'} \gamma_3(r)d\tau = V^{en}(x(t_1)) - \sum_{j=1}^{i}(t_j' - t_j)\gamma_3(r)$$

Since the right hand side of the above inequality will become negative for some $N = i, t_N > t_1$, the inequality contradicts the assumption, and thus $||x|| < r, \forall t > t_N$. ■

The above Theorem is used to prove the global asymptotical stability of a double integrator system with saturated linear control laws (Proposition 5.2).

C.4 Proof of Proposition 5.2

Proposition 5.2 states that the dynamics of the averaged model of an SSID (double integrator) with saturated linear control law is globally asymptotically stable.

Proof: The proof proceeds by showing that all the conditions of the Theorem C.1 are met. The state space is divided into three zones: (i) Z^0, $|-k_pe_1 - k_ve_2| < \bar{M}F_c$, $\nu = -k_pe_1 - k_ve_2$, (ii) Z^1, $-k_pe_1 - k_ve_2 \geq \bar{M}F_c$, $\nu = \bar{M}F_c$ and (iii) Z^2, $-k_pe_1 - k_ve_2 \leq -\bar{M}F_c$, $\nu = -\bar{M}F_c$ (Figure C.1), where $\bar{M} = \frac{m+M}{M}$. Note that the equilibrium point is in Z^0.

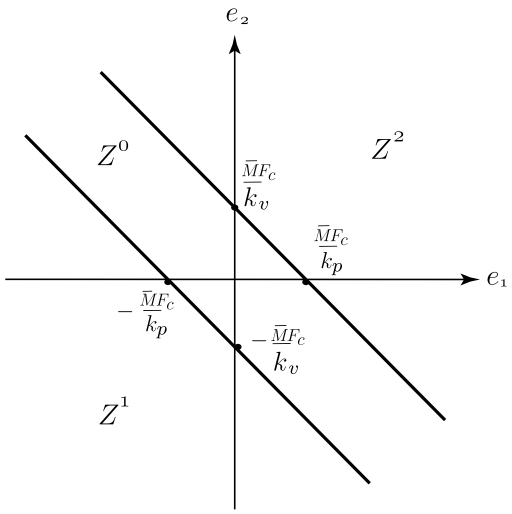

Figure C.1: State space and the Zones

I. Consider the Lyapunov function candidate $V = \frac{1}{2}e^T Pe$, where $e = (e_1, e_2)^T$ and

$$P = \begin{pmatrix} k_p + \frac{k_p^2+k_v^2}{m+M} & k_v \\ k_v & m+M+k_p \end{pmatrix}$$

$V(x)$ satisfies $c_1||e||^2 \leq V(e) \leq c_2||e||^2$, with $c_1 = \lambda_{\min}(P)$ and $c_1 = \lambda_{\max}(P)$.

II. Consider the system in Z^1. It will be shown that it comes to Z^0 after a finite time. The evolution of the states in Z^1 is given by:

$$e_2(t) = e_2(0) + \frac{F_c}{M}t \quad \text{(C.10)}$$

$$e_1(t) = e_1(0) + e_2(0)t + \frac{1}{2}\frac{F_c}{M}t^2 \quad \text{(C.11)}$$

$$-k_pe_1(t) - k_ve_2(t) = (-k_pe_1(0) - k_ve_2(0)) - \left(\frac{k_vF_c}{M} + k_pe_2(0)\right)t - \frac{1}{2}\frac{k_pF_c}{M}t^2 \tag{C.12}$$

For large values of t, it can be seen from (C.12) that $(-k_pe_1(t) - k_ve_2(t))$ is dominated by the last term and hence is negative. Therefore, the system cannot stay in Z^1 and will come to Z^0. A similar argument can be applied to Z^2.

III. Consider the case when exiting Z^0 implies entering Z^1 and vice versa. Let $({e_1}^{ex}, {e_2}^{ex})$ and $({e_1}^{en}, {e_2}^{en})$ be the states where the system leaves and reenters Z^0. Then,

$$-k_p{e_1}^{ex} - k_v{e_2}^{ex} = -k_p{e_1}^{en} - k_v{e_2}^{en} = \bar{M}F_c \tag{C.13}$$

The condition (C.13) can be used in (C.12) to compute the time spent in Z^1, τ_1:

$$\tau_1 = -2\left(\frac{k_v}{k_p} + \frac{M{e_2}^{ex}}{F_c}\right) \tag{C.14}$$

From (C.10) and (C.14), ${e_2}^{en}$ can be computed as:

$${e_2}^{en} = -{e_2}^{ex} - 2\frac{k_v}{k_p}\frac{F_c}{M} \tag{C.15}$$

Using the condition (C.13), the value of V at the entrance and exit can be calculated:

$$V^{en} = \frac{1}{2k_p^2(m+M)}\left(A({e_2}^{en})^2 + B{e_2}^{en} + C\right) \tag{C.16}$$

$$V^{ex} = \frac{1}{2k_p^2(m+M)}\left(A({e_2}^{ex})^2 + B{e_2}^{ex}_{M} + C\right) \tag{C.17}$$

with $A = k_v^4 + k_p^3(m+M) + 3k_pk_v^2(m+M) + k_p^2k_v^2 + k_p^2(m+M)^2$, $B = 2k_v\bar{M}F_c(k_p^2 + k_v^2 + 2k_p(m+M))$ and $C = \bar{M}F_c^2(k_v^2 + k_p^2 + k_p(m+M))$.

