ASPECTS OF TWO DIMENSIONAL MAGNETIC SCHRÖDINGER OPERATORS: QUANTUM HALL SYSTEMS AND MAGNETIC STARK RESONANCES

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Abstract

In this PhD thesis we deal with two mathematical problems arising from quantum mechanics. We consider a spinless non relativistic quantum particle whose configuration space is a two dimensional surface $\mathcal{S}$. We also suppose that the particle feels the effect of an homogeneous magnetic field perpendicular to the surface $\mathcal{S}$. In the first case $\mathcal{S} = \mathbb{R} \times \mathbb{S}^1_L$, the infinite cylinder of circumference L, corresponding to periodic boundary conditions, while in the second one $\mathcal{S} = \mathbb{R}^2$. In both cases the particle feels the effect of an additional suitable potential. We are thus left with the study of two specific classes of Schrödinger operators.

The operator of the first class generates the dynamics of the particle when it is submitted to an Anderson-type random potential, as well as to a non random potential confining the particle along the cylinder axis in an interval of length L. In this case we describe the spectrum and classify it by the quantum mechanical current carried by the corresponding eigenfunctions. We prove that there are spectral regions in which all the eigenvalues have an order one current with respect to L, and spectral regions where eigenvalues with order one current and eigenvalues with infinitesimal current with respect to L are intermixed. These results are relevant for the theory of the integer quantum Hall effect.

The second Schrödinger operator class corresponds to the physical situation where the potential is the sum of a "local" potential and of a potential due to a weak constant electric field F. In this case we show that the resonant states, induced by the electric field, decay exponentially at a rate given by the imaginary part of the eigenvalues of some non self-adjoint operator. Moreover we prove an upper bound on this imaginary part that turns out to be of order $\exp(-1/F^2)$ as F goes to zero. Therefore the lifetime of the resonant states is at least of order $\exp(1/F^2)$.

Versione abbreviata

In questo lavoro di dottorato studiamo due problemi matematici derivanti dalla meccanica quantistica. Consideriamo una particella quantica, senza spin e non relativista, che si muove su di una superficie bidimesionale $\mathcal{S}$. In un primo problema $\mathcal{S} = \mathbb{R} \times \mathbb{S}^1_L$ (il cilindro infinito di circonferenza L, ciò che induce delle condizioni al bordo periodiche), mentre nel secondo caso $\mathcal{S} = \mathbb{R}^2$ (il piano infinito). La particella subisce pure l'influsso di un campo magnetico omogeneo, perpendicolare alla superficie $\mathcal{S}$. In entrambi i casi essa è pure sottomessa all'effetto di un potenziale esterno appropriato. Dobbiamo quindi studiare due operatori di Schrödinger particolari.

Il primo operatore considerato genera la dinamica di una particella sottomessa ad un potenziale aleaotorio di tipo Anderson, ed un potenziale deterministico confinante la particella, lungo l'asse del cilindro, su una lunghezza L. In questo caso si localizza lo spettro, che viene poi classificato via la corrente quantomeccanica portata dalle rispettive autofunzioni. Dimostriamo che esistono delle regioni spettrali dove tutte gli autovalori hanno una corrente di ordine uno rispetto ad L, come pure regioni spettrali dove sono mescolati autovalori con corrente di ordine uno e autovalori con corrente infinitesimale rispetto ad L. Questi risultati hanno un'importanza nel quadro dell'effetto Hall quantistico.

Il secondo operatore di Schrödinger studiato, corrisponde alla situazione fisica in cui il potenziale è dato dalla somma di un potenziale "locale" e di un potenziale dovuto ad un piccolo campo elettrico costante F. In questo caso dimostriamo che gli stati risonanti indotti dal campo eletrico decandono esponenzialmente, con un tasso di decrescita dato dalla parte immaginaria degli autovalori di un certo operatore non auto-aggiunto. Dimostriamo poi un limite superiore, per questa parte immaginaria, dell'ordine di $\exp(-1/F^2)$ per i valori di F che tendono a zero. Dunque il tempo di vita di questi stati risonanti è almeno dell'ordine di $\exp(1/F^2)$.

Version abrégée

Cette thèse de doctorat concerne deux problèmes mathématiques issus de la mécanique quantique. On considère une particule quantique, non relativiste et sans spin, astreinte à se mouvoir sur une surface bidimensionnelle $\mathcal{S}$, plongée dans un champ magnétique homogène qui lui est perpendiculaire. Dans un premier problème, $\mathcal{S} = \mathbb{R} \times \mathbb{S}^1_L$, qui est un cylindre infini de circonférence L, ce qui correspond à des conditions aux bords periodiques. Dans le deuxième cas, $\mathcal{S} = \mathbb{R}^2$. En fonction du problème étudié, on ajoute un potentiel convenable. On est ainsi amené à étudier deux opérateurs de Schrödinger.

Le premier opérateur analysé génère la dynamique d'une particule soumise à un potentiel aléatoire de type Anderson ainsi qu'un potentiel non aléatoire dont le but est de confiner la particule le long de l'axe du cylindre, sur une longueur L. Dans ce cas, on localise le spectre et on le classifie par le courant quantique porté par les fonctions propres correspondantes. On montre qu'il y a des régions spectrales où n'existent que des valeurs propres avec courant d'ordre un par rapport à L, et des régions spectrales où sont mélangées valeurs propres avec courant d'ordre un et valeurs propres avec courant infinitésimal par rapport à L. Ces resultats on un intétet physique dans le cadre de l'effect Hall entier.

Le deuxième opérateur de Schrödinger étudié, correspond à la situation physique où le potentiel est donné par la somme d'un potentiel "local" et d'un potentiel dû à un petit champ électrique F constant. Dans ce cas on montre que les états résonants induits par le champ électrique décroissent exponentiellement avec un taux donné par la partie imaginaire des valeurs propres d'un certain opérateur non auto-adjoint. On montre de plus que cette partie imaginaire possède une borne supérieure de l'ordre de $\exp(-1/F^2)$, pour F tendant vers zéro. Ainsi, le temps de vie de l'état résonant en question est au moins de l'ordre de $\exp(1/F^2)$.

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Contents

Chapter 1

Introduction

The general domain of this thesis dissertation is the spectral analysis of Schrödinger operators. Schrödinger operators are a specific class of linear operators, acting in a separable Hilbert space over the field $\mathbb{C}$, that arise from quantum physics. In this introduction we first try to explain how non-relativistic quantum physics is characterized by a Hilbert space structure. Then we discuss the particular physical system consisting in an elementary quantum particle[1], which leads to Schrödinger operators. Finally we give a brief overview of the contents of this thesis.

Hilbertian structure of quantum physics

There are two suitable approaches to "endow" quantum mechanics with a Hilbert space structure. The first is based on a lattice-theoretical formulation of the properties of a quantum system, the so called propositional calculus of quantum mechanics. The second consists in an algebraic formulation of quantum mechanics extending the von Neumann synthesis of the quantum theories of Heisenberg, Born and Jordan (matrix mechanics – 1925), and Schrödinger (wave mechanics – 1926). In the latter approach there is a "direct" Hilbert space formulation of quantum mechanics [vN46]. Here we will deal only with *pure quantum systems* (but both approaches apply in a more general context).

The main idea of the propositional calculus of quantum mechanics is that a physical system can be described in term of so called "*yes-no experiments*". The latter are tests on the systems which permit only one of two alternatives as an answer. We define a *proposition* as a property of the system tested by an equivalence class of physical "yes-no experiments" (all the elements in the same equivalence class test the same property). Moreover the system exists independently of our knowledge of its propositions, and we

[1]See below for the definition of elementary quantum particle.

investigate the *properties of the propositions* of a physical system which are *independent of the state* of the system.
The set of all propositions of a physical system is supposed to have the mathematical structure of a complete orthocomplemented lattice $\mathcal{L}$. This means that $\mathcal{L}$ is a partially ordered set, that each subset of $\mathcal{L}$ admits a greatest lower bound and a smallest upper bound and that there exists an orthocomplementation. The structure of this lattice is independent of the state of the physical system, in others words $\mathcal{L}$ describes the *intrinsic structure* of the system.
As an example of "yes-no experiment", consider a particle in $\mathbb{R}^d$. A "yes-no experiment" is, for example, a test T_Δ on the particle (realized with a particle counter located in Δ) that has the answer "yes" if the particle is detected in a given subset $\Delta \subset \mathbb{R}^d$ and "no" otherwise. Denote by P_Δ the proposition (in $\mathcal{L}$) associated to T_Δ. P_Δ is "*true*" if the answer is "yes" with *certitude* (all repetitions of the experiment always yields the same result or equivalently the answer "no" is impossible) and "*not true*" otherwise. Clearly if $\Delta' \subset \Delta''$, whenever the response to $T_{\Delta'}$ is "yes", the response to $T_{\Delta''}$ must also be "yes". Therefore there exists a relation between certain pairs of propositions : if $P_{\Delta'}$ is "true" then $P_{\Delta''}$ must be "true" ($\mathcal{L}$ is partially ordered). Moreover, to the proposition $P_\Delta \in \mathcal{L}$ corresponds the orthocomplemented proposition $P_\Delta^{'} \in \mathcal{L}$ tested by T_{Δ^c} ($\Delta^c = \mathbb{R}^n \backslash \Delta$). In this case, if P_Δ is "true", then $P_\Delta^{'}$ is "*false*" (distinguished *in general* from "not true"), and viceversa.
Under five axioms, $\mathcal{L}$ can be represented as the set of all closed subspaces of a separable complex Hilbert space $\mathcal{H}$, denoted by $\mathcal{P}(\mathcal{H})$.
$\mathcal{P}(\mathcal{H})$ has clearly the structure of a complete orthocomplemented lattice, where the orthocomplemetation is the orthogonal $^\perp$ in the usual sense of the "geometry" of Hilbert spaces.
Each proposition corresponds to one of such closed subspaces, or equivalently to an orthogonal projector on $\mathcal{H}$ (bijection between $\mathcal{P}(\mathcal{H})$ and the orthogonal projectors on $\mathcal{H}$). In this framework *observables* are represented by spectral measures or equivalently, via the Spectral Theorem [RS72, Thm. VIII.6], as self-adjoint operators. The *states* are represented by the self-adjoint positive trace class operators ρ with $\mathrm{Tr}\,\rho = 1$. In particular the pure states of the system correspond to the one-dimensional projectors, or equivalently the closed one dimensional subspaces of $\mathcal{H}$ (the atoms of the lattice $\mathcal{P}(\mathcal{H})$) [Jau68], [Pir90], [RS98].

We now switch to the C$^*-$algebraic approach, following [Emc84, Chap. 9]. The fundamental postulate in this approach is the C^*-algebraic postulate: A physical system is characterized by a triple $\{\mathcal{E}, \mathcal{A}, \langle\cdot;\cdot\rangle\}$ where: $\mathcal{A}$, the set of its *observables*, is the collection of all the self-adjoint elements A of a C^*-algebra $\mathcal{B}$ with identity I; $\mathcal{E}$, the set of its *states*, is the collection of all real-valued, positive linear functionals ρ on $\mathcal{A}$, normalized by the condition $\langle\rho, I\rangle = 1$; and $\langle\cdot;\cdot\rangle$ is the prediction rule which attributes,

to every pair $(\rho, A) \in \mathcal{E} \times \mathcal{A}$, the value $\langle \rho; A \rangle$ of ρ at A, interpreted as the expectation of the observable A when the system is in the state ρ.
When we deal with a quantum system the C^*-algebra $\mathcal{B}$ is non-commutative and can be represented as a non-commutative subalgebra of the algebra of bounded linear operators on a Hilbert space [Dix69, Thm. 2.6.1]. That is, there exists an abstract Hilbert space $\mathcal{H}$ and an injective map $\pi : \mathcal{B} \longrightarrow \mathcal{L}(\mathcal{H})$ ($\mathcal{L}(\mathcal{H})$ is the C^*-algebra of bounded linear operators on $\mathcal{H}$) that is an algebraic $*-$homomorphism. In this framework the elements of $\mathcal{E}$ can be represented as density matrices, and pure states as one dimensional projectors, that are equivalence classes of vectors in $\mathcal{H}$.

We see that in both cases one can put a Hilbertian structure on a quantum system. But until now we only know that *there exists* an *abstract Hilbert space* $\mathcal{H}$, that observables are represented as self-adjoint operators acting in $\mathcal{H}$, that pure states are represented as rays in $\mathcal{H}$ and mixed states as positive self-adjoint operators with trace one (density matrices). Below we sketch to show how we can get a concrete realization of the abstract Hilbert space for one of the simplest physical systems.

Elementary quantum particle

We consider a *special physical system*: a (quantum) elementary particle in the configuration space $\mathbb{R}^d$, $d = 2, 3$ (without spin). The quantum elementary particle is the analogous of the classical point particle, in the sense that it is the simplest system for which at each time t two *observables* are defined: the *position* and the *momentum*. In this context, the word "elementary" means that there are no other non-trivial observables which are independent of the position and the momentum.

The problem is to give a concrete realization of the abstract Hilbert space for this physical system. To do this we need to consider the basic properties of physical space: its *homogeneity* and its *isotropy*. Both of these properties express the fact that the physical space has no observable physical properties: different points in the physical space are physically indistinguishable.
Consider first the observable position. The key concept is the *localisability* of the particle in some (Borel) subset Δ of the configuration space $\mathbb{R}^d$. To each $\Delta \in \mathcal{B}(\mathbb{R}^d)$ we associate a closed subspace $\mathcal{E}_\Delta \in \mathcal{P}(\mathcal{H})$, or equivalently an orthogonal projector E_Δ. We assume that the map $\mathcal{B}(\mathbb{R}^d) \longrightarrow \mathcal{P}(\mathcal{H})$ is a $\sigma-$homomorphism, that is an observable (see [Jau68, p.98]). The Borel sets Δ are subsets of $\mathbb{R}^d$ and the projectors E_Δ are projectors representing the "yes-no experiments" corresponding to find the particle in the subset Δ.
By space translations ($x \mapsto x + \alpha$) and rotations ($x \mapsto Rx$) we can associate to each Δ a subset $(R, \alpha) \circ \Delta := R\Delta + \alpha$. The set of all (R, α) forms the group of motions

of $\mathbb{R}^d$. Space homogeneity and space isotropy imply that translations and rotations are symmetries of the system and lead to the condition $\mathcal{E}_{(R,\alpha)\circ\Delta} = U(R,\alpha)\mathcal{E}_\Delta$ (or equivalently $E_{(R,\alpha)\circ\Delta} = U(R,\alpha)E_\Delta U(R,\alpha)^{-1}$), where, by the Wigner Theorem [Amr98], $U(R,\alpha)$ is a (projective) representation of the group $\{(R,\alpha)\}$ of motions of $\mathbb{R}^d$. This leads to the following commutative diagram, called *system of imprimitivity* for the position

$$\begin{array}{ccc} \mathcal{B}(\mathbb{R}^d) \ni \Delta & \longrightarrow & \mathcal{E}_\Delta \in \mathcal{P}(\mathcal{H}) \\ \big\downarrow {\scriptstyle (R,\alpha)} & & \big\downarrow {\scriptstyle U(R,\alpha)} \\ \mathcal{B}(\mathbb{R}^d) \ni (R,\alpha)\circ\Delta & \longrightarrow & \mathcal{E}_{(R,\alpha)\circ\Delta} = U(R,\alpha)\mathcal{E}_\Delta \in \mathcal{P}(\mathcal{H}) \end{array} .$$

A similar system of imprimitivity can be written for the momentum observable. To each $\Omega \in \mathcal{B}(\mathbb{R}^d)$ in the "momentum" space we associate a closed subspace $\mathcal{F}_\Omega \in \mathcal{P}(\mathcal{H})$, or equivalently an orthogonal projector F_Ω. The group symmetry is here that of momentum translations ($p \mapsto p + w$) and momentum rotations ($p \mapsto Rp$) (R is the same as for the position, since the classical direction of the momentum and position vectors refers to the configuration space).

$$\begin{array}{ccc} \mathcal{B}(\mathbb{R}^d) \ni \Omega & \longrightarrow & \mathcal{F}_\Omega \in \mathcal{P}(\mathcal{H}) \\ \big\downarrow {\scriptstyle (R,w)} & & \big\downarrow {\scriptstyle U(R,w)} \\ \mathcal{B}(\mathbb{R}^d) \ni (R,w)\circ\Omega & \longrightarrow & \mathcal{F}_{(R,w)\circ\Omega} = U(R,w)\mathcal{F}_\Omega \in \mathcal{P}(\mathcal{H}) \end{array} .$$

From the theory of the systems of imprimitivity, we can prove that the above imprimitivity systems determine completely the model of the elementary quantum particle. This determination is up to unitary equivalence and to an arbitrary parameter denoted by $\hbar$ [Pir90], [Jau68], [RS98].
The results are: the *Hilbert space* is $\mathcal{H} = L^2(\mathbb{R}^d, \mathrm{d}x)$. The *unitary representations* act on $\psi \in L^2(\mathbb{R}^d)$ as $[U_\alpha\psi](x) = \psi(x-\alpha)$, $[U_w\psi](x) = e^{i\,x\cdot w/\hbar}\psi(x)$ and $[U_R\psi](x) = \psi(R^{-1}x)$, and satisfy the *Weyl relations* $U_wU_\alpha = e^{iw\cdot\alpha/\hbar}U_\alpha U_w$. The *position operator* X acts as $(X\psi)(x) = x\psi(x)$, while the *momentum operator* P acts as $(P\psi)(x) = (-i\hbar\nabla\psi)(x)$. X and P are essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$ and satisfy the *Heisenberg canonical commutation relations* $[X_k, P_\ell] = i\hbar\delta_{k\ell}I$ defined on a dense set of vectors in $L^2(\mathbb{R}^d)$. P is the infinitesimal generator of the space translations $U_\alpha = e^{-i\alpha\cdot P/\hbar}$ and X is the infinitesimal generator of the momentum translations $U_w = e^{iw\cdot X/\hbar}$.

We are interested in the dynamical structure of our physical system consisting in a quantum particle. The dynamical structure contains the law which governs the time evolution of the states. We shall assume that we are dealing with a *conservative system*. Such systems interact with the external world only through constant forces (no time dependence) and do not react back on this world at all. We suppose that the state at one instant of time determines uniquely the state at any other time and that this transformation of states is *continuous*. In other words we suppose that the *time evolution* is

deterministic (as it is in classical mechanics). Finally, we also suppose *homogeneity of time*, or equivalently that time evolution is a symmetry transformation of the system.
According to the hypothesis above the time evolution is described by a group homomorphism which maps the real line continuously to a one-parameter group of unitary operators acting in $\mathcal{H}$ (Wigner Theorem). The homomorphism $t \longrightarrow U_t$ is called the *dynamical* (*or evolution*) *group* of the system. Consider at time $t = 0$ that the system is in a pure state given by a vector $\psi_0 \in \mathcal{H}$, then the state at time t is given by the vector $\psi_t = U_t\psi_0$.
We now look at the infinitesimal generator of the dynamical group $\{U_t\}_{t\in\mathbb{R}}$. The set of vectors $\psi \in \mathcal{H}$ for which

$$\text{s-lim}_{t\to 0} \tfrac{i\hbar}{t} [U_t - I]\psi = H\psi$$

exists is the domain $\mathcal{D}(H)$ of a *self-adjoint* operator H as defined above, moreover $\mathcal{D}(H)$ is dense in $\mathcal{H}$ (Stone Theorem [RS72, Thm. VIII.8]). H is called the *Hamiltonian*, it generates the dynamics of the system and it represents the energy observable of the system.

It remains to determine the form of the Hamiltonian. For this we need to introduce the *velocity*. It is defined as the formal differentiation of the position operator in the Heisenberg picture (at $t = 0$): $\dot{X} = \frac{i}{\hbar}[H, X]$. From the latter equation it follows that if we impose certain properties on $\dot{X}$, we must expect that they will restrict the possibilities for H. We will impose the *principle of Galilei invariance*, meaning that the Galilei transformations $X \longrightarrow X$ and $\dot{X} \longrightarrow \dot{X} + v$ are symmetries for the system (in classical mechanics these transformations leave the equations of motion invariant). One can prove (using Weyl relations) that the unitary group associated to this transformation is $W_v = e^{-i\,m\,v\cdot X/\hbar}$, with m a parameter [Jau68]. Thus

$$W_v\dot{X}W_v^{-1} = \dot{X} + v \qquad \text{and also} \qquad \frac{1}{m}W_vPW_v^{-1} = \frac{P}{m} + v\ .$$

Taking the difference and considering the form of W_v it follows that $m\dot{X} = P - A(X)$. Then, by the canonical commutation relations, $m[X_k, \dot{X}_\ell] = i\hbar\delta_{k\ell}I$, it follows that the operator $H_0 = \frac{m}{2}\dot{X}^2$ satisfies $\frac{i}{\hbar}[H_0, X] = \dot{X}$. We can thus conclude that the most general form of H is

$$H = \frac{(P - A(X))^2}{2m} + V(X) \tag{$\star$}$$

where m is interpreted as the particle mass. The identification of m with the particle mass follows if we identify the classical motion of the particle with the motion of the expectation value of the position operator. V represents an external potential and A represents a gauge field that is identified with a vector potential associated to an external magnetic field (more precisely we identify it with $\frac{1}{q}A$, q being the electric charge of the particle).
The aim of this discussion was to motivate our interest in the study of this specific class

of self-adjoint operators given by (non relativistic) Hamiltonian operators of the form ($\star$) acting on the Hilbert space $L^2(\mathbb{R}^d)$, the so called *Schrödinger operators*. Of course, the form of $A(X)$, $V(X)$ and the dimension d of the configuration space depend on the specific physical model under consideration.

Overview of the thesis

Let us briefly describe the general model studied in this thesis. We consider a non relativistic spinless quantum particle moving on a two dimensional surface $\mathcal{S} \subseteq \mathbb{R}^2$, and suppose that there is an homogenous magnetic field B perpendicular to $\mathcal{S}$ with an associated vector potential A. This forms the common background for the two problems studied during this work and motivates our title. For such systems the Hamiltonian generating the dynamics is just the kinetic Hamiltonian $(P - A)^2$ (in which the particle mass and the electric charge are taken equal to 1/2 and 1 respectively). The different Schrödinger operators studied in Part I and Part II differ for in choice of the configuration space $\mathcal{S}$ and in the potential added to the kinetic Hamiltonian.

In the first part we study a mathematical model inspired from the physics of two dimensional magnetic systems. We consider the case where $\mathcal{S}$ is the surface of an infinitely long cylinder of circumference L, $\mathcal{S} = \mathbb{R} \times \mathbb{S}^1_L$. To the kinetic Hamiltonian we add two confining potentials along the cylinder axis separated by a distance L, as well as a disordered potential in between. This choice, motivated from a physical point of view, corresponds to a random Hamiltonian H_ω that describes the dynamics of an electron in a disordered confined two dimensional device of "effective" size $L \times L$.
We study the spectral properties of H_ω. The spectrum is discrete, due to the periodic boundary conditions taken along the cylinder. We classify the eigenvalues in two classes which are characterized by the quantum mechanical current carried by the corresponding eigenfunctions. We study the spectrum of H_ω in two different energy intervals. The first lies in the spectral gap of the Hamiltonian H^b_ω that corresponds to H_ω in which the confining potentials have been removed. The second one lies in the spectrum of H^b_ω (in a Landau band). We first show that, in the spectral gap of H^b_ω, all the eigenfunctions of H_ω have a quantum mechanical current of order one with respect to the parameter L. On the other hand, in the Landau bands of H^b_ω, an intermixture of two types of eigenvalues of H_ω can be found: the first ones have eigenfunctions with associated quantum mechanical current of order one, the second ones have infinitesimal current for L large. In both cases, the above spectral properties are proved for realizations of the random potential that are typical, in the sense that this set of realizations has large probability. Finally, the information about the current is used to discuss the quantization of the Hall conductivity. The plan of this first part of the thesis is as follows: In Chapter 2 we motivate the

model from a physical point of view. In particular we briefly explain how one can have physical realization of a two dimensional system and why it is interesting to work with a random Hamiltonian. Then we shortly present the physics of the integer quantum Hall effect. In Chapter 3 we motivate our study in connection with previous theoretical and mathematical studies. In particular we expose Halperin's argument on the so called edge states, and we review recent results on edge states for systems with only one boundary. The goal of Chapter 4 is to present the contents of articles [FM03a] and [FM02] reproduced in Chapter 5 and Chapter 6. We expose in detail the model and the basic background necessary to understand the main results reported in a second step. After the main strategy of the proof, we briefly discuss the physical contents of our results in connection with the quantum Hall effect. Finally we present the main technical tools used in the proofs of the most important theorems.

In the second part we are concerned with a problem whose interest is mainly of mathematical nature. In this case the configuration space for the system is the two dimensional plane, $\mathcal{S} = \mathbb{R}^2$. We consider the kinetic operator $(P - A)^2$ and add a potential V that decays sufficiently rapidly at infinity, so that generically $(P - A)^2 + V$ has only a pure point spectrum. Adding a weak constant electric field F, our goal is to study the quantum resonances induced by the electric field, in particular to obtain some information on the resonance width or equivalently on the lifetime of a resonant state. From the analogy of the same problem without magnetic filed, these are called Stark resonances. Our study leads basically to two results. The first one consists in the proof that for sufficiently large times a magnetic Stark resonant state decays exponentially with a rate (the resonance width) given by the imaginary part of the eigenvalues of a certain non self-adjoint operator. The second result consists of an upper bound on the above mentioned imaginary part, or equivalently on a lower bound on the lifetime of the resonant state. In particular we prove that the lifetime of a resonant state is at least of order $\exp(1/F^2)$ as the electric field tends to zero. The main mathematical tool used in this analysis is the complex translation version of the spectral deformation theory.
The plan of the second part is as follows: In Chapter 7 we give a short introduction of the mathematical theory of quantum resonances. We discuss the different possible definitions and present the main technical tools used for their study, that is the spectral deformation theory. Chapter 8 contains a first section where we expose a previous study of magnetic Stark resonances in which the impurity potential is a point interaction. Then follows an introductory section to the articles [FK03a] and [FK03b]. We present the model and explain some aspects of the complex translation method, and we state our main results with a short sketch of the proof. Finally we briefly discuss our results in relation to the usual Stark effect. In Chapters 9 and 10 the articles [FK03a] and [FK03b] are reproduced.

We conclude this thesis with an outlook (Chapter 11) on open problems related with those studied in Part I and Part II.

Part I

Macroscopic Quantum Hall Systems

Chapter 2

$2D$ systems, disorder and integer quantum Hall effect

In this chapter we introduce the background for the physical model studied mathematically in the next chapters. The goal of this chapter is to motivate, from a physical point of view, the choice of our model, that should describe the dynamics of an electron in a two dimensional disordered sample.
We first explain how to create electron fluids that are effectively two dimensional and then introduce an important ingredient for the understanding of the physical behavior of these systems, that is the disorder. Finally we would like to give a concrete example of a beautiful effect, called Integer Quantum Hall Effect (IQHE), that occurs in these systems when they are submitted to a strong magnetic field and the temperature is very low. Most of the first section is based on the Nobel Lectures 1998 [Sto99], [Lau99].

2.1 Why $2D$? Quantum devices

In our three dimensional world, the creation of a two dimensional system usually requires a surface of an object or the interface between two substances and a force to keep things there. For example a billiard table confines the balls on a two dimensional plane. In our systems what we would like to confine are quantum particles, and more precisely electrons.
A successful method to create two dimensional electron systems (2DES) is to confine them within a solid to the interface between a semiconductor and an insulator, the so-called MOSFET (metal-oxide-semiconductor field-effect transistor). In a MOSFET the electrons are confined to the interface between silicon and silicon oxide (see Figure 2.1(a)). A similar method consists in confining the electrons to the interface between two different semiconductors (see Figure 2.1(b)). In both cases the force that holds electrons against one of the two substances is an electric field perpendicular to the interface. The two

dimensional character of the electrons in all these devices result from the quantization of the motion in the direction perpendicular to the interface.

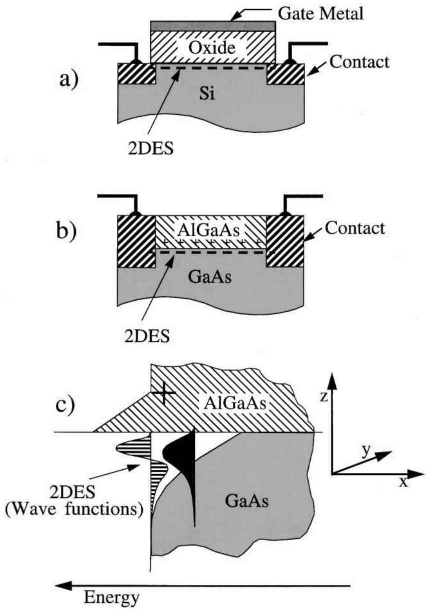

Figure 2.1: (a) Schematic drawing of a MOSFET. The 2DES resides at the interface between silicon and silicon oxide, electrons are held against the oxide by the electric field from the gate metal. (b) Schematic drawing of a GaAs/AlGaAs heterojunction. The 2DES resides at the interface between GaAs and AlGaAs, electrons are held against the AlGaAs by the electric field from the charged silicon dopants (+) in the AlGaAs. (c) Energetic condition in the GaAs/AlGaAs heterojunction structure (very similar to the condition in a MOSFET). Electrons are trapped in the triangular-shaped quantum well at the interface, they assume discrete energy states in the z direction. At low temperatures and low electron concentration only the lowest (black) electron state is occupied, the electrons are totally confined in the z direction but can move in the $x - y$ plane. From [Sto99]

In a MOSFET the electric field pushes the electrons so strongly against the silicon oxide and are so strongly entrapped along its direction that only a set of discrete states are quantum mechanically allowed in the direction perpendicular to the interface. At low temperatures and at low density all electrons reside in the lowest of these states. Their behavior is free in the interface-plane and rigidly confined in the third direction (see Figure 2.1(c)).
For the semiconductor heterostructures high mobility materials like GaAs/AlGaAs are used. By adding of a small number of impurities (silicon dopants) in the AlGaAs, and separating the mobile electrons from their parent impurities by confining them to different neighboring planes, one gets a junction between two semiconductors that have practically identical atom-to-atom spacing and differ slightly in the energies of their free electrons (electron affinity). The almost identical lattice constant guarantees a virtually defect-free

and stress-free interface of high quality, while the difference in electron affinity allows one to keep electrons away from their highly scattering parent impurities. We now describe briefly the implementation of this procedure done by using the technique of molecular beam epitaxy (MBE). A several μm-thick GaAs layer is grown onto a 1/2-mm-thick GaAs substrate. The GaAs layer is then covered by an approximately 0.5 μm-thick layer of AlGaAs. During the atomic-layer-by-atomic-layer growth process, silicon impurities are introduced into the AlGaAs material at a distance of about 0.1 μm from the interface. Each silicon impurity has one more outer-shell electron than the gallium atom, which it replaces in the solid. It easily looses this additional electron, which wanders around the solid as a conduction electron. Seeking the energetically lowest state, the electron ventures over the energetic cliff and falls "down" into the GaAs material, only 0.1 μm away. In the highly pure GaAs layer such conduction electrons can move practically unimpeded by their parent silicon impurities, which remain in the AlGaAs layer, on the other side of the barrier. The attraction from all those positively charged (loss of one electron) stationary silicon ions pulls the mobile electrons against the AlGaAs barrier of the interface. As for the MOSFET the same quantization perpendicular to the interface takes place and the electrons remain mobile within the interface plane (see Figure 2.1(c)).
Finally, using the procedures described above we can get a device in which the dynamics of the electrons is effectively two dimensional even if the quantum well created at the interface is not exactly two dimensional.

2.2 Clean samples do not exist: disorder

The two dimensional electron gas that can be created in a silicon MOSFET or a GaAs/AlGaAs heterostructure, as explained in Section 2.1, is of high purity but not perfect, there are in any case a small amount of imperfections.
In the MOSFET a source of impurity is that the Si and SiO_2 lattice parameters do not match, this create disorder at the interface. From this point of view GaAs/AlGaAs heterostructures are better but the Al atoms are still substituted at random in the GaAs lattice and are thus scattering centers. Moreover there are chemical impurities gathered at the interface in unknown amounts. Finally, modern heterostructures have huge mobility, but they are not perfect.
Paradoxically these imperfections in the $2D$ devices are of fundamental importance for the explanation of the integer quantum Hall effect, that we will discuss in the next section. But before turning to that subject we would like to explain what is the effect of the disorder on the 2DEG (two dimensional electron gas) and how we can model it.
Since Anderson [And58] it is known that disorder can create localization, that is, if the amount of disorder is large enough the electron states remain localized in a small domain of space for all time and no transport occurs. A lot of work was done in the last two decades to understand in which situation Anderson localization occurs, we will not enter

in this specific field, see for example [Hur00] and references therein.
We now look at the question how to model the disorder. There are basically two type of models that correspond to different kinds of disorder [LGP88]. The first consist to model the impurity potential as a sum of identical local perturbation located randomly in the plane (or more generally in the configuration space), this kind of model is convenient to describe amorphous matter where the disorder is of topological nature. The second kind of model consists to take as impurity potential a sum of local perturbations located on a regular lattice but where the local perturbations are different. This second possibility describes a perfect crystal where there exists a compositional disorder. In our concrete model we will use this second kind of model.

2.3 A beautiful phenomenon: integer quantum Hall effect

We now present a beautiful phenomenon in which a 2DEG created by a MOSFET or a GaAs/GaAlAs heterostructure shows a very remarkable behavior, the integer quantum Hall effect. But before explaining it we just look at the classical Hall effect discovered in 1879 by Edwin Hall.

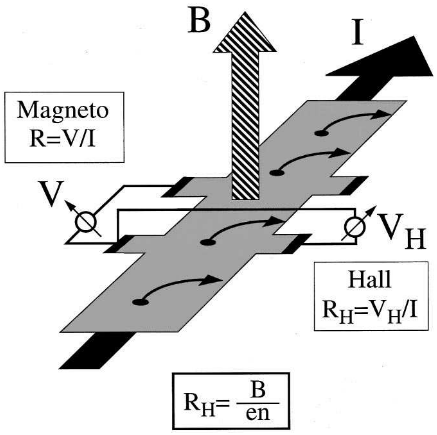

Figure 2.2: A typical Hall bar for measurement of the magnetoresitance R and the Hall resistance R_H. From [Sto99]

Hall considered a thin metal sheet submitted to a strong perpendicular magnetic field B and where a running current I was flowing along it (see Figure 2.2). He measured two different voltages, first the longitudinal one V (same direction than the current) and second the perpendicular one V_H (perpendicular with respect to the current). Hall discovered that at room temperature V_H is proportional to I and B, hence $R_H = V_H/I$, called *Hall resistance*, is just proportional to the strength of the perpendicular magnetic field, more precisely one gets $R_H = B/(ne)$ where n is the electron density (surface density) and e the the elementary charge of an electron.

If we take the same rectangular device and put it in a very high magnetic field (around 15 Tesla) and at very low temperature (around 4 Kelvin) we get a very different behavior of R_H in term of B and n. This was the sensational discovery of K. von Klitzing and coworker [vKDP80]. They find a stepwise dependance of the Hall resistance R_H with respect the magnetic field, for a fixed value of the electron density n. More surprisingly the value of R_H at the position of the plateaus of the steps is quantized (to a few part per billion) as $R_H = h/(ie^2) \equiv \sigma_H^{-1}$ where i is an integer and h Planck's constant (see Figure 2.3). The plateaus occur around precise values of the magnetic field (for a fixed electron density): these values are given by $B_i = (nh/e)/i$. Using the expression of $R_H = B/(ne)$ and replacing B with B_i we get $R_H = h/(ie^2)$, i an integer. For this discovery K. von Klitzing won the 1985 Physics Nobel Price.

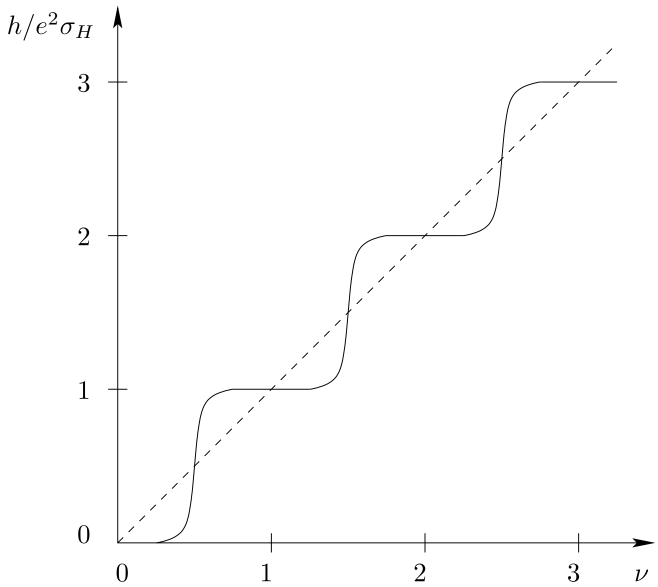

Figure 2.3: Hall conductance $\sigma_H = R_H^{-1}$ as function of the filling factor $\nu = \frac{nh}{eB}$. Plain line: Quantum measurement, Dotted line: Classical prediction.

The integer quantum Hall effect can be understood in the framework of a one particle quantum theory, that is on the basis of the dynamics of a single electron moving in a $2D$ plane in the presence of a magnetic field and a random potential simulating the disorder at the surface. The presence of all the electrons enters only when one fills up the electronic states according to the Pauli principle. The Coulomb interaction between the electrons carriers is irrelevant, but we just mention that this effect can no longer be neglected in order to understand the fractional quantum Hall effect.

We now briefly explain, from a quantum point of view, the behavior of the Hall conduc-

tance with respect to the magnetic field. If we solve [vK85], [PG87] the quantum problem of a particle in two dimensions submitted to a magnetic field we get a discrete set of infinitely degenerate energies, with gap proportional to B, called Landau energy levels. Confining the particle in a rectangular box each energy level is highly degenerate: per unit area there are $n_B = eB/h$ available states. Remark that n_B does not depend on any semiconductor parameter. We introduce the filling factor $\nu = n/n_B = nh/(eB)$, this is the important parameter for which at special values the Hall conductance is quantized. We have two different ways to change ν: vary the electron density n or vary the magnetic field B. Here we always consider n fixed. Fixing n we see that the special values of the magnetic field given above are exactly those for which $\nu = i$ (an integer). Since ν measures the filling of the Landau energy levels we get that a quantized Hall resistance is expected for values of B for which the first i^{th} lowest Landau levels are exactly filled.
In reality the Hall resistance takes the quantized values over extended regions of B around each B_i. The origin for plateau formation lies in electron localization due to the disorder, indeed the disorder broadens the Landau levels in bands with localized states at the band edges and at least one extended state at the center (see for example [Pra81] who deal with a delta impurity). Noting that at very low temperatures only the extended states carry current, we can understand the behavior of the Hall conductivity as follows. While the magnetic field decreases (for a fixed n) ν increases so the Landau levels are gradually filled up. When localized states are filled σ_H does not change, while when extended states are filled σ_H changes and makes a transition from one plateau to the next. A lot of theoretical physics and mathematical physics work has been done on the subject of the quantum Hall effect; it is not our purpose to review all this work, see for example [Hur00] and the papers [Lau81], [Hal82], [TKNdN82], [Kun86], [Kun87], [Hat93], [BvESB94], [ASS94], [Tho94], [AG98], [KRSB02], [EG02], [Mac03a] and the first section of the next chapter.
We have just given above a short explanation based on basic quantum mechanics. In the next chapter we will look briefly at Halperin's picture of the quantum Hall effect based on the notion of edge states. We shall not give a full explanation, but just focus on the ingredients that motivate the mathematical work.

Chapter 3

Current carrying edge states

In this chapter we first discuss Halperin's picture of the integer quantum Hall effect. This approach is based on the notion of current carrying edge states. We will explain the importance of these edge states without entering in the whole explanation of the integer quantum Hall effect. Edge states provide the physical motivation for the mathematical study of spectral properties of Hamiltonians describing the dynamics of a particle constrained to move in a semi-infinite plane, submitted to a strong perpendicular magnetic field and a weak disorder. These mathematical studies are the subject of the second part of this chapter.

3.1 Halperin's picture of the IQHE: edge states

For the explanation of the integer quantum Hall effect there are three main theoretical approaches. The first is based on the Laughlin gauge argument [Lau81] and was rigorously analyzed in [ASS94], the second uses the Kubo-Chern formula for the Hall conductivity and was introduced in [TKNdN82], then generalized in [Kun87], [BvESB94]. The connection between these two approach is well understood, see for example [AG98].
The third approach is that based on current carrying edge states introduced by Halperin in his famous paper [Hal82] briefly discussed below. Note that recently, the connection between the boundary current picture for the Hall conductivity and the one based on the first two approaches has been elucidated in [KRSB02] (generalizing [Hat93]), [EG02] and [Mac03a].

The main idea of Halperin's paper is the following. In a confined two dimensional electron gas submitted to a strong magnetic field there exist electronic states extended along the boundaries. These states are current carrying and contribute to the quantized Hall conductivity if at the two edges of the sample the Fermi levels are different. Moreover these states remain extended when a weak disorder is added.

The Halperin geometry corresponds to the domain in the plane $\mathbb{R}^2$ given by

$\{(x,y) : r_1^2 \leq x^2 + y^2 \leq r_2^2\}$ where we take Dirichlet boundary conditions (hard edges) at the two concentric circles of radii r_1 and r_2. This geometry corresponds to that of an annulus, called Corbino disk. (see Figure 3.1).

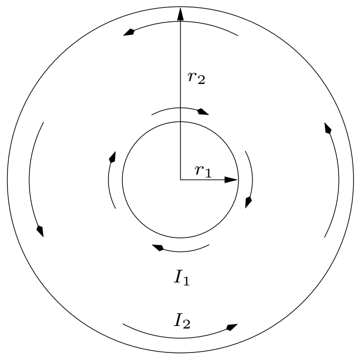

Figure 3.1: The Corbino disk. The magnetic field is constant between r_1 and r_2 and zero elsewhere. The curved arrows show the boundary currents.

We report Halperin's analysis based on [Hal82]. We choose the polar coordinates and the gauge $A = (0, 1/2Br)$. The Hamiltonian reads

$$H_0 = \tfrac{1}{2M} p_r^2 + \tfrac{1}{2M}\left(p_\vartheta - \tfrac{1}{2}eBr\right)^2 \tag{3.1}$$

where M is the electron mass, and at $r = r_1, r_2$ we take Dirichlet boundary conditions. Remark that here there is no electric field, in contrast to other analysis of the quantum Hall effect.

Since H_0 commutes with the angular momentum L_z (associated quantum number m) the electronic states are given by

$$\phi_{n,m}(r,\vartheta) = \frac{e^{im\vartheta}}{\sqrt{2\pi}}\psi_n(r) \tag{3.2}$$

with ψ_n the eigenfunctions of the one dimensional problem associated to the Hamiltonian

$$H_0(m) = \tfrac{1}{2M} p_r^2 + \tfrac{1}{2M}\left(\tfrac{m\hbar}{r} - \tfrac{1}{2}eBr\right)^2 \tag{3.3}$$

with Dirichlet boundary conditions at $r = r_1, r_2$.
Away from the edges we can write $\psi_n(r) = \varphi_n(r - r_m)$ with φ_n the eigenfunctions of the "approximate" Hamiltonian

$$\tilde{H}_0(m) = \tfrac{1}{2M} p_r^2 + \tfrac{e^2B^2}{2M}\left(r - r_m\right)^2 \simeq \tfrac{1}{2M} p_r^2 + \tfrac{e^2B^2}{2M}\left(r - r_m\right)^2 \left(\tfrac{r_m + r}{2r}\right)^2 = H_0(m) \tag{3.4}$$

with $r_m = \sqrt{\frac{2|m|\hbar}{eB}}$. This analysis holds provided that $r_1 < r_m < r_2$ and $|r_i - r_m| \gg \ell = \sqrt{\frac{\hbar}{eB}}$ for $i = 1, 2$ (we suppose $\ell \ll r_1, r_2 - r_1$). These states are localized in the

radial direction near r_m, that (by assumption) is well away from the boundaries. The associated energies are (approximatively) $E_n \equiv E_{n,m} = (n+1/2)\hbar\omega_c$ (Landau levels) with $\omega_c = eB/M$.
Let us now consider the situation where $r_m \simeq r_i$ ($r_m = r_i - \alpha\ell$, α small), $i = 1, 2$. In this cases, of course, the edges cannot be neglected. The energies are clearly no longer given by the Landau levels but are monotonic branches. The latter property follows form the presence of the edges. For example at the outer edge ($r_m \simeq r_2$) the energy $E_{n,m}$ will increase monotonically as r_m increases. While at the inner edge the behavior is monotonically decreasing (see Figure 3.2).

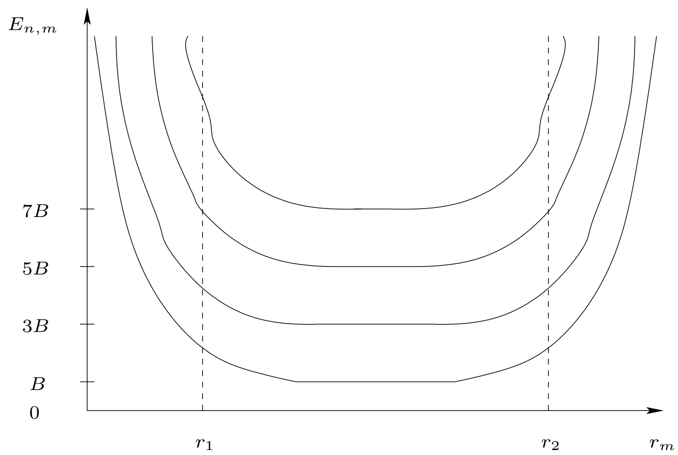

Figure 3.2: Energy spectral branches $E_{n,m}$. For each m or equivalently r_m (remark that $r_m \sim \sqrt{m}$) the energy levels are indexed by an integer n. To each fixed index n correspond a so-called Landau band.

We now calculate the azimuthal current carried by the states $\phi_{n,m}$. It is given by the relation

$$I_{n,m} = \frac{e}{M}\int_0^\infty |\phi_{n,m}(r,\vartheta)|^2 \left(\tfrac{m\hbar}{r} - \tfrac{1}{2}eBr\right)\, dr\ . \tag{3.5}$$

When r_m lies well inside the annulus ($r_1 \ll r_m \ll r_2$), we can write

$$\begin{aligned} I_{n,m} &= \frac{e^2B}{M}\int_0^\infty |\phi_{n,m}(r,\vartheta)|^2 (r_m - r)\frac{r_m + r}{2r}\, dr \\ &\simeq \frac{e^2B}{M}\int_0^\infty |\phi_{n,m}(r,\vartheta)|^2 (r_m - r)\, dr \end{aligned} \tag{3.6}$$

where we approximate $\frac{r_m+r}{2r} \simeq 1$ since for $|r_m - r| \gg \ell$ the density $|\phi_{n,m}|^2$ decreases rapidly. Using the latter property and the symmetry of $|\phi_{n,m}|^2$ with respect to $r = r_m$ the integral vanishes: the total current inside the annulus is zero.
Since for r_m close to the edges $|\phi_{n,m}|^2$ is no longer symmetric, we expect $I_{n,m} \neq 0$, that is, there are currents flowing along the edges. We can get this result starting from

$$I_{n,m} = \frac{e}{h}\frac{\partial E_{n,m}}{\partial m} \tag{3.7}$$

that follows from $E_{n,m} = (\phi_{n,m}, H_0\phi_{n,m})$ taking formally m as a continuous parameter. Therefore, for $r_m \simeq r_1$ we have $I_{n,m} < 0$ and for $r_m \simeq r_2$ we have $I_{n,m} > 0$ as we can see in Figure 3.2.
To find the total current carried by the electronic states close to the boundaries we have to sum up all the occupied states (n, m). The filling is submitted to the Pauli principle for fermions. Suppose the local Fermi level E_F lies in between the energies E_n of two Landau levels $n = N - 1$ and $n = N$ in the interior of the annulus, and takes the values $E_F^{(1)} < E_F^{(2)}$ for $r = r_1$ et $r = r_2$ (i.e. at the boundary). This difference of the Fermi energy at the two boundaries is, for example, due to a voltage drop. We can then calculate the total current flowing in the annulus

$$I = \frac{e}{h} \sum_{i=0}^{N-1} (E_{i,m_{\max}} - E_{i,m_{\min}}) = \frac{e}{h} N \left(E_F^{(2)} - E_F^{(1)} \right) \tag{3.8}$$

where the first equality follows from (3.7) (where $\frac{\partial E_{n,m}}{\partial m}$ as to be interpreted as a discrete derivative) and the second from $E_{i,m_{\max}} \equiv E_{i,m_{\max}(i)} \simeq E_F^{(2)}$ and $E_{i,m_{\min}} \equiv E_{i,m_{\min}(i)} \simeq E_F^{(1)}$. Finally we see that, if the Fermi energies at the edges differ, typically due to a small voltage drop (V_H) between the two edges, a net current flows inside the annulus. Moreover if $E_F^{(2)} - E_F^{(1)} = eV_H$ we get that the Hall conductivity $\sigma_H = I/V_H$ is given by $\sigma_H = Ne^2/h$. In this approach the Hall current is due to the (chiral) currents carried by the edge states.

Suppose now we add a disordered potential represented by a (random) potential V, we want to prove that, if the disorder is not too strong, there still exist states that are current carrying. As above, suppose that the Fermi energy lies in between two Landau levels E_{N-1} and E_N. The only states with energy near E_F are localized radially near r_1 and r_2, indeed the only possible energy for states inside the annulus are the Landau levels. Develop one of such states on the $\{\phi_{n,m}\}$ basis defined above, with Fourier coefficients $c_{n,m}$. Consider, for example, the case $r \simeq r_2$. The coefficients $c_{n,m}$ with $n > N - 1$ will be small, of order $\frac{V}{B}$, while the others ($n \leq N - 1$) will be appreciable unless $|r_2 - r_m| \gg \ell$ (see for example [Fer99], page 30 for a mathematical proof in a similar context). The current carried by such a state ψ is given by

$$I_\psi = \frac{e}{M} \left(\psi, \left[p_\theta - \tfrac{1}{2} eBr \right] \psi \right) = \sum_{\substack{m \\ n,n'}} c^*_{m,n'} c_{m,n} I_{m;n,n'} \tag{3.9}$$

with

$$I_{m;n,n'} = \frac{e}{2\pi M} \int_0^\infty dr \int_0^{2\pi} d\vartheta\, \phi^*_{m,n'}(r,\vartheta) \phi_{m,n}(r,\vartheta) \left(\tfrac{m\hbar}{r} - \tfrac{1}{2} eBr \right) . \tag{3.10}$$

The diagonal terms ($n = n'$) are exactly the currents $I_{n,m}$ defined in (3.5) for the non random system, they are non vanishing as we have seen above, while the off-diagonal terms ($n \neq n'$) are very small when $\frac{V}{B}$ is small (see [Fer99], page 32 for a similar analysis). Therefore, if the disorder is small enough with respect to the magnetic field, there still exist electronic states localized radially close to the edges and that are current carrying: $I_\psi \neq 0$. This analysis only shows that in presence of a small amount of disorder there still exists current carrying edge states. Clearly the above argument does not show that the current carried by the edge state satisfy $I = Ne^2/hV_H$. For this one can proceed with Laughlin's gauge argument and the extension given by Halperin, see [Lau81] and [Hal82] (paragraph IV).

3.2 Mathematical study of the semi-infinite systems

We now switch to the rigorous study of Halperin's quantum Hall effect picture. In the last years many mathematical works [MMP99], [Fer99], [dBP99], [FGW00] have been done in connection with the so-called edge states for quantum Hall systems. The first step consists in the study of a quantum particle, submitted to a magnetic field and a random potential, that is constrained to move in a semi-infinite system. This geometry is not exactly that of a Corbino disk, where two edges are present, but focuses on the dynamics when there is only one boundary. These first works are an important step for understanding the case corresponding to the Corbino disk. The Hall system with two boundaries is the subject of the first part of the present thesis and we will come to it in the next chapters. We now explain the results for the semi-infinite system, but before remark that the notion of current carrying edge states for semi-infinite systems is related to the continuity of the spectrum.

Macris, Martin and Pulé [MMP99] consider a confining soft wall given by

$$U(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \mu x^\gamma & \text{if } x \geq 0 \end{cases} \tag{3.11}$$

where $\mu > 0$ and $\gamma \geq 1$. They assume in addition that the particle is also submitted to a bounded and differentiable impurity potential V such that
(V1) : $\sup_{\mathbf{x}} |V(\mathbf{x})| = V_0 < \infty$
(V2) : $\sup_{\mathbf{x}} |\partial_x V(\mathbf{x})| = V_0' < \infty$.
It is easily found (see Chapter 4 for more details) that the Hamiltonian (that is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$)

$$H_0 = \tfrac{1}{2}p_x^2 + \tfrac{1}{2}\left(p_y - Bx\right)^2 + U \tag{3.12}$$

has only purely absolutely continuous spectrum corresponding to the interval $[B/2, +\infty)$. The main question addressed in [MMP99] is the stability of this continuous spectrum when the impurity potential V is added.
In the context of the quantum Hall effect it is interesting to take for V an Anderson-like

random potential. It consists of a sum of local perturbations located at the site of the lattice $\mathbb{Z}^2$ and whose coupling constants are random variables varying in $[-1, 1]$. For a given realization $\omega \in [-1, 1]^{\mathbb{Z}^2}$ the potential reads

$$V_\omega(\mathbf{x}) = \sum_{\mathbf{n}\in\mathbb{Z}^2} X_{\mathbf{n}}(\omega) v(\mathbf{x} - \mathbf{n}) \tag{3.13}$$

where $v(\mathbf{x}) = 0$ for $|\mathbf{x}| > 1/2$ and $X_{\mathbf{n}}$ are independent identically distributed random variables with common continuous density supported in $[-1, 1]$.
By standard arguments of the random Schrödinger operator theory the authors have shown that the spectrum of the family of random Hamiltonians $H_\omega = H_0 + V_\omega$ contains the interval $[B/2, +\infty)$ with probability one (see appendix B in [MMP99]). Macris, Martin, Pulé then show that if V_0 and V_0' are small enough (depending on B and the steepness of the wall), then H_ω cannot have point spectrum in the intervals

$$\Delta_n(B, \delta) =](n + 1)B - \delta, (n + 1)B + \delta[\tag{3.14}$$

of size $2\delta > B - V_0$ in between the Landau levels. Moreover, as told above, these intervals are in the spectrum of H_ω with probability one. We can summarize this result in

Theorem 3.1. [MMP99] *If $B/2 - V_0 > \delta$ for some $\delta > 0$ and V_0' is sufficiently small, then H_ω has no eigenvalues (i.e. no point spectrum) in the intervals $\Delta_n(B, \delta)$ of size $2\delta > B - V_0$ in between the Landau levels. The whole interval $\Delta_n(B, \delta)$ is included, with probability one, in the spectrum of H_ω. Thus, the spectrum of H_ω on $\Delta_n(B, \delta)$ is purely continuous almost surely.*

To conclude with this work we mention the idea involved in the proof of the absence of eigenvalues. The authors suppose that for a given energy E in $\Delta_n(B, \delta)$ there is a function $\psi \in L^2(\mathbb{R}^2)$ such that $H_\omega\psi = E\psi$. By the virial theorem it follows that $(\psi, [iA, H]\psi) = 0$ for a self-adjoint operator A. This implies, with $A = p_x - By$, that $(\psi, [\partial_x U + \partial_x V_\omega]\psi)$ should be zero, so that

$$(\psi, \partial_x U\psi) = -(\psi, \partial_x V_\omega\psi) \le V_0' \ . \tag{3.15}$$

For energies away from the Landau levels the corresponding eigenfunctions should be supported in region where the wall potential U is large. Indeed, if ψ were essentially localized in the bulk region, the wall would not contribute to the energy which would then lie in the vicinity of a Landau level for small V_0. Therefore, $(\psi, \partial_x U\psi)$ should be large, which contradicts (3.15) if V_0' is small enough. Then no eigenfunction can exist for such energies.
Finally, remark that using the same idea but taking for operator A the $-y$ coordinate of the particle in [Fer99] the author proved the same result without any restriction on the derivative of the impurity potential. The use of $-y$ instead of $p_x - By$ has a direct physical interest. Indeed, since the commutator $[-iy, H_\omega]$ gives the velocity operator

along the boundary, the value of $(\psi, [-iy, H_\omega]\psi)$ just gives the mean value of the velocity along the edge for a given state ψ. The fact that is not zero directly implies that there is a transport along the boundary, and the state ψ is thus a current carrying edge state.