Using the value of $({e_2}^{ex} + {e_2}^{en})$ computed from (C.15), the difference of V at the enter and the exit is given by:

$$V^{en} - V^{ex} = -\left({e_2}^{en} - {e_2}^{ex}\right) \frac{F_c k_v \left\{ \left(k_v^2 - k_p(m+M)\right)^2 + k_p^2 k_v^2 \right\}}{M k_p^3 (m+M)} \tag{C.18}$$

Since from (C.10), $e2_M^{en} > e2_M^{ex}$, $V^{en} - V^{ex} < 0$. Following similar arguments, when the system passes through Z^2, it can also be concluded that $(V^{en} - V^{ex}) < 0$.

IV. In Z^0, $\dot{V} = -\frac{k_p k_v}{m+M}(e_1 e_2^2 + e_2^2) \leq -c_3 ||x||^2$, with $c_3 = \frac{k_p k_v}{m+M}$.

Since all the conditions of Theorem C.1 are satisfied, global asymptotic stability of the origin follows.

■

C.5 Proof of Proposition 5.3

Proof: The proof is based on verifying the conditions of Theorem 4.3.

i) It can be easiliy verified that the vector fields f_i, $i = 1, 2$ satisfy $||f_1(e)|| \leq L_1 ||e|| + \delta_1$,with $\delta_1 = 0$, and $||f_2(e)|| \leq L_2 ||e|| + \delta_2$ with $L_2 = 1$ and $\delta_2 = \frac{F_c}{M}$.

ii) The averaged model (5.19) is exponentially stable in a domain $D \subset \mathbb{R}^2$ containing the origin, i.e. a subset of the non-saturated region, and letting the Lyapunov function candidate for the averaged model be

$$V = \frac{1}{2} e^T P e \tag{C.19}$$

where $e = (e_1, e_2)^T$ and

$$P = \begin{pmatrix} k_p + \frac{k_p^2 + k_v^2}{m+M} & k_v \\ k_v & m + M + k_p \end{pmatrix} \tag{C.20}$$

then it can be verified that V satisfies

$$c_1||e||^2 \leq V(e) \leq c_2||e||^2 \tag{C.21}$$

$$\dot{V}(e) = -\frac{k_p k_v}{m+M}(e_1^2 + e_2^2) \leq -c_3||e||^2 \tag{C.22}$$

$$\left|\left|\frac{\partial V(e)}{\partial e}\right|\right| \leq c_4||e|| \tag{C.23}$$

with $c_1 = \frac{1}{2}\lambda_{\min}(P)$ and $c_2 = \frac{1}{2}\lambda_{\max}(P)$, $c_3 = \frac{k_p k_v}{m+M}$ and $c_4 = \lambda_{\max}(P)$.

From Theorem 4.3 it can be seen that for a given ε, i.e. given parameters m and k, there exists $\varepsilon^\star$ such that for $\varepsilon < \varepsilon^\star$ the trajectories of the closed-loop dynamics are ultimately bounded by a bound which shrinks as ε shrinks. Note that $\varepsilon^\star$ depends on parameters c_1 to c_4 and consequently these parameters can be adjusted through control parameters k_p and k_v. From this observation it can be deduced that k_p and k_v can be chosen such that the condition $\varepsilon < \varepsilon^\star$ is verified and hence yield an ultimate boundedness of the solutions. ■

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Notations

$\mathcal{H}$	hybrid system
x	continuous state variables
y	continuous output
q	discrete state variables (modes)
$q^\star$	recurrent mode
u	continuous input
σ	discrete input events
Q	set of discrete states
X	set of continuous states
Y	set of continuous outputs
O	set of discrete outputs
U	set of continuous inputs
Σ	set of discrete inputs
$(.)^+$	variables after switch
t	time variable
τ_k	switching times
$F(.)$	vector field
L	Lipschitz constant
$R(.)$	discrete transition rules
$F_q(.)$	vector field in mode q
$\mathsf{F}(.)$	averaged vector field
T_ℓ	cycle time in the ℓ'th cycle
α_q^ℓ	duty ratio for mode q in the ℓ'th cycle
ε	small parameter

$\mathcal{F}$	friction force
F_c	Coulomb friction force
$V(.)$	Lyapunov function
∇p	gradient vector
$\mathcal{S}$	hypersurface
$\mathbb{R}^n$	n-dimensional Euclidean space
sat(.)	saturation function
sgn(.)	signum function
$O(.)$	order of magnitude notation
$\|\|p\|\|$	norm of vector p
$(.)^T$	transpose of matrix or vector
max	maximum
min	minimum
mod	modulus function
$\sum$	summation
$\leq$ ($\geq$)	less (greater) than or equal to
$\Rightarrow$	implies
$\in$	belongs to
$\subset$	subset of
$\blacksquare$	designation of the end of proofs

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