De Bièvre and Pulé [dBP99] are interested in the propagation of the edge states. They consider the Hamiltonian

$$H_0 = \tfrac{1}{2}p_x^2 + \tfrac{1}{2}(p_y - Bx)^2 \tag{3.16}$$

with a Dirichlet boundary condition at $x = 0$, thus the particle moves in the half-space $x > 0$. In a second stage they also suppose that the particle feel the effect of a bounded impurity potential V_B that satisfies $\|V_B\|_\infty \leq cB$ for some constant c. When there is no disorder the authors introduce a separation of the Hilbert space in two components. This separation is based on the idea that for each Landau band one can get two H_0 invariant subspaces related to a so-called edge and bulk spaces. After that they look at the effect of the impurity potential on the propagating properties of the edge states.
One can remark that, by translation invariance in the $y-$direction H_0 is unitarily equivalent to a direct integral over the momentum k, that is $H_0 \simeq \int_{\mathbb{R}}^{\oplus} H(k)\,dk$, where $H(k) = \frac{1}{2}p_x^2 + (k - Bx)^2$ act in $L^2(\mathbb{R}_+, dx)$. The spectrum of $H(k)$ is given by discrete eigenvalues $E_n(k)$ with corresponding eigenfunction $\varphi_n(x, k)$. The n^{th} band space $\mathcal{H}_n$ is defined as the space consisting of functions of the form $f(k)\varphi_n(x, k)$, with $f \in L^2(\mathbb{R}, dk)$. Similarly as in the Macris, Martin, Pulé work the spectrum of H_0 is absolutely continuous and given by $[B/2, +\infty)$. Define within $\mathcal{H}_n$ the edge and bulk spaces

$$\mathcal{H}_{n,e}(\sigma, \gamma) = L^2((-\infty, \sigma B^\gamma], dk) \subset \mathcal{H}_n \tag{3.17}$$

and

$$\mathcal{H}_{n,b}(\sigma, \gamma) = L^2([\sigma B^\gamma, +\infty), dk) \subset \mathcal{H}_n \tag{3.18}$$

so that $\mathcal{H}_n = \mathcal{H}_{n,e}(\sigma, \gamma) \oplus \mathcal{H}_{n,b}(\sigma, \gamma)$. $\mathcal{H}_{n,e}(\sigma, \gamma)$ is called an edge space for all $\gamma \leq 1/2$ and $\mathcal{H}_{n,b}(\sigma, \gamma)$ a bulk space for all $\gamma > 1/2$. We can understand these definitions with

Theorem 3.2. [dBP99] *If $k \in (-\infty, k_B)$ where k_B is of order $\sqrt{B}$, then the wave packet $f(k)\varphi_n(x, k)$ belongs to the edge space $\mathcal{H}_{n,e}(\sigma, 1/2)$ and the wave packet speeds along the edge in the $y-$direction with velocity of order $\sqrt{B}$. The wave packet is exponentially small for x greater than $1/\sqrt{B}$.*
If $k \in [k_B, +\infty)$ with k_B of order B^γ with $\gamma > 1/2$, then the group velocity is exponentially small in B and the wave packet is exponentially small within the region $0 \leq x \leq 1/\sqrt{B}$ (i.e. close to the edge).

When a weak impurity potential is added de Bièvre and Pulé show that, in spectral intervals of size of order B between the Landau levels, there are no bound states and that the speed in the $y-$direction, for the (extended) state localized in energy in such intervals, is still of order $\sqrt{B}$, consequently therein the spectrum is absolutely continuous. The tool used to prove this result is a Mourre estimate that consists to prove an estimate of the form

$$E_\Delta(H)[H, iA]E_\Delta(H) \geq \alpha E_\Delta(H) \tag{3.19}$$

for some $\alpha > 0$, where $E_\Delta(H)$ denote the spectral projector of the Hamiltonian H on the energy interval Δ and A is a self-adjoint operator called conjugate operator. Under some regularity conditions on H, $[H, iA]$ and $[[H, iA], iA]$ an estimate of the type of (3.19) implies that the spectrum of H in Δ is purely absolutely continuous, see [FGW00] or [Mou81].

In [dBP99] the conjugate operator is y, that is the coordinate of the particle along the boundary. As we already noted the commutator $[H, iy]$ is just the velocity operator along the edge. A Mourre estimate directly implies that a given state ψ localized in energy in Δ propagates with a velocity whose value is bounded from below by the constant α, that in [dBP99] turns out to be of order $\sqrt{B}$.

Using the Mourre technique, with a different conjugate operator, Fröhlich, Graf and Walcher proved a similar result as in [dBP99]. In [FGW00] the authors consider the random Hamiltonian

$$H_\omega = \tfrac{1}{2}p_x^2 + \tfrac{1}{2}\left(p_y - Bx\right)^2 + V_\omega \tag{3.20}$$

to which is also added or a soft wall potential U or a hard wall given by Dirichlet boundary condition, this in order to confine the particle in the half-space $x < 0$.

The conjugate operator is the same as that used in [MMP99], that is $p_x - By$, for the case of a soft wall the commutator is $i[H, p_x - By] = -\partial_x(V_\omega + U)$. Under the following assumption on the confining potential U

(U1) : $U(x) = 0$ for $x < 0$

(U2) : $\partial_x U(x) \geq 0$ for all x

(U2) : $\inf_{y \geq b} \partial_x U(x) > 0$ for all $b > 0$

Fröhlich, Graf and Walcher have proven

Theorem 3.3. [FGW00] *Assume $E \notin \{(n + 1/2)B, n \in \mathbb{N}\}$. If the disorder potential satisfies $\|V_\omega\|_\infty \leq \delta$, there is an open interval Δ that contains E and a positive constant α such that*

$$-E_\Delta(H)[H, i(p_x - By)]E_\Delta(H) \geq \alpha E_\Delta(H) \tag{3.21}$$

where $H = \frac{1}{2}p_x^2 + \frac{1}{2}\left(p_y - Bx\right)^2 + V_\omega + U$. Therefore the spectrum of H is absolutely continuous in Δ.

For the case with Dirichlet boundary condition they have proven

Theorem 3.4. [FGW00] *Assume $E \notin \{(n + 1/2)B, n \in \mathbb{N}\}$. If the disorder potential satisfies $\|V_\omega\|_\infty \leq \delta$, $\|\partial V_\omega\|_\infty \leq \delta'$ and $\|\partial^2 V_\omega\|_\infty \leq \delta''$ there is an open interval Δ that contains E and a positive constant α such that*

$$-E_\Delta(H)[H, i(p_x - By)]E_\Delta(H) \geq \alpha E_\Delta(H) \tag{3.22}$$

where $H = \frac{1}{2}p_x^2 + \frac{1}{2}(p_y - Bx)^2 + V_\omega + U$. *Therefore the spectrum of* H *is absolutely continuous in* Δ.

Remark that, the connection between the Mourre estimate and the absolutely continuity of the spectrum is not direct, indeed the conjugate operator here is only symmetric and not self-adjoint.
This last result is close to the one of de Bièvre and Pulé, but for these authors the assumption on the derivatives of the disorder potential is not necessary, since they use as conjugate operator y (that is self-adjoint) instead of $p_x - By$.

Finally, all these works, that deal with the spectral properties of a random magnetic Hamiltonian with one edge, show that if the disorder potential is small enough, then in between the Landau levels the Hamiltonian has spectral components of absolutely continuous spectrum. This implies that the states localized in energy in these intervals are extended states propagating along the edge and thus carry a non zero current and contribute to the transport.
In the next chapter we will look at the same problem but where two boundaries are present and where the geometry (configuration space) is that of a cylinder.

Chapter 4

Spectral properties of finite quantum Hall systems

The goal of the present chapter is to introduce the two articles that are the content of the next chapters, for this we partially follow the proceeding [FM03b].
Here we give the precise statement of the model that we study in Chapters 5 and 6, then introduce the notion of current carrying edge states. We also give some preliminary result based on previous study, in particular we look at the properties of three Hamiltonians that are important for the implementation of our proof strategy. The latter will be discussed in this chapter where the main mathematical tools are presented. Finally, also an overview of the results is given and discussed from a physical point of view.

4.1 The model

In the two next chapters we are interested in the study of the spectral properties of the family of random Schrödinger operators H_ω consisting in the sum of the kinetic term, a random potential and a confining deterministic potential. Below we define precisely this Hamiltonian. Moreover we investigate these spectral properties in connection to the notion of current carrying states.

Geometry and Hilbert space
We consider a spinless non relativistic quantum particle, whose configuration space is two dimensional, and given by the surface of an infinitely long cylinder whose circumference is L. The parameter L will be supposed large (macroscopic) but finite. The Hilbert space describing the pure states of this particle is

$$\mathcal{H} = L^2(\mathbb{R} \times \mathbb{S}^1_L,\, \mathrm{d}x\, \mathrm{d}y) \tag{4.1}$$

where $\mathbb{S}^1_L$ is the circle of circumference L. In the following we will write for $\mathcal{H}$

$$\mathcal{H} = L^2\left(\mathbb{R} \times \left[-\tfrac{L}{2}, \tfrac{L}{2}\right],\, \mathrm{d}x\, \mathrm{d}y\right) \tag{4.2}$$

where the points $(x, y = -\frac{L}{2})$ and $(x, y = \frac{L}{2})$ $(x \in \mathbb{R})$ have to be identified.
The cylindrical geometry is equivalent to take periodic boundary conditions along the $y-$direction for the functions in the Hilbert space when we define the Hamiltonians acting in $\mathcal{H}$. It is required that for any $\psi \in \mathcal{H}$ in the domain of the Hamiltonians under consideration

$$\psi\left(x, -\tfrac{L}{2}\right) = \psi\left(x, \tfrac{L}{2}\right) \qquad \text{for all } x \in \mathbb{R}\,. \tag{4.3}$$

Landau Hamiltonian

We suppose, that perpendicular to that surface, there is a constant magnetic field B with associated vector potential A. Since the particle is considered spinless, the spin-field term is not taken into account (equivalently we can suppose that the particle has a spin $\frac{1}{2}$ and is fully polarized, this gives only a shift in the energy and reduces the Hilbert space to the subspace with fixed spin).
We will consider units in which $M = 1$, $e = 1$ and $\hbar = 1$. In this case the dynamics of this particle is generated by the self-adjoint operator $H_0 = \frac{1}{2}(p - A)^2$. If we chose the Landau gauge, for which $A = (0, Bx)$, we have the Hamiltonian called *Landau Hamiltonian*[1]

$$H_0 = \tfrac{1}{2}p_x^2 + \tfrac{1}{2}(p_y - Bx)^2\,. \tag{4.4}$$

Pure edge Hamiltonian

Now we would like the particle to move only on a finite part of the cylinder, for this we add two confining soft walls along the cylinder axis whose support is at a distance L. We model this by two twice differentiable, strictly monotonic potentials U_ℓ (ℓ for left) and U_r (r for right) that satisfy

$$c_1|x + \tfrac{L}{2}|^{m_1} \le U_\ell(x) \le c_2|x + \tfrac{L}{2}|^{m_2} \qquad \text{for } x \le -\tfrac{L}{2} \tag{4.5}$$

$$c_1|x - \tfrac{L}{2}|^{m_1} \le U_r(x) \le c_2|x - \tfrac{L}{2}|^{m_2} \qquad \text{for } x \ge \tfrac{L}{2} \tag{4.6}$$

for some constants $0 < c_1 < c_2$, $2 \le m_1 < m_2 < \infty$ and $U_\ell(x) = 0$ for $x \ge -\frac{L}{2}$, $U_r(x) = 0$ for $x \le \frac{L}{2}$. We could allow steeper confinements (for example subexponential) but the present polynomial conditions turn out to be technically convenient.

The Landau Hamiltonian with one of the two edge potentials added

$$H^\alpha = H_0 + U_\alpha \qquad (\alpha = \ell, r) \tag{4.7}$$

is called *edge Hamiltonian* or *pure edge Hamiltonian*[2]. Remark that these Hamiltonians correspond to those studied in the previous chapter where the impurity potential is removed and where the geometry is that of a cylinder.

[1] All the Hamiltonians defined in this section are defined in the Hilbert space $L^2\left(\mathbb{R} \times \left[-\frac{L}{2}, \frac{L}{2}\right], \mathrm{d}x\,\mathrm{d}y\right)$ with periodic boundary conditions along y.

[2] These Hamiltonians are denoted by H_α in [FM02] and by H_α^0 in [FM03a].

Bulk Hamiltonian

We finally turn to the description of the disorder. As we mentioned in the Section 2.2, the disorder is modelled as a sum of local perturbations V located at the sites of a regular lattice Λ, but where these local perturbations have random coupling constants $X_{n,m}$. Thus the random potential V_ω have the form

$$V_\omega(x,y) = \sum_{(n,m)\in\Lambda} X_{n,m}(\omega) V(x-n, y-m) \qquad \omega \in \Omega\ . \tag{4.8}$$

For our purpose the local perturbations satisfy $V \in C^2$, $0 \le V(x,y) \le V_0 < \infty$, $\operatorname{supp} V \subset \mathbb{B}(\mathbf{0}, \frac{1}{4})$ (the open ball of radius $\frac{1}{4}$ centred in $(0,0)$). The lattice is $\Lambda = \mathbb{Z}^2 \cap \big[X \times \big[-\frac{L}{2}, \frac{L}{2}\big]\big]$, where the set $X \subset \mathbb{R}$ defines the support of the random potential along the $x-$direction and will be defined later. $\Omega = [-1,1]^\Lambda$ is the probability space for the model (the set of all possible realizations) on which are defined the random variables $X_{n,m}$ (coupling constants). These random variables are supposed independent and identically distributed (i.i.d.) with bounded probability density $h \in C^2([-1,1])$. We will denote by $\mathbb{P}_\Lambda$ the probability measure (product measure) defined on $\Omega = [-1,1]^\Lambda$. Clearly, for all $\omega \in \Omega$ we have $\|V_\omega\| \le V_0$ and we will assume that $V_0 \ll B$. This choice of ratio between the strength of the random potential V_0 and the magnetic field B corresponds to work in a strong magnetic field regime or, equivalently, in a weak disorder regime.

The Landau Hamiltonian with the random potential added

$$H^b_\omega = H_0 + V_\omega \tag{4.9}$$

is called *bulk Hamiltonian*[3].

Random edge Hamiltonian

There are two other "auxiliary" Hamiltonians that we need to consider. The Landau Hamiltonian with one of the two boundaries and a strip (denoted by Λ_ℓ resp. Λ_r) of random potential along the edge

$$H^\alpha_\omega = H_0 + U_\alpha + V_\omega|_{\Lambda_\alpha} \tag{4.10}$$

is called *random edge Hamiltonian*[4].

Full Hamiltonian

Finally we have the family of random Schrödinger operators

$$H_\omega = \tfrac{1}{2}p_x^2 + \tfrac{1}{2}(p_y - Bx)^2 + U_\ell + U_r + V_\omega \quad , \quad \omega \in \Omega \tag{4.11}$$

that are densely defined self-adjoint operators acting in the Hilbert space $\mathcal{H}$ defined above. H_ω describe the dynamics of a particle lying on a confined cylinder with magnetic field and disorder. In Figure 4.1 we sketch the potentials along the $x-$axis.

[3] It is denoted by H_b in [FM03a] and [FM02].
[4] It is denote by H_α in [FM03a].

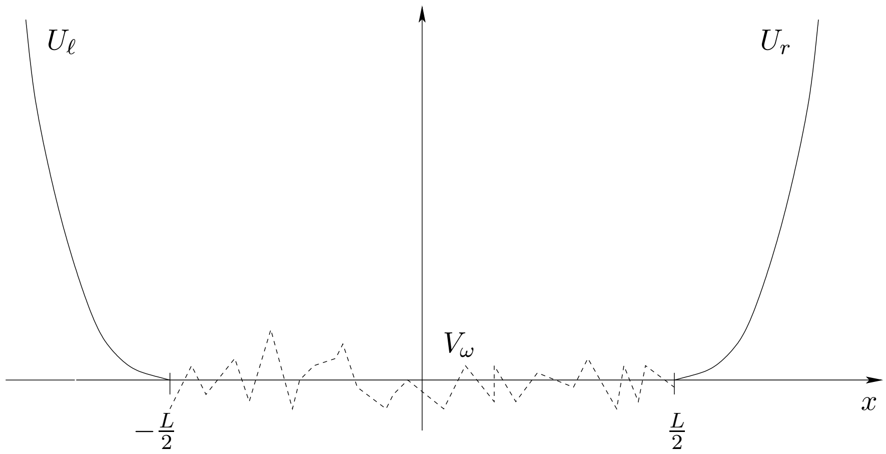

Figure 4.1: The potentials along the $x-$axis.

Remark that with two boundaries and random potential we have a model that is "topologically" equivalent to Halperin's system. Indeed the Corbino disk geometry can be easily mapped onto the cylindrical one. In Figure 4.2 we sketch the geometry of the system.

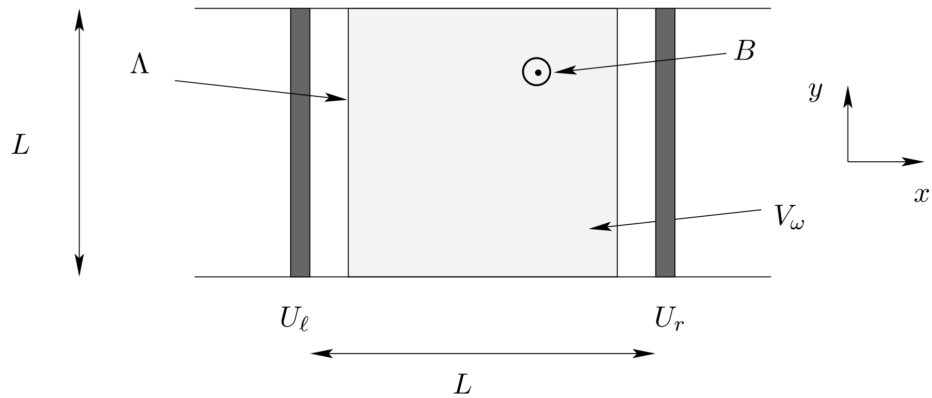

Figure 4.2: The geometry and a schematic representation of the potentials (edges and random). The upper and the lower boundaries of the strip have to be identified.

We now introduce the notion of current carrying states, that play an important role in our work.
Since our system is confined the spectrum is made of discrete eigenvalues. We introduce a natural classification of the eigenvalues of H_ω via the quantum mechanical current along the periodic direction. If ψ satisfies the eigenvalue equation $H_\omega\psi = E\psi$ the current is

defined (here) as[5]

$$J_E \equiv (\psi, v_y \psi) \tag{4.12}$$

where $v_y = (p_y - Bx)$ is the velocity operator in the $y-$direction (recall that we choose the mass $M = 1$ and the electric charge $e = 1$). Thanks to J_E we can classify the eigenvalues in two classes. The first consists on those which have $|J_E| > C$ with C a positive constant uniform in L, these states are called *current carrying states*. The second class consists on the states for which $|J_E| < \epsilon(L)$ with $\epsilon(L) \to 0$ as $L \to \infty$ (we stress that here L is finite but macroscopic, the limit means that $\epsilon(L)$ is infinitesimally small with L). The current carrying states are in this context also called extended states while the others are also called localized states.

4.2 Spectral properties of H_0, H^α, H^b_ω and H^α_ω

Landau Hamiltonian

It it well known that the spectrum of the *Landau Hamiltonian* H_0 is given by the Landau levels, that are infinitely degenerate

$$\sigma(H_0) = \left\{ (n + \tfrac{1}{2})B \,:\, n \in \mathbb{N} \right\} . \tag{4.13}$$

Pure edge Hamiltonian

Since the edge Hamiltonians $H^\alpha = H_0 + U_\alpha$ commute with p_y, they are unitarily equivalent to a direct sum

$$H^\alpha \simeq \sum_{k \in \frac{2\pi}{L}\mathbb{Z}}^{\oplus} H^\alpha(k) = \sum_{k \in \frac{2\pi}{L}\mathbb{Z}}^{\oplus} \left[\tfrac{1}{2} p_x^2 + \tfrac{1}{2}(k - Bx)^2 + U_\alpha \right] . \tag{4.14}$$

For each k the one dimensional Hamiltonian $H^\alpha(k)$ has a compact resolvent, thus it has discrete eigenvalues and by standard arguments one can show that they are not degenerate ([Fer99], Theorem 2.1). The corresponding eigenfunctions are denoted φ^α_{nk}. If the $y-$direction were infinitely extended, k would vary over the real axis and the eigenvalues of $H^\alpha(k)$ would form analytic spectral branches $\varepsilon^\alpha_n(\hat{k})$, $\hat{k} \in \mathbb{R}$ [RS78, Thm. XII.8], labelled by the Landau level index n.

These spectral branches are strictly monotonic. Indeed, for each $\hat{k} \in \mathbb{R}$ and each $n \in \mathbb{N}$ we have, by the Helman-Feynman theorem,

$$\partial_{\hat{k}} \varepsilon^\alpha_n(\hat{k}) = \left(\varphi^\alpha_{n\hat{k}}, (\hat{k} - Bx) \varphi^\alpha_{n\hat{k}} \right) = \tfrac{1}{B} \left(\varphi^\alpha_{n\hat{k}}, \partial_x U_\alpha \varphi^\alpha_{n\hat{k}} \right) . \tag{4.15}$$

This quantity is strictly positive for $\alpha = r$ and strictly negative for $\alpha = \ell$. Moreover we have the properties $\varepsilon^\ell_n(-\infty) = +\infty$, $\varepsilon^\ell_n(+\infty) = (n + \frac{1}{2})B$ and $\varepsilon^r_n(-\infty) = (n + \frac{1}{2})B$,

[5] In principle the physical current (in our units) is $L^{-1}(\psi, v_y \psi)$, but here we will call current the average velocity $(\psi, v_y \psi)$.

$\varepsilon_n^r(+\infty) = +\infty$. This can be seen by applying the unitary transformation $U(\hat{k}) = \exp(-ip_x[-\hat{k}/B])$ to $H^\alpha(\hat{k})$:

$$U(\hat{k})H^\alpha(\hat{k})U(\hat{k})^{-1} = \tfrac{1}{2}p_x^2 + \tfrac{1}{2}Bx^2 + U_\alpha\left(x + \tfrac{\hat{k}}{B}\right) . \tag{4.16}$$

For $\alpha = r$ we remark that for $\hat{k} \to +\infty$ we have $U_r\left(x + \frac{\hat{k}}{B}\right) \to \infty$ and for $\hat{k} \to -\infty$ we get $U_r\left(x + \frac{\hat{k}}{B}\right) = 0$ that leads to the harmonic oscillator, while for $\alpha = \ell$ the situation is similar. Moreover, for the infinite system the spectrum of H^α is absolutely continuous and given by $\sigma(H^\alpha) = \left[\frac{1}{2}B, +\infty\right)$.
Here, because of the periodic boundary conditions, the set of k is discrete so that the spectrum of H^α

$$\sigma(H^\alpha) = \left\{E_{nk}^\alpha; n \in \mathbb{N}, k \in \tfrac{2\pi}{L}\mathbb{Z}\right\} \tag{4.17}$$

consists of isolated points on the spectral branches $E_{nk}^\alpha = \varepsilon_n^\alpha(k)$, $k \in \frac{2\pi}{L}\mathbb{Z}$ with accumulation points at the Landau levels (see Figure 4.3). The corresponding eigenfunctions ψ_{nk}^α have the form

$$\psi_{nk}^\alpha(x, y) = \frac{1}{\sqrt{L}}e^{iky}\varphi_{nk}^\alpha(x) \tag{4.18}$$

where φ_{nk}^α are the normalized eigenfunctions of the one-dimensional Hamiltonian $H^\alpha(k)$.

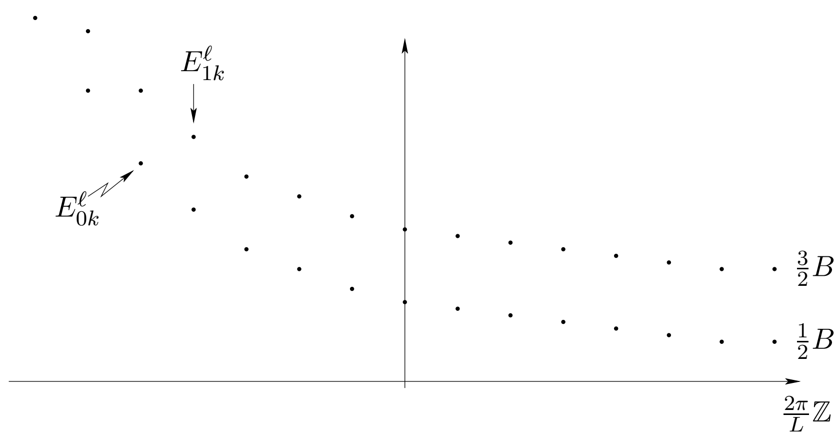

Figure 4.3: The spectrum of H^ℓ lies on monotonic decreasing branches. That of H^r lies on similar, but monotonic increasing, branches.

By definition, the current of the state ψ_{nk}^α in the $y-$direction is given by the expectation value of the velocity $v_y = p_y - Bx$,

$$J_{nk}^\alpha = (\psi_{nk}^\alpha, v_y\psi_{nk}^\alpha) = \partial_{\hat{k}}\varepsilon_n^\alpha(\hat{k})\Big|_{\hat{k}=\frac{2\pi m}{L}} . \tag{4.19}$$

From (4.19) we notice that for any $\varepsilon > 0$, one can find $j(\varepsilon) > 0$ and $L(\varepsilon)$ such that for $L > L(\varepsilon)$ the currents associated to the states of the two branches $n = 0$, $\alpha = \ell, r$ with energies $E^\alpha_{0k} \geq \frac{1}{2}B + \varepsilon$ satisfy

$$J^\ell_{0k} \leq -j(\varepsilon) < 0 \qquad J^r_{0k} \geq j(\varepsilon) > 0 \; . \tag{4.20}$$

In other words the eigenstates of the edge Hamiltonians carry an appreciable current and by our definition are extended. The spacing of two consecutive eigenvalues, on the first spectral branches, greater than $\frac{1}{2}B + \varepsilon$ satisfies

$$\left| E^\alpha_{0\frac{2\pi(m+1)}{L}} - E^\alpha_{0\frac{2\pi m}{L}} \right| > \frac{j(\varepsilon)}{L} \qquad \alpha = \ell, r \; . \tag{4.21}$$

Bulk Hamiltonian

The study of the *bulk Hamiltonian* H^b_ω defined in $\mathbb{R}^2$ is a large subject on Anderson localization theory, there are many works about it. We refer for a short overview to the introduction in [FM02]. Here we only discuss the properties that we need for our purpose. The spectrum of the bulk Hamiltonian $H^b_\omega = H_0 + V_\omega$ is contained in Landau bands around each Landau level

$$\sigma(H^b_\omega) \subset \bigcup_{n \geq 0} \left[\left(n + \tfrac{1}{2}\right) B - V_0, \left(n + \tfrac{1}{2}\right) B + V_0 \right] \tag{4.22}$$

and if, as we have supposed $V_0 \ll B$, there are open spectral gaps

$$G_n \supseteq \left(\left(n + \tfrac{1}{2}\right) B + V_0, \left(n + \tfrac{3}{2}\right) B - V_0 \right) \; , \quad n \in \mathbb{N} \; . \tag{4.23}$$

Random edge Hamiltonian

Finally we describe the *random edge Hamiltonians*

$$H^\alpha_\omega = H_0 + U_\alpha + V^\alpha_\omega \tag{4.24}$$

where $V^\alpha_\omega = V_\omega|_{\Lambda_\alpha}$. The supports of the random potential along the two edges are

$$\Lambda_r = \left\{ (n, m) \in \mathbb{Z}^2 ; n \in [\tfrac{L}{2} - \tfrac{3D}{4} - 1, \tfrac{L}{2}], m \in [-\tfrac{L}{2}, \tfrac{L}{2}] \right\} \tag{4.25}$$

$$\Lambda_\ell = \left\{ (n, m) \in \mathbb{Z}^2 ; n \in [-\tfrac{L}{2}, -\tfrac{L}{2} + \tfrac{3D}{4} + 1], m \in [-\tfrac{L}{2}, \tfrac{L}{2}] \right\} \; . \tag{4.26}$$

The choice of D has only the restriction $D \geq c \log L$ ($c > 0$). In [FM03a] we choose $D = \sqrt{L}$ but, as we will see in the complement of this article at the end of Chapter 5, one can also take $D = c \log L$ and get essentially the same results.

Since the perturbation has compact support and the essential spectrum of H^α is given by the Landau levels, the spectrum of H^α_ω is discrete with the Landau levels as only accumulation points. We denote it by $\sigma(H^\alpha_\omega) = \{E^\alpha_\kappa : \kappa \in \mathcal{I}\}$, $\mathcal{I}$ being the appropriate index set. One can prove [Mac03b] that, for each $\omega \in \Omega_{\Lambda_\alpha} = [-1, 1]^{\Lambda_\alpha}$ (the restriction of the configurations ω to the sublattice Λ_α) and for each κ such that E^α_κ lies in a suitable interval Δ[6] of the spectral gap of H_ω^b, the distance between two consecutive eigenvalues satisfies, for L large enough and V_0 small enough,

$$\left|E_{\kappa+1}^\alpha - E_\kappa^\alpha\right| \geq \frac{C}{L} \qquad \alpha = \ell, r \tag{4.27}$$

where $C > 0$ is uniform in κ, ω. Moreover for each $E_\kappa^\ell \in \Delta$ (resp. $E_\kappa^r \in \Delta$) the average velocity associated to the corresponding eigenfunctions is strictly negative (resp. positive) uniformly in L

$$|J_\kappa^\alpha| \geq C' > 0 \qquad \alpha = \ell, r \tag{4.28}$$

with $C' = \mathcal{O}(\sqrt{B})\left[1 - \mathcal{O}\left(\frac{V_0}{B}; \frac{V_0^2}{B^2}\right)\right]$.

4.3 Overview of the results

Before giving the precise statement of the two theorems we have to define the set X, that defines the support of the random potential along the $x-$direction. Indeed, the choice of X depend on the energy interval where the spectral analysis is done, see Figure 4.4.

We have two different definitions of X. Our first result [FM03a] (Chapter 5) concerns the study of $\sigma(H_\omega)$ in the energy interval ($\varepsilon > 0$)

$$\Delta = (B - \delta, B + \delta) \subset \left(\tfrac{1}{2}B + V_0 + \varepsilon, \tfrac{3}{2}B - V_0 - \varepsilon\right) \ . \tag{4.29}$$

Δ lies inside the first spectral gap of the bulk Hamiltonian defined in the infinite plane $\mathbb{R}^2$, see Figure 4.4. In this case the random potential is supposed to fill the whole space in between the confining walls, and $X = \left[-\frac{L}{2}, \frac{L}{2}\right]$.
The second result [FM02] (Chapter 6) is about $\sigma(H_\omega)$ in the energy interval

$$\Delta_\varepsilon = \left[\tfrac{1}{2}B + \varepsilon, \tfrac{1}{2}B + V_0\right] \quad , \quad \varepsilon > 0 \ . \tag{4.30}$$

Δ_ε lies inside the first Landau band of the bulk Hamiltonian defined in $\mathbb{R}^2$, see Figure 4.4. In this case the interval X is $\left[-\frac{L}{2} + \log L, \frac{L}{2} - \log L\right]$: we leave a thin strip of size $\log L$ without random potential along each confining wall.

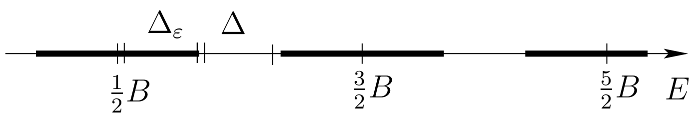

Figure 4.4: The energy axis with the first three Landau levels and the corresponding Landau bands (thick line) and spectral gaps, associated to H_ω^b. The intervals Δ lies in the first spectral gap, while Δ_ε lies in the first Landau band.

[6] See next Section for a precise definition of Δ.

We now turn to the Hypothesis under which we have proved the theorems below. The first concerns the spectrum of the pure edge Hamiltonians H^ℓ and H^r, while the second one deals with the eigenfunctions associated to the bulk Hamiltoninan H^b_ω.

We have seen that the spacing between the consecutive eigenvalues of H^α (for a fixed $\alpha = \ell$ or r) in Δ_ε is of order $\frac{1}{L}$. However the spacing between the energies of $\sigma(H^\ell)$ and $\sigma(H^r)$ is a priori arbitrary. We will assume that the confining potentials U_ℓ and U_r are such that the following hypothesis[7], is fulfilled.

Hypothesis 4.1. *Fix any $\varepsilon > 0$. There exist $L(\varepsilon)$ and $d(\varepsilon) > 0$ such that for all $L > L(\varepsilon)$*

$$\operatorname{dist}\left(\sigma(H^\ell) \cap \Delta_\varepsilon, \sigma(H^r) \cap \Delta_\varepsilon\right) \geq \frac{d(\varepsilon)}{L} \tag{4.31}$$

and, there exist L_0 and $d_0 > 0$ such that for all $L > L_0$

$$\operatorname{dist}\left(\sigma(H^\ell) \cap \Delta, \sigma(H^r) \cap \Delta\right) \geq \frac{d_0}{L}\ . \tag{4.32}$$

This hypothesis is important because a minimal amount of non-degeneracy between the spectra of the two edge systems is needed in order to control backscattering effects induced by the random potential. Indeed in a system with two boundaries backscattering favors localization and has a tendency to destroy currents. This hypothesis can easily be realized by taking symmetric confining potentials U_ℓ and U_r and adding a flux tube of suitable intensity Φ (see also [FM03a] for a detailed discussion).

We now switch to the hypothesis concerning H^b_ω. From the theory of localization we expect that the eigenfunctions of H^b_ω with energy not too close to the Landau levels are exponentially localized on a scale of order one with respect to L. For our purpose we will assume the following hypothesis[8], that is a weaker version of the above statement.

Hypothesis 4.2. *Fix any $\varepsilon > 0$. Then there exist $\mu(\varepsilon)$ a strictly positive constant and $L(\varepsilon)$ such that for all $L > L(\varepsilon)$ one can find a set of realizations of the random potential Ω'_Λ with $\mathbb{P}_\Lambda(\Omega'_\Lambda) \geq 1 - L^{-\theta}$, $\theta > 0$, with the property that if $\omega \in \Omega'_\Lambda$ the eigenfunctions corresponding to $E^b_\beta \in \sigma(H^b_\omega) \cap \Delta_\varepsilon$ satisfy*

$$|\psi^b_\beta(x, \bar{y}_\beta)| \leq e^{-\mu(\varepsilon)L} \qquad , \qquad |\partial_y \psi^b_\beta(x, \bar{y}_\beta)| \leq e^{-\mu(\varepsilon)L} \qquad \forall\, x \in \mathbb{R} \tag{4.33}$$

for some $\bar{y}_\beta$ depending on ω and L.

The main physical consequence of this hypothesis is that a state satisfying (4.33) does not carry any appreciable current in the sense that

$$J^b_\beta = (\psi^b_\beta, v_y \psi^b_\beta) = \mathcal{O}(e^{-\mu(\varepsilon)L})\ . \tag{4.34}$$

[7] This Hypothesis is called $(H1)$ in [FM03a] and in [FM02]

[8] This Hypothesis is called $(H2)$ in [FM02]

Following our definition these states are localized.

We are now ready to state the two main results of this first part of the thesis, they are the contents of the two articles [FM03a] and [FM02].

Theorem 4.1. [FM03a] *Let V_0 small enough with respect to B, fix $\varepsilon > 0$ and let $0 < \delta < B - V_0 - \varepsilon$. Suppose that $(H1)$ holds. Then there exists $\mu > 0$, $\bar{L} \geq L(\varepsilon)$ such that if $L > \bar{L}$ one can find a set $\hat{\Omega} \subset \Omega$ of realizations of the random potential V_ω with $\mathbb{P}_\Lambda(\hat{\Omega}) \geq 1 - L^{-\nu}$ $(\nu \gg 1)$ such that for all $\omega \in \hat{\Omega}$ the spectrum of H_ω in $\Delta = (B - \delta, B + \delta)$ is the union of two sets Σ'_ℓ and Σ'_r, each depending on ω and L, with the following properties:*

a) $\mathcal{E}^\alpha_\kappa \in \Sigma'_\alpha$ $(\alpha = \ell, r)$ *are a small perturbation of* $E^\alpha_\kappa \in \sigma(H^\alpha_\omega) \cap \Delta$ *with*

$$|\mathcal{E}^\alpha_\kappa - E^\alpha_\kappa| \leq e^{-\mu\sqrt{B}\sqrt{L}} \ . \tag{4.35}$$

b) For $\mathcal{E}^\alpha_\kappa \in \Sigma'_\alpha$ *the current* $\mathcal{J}^\alpha_\kappa$ *of the associated eigenstate satisfies*

$$|\mathcal{J}^\alpha_\kappa - J^\alpha_\kappa| \leq e^{-\mu\sqrt{B}\sqrt{L}} \ . \tag{4.36}$$

Theorem 4.2. [FM02] *Fix $\varepsilon > 0$ and assume that $(H1)$ and $(H2)$ are fulfilled. Assume $B > 4V_0$. Then there exists a numerical constant $\gamma > 0$ and an $\bar{L} \geq L(\varepsilon)$ such that for all $L > \bar{L}$ one can find a set $\hat{\Omega}$ of realizations of the random potential with $\mathbb{P}_\Lambda(\hat{\Omega}) \geq 1 - L^{-s}$ $(s \gg 1)$ such that for any $\omega \in \hat{\Omega}$, $\sigma(H_\omega) \cap \Delta_\varepsilon$ is the union of three sets $\Sigma_\ell \cup \Sigma_b \cup \Sigma_r$, each depending on ω and L, and characterized by the following properties:*

a) $\mathcal{E}^\alpha_k \in \Sigma_\alpha$ $(\alpha = \ell, r)$ *are a small perturbation of* $E^\alpha_{0k} \in \sigma(H^\alpha) \cap \Delta_\varepsilon$ *with*

$$|\mathcal{E}^\alpha_k - E^\alpha_{0k}| \leq e^{-\gamma B (\log L)^2}, \qquad \alpha = \ell, r \ . \tag{4.37}$$

b) For $\mathcal{E}^\alpha_k \in \Sigma_\alpha$ *the current* $\mathcal{J}^\alpha_k$ *of the associated eigenstate satisfies*

$$|\mathcal{J}^\alpha_k - J^\alpha_{0k}| \leq e^{-\gamma B (\log L)^2}, \qquad \alpha = \ell, r \ . \tag{4.38}$$

c) Σ_b *contains the same number of energy levels as* $\sigma(H^b_\omega) \cap \Delta_\varepsilon$ *and* $(p \gg 1)$

$$\operatorname{dist}(\Sigma_b, \Sigma_\alpha) \geq L^{-p}, \qquad \alpha = \ell, r \ . \tag{4.39}$$

d) The current associated to each level $\mathcal{E}_\beta \in \Sigma_b$ *satisfies*

$$|\mathcal{J}_\beta| \leq e^{-\gamma B (\log L)^2} \ . \tag{4.40}$$

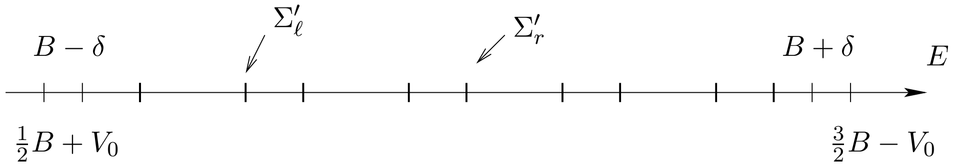

Figure 4.5: The spectral interval Δ with represented schematically the spectrum of H_ω. In this case the spectrum consists in the union of two sets Σ'_ℓ, Σ'_r of current carrying states.

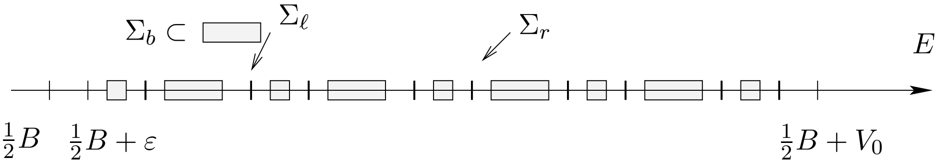

Figure 4.6: The spectral interval Δ_ε with represented schematically the spectrum of H_ω. It consists on the two sets Σ_ℓ, Σ_r of current carrying states, and the set Σ_b, intermixed in between the points of $\Sigma_\ell \cup \Sigma_r$, corresponding to states carrying an infinitesimal current.

In Figure 4.5 and Figure 4.6 we report two schematic representations of the spectrum of H_ω in the spectral interval Δ, that correspond to Theorem 4.1, and in the spectral interval Δ_ε, that correspond to Theorem 4.2.

The idea of the proofs of Theorems 4.1 and 4.2 is to link the resolvent of the full Hamiltonian H_ω to those of the easier Hamiltonians H^ℓ (resp. H^ℓ_ω), H^r (resp. H^r_ω) and H^b_ω. This is achieved via a *geometric resolvent equation* formula. Using it we can do deterministic estimates on the norm difference between the projector $P_{H_\omega}(\Gamma)$, associated to H_ω into the disc with boundary Γ, and the projector associated to one of the easier Hamiltonians. This is done for suitable circles Γ in the complex plane and a suitable set $\hat{\Omega}$ of realizations of the random potential. Using *Wegner estimates* on H^b_ω (resp. H^α_ω) we control the probability of $\hat{\Omega}$ and show that it can be made large.

Our classification of the spectrum via the quantum mechanical current leads to a well defined notion of *extended edge states* and *localized bulk states*. The former are those belonging to Σ_α (resp. Σ'_α), they are small perturbations of the eigenvalues of $\sigma(H^\alpha)$ (resp. $\sigma(H^\alpha_\omega)$) and have a quantum mechanical current of order $\mathcal{O}(1)$ with respect to L. The latter are those belonging to Σ_b, and have a infinitesimal current with respect to L (of order $\mathcal{O}\left(e^{-\gamma B(\log L)^2}\right)$), they "arise" from the spectrum of H^b_ω. It is interesting to note that in the interval inside the first Landau band, our description leads to a spectrum in which extended edge and localized bulk states are intermixed: in some sense there is no "mobility edge". On the other hand in the interval inside the spectral gap there exists only extended edge states.

4.4 Physical contents of the two Theorems

Here we want to briefly discuss the physical interest related to Theorem 4.1, in its version found in Section 5.D (Theorem 5.2), and Theorem 4.2.

The model studied here has a direct relevance for the physics of the quantum Hall effect. Indeed, as mentioned in Chapter 2, to describe the physics of the quantum Hall effect we have to understand the dynamics of an effective two dimensional electron gas. Our special choice of the geometry provides a description of the dynamics of an electron moving in a two dimensional system of size $L \times L$. Since the parameter L, as already remarked, has to be chosen macroscopic, this could simulate a real sample. Therefore, the study of the spectral properties of the Hamiltonian H_ω in connection to the quantum mechanical current is of great interest from a physical point of view. Indeed, the knowledge of the quantum mechanical currents associated to the eigenstates has a direct relevance for the Hall conductivity σ_H of the many non interacting electrons system. We look how we can get the Hall conductivity from the results of Theorem 4.2.

In the formulation advocated by Halperin [Hal82] the Hall conductivity is computed as the ratio of the net equilibrium current I and the difference of chemical potentials between the two edges $\Delta\mu = \mu_r - \mu_\ell$ (μ_α being the chemical potential on the edge $\alpha = \ell, r$)

$$\sigma_H = \frac{I}{\Delta\mu} \,. \tag{4.41}$$

Consider the many fermion state $\Psi(\mu_\ell, \mu_r, E_F)$ obtained by filling the energy levels of H_ω (one particle per state) in $\Sigma_\ell \cap \left[\frac{B}{2} + \varepsilon, \mu_\ell\right]$, $\Sigma_r \cap \left[\frac{B}{2} + \varepsilon, \mu_r\right]$ and $\Sigma_b \cap \left[\frac{B}{2} + \varepsilon, E_F\right]$ with $\frac{B}{2} + \varepsilon < \mu_\ell < E_F < \mu_r < \frac{B}{2} + V_0$. The total current $I(\mu_\ell, \mu_r, E_F)$ of this state – a stationary state of the many particle Hamiltonian – is given by the sum of the individual physical currents of the filled levels (given by $L^{-1}(\psi, v_y\psi)$). From the estimates (4.38) and (4.40) in Theorem 4.2

$$\sum_k J_k^\ell + \sum_k J_k^r + \sum_\beta J_\beta = \sum_k J_{0k}^\ell + \sum_k J_{0k}^r + \mathcal{O}(e^{-(\log L)^2} L^2) \tag{4.42}$$

and from (4.19) we get

$$\frac{1}{L} \sum_k J_{0k}^r = \frac{1}{2\pi} \int_{\frac{B}{2}+\varepsilon}^{\mu_r} \mathrm{d}E + \mathcal{O}(L^{-1}) \,, \tag{4.43}$$

$$\frac{1}{L} \sum_k J_{0k}^\ell = \frac{1}{2\pi} \int_{\mu_\ell}^{\frac{B}{2}+\varepsilon} \mathrm{d}E + \mathcal{O}(L^{-1}) \,. \tag{4.44}$$

It follows that up to finite size effects

$$I(\mu_\ell, \mu_r, E_F) \simeq \frac{1}{2\pi}(\mu_r - \mu_\ell) \,. \tag{4.45}$$

Finally form (4.41) we get, up to finite size effects, the quantization of the Hall conductivity, namely, restoring all the physical constants,

$$\sigma_H \simeq \frac{e^2}{h} \; , \tag{4.46}$$

e being the electron charge and h the Plank constant.

Let us comment this result. In (4.46) the Hall conductance is equal to e^2/h, this is because we have considered only the first band. It is interesting to note that, when μ_ℓ and μ_r vary, the density of particles in the state $\Psi(\mu_\ell, \mu_r, E_F)$ does not change since the number of levels in Σ_α $(\alpha = \ell, r)$ is of order $\mathcal{O}(L)$, see (4.21). However if E_F is increased the particle density increases since the number of levels in Σ_b is of order $\mathcal{O}(L^2)$. Recall that the filling factor is given by $\nu = \frac{nh}{eB}$, n the electron density. Thus, we see that as the Fermi energy increases ν increases but the Hall conductance does not change and hence has a plateau (we add only localized states). In other words the edge states contribute to the Hall conductance but not to the density of states of the sample in the thermodynamic limit.

We now briefly look at the physical interest of Theorem 4.1, in its version given in Theorem 5.2. In the context of this results the geometry is slightly modified, we consider the same model as above but where the square box $L \times L$ is replaced with a rectangular box $D \times L$. The goal is to explore which condition has to satisfy the width D of the sample in relation to its length L, this in view to have current carrying states for energies in the first spectral gap. From Theorem 5.2 we have that current carrying states can exist if D is a function of L that fulfill the condition

$$D(L) \geq c \log L \; , \tag{4.47}$$

for a suitable constant $c > 0$. We see that current carrying states do not exist for all rectangular samples, but only for box $D(L) \times L$ with $D(L)$ satisfying the geometrical condition (4.47). In particular for a disordered infinitely long strip of fixed width D_0 current carrying states do not exists. This seems realistic from a physical point of view, because of the tunnelling induced between the two edges by the disorder present in the sample.

4.5 Technical tools

In this section we look at the technical tools that we use in the proofs of Theorems 4.1 and 4.2. First we show how to relate the full resolvent to the resolvent of simpler Hamiltonians, then we present the concept of Wegner estimate.

We present here the ideas of the *Geometric Resolvent Equations* (GRE) in the general context where the Hilbert space is $L^2(\mathbb{R}^d)$. Remark that there are many version of GRE, here we present the form which will be of interest for us, see also [HS96]. The resolvent geometric equations provide a powerful tool for comparing the resolvents of operators that are the same when acting on functions localized to certain region of $\mathbb{R}^d$, but differ in others regions where the resolvents can be controlled. The main idea of the geometric perturbation theory is to estimate $H = H_0 + V$ by simpler Hamiltonians $H_i = H_0 + V_i$ ($i = 1, \ldots, N$) with V_i differing from V in "suitable" regions of $\mathbb{R}^d$. Typically, the local potentials V_i are obtained as follows. We introduce two set of functions: first a partition of the unity for $\mathbb{R}^d$, $\{\tilde{J}_i\}_{i=1}^N$ with $\sum_{i=1}^N \tilde{J}_i = 1$, and then a set of bounded, positive and $C^\infty(\mathbb{R}^d)$ functions $\{J_i\}_{i=1}^N$ such that $J_i\tilde{J}_i = \tilde{J}_i$. Then the operators $\{H_i\}_{i=1}^N$ are Schrödinger operators on $L^2(\mathbb{R}^d)$ with potentials V_i having the property that

$$V J_i = V_i J_i \qquad i = 1, \ldots, N \; . \tag{4.48}$$

Each H_i is simple in the sense that the associated resolvent $R_i(z) = (z - H_i)^{-1}$ can be analyzed. We introduce the following first order differential operators

$$W_i = [H_0, J_i] \tag{4.49}$$

the most general expression for H_0 is $H_0 = \sum_{i=1}^d \frac{1}{2}(p_i - A_i)^2$, A_i representing the vector potential associated to a magnetic field B (clearly if $B \neq 0$, $d \geq 2$). We relate $R_i(z)$ to $R(z) = (z - H)^{-1}$ by the GRE.

Proposition 4.1. *Let H and $\{H_i\}_{i=1}^N$ be constructed as above using the two set of functions $\{J_i\}_{i=1}^N$ and $\{\tilde{J}_i\}_{i=1}^N$. Then for all z in the resolvent sets of H and of each H_i,*

$$R(z) = \left(\sum_{i=1}^N J_i R_i(z) \tilde{J}_i \right) (1 - \mathcal{K}(z))^{-1} \tag{4.50}$$

where

$$\mathcal{K}(z) = \sum_{i=1}^N K_i(z) = \sum_{i=1}^N W_i R_i(z) \tilde{J}_i \; . \tag{4.51}$$

Proof. For $i = 1, \ldots, N$ we have $HJ_i = H_i J_i$ thus

$$(z - H) \sum_{i=1}^N J_i R_i(z) \tilde{J}_i = \sum_{i=1}^N (z - H_i) J_i R_i(z) \tilde{J}_i = 1 - \mathcal{K}(z) \; . \tag{4.52}$$

To obtain the second equality one commutes $(z - H_i)$ and J_i: $(z - H_i)J_i = J_i(z - H_i) + [(z - H_i), J_i]$, then uses the identity $\sum_{i=1}^N J_i \tilde{J}_i = \sum_{i=1}^N \tilde{J}_i = 1$. From (4.52) we deduce the decoupling formula. ∎

The main work to do is to give an estimate of the operator norm of $\mathcal{K}(z)$. In particular to prove that $\|\mathcal{K}(z)\| < 1$, which permits to invert $1 - \mathcal{K}(z)$.
In our context the GRE is used to "decouple" the full Hamiltonian in three parts: the left (random) edge Hamiltonian, the bulk Hamiltonian and the right (random) edge Hamiltonian, see Figure 4.7.

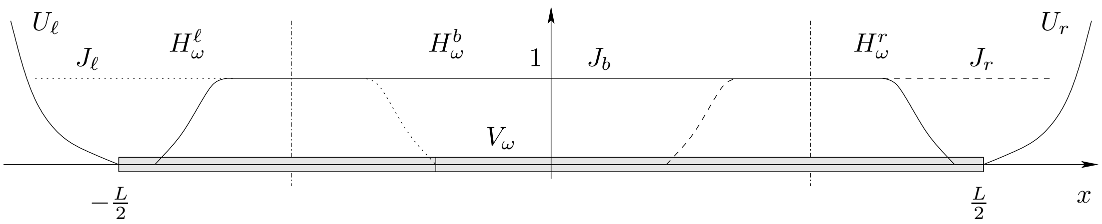

Figure 4.7: A schematic representation of the partition of the configuration space along the $x-$axis. The left and right parts correspond to an edge systems, while the central part to a bulk system.

We now switch to the *Wegner estimate*, we introduce it in the general context of the localization theory. Consider the family of random self-adjoint Schrödinger operators $\{H_\omega, \omega \in \Omega\}$, $(\Omega, \mathcal{F}, \mathbb{P})$ being the probability space for the model. Typically, $\Omega = [a, b]^{\mathbb{Z}^d}$, $\mathcal{F}$ the $\sigma-$algebra defined on Ω generated by the cylinders

$$\{\omega \in \Omega \, : \, \omega_{n_1} \in \mathcal{A}_1, \ldots, \omega_{n_\ell} \in \mathcal{A}_\ell\} \tag{4.53}$$

with $n_i \in \mathbb{Z}^d$ and $\mathcal{A}_i$ a Borel set in $[a, b]$. For an Anderson-type random potential (see (4.8) for $d = 2$) where the random variables X_n, defined as $X_n(\omega) = \omega_n$, are i.i.d. the probability measure is simply the product measure $\mathbb{P} = \prod_{n \in \mathbb{Z}^d} P_0$ with $P_0(\mathcal{A}) = \mathbb{P}(X_n(\omega) \in \mathcal{A})$ for all Borel set $\mathcal{A} \subset [a, b]$, $n \in \mathbb{Z}^d$. Finally there exists a group $\{T_n \, : \, n \in \mathbb{Z}^d\}$ of ergodic transformations of Ω, explicitly $(T_m\omega)_n = \omega_{n+m}$. We remark that the ergodicity of the family $\{H_\omega\}$ implies that the spectrum of this family of random Schrödinger operators is almost surely non random.

One of the goals of the localization theory [CHN01] is to prove that the deterministic spectrum of the family $\{H_\omega\}$ is pure point almost surely in certain energy intervals. The proof of localization for random Hamiltonians acting in $L^2(\mathbb{R}^d)$ is based on the analysis of finite-volume perturbations $H_\Lambda = H_0 + V_\Lambda$, for a bounded region $\Lambda \subset \mathbb{R}^d$, of a self-adjoint background operator H_0 (usually H_0 represent the kinetic energy plus some background potential). Two estimates on H_Λ are needed. First a decay estimate on the Green's function of H_Λ at far separated points holding with a probability converging to one as $\Lambda \to \mathbb{R}^d$. The second estimate is a probabilistic estimate on the location of the eigenvalues of H_Λ, called *Wegner estimate*. A Wegner estimate is an upper bound on the probability that the spectrum of the local Hamiltonian H_Λ lies within a $\delta-$neighborhood of a given (non random) energy E. A good Wegner estimate is one for which the upper bound depends linearly on the volume $|\Lambda|$ and vanishes as the size of the energy neighborhood δ shrinks to zero, for example linearly in δ. That is

$$\mathbb{P}_\Lambda \{\text{dist}(\sigma(H_\Lambda), E) \leq \delta\} \leq C(E)\delta|\Lambda| \tag{4.54}$$

where $\mathbb{P}_\Lambda$ is the probability measure restricted to the variables in Λ.
In our context the Wegner estimate is used to "localize" with high probability the spectrum of the bulk Hamiltonian (resp. random edge Hamiltonian).

In the next two chapters we report the articles [FM03a] (Chapter 5) and [FM02] (Chapter 6) without the references that are included in the bibliography of this thesis. At the end of the article [FM03a] we add a short paragraph that deals with the question, briefly discussed in Section 4.4, of the geometrical condition for the existence of extended states.

Chapter 5

Extended Edge States

In this chapter we report the article [FM03a]: J. Math. Phys. (to appear).

Extended Edge States in Finite Hall Systems

Christian Ferrari and Nicolas Macris

Abstract

We study edge states of a random Schrödinger operator for an electron submitted to a magnetic field in a finite macroscopic two dimensional system of linear dimensions equal to L. The $y-$direction is L-periodic and in the $x-$direction the electron is confined by two smoothly increasing parallel boundary potentials. We prove that, with large probability, for an energy range in the first spectral gap of the bulk Hamiltonian, the spectrum of the full Hamiltonian consists only on two sets of eigenenergies whose eigenfuntions have average velocities which are strictly positive/negative, uniformly with respect to the size of the system. Our result gives a well defined meaning to the notion of edge states for a finite cylinder with two boundaries, and extends previous studies on systems with only one boundary.

5.1 Introduction

In this paper we investigate spectral properties of random Hamiltonians describing the dynamics of a spinless quantum particle on a cylinder of circumference L and confined along the cylinder axis by two boundaries separated by the distance L. The particle is subject to an external homogeneous magnetic field and a weak random potential. A precise statement of the model is given in Section 5.2. The physical interest of the model comes from the integral quantum Hall effect occurring in disordered two dimensional electronic systems subject to a uniform magnetic field, for example, in the interface of an heterojunction [vKDP80], [PG87]. In his treatment of this effect Halperin [Hal82] pointed out the fundamental role played by edge states carrying boundary diamagnetic currents, and it is therefore important to understand the spectral properties of finite but macroscopic quantum Hall samples with boundaries. A short review of the spectral properties of finite quantum Hall systems can be found in [FM03a].

Random Landau Hamiltonians on an infinite plane have been analyzed in the last decade [DMP95], [DMP96], [CH96],[BCH97], [Wan97], [DMP97], [DMP99] and [GK02].

The study of random magnetic Hamiltonians with boundaries is more recent and, before we address the case of a (finite) cylinder, we wish to briefly discuss a few existing results. The case of a semi-infinite plane with one planar boundary, modelled by a smooth confining potential U or a Dirichlet condition at $x = 0$, is satisfactorily understood. In this case it is proven that the spectrum of the Hamiltonian $H_\omega^e = H_0 + U + V_\omega$, H_0 being the Landau Hamiltonian for a uniform magnetic field B and V_ω an Anderson-type random potential, has absolutely continuous components inside the complement of Landau bands, for $\|V_\omega\|_\infty \ll B$ ([FGW00], [dBP99] and [MMP99]). The proof of this statement is essentially based on Mourre theory with conjugate operator y. The positivity of $i[H_\omega^e, y]$ in suitable spectral subspaces of H_ω^e leads to the absolutely continuous nature of the spectrum. Since this commutator is equal to the velocity v_y this means that states in the corresponding spectral subspaces propagate in the $y-$direction along the edge with positive velocity.

For the case of an infinite strip with two boundaries, separated by a distance L, few results are known. For a general (random) potential we expect that there is no absolutely continuous component in the spectrum, because the impurities may induce a tunnelling (or backscattering) between the two boundaries and thus propagating edge states along each boundary cannot persist for an infinite time. In [CHS02] the authors have shown that such states survive, for a finite time related to the quantum tunnelling time between the two edges. In [EJK01] the authors consider a parabolic channel in the $y-$direction. They show that if the perturbation V is periodic, or if V is small enough and decays fast enough in the $y-$direction, then the absolutely continuous spectrum survives in certain intervals, but their analysis does not cover true Anderson like potentials.

In this work, as in our previous work [FM02], we address the case of a macroscopic finite systems with two confining walls separated by a distance L along the $x-$direction and with the $y-$direction of length L made periodic (i.e. the geometry is that of a cylinder). The *left* (resp. *right*) *walls* are modelled by a smooth confining potential U_ℓ (resp. U_r) separated by a distance L, and the *bulk* between them contains impurities modelled by a random Anderson-like potential V_ω. Although the spectrum consists of discrete isolated eigenvalues, we show that there is a well defined notion of edge states associated to each boundary.

Let us explain our main new result expressed in Theorem 5.1 and compare it with that of [FM02]. We show that, with large probability, the spectrum of the random Hamiltonian

$$H_\omega = H_0 + V_\omega + U_\ell + U_r$$

in an energy interval $\Delta \subset \left(\frac{1}{2}B + \|V_\omega\|_\infty, \frac{3}{2}B - \|V_\omega\|_\infty\right)$ consists in the union of two sets Σ_ℓ and Σ_r, which are small perturbations of the spectra $\sigma(H_0+U_\ell+V_\omega^\ell)$ and $\sigma(H_0+U_r+V_\omega^r)$, of the two edge random Hamiltonians (see Section 5.2 for their precise definition). The eigenvalues in Σ_ℓ and Σ_r are characterized by their average velocity along the periodic direction $J_E = (\psi_E, v_y\psi_E)$: the eigenfunctions corresponding to the eigenvalues in Σ_ℓ (resp. Σ_r) have a uniformly, negative (resp. positive) velocity, with respect to L. These are the so-called edge states and from the constructions in the proofs it is possible to see that the eigenvalues in Σ_ℓ (resp. Σ_r) correspond to eigenfunctions localized in the $x-$ direction near the left (resp. right) boundary. The number of eigenvalues in Σ_ℓ and Σ_r is of order $O(L)$.

We briefly comment about our paper [FM02] where energies inside the Landau bands are considered. We proved that with large probability, for a similar model (where no disorder is present in a thin strip along the boundaries) the spectrum of H_ω in $\Delta_\varepsilon = [\frac{1}{2}B+\varepsilon, \frac{1}{2}B+V_0]$ is given by $\Sigma_\ell \cup \Sigma_b \cup \Sigma_r$. The eigenfunctions corresponding to the eigenvalues in $\Sigma_\ell \cup \Sigma_r$ have strictly positive/negative velocity, and Σ_b is intermixed in between $\Sigma_\ell \cup \Sigma_r$ and the corresponding eigenfunctions have an infinitesimal velocity of order $O\left(e^{-B(\log L)^2}\right)$. The number of eigenvalues in Σ_ℓ and Σ_r is $O(L)$ while that in Σ_b is $O(L^2)$.

Although our analysis is presented for a sample of size $L \times L$ the same results can be straightforwardly extended to all geometries where the two boundaries are separated by any distance D at least $O(\ln L)$ (assuming the length of the periodic direction is fixed to L). For distances $D = O(1)$ our analysis does not hold, a fact which is consistent with [CHS02]. In fact, we expect that by using the results in the present paper one could prove that a wave packet localized on the left boundary and with appropriate energy, will propagate along the left boundary up to a finite tunnelling time and then, backscatter and propagate along the right boundary and so forth. The tunnelling time is set by V_ω and the distance D between the two boundaries. Thus if $D = O(1)$ with respect to L, this tunnelling time is also $O(1)$, and always remains much smaller than $O(L)$ which is the time needed for a ballistic flight around the whole periodic direction y. In [CHS02] the randomness of the potential is not needed. We suspect that this may also be the case in the present problem, but in order to study the non-random situation one should appeal to other arguments not relying on the Wegner estimate.

The paper is organized as follows. In Section 5.2 we present the precise definition of the model and state the main theorem. Section 5.3 is concerned with the main mathematical tools used in our analysis: a Wegner estimate and a decoupling scheme of the cylinder into two semi-infinite ones. The proof of the main theorem is then completed in Section 5.4. Some useful estimates and more technical material are collected in the appendices.

5.2 The Model and Main Result

We study the spectral properties of the family of random Hamiltonians

$$H_\omega = H_0 + U_\ell + U_r + V_\omega\,, \quad \omega \in \Omega_\Lambda \tag{5.1}$$

acting in the Hilbert space $L^2(\mathbb{R}\times[-\frac{L}{2},\frac{L}{2}])$ with periodic boundary conditions along y: $\psi(x,-\frac{L}{2}) = \psi(x,\frac{L}{2})$. The Hamiltonians H_ω, and all the Hamiltonians defined below, are densely defined self-adjoint operators.
We choose the Landau gauge in which the kinetic part has the form $H_0 = \frac{1}{2}p_x^2 + \frac{1}{2}(p_y - Bx)^2$ with spectrum given by the Landau levels: $\sigma(H_0) = \{(n+\frac{1}{2})B; n \in \mathbb{N}\}$. The potentials U_ℓ and U_r representing the confinement along the $x-$direction at $x = \pm\frac{L}{2}$ are independent of y and are supposed strictly monotonic, twice differentiable and satisfy

$$c_1|x+\tfrac{L}{2}|^{m_1} \le U_\ell(x) \le c_2|x+\tfrac{L}{2}|^{m_2} \qquad \text{for } x \le -\tfrac{L}{2} \tag{5.2}$$

$$c_1|x-\tfrac{L}{2}|^{m_1} \le U_r(x) \le c_2|x-\tfrac{L}{2}|^{m_2} \qquad \text{for } x \ge \tfrac{L}{2} \tag{5.3}$$

for some constants $0 < c_1 < c_2$, $2 \le m_1 < m_2 < \infty$ and $U_\ell(x) = 0$ for $x \ge -\frac{L}{2}$, $U_r(x) = 0$ for $x \le \frac{L}{2}$. The random potential V_ω is given by the sum of local perturbations located at the sites of a finite lattice $\Lambda = \{(n,m) \in \mathbb{Z}^2; n \in [-\frac{L}{2},\frac{L}{2}], m \in [-\frac{L}{2},\frac{L}{2}]\}$. Let $V \ge 0$, with $V \in C^2$, $\|V\|_\infty \le V_0$, $\operatorname{supp} V \subset \mathbb{B}(\mathbf{0},\frac{1}{4})$ (the open ball centered at $(0,0)$ of radius $\frac{1}{4}$) and $X_{n,m}$ i.i.d. random variables with common bounded density $h \in C^2([-1,1])$ representing the random strength of each local perturbation. Then V_ω has the form

$$V_\omega(x,y) = \sum_{(n,m)\in\Lambda} X_{n,m}(\omega) V(x-n, y-m) \tag{5.4}$$

We denote by $\mathbb{P}_\Lambda$ the product measure defined on the set of all possible realizations $\Omega_\Lambda = [-1,1]^\Lambda$. Clearly for each realization $\omega \in \Omega_\Lambda$ we have $\|V_\omega\| \le V_0$ and we suppose $V_0 \ll B$.

For future use we collect some properties of three simpler random Hamiltonians. Let us first consider the pure edge Hamiltonians

$$H_\alpha^0 = H_0 + U_\alpha \qquad\qquad \alpha = \ell, r\,. \tag{5.5}$$

In the half-plane case studied in [MMP99] (H_α^0 acting in $L^2(\mathbb{R}^2)$ with U_α a confining wall at $x=0$) we deduce, from translation invariance along y, that the spectrum consists of analytic and monotone decreasing (resp. increasing) branches $\varepsilon_n^\ell(k)$ (resp. $\varepsilon_n^r(k)$) where $k \in \mathbb{R}$

is the quantum number associated to p_y. One has $\lim_{k\to+\infty}\varepsilon_n^\ell(k) = \lim_{k\to-\infty}\varepsilon_n^r(k) = (n+\frac{1}{2})B$ and $\lim_{k\to-\infty}\varepsilon_n^\ell(k) = \lim_{k\to+\infty}\varepsilon_n^r(k) = +\infty$. For the present case (5.5) because of periodic boundary conditions along y the quantum number k takes discrete values $\frac{2\pi m}{L}$, $m\in\mathbb{Z}$. For L finite the spectrum consists of discrete eigenvalues $E_{n,m}^\alpha = \varepsilon_n^\alpha(\frac{2\pi m}{L})$ on the spectral branches. Moreover from the mean value theorem we deduce

$$\left|E_{0,m+1}^\alpha - E_{0,m}^\alpha\right| \geq \frac{C_0}{L} \qquad \alpha = \ell, r \tag{5.6}$$

for each m such that $E_{0,m}^\alpha \in \Delta_\varepsilon = \left(\frac{1}{2}B + V_0 + \varepsilon, \frac{3}{2}B - V_0 - \varepsilon\right)$, where $C_0 > 0$ is independent of m and depends only on the spectral branch ε_0^α.

We will suppose that the following hypothesis is fulfilled

Hypothesis 5.1. *There exists L_0 and $d_0 > 0$ such that for all $L > L_0$*

$$\operatorname{dist}\left(\sigma(H_\ell^0)\cap\Delta_\varepsilon, \sigma(H_r^0)\cap\Delta_\varepsilon\right) \geq \frac{d_0}{L}\ . \tag{5.7}$$

In order to fulfill this hypothesis one must take non-symmetric boundary potentials U_ℓ and U_r. We expect that in fact our result still holds for $U_\ell(x) = U_r(-x)$ because physically the random potential V_ω removes with high probability any degeneracy, but in order to control this case one should improve the Wegner estimate in Section 5.3. In Appendix 5.C we give an example for a situation where this hypothesis is satisfied.

We will make use of the random edge Hamiltonians

$$H_\alpha = H_0 + U_\alpha + V_\omega^\alpha \tag{5.8}$$

where $V_\omega^\alpha = V_\omega|_{\Lambda_\alpha}$ with $\Lambda_r = \left\{(n,m)\in\mathbb{Z}^2; n\in[\frac{L}{2}-\frac{3D}{4}-1,\frac{L}{2}], m\in[-\frac{L}{2},\frac{L}{2}]\right\}$ and $\Lambda_\ell = \left\{(n,m)\in\mathbb{Z}^2; n\in[-\frac{L}{2},-\frac{L}{2}+\frac{3D}{4}+1], m\in[-\frac{L}{2},\frac{L}{2}]\right\}$, where $D=\sqrt{L}$. This choice of D turns out to be convenient in the next sections, but (5.9) and (5.10) below are still true for $D = O(L)$.

Since the perturbation has compact support and the essential spectrum of H_α^0 is given by the Landau levels, the spectrum of H_α is discrete with the Landau levels as only accumulation points. We denote it by $\sigma(H_\alpha) = \{E_\kappa^\alpha\}$. One can prove [Mac03b] that, for each $\omega\in\Omega_{\Lambda_\alpha} = [-1,1]^{\Lambda_\alpha}$ (the restriction of the configurations ω to the sublattice Λ_α) and for each κ such that $E_\kappa^\alpha \in \Delta = (B-\delta, B+\delta)\subset\Delta_\varepsilon$, for L large enough and $\frac{V_0}{B}$ small but independent of L, the distance between two consecutive eigenvalues satisfies

$$\left|E_{\kappa+1}^\alpha - E_\kappa^\alpha\right| \geq \frac{C}{L} \qquad \alpha = \ell, r \tag{5.9}$$

where $C > 0$ is uniform in κ, ω. Moreover for each $E_\kappa^\ell \in \Delta$ (resp. $E_\kappa^r\in\Delta$) the average velocity associated to the corresponding eigenfunctions is strictly negative (resp. positive) uniformly in L

$$\left|J_{E_\kappa^\alpha}\right| \geq C' > 0 \qquad \alpha = \ell, r\ . \tag{5.10}$$

The constant C' is estimated in Appendix 5.B (5.101) in terms of the parameters of the model.

Finally we remark that the Hamiltonian $H_0 + V_\omega|_{\tilde{\Lambda}}$ ($\tilde{\Lambda} \subset \Lambda$) has a point spectrum contained in Landau bands (since $V_\omega|_{\tilde{\Lambda}}$ has bounded support and $\|V_\omega|_{\tilde{\Lambda}}\| = V_0$)

$$\sigma(H_0 + V_\omega|_{\tilde{\Lambda}}) \subset \bigcup_{n\geq 0} \left[(n+\tfrac{1}{2})B - V_0, (n+\tfrac{1}{2})B + V_0\right] \ . \tag{5.11}$$

When $\tilde{\Lambda}$ is given by

$$\Lambda_b \equiv \tilde{\Lambda} = \left\{(n,m) \in \mathbb{Z}^2; n \in [-\tfrac{L}{2} + (\tfrac{D}{4} - 1), \tfrac{L}{2} - (\tfrac{D}{4} - 1)], m \in [-\tfrac{L}{2}, \tfrac{L}{2}]\right\}$$

we call the Hamiltonian $H_b \equiv H_0 + V_\omega|_{\Lambda_b}$ the bulk Hamiltonian.

We now state the main result of this paper.

Theorem 5.1. *Let V_0 small enough, fix $\varepsilon > 0$ and let $0 < \delta < \frac{B}{2} - V_0 - \varepsilon$. Suppose that $(H1)$ hold. Then there exists $\mu > 0$, $\bar{L}$ such that if $L > \bar{L}$ one can find a set $\hat{\Omega} \subset \Omega_\Lambda$ of realizations of the random potential V_ω with $\mathbb{P}_\Lambda(\hat{\Omega}) \geq 1 - L^{-\nu}$ ($\nu \gg 1$) such that for all $\omega \in \hat{\Omega}$ the spectrum of H_ω in $\Delta = (B - \delta, B + \delta)$ is the union of two sets Σ_ℓ and Σ_r with the following properties:*

a) $\mathcal{E}^\alpha_\kappa \in \Sigma_\alpha$ ($\alpha = \ell, r$) are a small perturbation of $E^\alpha_\kappa \in \sigma(H_\alpha) \cap \Delta$ with

$$|\mathcal{E}^\alpha_\kappa - E^\alpha_\kappa| \leq e^{-\mu\sqrt{B}\sqrt{L}} \ . \tag{5.12}$$

b) For $\mathcal{E}^\alpha_\kappa \in \Sigma_\alpha$ the average velocity $J_{\mathcal{E}^\alpha_\kappa}$ of the associated eigenstate satisfies

$$|J_{\mathcal{E}^\alpha_\kappa} - J_{E^\alpha_\kappa}| \leq e^{-\mu\sqrt{B}\sqrt{L}} \ . \tag{5.13}$$

That is the eigenfunctions associated to the eigenvalues (of H_ω) in Δ have an $\mathcal{O}(1)$ velocity.

The main tools for the proof of Theorem 5.1 are developed in Section 5.3. Basically they consist in a Wegner estimate for the random Hamiltonians H_α ($\alpha = \ell, r$) and a decoupling scheme that links the resolvent of the full Hamiltonian H_ω with those of H_ℓ, H_r and H_b. In Section 5.4 we prove two propositions that lead to parts a) and b) of Theorem 5.1. Finally in Appendix 5.A we prove some technical results, in Appendix 5.B we prove (5.10) and in Appendix 5.C we discuss the Hypothesis 5.1.

Let $\boldsymbol{x}, \boldsymbol{x}' \in \mathbb{R} \times \left[-\frac{L}{2}, \frac{L}{2}\right]$, then one can check that

$$|\boldsymbol{x} - \boldsymbol{x}'|_\star \equiv \inf_{n\in\mathbb{Z}} \sqrt{(x-x')^2 + (y - y' - nL)^2} \tag{5.14}$$

has the properties of a distance on $\mathbb{R} \times \mathbb{S}_L$ ($\mathbb{S}_L$ being the circle of circumference L) and that it is related to the Euclidian distance $|\boldsymbol{x} - \boldsymbol{x}'| \equiv \sqrt{(x-x')^2 + (y-y')^2}$ by

$$|\boldsymbol{x} - \boldsymbol{x}'|_\star \leq |\boldsymbol{x} - \boldsymbol{x}'| \, . \tag{5.15}$$

The interest of $|\cdot|_\star$ is that, since we are working with a cylindrical geometry all decay estimates are naturally expressed in terms of this distance. In particular, it permits to express in a convenient way decay in the $y-$direction that occurs on a scale much smaller than L.

5.3 Wegner Estimates and Decoupling Scheme

We first give a Wegner estimate for the Hamiltonians H_α ($\alpha = \ell, r$). Denote by $P^\alpha_{0,m}$ the projector of H^0_α onto the eigenvalue $E^\alpha_{0,m}$ and by $P_\alpha(I)$ the projector of H_α on an interval I. Let $I_m = \left(E^\alpha_{0,m-1} + \delta_0, E^\alpha_{0,m} - \delta_0\right)$ and $\Delta_\alpha = \bigcup_{m_0 \leq m \leq m_1} I_m$, for some $-\infty \ll m_0 < m_1 \ll \infty$ and $\delta_0 \ll \frac{C_0}{L}$, where C_0 is the constant defined in (5.6). The local potentials $V(x-n, y-m)$ will also be denoted by $V_{\boldsymbol{i}}$, $\boldsymbol{i} = (m,n) \in \Lambda$.

Proposition 5.1. *Let $V_0 = \|V_\omega\|$ sufficiently small with respect to B, $E \in \Delta_\alpha \cap \Delta_\varepsilon$ and $I = [E - \bar\delta, E + \bar\delta] \subset I_m$. Then*

$$\mathbb{P}_{\Lambda_\alpha}\left\{\operatorname{dist}(\sigma(H_\alpha), E) < \bar\delta\right\} \leq \|h\|_\infty \bar\delta \operatorname{dist}(I, E^\alpha_{0,\bar m})^{-2} V_0^2 L^4 \tag{5.16}$$

where $E^\alpha_{0\bar m}$ is the closest eigenvalue of $\sigma(H^0_\alpha)$ to the interval I.

Proof. We first observe that $V_{\boldsymbol{i}}^{1/2} P^\alpha_{0,m} V_{\boldsymbol{j}}^{1/2}$ is trace class. Indeed, using $\|AB\|_i \leq \|A\|\|B\|_i$ ($i = 1,2$) and $\|AB\|_1 \leq \|A\|_2 \|B\|_2$ we get $\|V_{\boldsymbol{i}}^{1/2} P^\alpha_{0,m} V_{\boldsymbol{j}}^{1/2}\|_1 \leq \|V_{\boldsymbol{i}}^{1/2} P^\alpha_{0,m}\|_2 \|P^\alpha_{0,m} V_{\boldsymbol{j}}^{1/2}\|_2 \leq V_0 \|P^\alpha_{0,m}\|_1^2 \leq V_0$.

We have $E \in \Delta_\alpha \cap \Delta_\varepsilon$, and $I = [E - \bar\delta, E + \bar\delta]$ for $\bar\delta$ small enough (we require that $I \subset \Delta_\alpha \cap \Delta_\varepsilon$). By the Chebyshev inequality we have

$$\mathbb{P}_{\Lambda_\alpha}\left\{\operatorname{dist}(\sigma(H_\alpha), E) < \bar\delta\right\} = \mathbb{P}_{\Lambda_\alpha}\left\{\operatorname{Tr} P_\alpha(I) \geq 1\right\} \leq \mathbb{E}_{\Lambda_\alpha}\{\operatorname{Tr} P_\alpha(I)\} \tag{5.17}$$

where $\mathbb{E}_{\Lambda_\alpha}$ is the expectation with respect to the random variables in Λ_α.

We first give an estimate on $\operatorname{Tr} P_\alpha(I)$. Let $E^\alpha_{0,\bar m}$ the closest eigenvalue of $\sigma(H^0_\alpha)$ to I and m_i ($i = 0,1$) s.t. $\operatorname{dist}(E^\alpha_{0,\bar m}, E^\alpha_{0,m_i}) = \mathcal{O}(B)$. Let also $P^\alpha_> = \sum_{m > m_1} P^\alpha_{0,m}$ and $P^\alpha_< = \sum_{m < m_0} P^\alpha_{0,m}$.
Using $P^\alpha_>(H^0_\alpha - E)P^\alpha_> \geq 0$ and $P^\alpha_> R^0_\alpha(E) P^\alpha_> \leq \operatorname{dist}(E^\alpha_{0,m_1+1}, E)^{-1} P^\alpha_>$ we can write

$$\begin{aligned} P_\alpha(I) P^\alpha_> P_\alpha(I) &= P_\alpha(I) P^\alpha_> (H^0_\alpha - E)^{1/2} R^0_\alpha(E) (H^0_\alpha - E)^{1/2} P^\alpha_> P_\alpha(I) \\ &\leq \operatorname{dist}(E^\alpha_{0,m_1+1}, E)^{-1} \left[P_\alpha(I)(H_\alpha - E) P^\alpha_> P_\alpha(I) - P_\alpha(I) V^\alpha_\omega P^\alpha_> P_\alpha(I)\right] \end{aligned} \tag{5.18}$$

and thus

$$\|P_\alpha(I) P^\alpha_> P_\alpha(I)\| \leq \operatorname{dist}(E^\alpha_{0,m_1+1}, E)^{-1} \left(\tfrac{|I|}{2} + V_0\right) \leq \tfrac{1}{4} \tag{5.19}$$

if, as we can suppose, V_0 is sufficiently small ($\mathrm{dist}(E^\alpha_{0,m_1+1}, E)^{-1} V_0 = \mathcal{O}\left(\frac{V_0}{B}\right)$). In a similar way we get

$$\|P_\alpha(I) P^\alpha_< P_\alpha(I)\| \le \mathrm{dist}(E^\alpha_{0,m_0-1}, E)^{-1} \left(\frac{|I|}{2} + V_0\right) \le \frac{1}{4} \ . \tag{5.20}$$

Now

$$\mathrm{Tr}\, P_\alpha(I) P^\alpha_< = \mathrm{Tr}\, P_\alpha(I) P^\alpha_< P_\alpha(I) \le \|P_\alpha(I) P^\alpha_< P_\alpha(I)\| \,\mathrm{Tr}\, P_\alpha(I) \tag{5.21}$$

and similarly for $\mathrm{Tr}\, P_\alpha(I) P^\alpha_>$. Therefore, using $1 = P^\alpha_< + P^\alpha_> + \sum_{m_0 \le m \le m_1} P^\alpha_{0,m}$, together with (5.19) and (5.20) we obtain

$$\mathrm{Tr}\, P_\alpha(I) \le 2 \sum_{m_0 \le m \le m_1} \mathrm{Tr}\, P_\alpha(I) P^\alpha_{0,m} P_\alpha(I) \ . \tag{5.22}$$

Since

$$\mathrm{dist}(I, E^\alpha_{0,m})^2 P_\alpha(I)^2 \le \big(P_\alpha(I)(H_\alpha - E^\alpha_{0,m}) P_\alpha(I)\big)^2 \tag{5.23}$$

and $\mathrm{dist}(I, E^\alpha_{0,m})^{-1} \le \mathrm{dist}(I, E^\alpha_{0,\bar m})^{-1}$ for all $m_0 \le m \le m_1$, it follows that

$$\begin{aligned} \mathrm{Tr}\, P^\alpha_{0,m} P_\alpha(I) P^\alpha_{0,m} &\le \mathrm{dist}(I, E^\alpha_{0,\bar m})^{-2} \times \\ &\times \mathrm{Tr}(P^\alpha_{0k} P_\alpha(I)(H_\alpha - E^\alpha_{0,m}) P_\alpha(I)(H_\alpha - E^\alpha_{0,m}) P_\alpha(I) P^\alpha_{0,m}) \\ &= \mathrm{dist}(I, E^\alpha_{0,\bar m})^{-2}\, \mathrm{Tr}(P^\alpha_{0,m} V^\alpha_\omega P_\alpha(I) V^\alpha_\omega P^\alpha_{0,m}) \ . \end{aligned} \tag{5.24}$$

Thus, taking the expectation value in (5.22) and using that there are $\mathcal{O}(L)$ m's between m_0 and m_1, we get

$$\mathbb{E}_{\Lambda_\alpha}\{\mathrm{Tr}\, P_\alpha(I)\} \le 2 \cdot \mathcal{O}(L) \cdot \mathrm{dist}(I, E^\alpha_{0,\bar m})^{-2} \sup_{m_0 \le m \le m_1} \mathbb{E}_{\Lambda_\alpha}\{\mathrm{Tr}(P^\alpha_{0,m} V^\alpha_\omega P_\alpha(I) V^\alpha_\omega P^\alpha_{0,m})\} \ . \tag{5.25}$$

It remains to estimate the expectation value in the right hand side of (5.25). Here we follows a method of Combes and Hislop [CH96]. Writing $V^\alpha_\omega = \sum_{\boldsymbol{i} \in \Lambda_\alpha} X_{\boldsymbol{i}}(\omega) V_{\boldsymbol{i}}$

$$\begin{aligned} \mathrm{Tr}\, P^\alpha_{0,m} V^\alpha_\omega P_\alpha(I) V^\alpha_\omega P^\alpha_{0,m} &= \sum_{\boldsymbol{i},\boldsymbol{j} \in \Lambda^2_\alpha} X_{\boldsymbol{i}}(\omega) X_{\boldsymbol{j}}(\omega)\, \mathrm{Tr}\, P^\alpha_{0,m} V_{\boldsymbol{i}} P_\alpha(I) V_{\boldsymbol{j}} P^\alpha_{0,m} \qquad (5.26) \\ &= \sum_{\boldsymbol{i},\boldsymbol{j} \in \Lambda^2_\alpha} X_{\boldsymbol{i}}(\omega) X_{\boldsymbol{j}}(\omega)\, \mathrm{Tr}\, V^{1/2}_{\boldsymbol{j}} P^\alpha_{0,m} V^{1/2}_{\boldsymbol{i}} V^{1/2}_{\boldsymbol{i}} P_\alpha(I) V^{1/2}_{\boldsymbol{j}} \ . \end{aligned}$$

Since $V^{1/2}_{\boldsymbol{j}} P^\alpha_{0,m} V^{1/2}_{\boldsymbol{i}}$ is trace class we can introduce the singular value decomposition

$$V^{1/2}_{\boldsymbol{j}} P^\alpha_{0,m} V^{1/2}_{\boldsymbol{i}} = \sum_{n=0}^{\infty} \mu_n (u_n, .) v_n \tag{5.27}$$

where $\sum_{n=0}^{\infty} \mu_n = \|V_{\boldsymbol{j}}^{1/2} P_{0,m}^{\alpha} V_{\boldsymbol{i}}^{1/2}\|_1$. Then

$$\begin{aligned}
\operatorname{Tr} V_{\boldsymbol{j}}^{1/2} P_{0k}^{\alpha} V_{\boldsymbol{i}}^{1/2} V_{\boldsymbol{i}}^{1/2} P_{\alpha}(I) V_{\boldsymbol{j}}^{1/2} &= \sum_{n=0}^{\infty} \mu_n (u_n, V_{\boldsymbol{i}}^{1/2} P_{\alpha}(I) V_{\boldsymbol{j}}^{1/2} v_n) \\
\leq \ & \sum_{n=0}^{\infty} \mu_n (v_n, V_{\boldsymbol{j}}^{1/2} P_{\alpha}(I) V_{\boldsymbol{j}}^{1/2} v_n)^{1/2} (u_n, V_{\boldsymbol{i}}^{1/2} P_{\alpha}(I) V_{\boldsymbol{i}}^{1/2} u_n)^{1/2} \\
\leq \ & \tfrac{1}{2} \sum_{n=0}^{\infty} \mu_n \left\{ (v_n, V_{\boldsymbol{j}}^{1/2} P_{\alpha}(I) V_{\boldsymbol{j}}^{1/2} v_n) + (u_n, V_{\boldsymbol{i}}^{1/2} P_{\alpha}(I) V_{\boldsymbol{i}}^{1/2} u_n) \right\} . \qquad (5.28)
\end{aligned}$$

An application of the spectral averaging theorem (see [CH96]) shows that

$$\mathbb{E}_{\Lambda_\alpha} \{ (v_n, V_{\boldsymbol{j}}^{1/2} P_{\alpha}(I) V_{\boldsymbol{j}}^{1/2} v_n) \} \leq \|h\|_\infty 2\bar{\delta} \qquad (5.29)$$

as well as for the term with $\boldsymbol{j}$ replacing $\boldsymbol{i}$ and v_n replacing u_n. Combining (5.25), (5.28), (5.29) and (5.26) we get

$$\begin{aligned}
\mathbb{E}_{\Lambda_\alpha} \{ \operatorname{Tr} P_\alpha(I) \} &\leq 4 \cdot \mathcal{O}(L) \cdot \|h\|_\infty \bar{\delta} \operatorname{dist}(I, E_{0,\bar{m}}^{\alpha})^{-2} V_0^2 \sum_{\boldsymbol{i},\boldsymbol{j} \in \Lambda_\alpha^2} \|V_{\boldsymbol{j}}^{1/2} P_{0,m}^{\alpha} V_{\boldsymbol{i}}^{1/2}\|_1 \\
&\leq 4 \cdot \mathcal{O}(L) \cdot \|h\|_\infty \bar{\delta} \operatorname{dist}(I, E_{0,\bar{m}}^{\alpha})^{-2} V_0^2 |\Lambda_\alpha|^2 . \qquad (5.30)
\end{aligned}$$

□

We now turn to the decoupling scheme. By a decoupling formula [BG91], [BCD89] the resolvent $R(z) = (z - H_\omega)^{-1}$ can be expressed, up to a small term, as the sum of $R_\alpha(z) = (z - H_\alpha)^{-1}$ ($\alpha = \ell, r$) and $R_b(z) = (z - H_b)^{-1}$. We set $D = \sqrt{L}$ and introduce the characteristic functions

$$\begin{aligned}
\tilde{J}_\ell(x) &= \chi_{]-\infty, -\frac{L}{2} + \frac{D}{2}]}(x) \qquad \tilde{J}_b(x) = \chi_{[-\frac{L}{2} + \frac{D}{2}, \frac{L}{2} - \frac{D}{2}]}(x) \\
\tilde{J}_r(x) &= \chi_{[\frac{L}{2} - \frac{D}{2}, +\infty[}(x) . \qquad (5.31)
\end{aligned}$$

We will also use three bounded $C^\infty(\mathbb{R})$ functions $|J_i(x)| \leq 1$, $i \in \mathcal{I} \equiv \{\ell, b, r\}$, with bounded first and second derivatives $\sup_x |\partial_x^n J_i(x)| \leq 2$, $n = 1, 2$, and such that

$$\begin{aligned}
J_\ell(x) &= \begin{cases} 1 & \text{if } x \leq -\frac{L}{2} + \frac{3D}{4} \\ 0 & \text{if } x \geq -\frac{L}{2} + \frac{3D}{4} + 1 \end{cases} \qquad J_b(x) = \begin{cases} 1 & \text{if } |x| \leq \frac{L}{2} - \frac{D}{4} \\ 0 & \text{if } |x| \geq \frac{L}{2} - \frac{D}{4} + 1 \end{cases} \\
J_r(x) &= \begin{cases} 1 & \text{if } x \geq \frac{L}{2} - \frac{3D}{4} \\ 0 & \text{if } x \leq \frac{L}{2} - \frac{3D}{4} - 1 \end{cases} . \qquad (5.32)
\end{aligned}$$

For $i \in \mathcal{I}$ we have $H_\omega J_i = H_i J_i$ and the decoupling formula is [BG91]

$$R(z) = \left(\sum_{i \in \mathcal{I}} J_i R_i(z) \tilde{J}_i \right) (1 - \mathcal{K}(z))^{-1} \qquad (5.33)$$

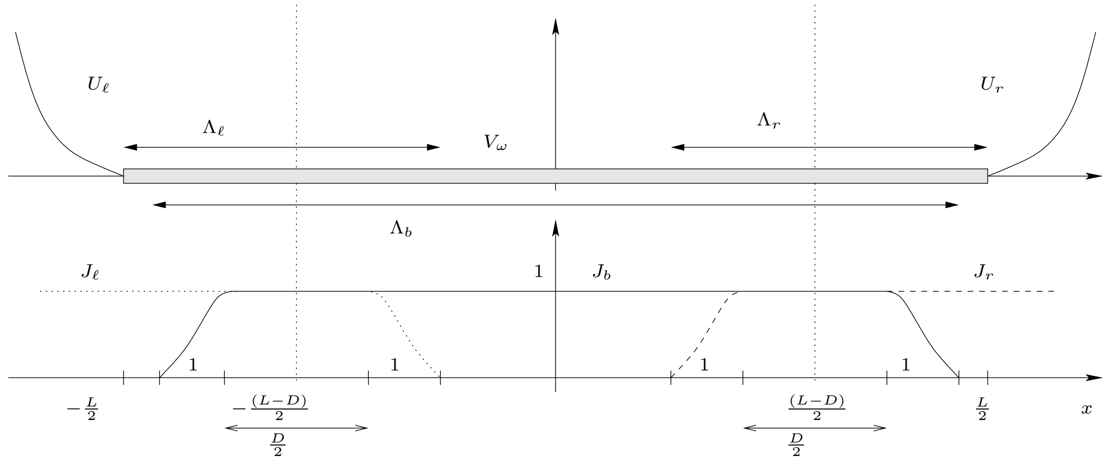

Figure 5.1: The system of decoupling functions J_i ($i \in \mathcal{I}$).

where

$$\mathcal{K}(z) = \sum_{i\in\mathcal{I}} K_i(z) = \sum_{i\in\mathcal{I}} \tfrac{1}{2}[p_x^2, J_i]R_i(z)\tilde{J}_i \ . \tag{5.34}$$

The main result of this part is a lemma about $\|\mathcal{K}(z)\|$ for z such that $\mathrm{dist}(z, \sigma(H_\alpha)) \geq e^{-\bar{\mu}\sqrt{B}\sqrt{L}}$, for a suitable $\bar{\mu} > 0$ and $\mathrm{dist}(z, \sigma(H_b)) \geq \varepsilon$.

Proposition 5.2. *Let $\varepsilon > 0$, and $z \in \Delta_\varepsilon$ such that* $\mathrm{dist}(z, \sigma(H_\ell) \cup \sigma(H_r)) \geq e^{-\bar{\mu}\sqrt{B}\sqrt{L}}$ *with $\bar{\mu} < \frac{1}{192}$. Then for L large enough there exists $C(B, V_0, \varepsilon) > 0$ and $\tilde{\gamma} > 0$ independent of L such that*

$$\|\mathcal{K}(z)\| \leq C(B, V_0, \varepsilon)e^{-\tilde{\gamma}\sqrt{B}\sqrt{L}} \ . \tag{5.35}$$

Proof. Computing the commutator in the definition of $K_i(z)$ we have

$$K_i(z) \ = \ -\tfrac{1}{2}(\partial_x^2 J_i)R_i(z)\tilde{J}_i - (\partial_x J_i)\partial_x R_i(z)\tilde{J}_i \ . \tag{5.36}$$

Then

$$\|K_b(z)\| \ \leq \ \tfrac{1}{2}\|(\partial_x^2 J_b)R_b(z)\tilde{J}_b\| + \|(\partial_x J_b)\partial_x R_b(z)\tilde{J}_b\| \tag{5.37}$$

$$\begin{aligned} \|K_\alpha(z)\| \ &\leq \ \tfrac{1}{2}\|(\partial_x^2 J_\alpha)R_\alpha^b(z)\tilde{J}_\alpha\| + \tfrac{1}{2}\|(\partial_x^2 J_\alpha)R_\alpha^b(z)U_\alpha\| \ \mathrm{dist}(z, \sigma(H_\alpha))^{-1} \\ &+ \ \|(\partial_x J_\alpha)\partial_x R_\alpha^b(z)\tilde{J}_\alpha\| + \|(\partial_x J_\alpha)\partial_x R_\alpha^b(z)U_\alpha\| \ \mathrm{dist}(z, \sigma(H_\alpha))^{-1} \end{aligned} \tag{5.38}$$

where for the the second term we used the second resolvent identity and where $R_\alpha^b(z) = (z - [H_0 + V_\omega^\alpha])^{-1}$.
We have to estimate norms of the form $\|f\partial_x^\alpha \tilde{R}(z)g\|$ ($\alpha = 0, 1$) where here $\tilde{R}(z)$ is $R_b(z)$ or $R_\alpha^b(z)$, $f = \partial_x^m J_i$ and $g = \tilde{J}_i$ or $g = U_\alpha$.
Using the second resolvent formula we develop $\tilde{R}(z)$ in its Neumann series, denote $V_\omega|_{\tilde{\Lambda}} \equiv W$ ($\tilde{\Lambda} = \Lambda_b$ or Λ_α)

$$\tilde{R}(z) = \sum_{n=0}^{\infty} R_0(z)[WR_0(z)]^n \tag{5.39}$$

where $R_0(z) = (z - H_0)^{-1}$. The norm convergence is ensured since we are in a spectral gap, indeed

$$\|WR_0(z)\| \le V_0 \operatorname{dist}(z, \sigma(H_0))^{-1} \le \frac{V_0}{V_0+\varepsilon} < 1 \,. \tag{5.40}$$

Therefore

$$\|f\partial_x^\alpha \tilde{R}(z) g\| \le \sum_{n=1}^{\infty} \|f\partial_x^\alpha R_0(z)\,[WR_0(z)]^n\, g\| \tag{5.41}$$

and we have to control the operator norms $\|f\partial_x^\alpha R_0(z)\,[WR_0(z)]^n\, g\|$.
For any vector $\varphi \in L^2(\mathbb{R}\times[-\frac{L}{2},\frac{L}{2}])$ with $\|\varphi\| = 1$

$$\|f\partial_x^\alpha R_0(z)\,[WR_0(z)]^n\, g\varphi\|^2 = \int_{\operatorname{supp} f} |f(\boldsymbol{x})|^2 \left|(\partial_x^\alpha R_0(z)\,[WR_0(z)]^n\, g\varphi)(\boldsymbol{x})\right|^2 \mathrm{d}\boldsymbol{x} \tag{5.42}$$

For the integrand in (5.42) we have

$$\begin{aligned} \mathcal{J} \;&\equiv\; |(\partial_x^\alpha R_0(z)\,[WR_0(z)]^n\, g\varphi)(\boldsymbol{x})| \le \int_{\operatorname{supp} g} \mathrm{d}\boldsymbol{x}' \int \mathrm{d}\boldsymbol{x}_1 \ldots \mathrm{d}\boldsymbol{x}_n \times \\ &\times\; |\partial_x^\alpha R_0(\boldsymbol{x},\boldsymbol{x}_1;z)||W(\boldsymbol{x_1})||R_0(\boldsymbol{x}_1,\boldsymbol{x}_2;z)|\ldots|W(\boldsymbol{x_n})||R_0(\boldsymbol{x}_n,\boldsymbol{x}';z)||g(\boldsymbol{x}')||\varphi(\boldsymbol{x}')| \,. \end{aligned} \tag{5.43}$$

Now, taking out $\|W\|_\infty$ and using Lemma 5.1, Appendix 5.A we get

$$\begin{aligned} \mathcal{J} \;&\le\; \left(cB^2 \tfrac{V_0}{V_0+\varepsilon}\right)^n \int_{\operatorname{supp} g} \mathrm{d}\boldsymbol{x}' \int \mathrm{d}\boldsymbol{x}_1 \ldots \mathrm{d}\boldsymbol{x}_n e^{-\bar{\gamma}\sqrt{B}\sum_{i=0}^n |\boldsymbol{x}_i - \boldsymbol{x}_{i+1}|_\star} \times \\ &\times\; |\Phi^1(|\boldsymbol{x}-\boldsymbol{x}_1|_\star)|\ldots|\Phi^0(|\boldsymbol{x}_n-\boldsymbol{x}'|_\star)||g(\boldsymbol{x}')||\varphi(\boldsymbol{x}')| \end{aligned} \tag{5.44}$$

where $\boldsymbol{x}_0 = \boldsymbol{x}$ and $\boldsymbol{x}_{n+1} = \boldsymbol{x}'$. Splitting the exponential and making the change of variables $\boldsymbol{x}-\boldsymbol{x}_1 = -\boldsymbol{z}_1, \ldots, \boldsymbol{x}_{n-1}-\boldsymbol{x}_n = -\boldsymbol{z}_n$ we get (with $\boldsymbol{x}_n = \boldsymbol{x}_n(\{\boldsymbol{z}_i\},\boldsymbol{x})$ and $A = cB^2\frac{V_0}{V_0+\varepsilon}$)

$$\begin{aligned} \mathcal{J} \;&\le\; A^n \sup_{\boldsymbol{z}_1\ldots\boldsymbol{z}_n} \left\{ \int_{\operatorname{supp} g} e^{-\frac{2}{3}\bar{\gamma}\sqrt{B}|\boldsymbol{x}-\boldsymbol{x}'|_\star} |g(\boldsymbol{x}')||\varphi(\boldsymbol{x}')||\Phi^0(|\boldsymbol{x}_n-\boldsymbol{x}'|_\star)| e^{-\frac{1}{3}\bar{\gamma}\sqrt{B}|\boldsymbol{x}_n-\boldsymbol{x}'|_\star} \mathrm{d}\boldsymbol{x}' \right\} \times \\ &\times\; \left[\int_{\mathbb{R}^2} |\Phi^1(|\boldsymbol{z}|)| e^{-\frac{1}{3}\bar{\gamma}\sqrt{B}|\boldsymbol{z}|} \mathrm{d}\boldsymbol{z}\right] \left[\int_{\mathbb{R}^2} |\Phi^0(|\boldsymbol{z}|)| e^{-\frac{1}{3}\bar{\gamma}\sqrt{B}|\boldsymbol{z}|} \mathrm{d}\boldsymbol{z}\right]^{n-1} \end{aligned} \tag{5.45}$$

$$\equiv\; A^n \sup_{\boldsymbol{z}_1\ldots\boldsymbol{z}_n} \{\mathcal{X}\}\,[\mathcal{Y}]\,[\mathcal{Z}]^{n-1} \,. \tag{5.46}$$

Splitting the exponential and using the Schwartz inequality we have the estimate

$$\begin{aligned} \sup_{\boldsymbol{z}_1\ldots\boldsymbol{z}_n} \mathcal{X} \;&\le\; \sup_{\boldsymbol{x}'\in\operatorname{supp} g} e^{-\frac{1}{3}\bar{\gamma}\sqrt{B}|\boldsymbol{x}-\boldsymbol{x}'|_\star} \left\{\int_{\mathbb{R}^2} |\Phi^0(|\boldsymbol{w}|)|^2 e^{-\frac{2}{3}\bar{\gamma}\sqrt{B}|\boldsymbol{w}|} \mathrm{d}\boldsymbol{w}\right\}^{1/2} \times \\ &\times\; \left(\sup_{\boldsymbol{x}'\in\operatorname{supp} g} e^{-\frac{2}{3}\bar{\gamma}\sqrt{B}|x-x'|} |g(x')|^2\right)^{1/2} \|\varphi\| \,. \end{aligned} \tag{5.47}$$

Now, since U_α do not grow to fast (see (5.2), (5.3)) $(\sup_{\boldsymbol{x}'\in \operatorname{supp} g} e^{-\frac{2}{3}\bar{\gamma}\sqrt{B}|x-x'|}|g(x')|^2)^{1/2}$ is bounded by a numerical constant. On the other and the term $\int_{\mathbb{R}^2} |\Phi^0(|\boldsymbol{w}|)|^2 e^{-\frac{2}{3}\bar{\gamma}\sqrt{B}|\boldsymbol{w}|}\,\mathrm{d}\boldsymbol{w}$ is bounded by a constant depending only on B.

Moreover the terms $\mathcal{Y}$ and $\mathcal{Z}$ are also bounded by a constant depending only on B and not on L. This leads to

$$\|f\partial_x^\alpha \left[R_0(z)\right]^n g\varphi\| \le \|f\|_\infty \hat{C}(B)(\tilde{C}(B)A)^n e^{-\frac{1}{12}\bar{\gamma}\sqrt{B}D}\|\varphi\| \,. \tag{5.48}$$

Therefore, if V_0 is small enough the series (5.41) converges and

$$\|f\partial_x^\alpha \tilde{R}(z)g\| \le \tilde{C}(B,V_0)\sqrt{L}e^{-\frac{1}{12}\bar{\gamma}\sqrt{B}D} \,. \tag{5.49}$$

This implies

$$\|K_b(z)\| \;\le\; \varepsilon^{-1}\sqrt{L}C(B,V_0)e^{-\frac{1}{12}\bar{\gamma}\sqrt{B}\sqrt{L}} \tag{5.50}$$

$$\|K_\alpha(z)\| \;\le\; \sqrt{L}e^{\bar{\mu}\sqrt{B}\sqrt{L}}C(B,V_0)e^{-\frac{1}{12}\bar{\gamma}\sqrt{B}\sqrt{L}} \qquad \alpha=\ell,r \tag{5.51}$$

thus $\|\mathcal{K}(z)\| \le C(B,V_0,\varepsilon)e^{-\tilde{\gamma}\sqrt{B}\sqrt{L}}$ where $2\tilde{\gamma} = \frac{\bar{\gamma}}{12} - \bar{\mu}$. Since $\bar{\gamma} = \frac{1}{16}$ in Lemma 5.1, Appendix 5.A we must take $\bar{\mu} < \frac{1}{192}$. □

We remark that in the proof above we have proved the following statement (see (5.49)) that will be useful in the next section

$$\|(1-\tilde{J}_\alpha)\tilde{R}_b(z)g\| \le \bar{C}(B,V_0,\varepsilon)e^{-\tilde{\gamma}\sqrt{B}\sqrt{L}} \,. \tag{5.52}$$

where $g = U_\alpha$ or $g = \chi_B$ ($B \subset \mathbb{R}\times[-\frac{L}{2},\frac{L}{2}]$) with $\operatorname{dist}(\operatorname{supp} g, \operatorname{supp}(1-\tilde{J}_\alpha)) = \mathcal{O}(D)$ and $\tilde{R}_b(z)$ a resolvent associated to a generic bulk Hamiltonian $(H_0 + V_\omega|_{\tilde{\Lambda}})$.

5.4 Projector estimates and the proof of Theorem 5.1

In this section we prove two propositions that lead to Theorem 5.1. Let $\mathcal{D}' = \{\kappa : E_\kappa^\alpha \in \Delta, \alpha=\ell,r\}$, $\operatorname{card}(\mathcal{D}') = \mathcal{O}(L)$, where $\Delta \subset \Delta_\varepsilon$ is given in Section 5.2.

Proposition 5.3. *For L large enough, with probability greater then $1-L^{-\nu}$ ($\nu \gg 1$), we have for all $\kappa \in \mathcal{D}'$*

$$\|P - P_\alpha(E_\kappa^\alpha)\| \le e^{-\gamma\sqrt{B}\sqrt{L}} \tag{5.53}$$

where $P_\alpha(E_\kappa^\alpha)$ is the projector associated to H_α onto E_κ^α and P is the projector associated to H_ω onto $\{z\in\mathbb{C} : |z-E_\kappa^\alpha| \le e^{-\bar{\mu}\sqrt{B}\sqrt{L}}\}$.

Proof. (1): Let $\mathcal{E} = \{m : E_{0,m}^\alpha \in \Delta, \alpha=\ell,r\}$, $\operatorname{card}(\mathcal{E}) = \mathcal{O}(L)$, and let

$$\hat{\Omega}_\ell = \{\omega \in \Omega_{\Lambda_\ell} : \operatorname{dist}(E_{0,m}^r, \sigma(H_\ell)) \ge L^{-\sigma}, \forall m \in \mathcal{E}\} \,, \tag{5.54}$$

with $\sigma > 11$, this set has probability

$$\mathbb{P}_{\Lambda_\ell}(\hat{\Omega}_\ell) \geq 1 - L^{-(\sigma-8)} \; . \tag{5.55}$$

Indeed for a fixed $m \in \mathcal{E}$, using Proposition 5.1 and $(H1)$ one gets

$$\begin{aligned} & \mathbb{P}_{\Lambda_\ell}\left\{\omega \in \Omega_{\Lambda_\ell} : \mathrm{dist}(E^r_{0,m}, \sigma(H_\ell)) \geq L^{-\sigma}, \text{ for one } m \in \mathcal{E}\right\} \\ \geq \;& 1 - C'(h, V_0) L^{-\sigma} L^4 \left(\tfrac{d_0}{L} - L^{-\sigma}\right)^{-2} \geq 1 - C(h, V_0) L^{6-\sigma} \; . \end{aligned} \tag{5.56}$$

For a given realization $\omega_\ell \in \hat{\Omega}_\ell$ let

$$\hat{\Omega}_r(\omega_\ell) = \{\omega \in \Omega_{\Lambda_r} : \mathrm{dist}(E^\ell_\kappa, \sigma(H_r)) \geq L^{-3\sigma}, \forall \kappa \in \mathcal{D}'\} \; , \tag{5.57}$$

this set has probability

$$\mathbb{P}_{\Lambda_r}(\hat{\Omega}_r(\omega_\ell) | \omega_\ell) \geq 1 - L^{-(\sigma-6)} \; . \tag{5.58}$$

uniformly with respect to the realizations of $\hat{\Omega}_\ell$. Indeed

$$\begin{aligned} & \mathbb{P}_{\Lambda_r}\left\{\omega \in \Omega_{\Lambda_r} : \mathrm{dist}(E^\ell_\kappa, \sigma(H_r)) \geq L^{-3\sigma}, \text{ for one } \kappa \in \mathcal{D}'\right\} \\ \geq \;& 1 - C'(h, V_0) L^{-3\sigma} L^4 \left(L^{-\sigma} - L^{-3\sigma}\right)^{-2} \geq 1 - C(h, V_0) L^{4-\sigma} \; . \end{aligned} \tag{5.59}$$

It follows that the set

$$\hat{\Omega}^{(\ell)} = \left\{\omega = (\omega_\ell, \omega_b, \omega_r) \in \Omega : \omega_\ell \in \hat{\Omega}_\ell, \omega_b \in \Omega_b, \omega_r \in \hat{\Omega}_r(\omega_\ell)\right\} \tag{5.60}$$

$\Omega_b = \Omega|_{\Lambda_b \setminus (\Lambda_\ell \cup \Lambda_r)}$ has probability

$$\begin{aligned} \mathbb{P}_\Lambda(\hat{\Omega}^{(\ell)}) &= \mathbb{P}_{\Lambda_b}(\hat{\Omega}_b) \mathbb{E}_{\Lambda_\ell}\left\{\mathbb{P}_{\Lambda_r}(\hat{\Omega}_r | \omega_\ell) \Big| \omega_\ell \in \hat{\Omega}_\ell\right\} \\ &\geq (1 - L^{-(\sigma-6)}) \mathbb{P}_{\Lambda_\ell}(\hat{\Omega}_\ell) \geq 1 - L^{-(\sigma-9)} \end{aligned} \tag{5.61}$$

(2): We now work with a given $\omega \in \hat{\Omega}^{(\ell)}$. Take $\bar{\mu} > 0$ as in Proposition 5.2 and L large enough such that for all $\kappa \in \mathcal{D}'$ $\Gamma_\kappa = \{z \in \mathbb{C} : |z - E^\ell_\kappa| \leq e^{-\bar{\mu}\sqrt{B}\sqrt{L}}\} \cap \sigma(H_r) = \emptyset$, and remark that $\mathrm{Tr}\, P_b(\Delta) = 0$ (P_b the projector associated to H_b).
We need to introduce two auxiliary Hamiltonians H_1 and H_2 defined as follows:

$$H_1 = H_0 + V^\ell_\omega|_{\Lambda_1} \tag{5.62}$$

$$H_2 = H_0 + V^\ell_\omega|_{\Lambda_2} + U_\ell \tag{5.63}$$

where $\Lambda_2 = \left\{(n, m) \in \mathbb{Z}^2; n \in [-\frac{L}{2}, -\frac{L}{2} + (\frac{D}{4} - 1)], m \in [-\frac{L}{2}, \frac{L}{2}]\right\}$, and $\Lambda_1 = \Lambda_\ell \backslash \Lambda_2$, of course $H_\ell = H_2 + V^\ell_\omega|_{\Lambda_1}$.

From the decoupling formula (5.33) we have

$$\begin{aligned} R(z) - R_\ell(z) &= \left(\sum_{i \in \mathcal{I}} J_i R_i(z) \tilde{J}_i\right)\left(\sum_{n=1}^{\infty} \mathcal{K}(z)^n\right) - (1 - J_\ell) R_\ell(z) \\ &- J_\ell R_\ell(z)(1 - \tilde{J}_\ell) + J_b R_b(z) \tilde{J}_b + J_r R_r(z) \tilde{J}_r \; . \end{aligned} \tag{5.64}$$

integrating over $\partial\Gamma_\kappa$ and taking the operator norm we get

$$\begin{aligned} \|P - P_\ell(E_\kappa^\ell)\| &\leq e^{-\bar{\mu}\sqrt{B}\sqrt{L}} \left(\sum_{i\in\mathcal{I}} \sup_{z\in\partial\Gamma_\kappa} \|R_i(z)\| \right) \frac{\sup_{z\in\partial\Gamma_\kappa} \|\mathcal{K}(z)\|}{1 - \sup_{z\in\partial\Gamma_\kappa} \|\mathcal{K}(z)\|} \\ &+ \|(1 - J_\ell)P_\ell(E_\kappa^\ell)\| + \|J_\ell P_\ell(E_\kappa^\ell)(1 - \tilde{J}_\ell)\| \\ &= a + b + c\,. \end{aligned} \tag{5.65}$$

For the first term we note that for L large enough $e^{-\bar{\mu}\sqrt{B}\sqrt{L}} \sup_{z\in\partial\Gamma_\kappa} \|R_i(z)\| \leq 1$ ($i \in \mathcal{I}$). Indeed, for $i = \ell$ we have $\sup_{z\in\partial\Gamma_\kappa} \|R_\ell(z)\| = e^{\bar{\mu}\sqrt{B}\sqrt{L}}$ by construction, for $i = b$ we have $\sup_{z\in\partial\Gamma_\kappa} \|R_b(z)\| = \varepsilon^{-1}$ and for $i = r$ $\sup_{z\in\partial\Gamma_\kappa} \|R_r(z)\| = \left(L^{-3\sigma} - e^{-\bar{\mu}\sqrt{B}\sqrt{L}}\right)^{-1}$. Then, applying Proposition 5.2 we get

$$a \leq 2C(B, V_0, \varepsilon)e^{-\tilde{\gamma}\sqrt{B}\sqrt{L}}\,. \tag{5.66}$$

For the second and third term we first observe that by the second resolvent formula

$$\frac{P_\ell(E_\kappa^\ell)}{(z - E_\kappa^\ell)} = (z - H_1)^{-1}P_\ell(E_\kappa^\ell) + (z - H_1)^{-1}[V_\omega^\ell|_{\Lambda_2} + U_\ell]\frac{P_\ell(E_\kappa^\ell)}{(z - E_\kappa^\ell)}\,. \tag{5.67}$$

and integrating (5.67) along $\partial\Gamma_\kappa$ we obtain (using $\sigma(H_1) \cap \Delta_\varepsilon = \emptyset$)

$$P_\ell(E_\kappa^\ell) = R_1(E_\kappa^\ell)[V_\omega^\ell|_{\Lambda_2} + U_\ell]P_\ell(E_\kappa^\ell) \tag{5.68}$$

$$= P_\ell(E_\kappa^\ell)[V_\omega^\ell|_{\Lambda_2} + U_\ell]R_1(E_\kappa^\ell)\,. \tag{5.69}$$

Therefore, using (5.68) for b and (5.69) for c we get

$$b \leq \|(1 - J_\ell)R_1(E_\kappa^\ell)[V_\omega^\ell|_{\Lambda_2} + U_\ell]\| \leq \|(1 - \tilde{J}_\ell)R_1(E_\kappa^\ell)[V_\omega^\ell|_{\Lambda_2} + U_\ell]\| \tag{5.70}$$

$$c \leq \|(1 - \tilde{J}_\ell)R_1(E_\kappa^\ell)[V_\omega^\ell|_{\Lambda_2} + U_\ell]\|\,. \tag{5.71}$$

Using (5.52) we get

$$\begin{aligned} b + c &\leq 2\left(V_0L^2\|(1 - \tilde{J}_\ell)R_1(E_\kappa^\ell)\chi_{\Lambda_2}\| + \|(1 - \tilde{J}_\ell)R_1(E_\kappa^\ell)U_\ell\|\right) \\ &\leq 2\bar{C}(B, V_0, \varepsilon)L^2e^{-\tilde{\gamma}\sqrt{B}\sqrt{L}}\,. \end{aligned} \tag{5.72}$$

Thus

$$\|P - P_\ell(E_\kappa^\ell)\| \leq e^{-\gamma\sqrt{B}\sqrt{L}}\,. \tag{5.73}$$

By repeating the above proof in a symmetrical way we get for ω in a set $\hat{\Omega}^{(r)}$ similar to $\hat{\Omega}^{(\ell)}$

$$\|P - P_r(E_\kappa^r)\| \leq e^{-\gamma\sqrt{B}\sqrt{L}}\,. \tag{5.74}$$

Finally we have both (5.73) and (5.74) for $\omega \in \hat{\Omega} = \hat{\Omega}^{(\ell)} \cap \hat{\Omega}^{(r)}$ with $\mathbb{P}_\Lambda \geq 1 - L^{-\nu}$, $\nu = \sigma - 10$. Note that we can take $\nu' \gg 1$ by taking $\sigma \gg 11$.

□

The estimate on the norm difference of the projectors implies that their dimensions are the same and that $\mathcal{E}_\kappa^\alpha \in \sigma(H_\omega)$ is a small perturbation of E_κ^α: this gives part *a*) of Theorem 5.1.

Proposition 5.4. *Let $\omega \in \hat{\Omega}$. Then there exists $\hat{\mu} > 0$ such that the velocity associated to each eigenvalue $\mathcal{E}_\kappa^\alpha$ of H_ω in Δ satisfies*

$$\left| J_{\mathcal{E}_\kappa^\alpha} - J_{E_\kappa^\alpha} \right| \leq e^{-\hat{\mu}\sqrt{B}\sqrt{L}} \ . \tag{5.75}$$

Proof. Let $J_{\mathcal{E}_\kappa^\alpha} = \operatorname{Tr} v_y P(\mathcal{E}_\kappa^\alpha)$ the average velocity associated to the eigenvalue $\mathcal{E}_\kappa^\alpha \in \sigma(H_\omega)$ and $J_{E_\kappa^\alpha} = \operatorname{Tr} v_y P_\alpha(E_\kappa^\alpha)$ that associated to the eigenvalue E_κ^α of H_α. First we observe that $v_y P(\mathcal{E}_\kappa^\alpha)$ is trace class. Indeed, $v_y P(\mathcal{E}_\kappa^\alpha) = v_y P(\mathcal{E}_\kappa^\alpha) P(\mathcal{E}_\kappa^\alpha)$ with $v_y P(\mathcal{E}_\kappa^\alpha)$ bounded and $\|P(\mathcal{E}_\kappa^\alpha)\|_1 = \operatorname{Tr} P(\mathcal{E}_\kappa^\alpha) = \operatorname{Tr} P_\alpha(E_\kappa^\alpha) = 1$.

$$\begin{aligned} \|v_y P(\mathcal{E}_\kappa^\alpha)\|_1^2 &\leq \|v_y P(\mathcal{E}_\kappa^\alpha)\|^2 \leq \|P(\mathcal{E}_\kappa^\alpha) v_y^2 P(\mathcal{E}_\kappa^\alpha)\| \\ &\leq 2\|P(\mathcal{E}_\kappa^\alpha)(H_\omega - V_\omega) P(\mathcal{E}_\kappa^\alpha)\| \leq (3B + 2V_0) \end{aligned} \tag{5.76}$$

To get the second inequality one has simply added positive terms to v_y^2. Similarly

$$\|v_y P_\alpha(E_\kappa^\alpha)\|_1^2 \leq (3B + 2V_0) \ . \tag{5.77}$$

With the help of the identity

$$\begin{aligned} P(\mathcal{E}_\kappa^\alpha) - P_\alpha(E_\kappa^\alpha) &= [P(\mathcal{E}_\kappa^\alpha) - P_\alpha(E_\kappa^\alpha)]^2 + [P(\mathcal{E}_\kappa^\alpha) - P_\alpha(E_\kappa^\alpha)] P_\alpha(E_\kappa^\alpha) \\ &+ P_\alpha(E_\kappa^\alpha)[P(\mathcal{E}_\kappa^\alpha) - P_\alpha(E_\kappa^\alpha)] \end{aligned} \tag{5.78}$$

we get

$$\begin{aligned} |J_{\mathcal{E}_\kappa^\alpha} - J_{E_\kappa^\alpha}| &= |\operatorname{Tr} v_y [P(\mathcal{E}_\kappa^\alpha) - P_\alpha(E_\kappa^\alpha)]| \leq \left|\operatorname{Tr} v_y [P(\mathcal{E}_\kappa^\alpha) - P_\alpha(E_\kappa^\alpha)]^2\right| \\ &+ |\operatorname{Tr} v_y [P(\mathcal{E}_\kappa^\alpha) - P_\alpha(E_\kappa^\alpha)] P_\alpha(E_\kappa^\alpha)| \\ &+ |\operatorname{Tr} v_y P_\alpha(E_\kappa^\alpha)[P(\mathcal{E}_\kappa^\alpha) - P_\alpha(E_\kappa^\alpha)]| \ . \end{aligned} \tag{5.79}$$

and then, from (5.76) and (5.77), we get

$$\begin{aligned} |J_{\mathcal{E}_\kappa^\alpha} - J_{E_\kappa^\alpha}| &\leq 2\left(\|v_y P(\mathcal{E}_\kappa^\alpha)\|_1 + \|v_y P_\alpha(E_\kappa^\alpha)\|_1\right) \|P(\mathcal{E}_\kappa^\alpha) - P_\alpha(E_\kappa^\alpha)\| \\ &\leq 4(3B + 2V_0)^{1/2} \|P(\mathcal{E}_\kappa^\alpha) - P_\alpha(E_\kappa^\alpha)\| \ . \end{aligned} \tag{5.80}$$

Combining this last inequality with Proposition 5.3 we get the result. □

From Proposition 5.4 and the result of Appendix 5.B given in (5.10) we obtain part *b*) of Theorem 5.1.

5.A Estimate of the Green function $R_0(\boldsymbol{x}, \boldsymbol{x}'; z)$

In this appendix we give the necessary decay property of the kernel $R_0(\boldsymbol{x}, \boldsymbol{x}'; z)$ with periodic boundary conditions along y. The exact formula for $R_0(\boldsymbol{x}, \boldsymbol{x}'; z)$ can be found in [FM02]. We introduce the following notation

$$\begin{aligned}&\Phi^{\alpha}(|\boldsymbol{x}-\boldsymbol{x}'|_\star)\\ =\ &\begin{cases}1+\left|\ln\left(\frac{B}{2}|\boldsymbol{x}-\boldsymbol{x}'|_\star^2\right)\right| & ,\ \alpha=0\\ 1+\left[\left|\ln\left(\frac{B}{2}|\boldsymbol{x}-\boldsymbol{x}'|_\star^2\right)\right|+\left(1+\left|\ln\left(\frac{B}{2}|\boldsymbol{x}-\boldsymbol{x}'|_\star^2\right)\right|\right)|\boldsymbol{x}-\boldsymbol{x}'|_\star^{-1}\right] & ,\ \alpha=1\ .\end{cases}\end{aligned} \tag{5.81}$$

Lemma 5.1. *If $|\mathcal{I}m\, z| \le 1$, $\mathcal{R}e\, z \in \left]\frac{1}{2}B, \frac{3}{2}B\right[$ then, for L large enough, there exists $C(z, B)$ positive constant independent of L such that* ($\alpha = 0, 1$)

$$\begin{aligned}|\partial_x^{\alpha} R_0(\boldsymbol{x}, \boldsymbol{x}'; z)| &\le C'(z,B)e^{-\frac{B}{8}|\boldsymbol{x}-\boldsymbol{x}'|_\star^2}\Phi^{\alpha}(|\boldsymbol{x}-\boldsymbol{x}'|_\star)\\ &\le C(z,B)e^{-\bar{\gamma}\sqrt{B}|\boldsymbol{x}-\boldsymbol{x}'|_\star}\Phi^{\alpha}(|\boldsymbol{x}-\boldsymbol{x}'|_\star)\end{aligned} \tag{5.82}$$

where $C(z, B) = cB^2 \operatorname{dist}(z, \sigma(H_0))^{-1}$ with c a numerical positive constant and $\bar{\gamma} = \frac{1}{16}$.

Proof. As in [FM02] we can prove that (for L large enough the logarithmic divergences appear only for $|m| \le 1$ and the sum over $|m| > 1$ converge)

$$|\partial_x^{\alpha} R_0(\boldsymbol{x}, \boldsymbol{x}'; z)| \le \tfrac{C'(z,B)}{3}e^{-\frac{B}{8}|\boldsymbol{x}-\boldsymbol{x}'|^2} + \sum_{|m|\le 1} |\partial_x^{\alpha} R_0^{\infty}(x\, y - mL, \boldsymbol{x}'; z)| \tag{5.83}$$

with

$$\begin{aligned}&|\partial_x^{\alpha} R_0^{\infty}(\boldsymbol{x}, \boldsymbol{x}'; z)|\\ \le\ &\begin{cases}\frac{C'(z,B)}{3}e^{-\frac{B}{8}|\boldsymbol{x}-\boldsymbol{x}'|^2}\left\{1+\mathbf{1}_{\mathbb{B}(\mathbf{0},\sqrt{2B^{-1}})}(|\boldsymbol{x}-\boldsymbol{x}'|)\left|\ln\left(\frac{B}{2}|\boldsymbol{x}-\boldsymbol{x}'|^2\right)\right|\right\}, & \alpha=0\\ \frac{C'(z,B)}{3}e^{-\frac{B}{8}|\boldsymbol{x}-\boldsymbol{x}'|^2}\left\{1+\mathbf{1}_{\mathbb{B}(\mathbf{0},\sqrt{2B^{-1}})}(|\boldsymbol{x}-\boldsymbol{x}'|)\left[\left|\ln\left(\frac{B}{2}|\boldsymbol{x}-\boldsymbol{x}'|^2\right)\right|\right.\right. & \\ \left.\left.+\left(1+\left|\ln\left(\frac{B}{2}|\boldsymbol{x}-\boldsymbol{x}'|^2\right)\right|\right)|\boldsymbol{x}-\boldsymbol{x}'|^{-1}\right]\right\}, \quad \alpha=1\ . & \end{cases}\end{aligned} \tag{5.84}$$

Now, using $|\boldsymbol{x} - \boldsymbol{x}'|_\star \le |\boldsymbol{x} - \boldsymbol{x}'|$, we can replace the Euclidean distance with the distance $|\cdot|_\star$ in all the terms in the RHS of (5.83), since all these functions are decreasing. To obtain the same bound for the terms $|m| \le 1$ in the sum we just drop the characteristic functions $\mathbf{1}_{\mathbb{B}(\mathbf{0},\sqrt{2B^{-1}})}$. □

5.B Average velocity of the eigenstate associated to E_κ^α

In this appendix we prove following [Fer99] that the eigenstates corresponding to the eigenvalues of H_α ($\alpha = \ell, r$) in a energy interval $\Delta = (B - \delta, B + \delta) \subset \Delta_\varepsilon$ have an average velocity that is strictly positive/negative uniformly in L, that is, if we have $H_\alpha \psi_\kappa^\alpha = E_\kappa^\alpha \psi_\kappa^\alpha$ then

$$|(\psi_\kappa^\alpha, v_y \psi_\kappa^\alpha)| \geq C' > 0 \, . \tag{5.85}$$

From the eigenvalue equation we have

$$\|(H_\alpha^0 - E_\kappa^\alpha)\psi_\kappa^\alpha\|^2 = \|V_\omega^\alpha \psi_\kappa^\alpha\|^2 \leq V_0^2 \, . \tag{5.86}$$

We now expand ψ_κ^α on the eigenfunctions of H_α^0 denoted $\left\{\phi_{n,m}(x,y) = \frac{e^{iky}}{\sqrt{L}}\varphi_{nk}(x)\right\}_{n\in\mathbb{N}, k\in\frac{2\pi}{L}\mathbb{Z}}$ where φ_{nk} is the solution on the eigenvalue problem $[\frac{1}{2}p_x^2 + \frac{1}{2}(k - Bx)^2 + U_\alpha]\varphi_{nk} = E_{nk}^\alpha \varphi_{nk}$.

$$\psi_\kappa^\alpha(x,y) = \sum_{n=0}^{\infty}\sum_{m\in\mathbb{Z}} \psi_n(m)\phi_{n,m}(x,y) \, , \tag{5.87}$$

and of course

$$\|\psi_\kappa^\alpha\|^2 = \sum_{n=0}^{\infty}\sum_{m\in\mathbb{Z}} |\psi_n(m)|^2 = 1 \, . \tag{5.88}$$

From (5.87) the equation (5.86) becomes

$$\sum_{n=0}^{\infty}\sum_{m\in\mathbb{Z}} |\psi_n(m)|^2 \left(E_{n,m}^\alpha - E_\kappa^\alpha\right)^2 \leq V_0^2 \tag{5.89}$$

thus since each term in the sum is positive we have

$$\sum_{m\in\mathbb{Z}} |\psi_0(m)|^2 \left(E_{0,m}^\alpha - E_\kappa^\alpha\right)^2 \leq V_0^2 \tag{5.90}$$

We remark that for $n \geq 1$ one has $|E_{n,m}^\alpha - E_\kappa^\alpha| \geq \frac{B}{2} - \delta$, this leads to

$$\|\psi_\star\|^2 \equiv \sum_{n=1}^{\infty}\sum_{m\in\mathbb{Z}} |\psi_n(m)|^2 \leq \frac{V_0^2}{(\frac{B}{2} - \delta)^2} \, . \tag{5.91}$$

Let $m^\star$ such that $|E_{0,m^\star}^\alpha - E_\kappa^\alpha|$ is minimal, and for a fixed a independent of L let $\mathcal{A} = [m^\star - a, m^\star + a]$. Then from (5.89)

$$\begin{aligned} V_0^2 &\geq \sum_{m\in\mathbb{Z}} |\psi_0(m)|^2 \left(E_{0,m}^\alpha - E_\kappa^\alpha\right)^2 \geq \sum_{m\in\mathcal{A}^c} |\psi_0(m)|^2 \left(E_{0,m}^\alpha - E_\kappa^\alpha\right)^2 \\ &\geq \inf_{m\in\mathcal{A}^c} \left(E_{0,m}^\alpha - E_\kappa^\alpha\right)^2 \sum_{m\in\mathcal{A}^c} |\psi_0(m)|^2 \end{aligned} \tag{5.92}$$

thus

$$\sum_{m\in\mathcal{A}^c} |\psi_0(m)|^2 \leq V_0^2 \sup_{m\in\mathcal{A}^c} \left(E_{0,m}^\alpha - E_\kappa^\alpha\right)^{-2} \, . \tag{5.93}$$

From (5.88) and (5.91) we get

$$1 \geq \sum_{m\in\mathbb{Z}} |\psi_0(m)|^2 \geq 1 - \tfrac{V_0^2}{(\frac{B}{2}-\delta)^2} \ . \tag{5.94}$$

Combining the last equation and (5.93) we get

$$\sum_{m\in\mathcal{A}} |\psi_0(m)|^2 \geq 1 - V_0^2 \left[\tfrac{1}{(\frac{B}{2}-\delta)^2} + \sup_{m\in\mathcal{A}^c} (E^\alpha_{0,m} - E^\alpha_\kappa)^{-2} \right] \ . \tag{5.95}$$

Decompose now ψ^α_κ as $\psi^\alpha_\kappa = \psi_0 + \psi_\star$, then

$$|(\psi^\alpha_\kappa, v_y\psi^\alpha_\kappa)| \geq |(\psi_0, v_y\psi_0)| - |(\psi_\star, v_y\psi_\star)| - 2|(\psi_\star, v_y\psi_0)| \tag{5.96}$$

the first term can be written as

$$\begin{aligned}
& \int_{\mathbb{R}} \mathrm{d}x \int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}y \left\{ \sum_{m'\in\mathbb{Z}} \psi_0^*(m') \frac{e^{-i\frac{2\pi m'}{L}y}}{\sqrt{L}} \varphi^*_{0,m'}(x) \sum_{m\in\mathbb{Z}} \psi_0(m) v_y \frac{e^{i\frac{2\pi m}{L}y}}{\sqrt{L}} \varphi_{0,m}(x) \right\} \\
= & \sum_{m\in\mathbb{Z}} |\psi_0(m)|^2 \int_{\mathbb{R}} \mathrm{d}x \, (k - Bx) \, |\varphi_{0,m}(x)|^2 \\
= & \sum_{m\in\mathbb{Z}} |\psi_0(m)|^2 \partial_k E^\alpha_0(k) \Big|_{k=\frac{2\pi m}{L}}
\end{aligned} \tag{5.97}$$

The partial derivative of E^α_0 is the average velocity $\partial_k E^\alpha_0(k)\Big|_{k=\frac{2\pi m}{L}} = J_{E^\alpha_{0,m}}$, thus

$$\begin{aligned}
|(\psi_0, v_y\psi_0)| & \geq \left| \sum_{m\in\mathbb{Z}} |\psi_0(m)|^2 J_{E^\alpha_{0,m}} \right| \\
& \geq |J_{E^\alpha_{0\bar{m}}}| \left\{ 1 - V_0^2 \left[\tfrac{1}{(\frac{B}{2}-\delta)^2} + \sup_{m\in\mathcal{A}^c} \left(E^\alpha_{0,m} - E^\alpha_\kappa\right)^{-2} \right] \right\}
\end{aligned} \tag{5.98}$$

for a suitable $\bar{m} \in \mathcal{A}$, and we have $|J_{E^\alpha_{0,\bar{m}}}| > 0$. The second term can be bounded as follows $|(\psi_\star, v_y\psi_\star)| \leq \|\psi_\star\| \|v_y\psi_\star\| \leq \frac{V_0}{\frac{B}{2}-\delta} \|v_y\psi_\star\|$ and

$$\begin{aligned}
\|v_y\psi_\star\|^2 & = 2\left(\psi_\star, \tfrac{1}{2}\left(p_y - Bx\right)^2 \psi_\star\right) \\
& \leq 2\left(\psi_\star, \left[\tfrac{1}{2}p_x^2 + \tfrac{1}{2}\left(p_y - Bx\right)^2 + U_\alpha\right] \psi_\star\right) \\
& + 2\left(\psi_0, \left[\tfrac{1}{2}p_x^2 + \tfrac{1}{2}\left(p_y - Bx\right)^2 + U_\alpha\right] \psi_0\right) = 2\left(\psi^\alpha_\kappa, H^0_\alpha \psi^\alpha_\kappa\right) \\
& = 2(\psi^\alpha_\kappa, H_\alpha\psi^\alpha_\kappa) - 2(\psi^\alpha_\kappa, V^\alpha_\omega\psi^\alpha_\kappa) \leq 2(E^\alpha_\kappa + V_0) \ .
\end{aligned} \tag{5.99}$$

This leads to the bound

$$|(\psi_\star, v_y\psi_\star)| \leq \tfrac{V_0}{\frac{B}{2}-\delta} \sqrt{2(E^\alpha_\kappa + V_0)} \tag{5.100}$$

A similar argument gives the same bound for the third term.
Finally

$$\begin{aligned} |(\psi_\kappa^\alpha, v_y\psi_\kappa^\alpha)| \;&\geq\; |J_{E^\alpha_{0,\bar m}}| \left\{1 - V_0^2\left[\tfrac{1}{(\frac{B}{2}-\delta)^2} + \sup_{m\in\mathcal{A}^c}\left(E^\alpha_{0,m} - E^\alpha_\kappa\right)^{-2}\right]\right\} \\ &-\; 3\tfrac{V_0}{\frac{B}{2}-\delta}\sqrt{2(E^\alpha_\kappa + V_0)}\,. \end{aligned} \tag{5.101}$$

The right hand side of (5.101) is greater than

$$J\left[1 - O\left(\frac{V_0^2}{B^2}\right)\right] - \sqrt{B}O\left(\frac{V_0}{B}\right) \tag{5.102}$$

where the strictly positive constant J depends only on B and U_α. For a sufficiently small $V_0 > 0$ the right hand side of (5.101) is strictly positive.

5.C Discussion of Hypothesis 5.1

In this section we indicate a way in which Hypothesis $(H1)$ can be achieved explicitly. We thank F. Bentosela for pointing out this possibility to one of us. We take two symmetric confining walls $U_\ell(-x) = U_r(x) \equiv U(x)$ and add a magnetic flux tube of intensity $0 \leq \Phi \leq 2\pi$ along the cylinder axis. Below we check that the magnetic flux lifts the degeneracy of the levels on the two sides of the sample.
In this case the pure edge Hamiltonians are

$$\begin{aligned} H_\ell^0[\Phi] &= \tfrac{1}{2}p_x^2 + \tfrac{1}{2}\left(p_y - Bx + \tfrac{\Phi}{L}\right)^2 + U(-x) && (5.103)\\ H_r^0[\Phi] &= \tfrac{1}{2}p_x^2 + \tfrac{1}{2}\left(p_y - Bx + \tfrac{\Phi}{L}\right)^2 + U(x)\,. && (5.104) \end{aligned}$$

The spectra of these Hamiltonians are

$$\sigma(H_\alpha^0[\Phi]) = \{E^\alpha_{n,m}(\Phi) : n \in \mathbb{N}, m \in \mathbb{Z}\}. \tag{5.105}$$

with $E^\alpha_{n,m}(\Phi) = \varepsilon^\alpha_n(\frac{2\pi m}{L} + \frac{\Phi}{L})$. We consider here only the first spectral branches and note that from the symmetry of the walls, for $\Phi = 0$

$$\varepsilon_0^\ell\left(-\tfrac{2\pi}{L}m\right) = \varepsilon_0^r\left(\tfrac{2\pi}{L}m\right) \quad \forall\, m \in \mathbb{Z} \tag{5.106}$$

We have

$$\begin{aligned} \varepsilon_0^\ell\left(-\tfrac{2\pi m}{L} + \tfrac{\Phi}{L}\right) &= \varepsilon_0^\ell\left(-\tfrac{2\pi m}{L}\right) + \partial_k\varepsilon_0^\ell(k_\ell)\frac{\Phi}{L} && (5.107)\\ \varepsilon_0^r\left(\tfrac{2\pi m}{L} + \tfrac{\Phi}{L}\right) &= \varepsilon_0^r\left(\tfrac{2\pi m}{L}\right) + \partial_k\varepsilon_0^r(k_r)\frac{\Phi}{L} && (5.108) \end{aligned}$$

for a suitable $\frac{2\pi}{L}(-m) \leq k_\ell \leq \frac{2\pi}{L}(-m) + \frac{\Phi}{L}$ and $\frac{2\pi}{L}m \leq k_r \leq \frac{2\pi}{L}m + \frac{\Phi}{L}$. Thus

$$\begin{aligned}\left|\varepsilon_0^\ell\left(-\tfrac{2\pi m}{L} + \tfrac{\Phi}{L}\right) - \varepsilon_0^r\left(\tfrac{2\pi m}{L} + \tfrac{\Phi}{L}\right)\right| &= \frac{\Phi}{L}\left|\partial_k\varepsilon_0^r(k_r) - \partial_k\varepsilon_0^\ell(k_\ell)\right| \\ &\geq 2\frac{\Phi}{L}|\partial_k\varepsilon_0^\ell(k_\ell)| \geq 2\mathcal{C}\frac{\Phi}{L} \qquad (5.109)\end{aligned}$$

where $\mathcal{C} > 0$. A similar argument shows that

$$\begin{aligned}&\left|\varepsilon_0^\ell\left(-\tfrac{2\pi(m+1)}{L} + \tfrac{\Phi}{L}\right) - \varepsilon_0^r\left(\tfrac{2\pi m}{L} + \tfrac{\Phi}{L}\right)\right| \\ = &\left|\frac{\Phi}{L}\left[\partial_k\varepsilon_0^\ell(k_\ell) - \partial_k\varepsilon_0^r(k_r)\right] - \tfrac{2\pi}{L}\partial_k\varepsilon_0^\ell(k_\ell)\right| \geq \left|2\frac{\Phi}{L}|\partial_k\varepsilon_0^\ell(k_\ell)| - \frac{2\pi}{L}|\partial_k\varepsilon_0^\ell(k_\ell)|\right| \\ \geq &\; 2\mathcal{C}\frac{|\Phi - \pi|}{L} \qquad (5.110)\end{aligned}$$

Then, by fixing $\Phi^\star$ such that $0 < \Phi^\star < \pi$ or $\pi < \Phi^\star < 2\pi$ we achive (5.7).

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5.D Geometrical condition for extended states

[This section is not included in [FM03a]]

Consider the same model than in the article above but where the two confining walls U_ℓ, U_r have supports at the distance D. In this case the mathematical model simulates a two dimensional quantum Hall device of size $D \times L$. When $D = L$, the study of the spectral properties of H_ω have just been presented.
The goal of the present section is to study the dependence of the width D of the sample as a function of its length L, this in view to have current carrying states. The physical relevance of this study is to give some geometrical condition between L and D for which the quantum Hall effect take place.

We first report the previous theorem for this geometry where D is fixed

Theorem 5.2. *Let L large enough. There exists $\hat{\Omega} \subset \Omega$ with the property $\mathbb{P}(\hat{\Omega}) > 1 - \frac{D^2}{L^s}$ $(s \gg 2)$ such that if $\omega \in \hat{\Omega}$ then*

$$\sigma(H_\omega) \cap \Delta = \Sigma_\ell \cup \Sigma_r$$

with Σ_α *such that if* $\mathcal{E}^\alpha_\kappa \in \Sigma_\alpha$ *then*

$$|\mathcal{E}^\alpha_\kappa - E^\alpha_\kappa| \leq \rho(L)$$

provided that $\mathcal{C}\rho^{-1}(L)Le^{-\frac{1}{384}\sqrt{B}D} < 1$, *with* $\rho(L) = o(L^{-\nu})$, *for* $\nu \gg 1$ *and* $\mathcal{C}$ *a generic positive constant.*
Moreover, there exist a constant $\mathcal{J} > 0$ (*uniformly in* L, D) *such that for any eigenstate* ψ^α_κ, $H_\omega \psi^\alpha_\kappa = \mathcal{E}^\alpha_\kappa \psi^\alpha_\kappa$

$$|(\psi^r_\kappa, v_y \psi^r_\kappa) - \mathcal{J}| \leq \mathcal{C}\rho^{-1}(L)Le^{-\frac{1}{384}\sqrt{B}D} \qquad |(\psi^\ell_\kappa, v_y \psi^\ell_\kappa) + \mathcal{J}| \leq \mathcal{C}\rho^{-1}(L)Le^{-\frac{1}{384}\sqrt{B}D}$$

where v_y *is the velocity operator along the* $y-$*direction.*

Now we are interested in current carrying eigenstates ψ^α_κ. Therefore we look at the condition $(\psi^\alpha_\kappa, v_y \psi^\alpha_\kappa)$ to be of order $\mathcal{O}(1)$ with respect to the size $D \times L$ of the confined system.

From Theorem 5.2 we can easily see that the condition to have current carrying states in the limit of large L is given by

$$\rho^{-1}(L)Le^{-\frac{1}{384}\sqrt{B}D} \to 0 \qquad \text{as } L \to \infty\ . \tag{5.111}$$

We immediately remark that $\rho^{-1}(L)L$ diverges for $L \to \infty$, thus for a fixed D independent of L we cannot expect existence of current carrying eigenstate for a infinitely long strip of width D. We then have to set $D = D(L)$ with $D(L) \to \infty$ for $L \to \infty$.

The condition (5.111) can be bounded from above using the hypothesis $\rho(L) = o(L^{-\nu})$. We then have the following condition for the existence of current carrying eigenstates:

$$\rho^{-2}(L)e^{-\frac{1}{384}\sqrt{B}D(L)} \to 0 \qquad \text{as } L \to \infty\ . \tag{5.112}$$

We now set $\rho(L) = L^{-p}$ for a sufficiently large $p > 0$. We get that if $D(L) = o(\log L)$ then the condition (5.112) is not satisfied.
Finally we can conclude that the borderline to get current carrying eigenstates is
Borderline:

$$D(L) = \mathcal{O}(\log L)\ . \tag{5.113}$$

Chapter 6

Intermixture of Extended and Localized Energy Levels

In this chapter we report the article [FM02]: J. Phys. A: Math. Gen. **35** (2002), 6339–6358.

Intermixture of Extended Edge and Localized Bulk Energy Levels in Macroscopic Hall Systems

Christian Ferrari and Nicolas Macris

Abstract

We study the spectrum of a random Schrödinger operator for an electron submitted to a magnetic field in a finite but macroscopic two dimensional system of linear dimensions equal to L. The $y-$direction is periodic and in the $x-$direction the electron is confined by two smooth increasing boundary potentials. The eigenvalues of the Hamiltonian are classified according to their associated quantum mechanical diamagnetic current in the $y-$direction. Here we look at an interval of energies inside the first Landau band of the random operator for the infinite plane. In this energy interval, with large probability, there exist $\mathcal{O}(L)$ eigenvalues with positive or negative currents of $\mathcal{O}(1)$. Between each of these there exist $\mathcal{O}(L^2)$ eigenvalues with infinitesimal current $\mathcal{O}(e^{-\gamma B(\log L)^2})$. We explain what is the relevance of this analysis of boundary diamagnetic currents to the integer quantum Hall effect.

6.1 Introduction

In this paper we are concerned about boundary currents in the integer quantum Hall effect, that occurs in disordered electronic systems subject to a uniform magnetic field and confined in a two dimensional interface of an heterojunction [PG87]. It was recognized by Halperin that boundary diamagnetic equilibrium currents play an important role in understanding the transport properties of such systems [Hal82]. Later on it was realized that there is an intimate connection between these boundary currents and the topological properties of the state in the bulk [FK91], [Wen91]. Here we will study only diamagnetic currents due to the boundaries, and not those produced by the adiabatic switching of an external infinitesimal electric field (as in linear response theory) which may exist in the bulk. Many features of the integral quantum Hall effect can be described in the framework one particle random magnetic Schrödinger operators and therefore it is important to understand their spectral properties for finite but macroscopic samples with boundaries. This problem has been approached recently for geometries where only one boundary is present and the operator is defined in a semi-infinite region [MMP99], [FGW00], [dBP99].

Here we will take a finite system: our geometry is that of a cylinder of length and circumference both equal to L. There are two boundaries at $x = \pm\frac{L}{2}$ modelled by two smooth confining potentials $U_\ell(x)$ (ℓ for left) and $U_r(x)$ (r for right), and we take periodic boundary conditions in the $y-$direction. These potentials vanish for $-\frac{L}{2} \leq x \leq \frac{L}{2}$ and grow fast enough for $|x| \geq \frac{L}{2}$. The Hamiltonian is of the form

$$H_\omega = H_0 + V_\omega + U_\ell + U_r \tag{6.1}$$

where H_0 is the pure Landau Hamiltonian for a uniform field of strength B and V_ω is a suitable weak random potential produced by impurities with $\sup|V_\omega(x,y)| = V_0 \ll B$ (see Section 6.2 for precise assumptions). Before explaining our results it is useful to describe what is known about the infinite and semi-infinite case.

In the case of the infinite plane $\mathbb{R}^2$ for the Hamiltonian $H_0 + V_\omega$ the spectrum forms "Landau bands" contained in $\bigcup_{\nu\geq 0}\left[(\nu+\frac{1}{2})B - V_0, (\nu+\frac{1}{2})B + V_0\right]$. It is proved that the band tails have pure point spectrum corresponding to exponentially localized wavefunctions [DMP95], [DMP96], [CH96], [BCH97], [Wan97]. There are no rigorous results for energies at the band centers, except for a special model where the impurities are point scatterers [DMP97], [DMP99]. As first shown in [Kun87] these spectral properties of random Schrödinger operators imply that the Hall conductivity – given by the Kubo formula – considered as a function of the filling factor (ratio of electron number and number of flux quanta) has quantized plateaux at values equal to $\nu e^2/h$ where ν is the number of filled Landau levels. The presence of the plateaux is a manifestation of Anderson localization while the quantization has a topological origin. The latter was first discovered in particular situations [TKNdN82], and it has been proved for more general models using non commutative geometry [BvESB94] and the index of Fredholm operators [ASS94] (see [AG98] for a review).

In a semi-infinite system where the particle is confined in a half plane with Hamiltonian $H_0 + V_\omega + U_\ell$ (here (x, y) belongs to $\mathbb{R}^2$) the spectrum includes all energies in $\left[\frac{B}{2}, +\infty\right[$. The lower edge of the spectrum is between $\frac{B}{2} - V_0$ and $\frac{B}{2}$ and in its vicinity the spectrum is pure point (this follows from techniques in [BCH97]). For energies in intervals inside the gaps of the bulk Hamiltonian $H_0 + V_\omega$ the situation is completely different. One can show that the average velocity $(\psi, v_y\psi)$ in the $y-$direction of an assumed eigenstate ψ does not vanish, but since the velocity v_y is the commutator between y and the Hamiltonian, this implies that the eigenstate cannot exist, and that therefore the spectrum is purely continuous [MMP99], [Fer99]. In fact Mourre theory has been suitably applied to prove that the spectrum is purely absolutely continuous [FGW00], [dBP99]. These works put on a rigorous basis the expectation that, because of the chiral nature of the boundary currents, the states remain extended in the $y-$direction even in the presence of disorder [Hal82]. The same sort of analysis shows that if the $y-$direction is made periodic of length L, the same energy intervals have discrete eigenstates which carry a current that is $\mathcal{O}(1)$ – say positive – with respect to L [FGW00]. Furthermore one can show that the eigenvalue spacing is of order $\mathcal{O}(L^{-1})$ [Mac03b].

The nature of the spectrum for a semi-infinite system for intervals inside the Landau bands of the bulk Hamiltonian $\bigcup_{\nu\geq 0}\left[(\nu+\frac{1}{2})B - V_0, (\nu+\frac{1}{2})B + V_0\right]$ has not yet been elucidated.

For the finite system on a cylinder with two boundaries the spectrum consists of finitely degenerate isolated eigenvalues. In [FM03a] the results of [MMP99], [FGW00] for energy intervals inside the gaps of the bulk Hamiltonian are extended to the present two boundary system. The eigenvalues can be classified in two sets distinguished by the sign of their associated current[1]. These currents are uniformly positive or uniformly negative with respect to L. For this result to hold it is important to take the circumference and the length of the cylinder both of the order L.

In the present work we study the currents of the eigenstates for eigenvalues in the interval $\Delta_\varepsilon = \left]\frac{B}{2} + \varepsilon, \frac{B}{2} + V_0\right[$ where ε is a small positive number independent of L. We limit ourselves to the first band to keep the discussion simpler. The content of our main result (Theorem 6.1) is the following. Given ε, for L large enough there is a ensemble of realizations of the random potential with probability $1-\mathcal{O}(L^{-s})$ for which the eigenvalues of H_ω can be classified into three sets that we call Σ_ℓ, Σ_r and Σ_b. The eigenstates of Σ_r (resp. Σ_ℓ) have uniformly positive (resp. negative) currents with respect to L, while those of Σ_b have a current of the order of $\mathcal{O}(e^{-\gamma B(\log L)^2})$. The number of eigenvalues in Σ_α ($\alpha = \ell, r$) is $\mathcal{O}(L)$ while that in Σ_b is $\mathcal{O}(L^2)$. This classification of eigenvalues leads to a well defined notion of extended edge and localized bulk states. The edge states are those which belong to Σ_α ($\alpha = \ell, r$) and are extended in the sense that they have a current of order $\mathcal{O}(1)$. The bulk states are those which belong to Σ_b and are localized in the

[1]In principle the physical current is $L^{-1}(\psi, v_y\psi)$, but here we will call current the average velocity $(\psi, v_y\psi)$.

sense that their current is infinitesimal. The energy levels of the extended and localized states are *intermixed* in the same energy interval. See also [FM03b] for a short review on spectral properties of systems defined on a cylinder.

Let us explain the mechanism that is at work. When the random potential is removed $V_\omega = 0$ in (6.1) the eigenstates with energies away from $\frac{B}{2}$ are extended in the $y-$direction and localized in the $x-$direction at a finite distance from the boundaries. Their energies form a sequence of "edge levels" and have a spacing of the order of $\mathcal{O}(L^{-1})$. When the potential of one impurity is added to H_0 it typically creates a localized bound state with energy between the Landau levels. Suppose now that *i*) a coupling constant in the impurity potential is *fine tuned* as a function of L so that the energy of the impurity level stays at distance greater than L^{-p} from the edge levels, *ii*) the position of the impurity is at a distance D from the boundaries. Then the mixing between the localized bound state and the extended edge states is controlled in second order perturbation theory by the parameter $L^p e^{-cBD^2}$. Therefore one expects that bound states of impurities that have $D \gg (\log L)^{1/2}$ are basically unperturbed and have an infinitesimal current. On the other hand bound states coming from impurities with $D \ll (\log L)^{1/2}$ will mix with edge states. Note that *even for impurities with* $D \gg (\log L)^{1/2}$ *the coupling constant* (equivalently the impurity level) *has to be fine tuned* as a function of L. Indeed, for a coupling constant with a fixed value the energy of the impurity level is independent of L, and surely for L large enough the energy difference between the impurity and the edge levels becomes much smaller than $\mathcal{O}(e^{-cBD^2})$. Remarkably for a random potential the absence of resonance is automatically achieved with large probability and no fine tuning is needed: this is why localized bulk states survive. We have analyzed this mechanism rigorously for a model (see also [Hal82]) where there are no impurities in a layer of thickness $(\log L)$ along the boundary. Then the edge levels are basically non random and the typical spacing between current carrying eigenvalues is easily controlled. Of course it is desirable to allow for impurities close to the boundary but then the edge levels become random and some further analysis is needed. However we expect that the same basic mechanism operates because the typical spacing between edge levels should still be $\mathcal{O}(L^{-1})$. In connection to the discussion above we mention that for a semi-infinite system the bound state of an impurity at any fixed distance from the boundary turns into a resonance. A similar situation has been analysed in [GM99].

We note that the spectral region close to $\frac{B}{2}$ that is left out in our theorem is precisely the one where resonances between edge and bulk states may occur because edge states become very dense. It is not clear what is the connection with the divergence of the localization length of the infinite system at the band center.

In the present work we have shown that in quantum Hall samples there exist well defined notions of extended edge states (current of $\mathcal{O}(1)$) and localized bulk states (infinitesimal current). Instead of classifying the energy levels according to their current one could try to use level statistics. We expect that the localized bulk states have

Poissonian statistics whereas the extended edge states should display a level repulsion. In fact such a strong form of level repulsion in proved in [Mac03b] for energies in the gap of the bulk Hamiltonian where only extended edge states exist. It is interesting to observe that in the present situation both kind of states have *intermixed* energy levels. In usual Schrödinger operators (e.g. the Anderson model on a $3D$ cubic lattice) it is accepted (but not proven) that they are separated by a well defined *mobilty edge* (results in this direction have recently been obtained [JL00] under a suitable hypothesis). The states at the band edge are localized in the sense that the spectrum is dense pure point for the infinite lattice and has Poisson statistics for the finite system [Min96]. At the band center the states are believed to be extended in the sense that the spectrum is absolutely continuous for the infinite lattice and has the statistics of the Gaussian Orthogonal Ensemble for the finite lattice.

Other ways of formulating the notion of edge states have been proposed in different contexts. In [AANS98] the authors consider a clean system with a novel kind of chiral boundary conditions. The Hilbert space then separate in two parts corresponding to edge and bulk states. The bulk states have exactly the Landau energy and the edge states a linear dispersion relation; the distinction between them being sharp because of the special nature of boundary conditions. It would be interesting to extend this definition to disordered systems. Recently in [HS02] (see also [HS01]) another approach has been used in the context of magnetic billiards. The authors study a magnetic billiard with mixed boundary conditions with mixing parameter Λ interpolating between Dirichlet and Neumann boundary conditions. They look at the sensibility of the eigenstates and eigenvalues under the variation of Λ and define in this way an edge state as a state that depends strongly on Λ. Let us note that our notion of edge state as well as the other ones all share the feature that an edge state carries a substantial current.

The characterization of the spectrum of (6.1) proposed here also has a direct relevance to the Hall conductivity of the many electron (non interacting) system. In the formulation advocated by Halperin [Hal82] the Hall conductivity is computed as the ratio of the net equilibrium current and the difference of chemical potentials between the two edges. Consider the many fermion state $\Psi(\mu_\ell, \mu_r, E_F)$ obtained by filling the levels of H_ω (one particle per state) in $\Sigma_\ell \cap \left[\frac{B}{2} + \varepsilon, \mu_\ell\right]$, $\Sigma_r \cap \left[\frac{B}{2} + \varepsilon, \mu_r\right]$ and $\Sigma_b \cap \left[\frac{B}{2} + \varepsilon, E_F\right]$ with $\frac{B}{2} + \varepsilon < \mu_\ell < E_F < \mu_r < \frac{B}{2} + V_0$. The total current $I(\mu_\ell, \mu_r, E_F)$ of this state – a stationary state of the many particle Hamiltonian – is given by the sum of the individual physical currents of the filled levels (given by $L^{-1}(\psi, v_y \psi)$). From the estimates (6.21) and (6.23) in Theorem 6.1

$$\sum_k J_k^\ell + \sum_k J_k^r + \sum_\beta J_\beta = \sum_k J_{0k}^\ell + \sum_k J_{0k}^r + \mathcal{O}(e^{-(\log L)^2} L^2) \tag{6.2}$$

and from (6.15) we get

$$\frac{1}{L}\sum_k J^r_{0k} = \frac{1}{2\pi}\int_{\frac{B}{2}+\varepsilon}^{\mu_r} \mathrm{d}E + \mathcal{O}(L^{-1}) \tag{6.3}$$

$$\frac{1}{L}\sum_k J^\ell_{0k} = \frac{1}{2\pi}\int_{\mu_\ell}^{\frac{B}{2}+\varepsilon} \mathrm{d}E + \mathcal{O}(L^{-1}) \tag{6.4}$$

It follows that to leading order

$$I(\mu_\ell, \mu_r, E_F) \simeq \frac{1}{2\pi}(\mu_r - \mu_\ell)\ . \tag{6.5}$$

In (6.5) the Hall conductance is equal to one (this is because we have considered only the first band). When μ_ℓ and μ_r vary the density of particles in the state $\Psi(\mu_\ell, \mu_r, E_F)$ does not change since the number of levels in Σ_α ($\alpha = \ell, r$) is of order $\mathcal{O}(L)$. However if E_F is increased the particle density (and thus the filling factor) increases since the number of levels in Σ_b is of order $\mathcal{O}(L^2)$, but the Hall conductance does not change and hence has a plateau. In other words the edge states contribute to the Hall conductance but not to the density of states of the sample in the thermodynamic limit.

In a more complete theory one should also take in account currents possibly flowing in the bulk due to the adiabatic switching of an external electric field, an issue that is beyond the scope of the present analysis. A related problem is the relationship between the conductance in the present picture, defined through (6.5), and the one using Kubo formula (see [KRSB02], [KRSB00], [EG02]).

The precise definition of the model and the statement of the main result (Theorem 6.1) are the subject of the next section.

6.2 The Structure of the Spectrum

We consider the family of random Hamiltonians (6.1) acting on the Hilbert space $L^2(\mathbb{R}\times[-\frac{L}{2},\frac{L}{2}])$ with periodic boundary conditions along y, $\psi(x,-\frac{L}{2}) = \psi(x,\frac{L}{2})$. In the Landau gauge the kinetic term of (6.1) is

$$H_0 = \tfrac{1}{2}p_x^2 + \tfrac{1}{2}(p_y - Bx)^2 \tag{6.6}$$

and has infinitely degenerate Landau levels $\sigma(H_0) = \left\{(\nu+\frac{1}{2})B; \nu\in\mathbb{N}\right\}$. We will make extensive use of explicit point-wise bounds, proved in Appendix 6.A, on the integral kernel of the resolvent $R_0(z) = (z-H_0)^{-1}$ with periodic boundary conditions along y.

The confining potentials modelling the two edges at $x = -\frac{L}{2}$ and $x = \frac{L}{2}$ are assumed to be strictly monotonic, differentiable and such that

$$c_1|x+\tfrac{L}{2}|^{m_1} \le U_\ell(x) \le c_2|x+\tfrac{L}{2}|^{m_2} \qquad \text{for } x \le -\tfrac{L}{2} \tag{6.7}$$

$$c_1|x-\tfrac{L}{2}|^{m_1} \le U_r(x) \le c_2|x-\tfrac{L}{2}|^{m_2} \qquad \text{for } x \ge \tfrac{L}{2} \tag{6.8}$$

for some constants $0 < c_1 < c_2$ and $2 \leq m_1 < m_2 < \infty$. Recall that $U_\ell(x) = 0$ for $x \geq -\frac{L}{2}$ and $U_r(x) = 0$ for $x \leq \frac{L}{2}$. We could allow steeper confinements but the present polynomial conditions turn out to be technically convenient.

We assume that each impurity is the source of a local potential $V \in C^2$, $0 \leq V(x,y) \leq V_0 < \infty$, $\operatorname{supp} V \subset \mathbb{B}\left(\mathbf{0}, \frac{1}{4}\right)$, and that they are located at the sites of a finite lattice $\Lambda = \left\{(n,m) \in \mathbb{Z}^2; n \in [-\frac{L}{2} + \log L, \frac{L}{2} - \log L], m \in [-\frac{L}{2}, \frac{L}{2}]\right\}$. The random potential V_ω has the form

$$V_\omega(x,y) = \sum_{(n,m)\in\Lambda} X_{n,m}(\omega) V(x-n, y-m) \tag{6.9}$$

where the coupling constants $X_{n,m}$ are i.i.d. random variables with common density $h \in C^2([-1,1])$ that satisfies $\|h\|_\infty < \infty$, $\operatorname{supp} h = [-1,1]$. We will denote by $\mathbb{P}_\Lambda$ the product measure defined on the set of all possible realizations $\omega \in \Omega_\Lambda = [-1,1]^\Lambda$. Clearly for any realization we have $|V_\omega(x,y)| \leq V_0$. Furthermore it will be assumed that the random potential is weak in the sense that $4V_0 < B$.

We will think of our system as being constituted of three pieces corresponding to the *bulk system* with the random Hamiltonian

$$H_b = H_0 + V_\omega \tag{6.10}$$

and the *left* and *right edge systems* with non random Hamiltonians

$$H_\alpha = H_0 + U_\alpha, \qquad \alpha = \ell, r\ . \tag{6.11}$$

All the Hamiltonians considered above have periodic boundary conditions along the $y-$direction and are essentially self-adjoint on $C_0^\infty(\mathbb{R} \times [-\frac{L}{2}, \frac{L}{2}])$. For each realization ω and size L the spectrum $\sigma(H_\omega)$ of (6.1) (it depends on L) consists of isolated eigenvalues of finite multiplicity. In order to state our main result characterizing these eigenvalues we first have to describe the spectra of (6.10) and (6.11).

Let us begin with the edge Hamiltonians (6.11). Here we state their properties without proofs and refer the reader to [MMP99], [Fer99] for more details. Since the edge Hamiltonians H_α commute with p_y, they are decomposable into a direct sum

$$H_\alpha = \sum_{k\in\frac{2\pi}{L}\mathbb{Z}}^{\oplus} H_\alpha(k) = \sum_{k\in\frac{2\pi}{L}\mathbb{Z}}^{\oplus} \left[\tfrac{1}{2}p_x^2 + \tfrac{1}{2}(k - Bx)^2 + U_\alpha\right]\ . \tag{6.12}$$

For each k the one dimensional Hamiltonian $H_\alpha(k)$ has a compact resolvent, thus it has discrete eigenvalues and by standard arguments one can show that they are not degenerate. If the $y-$direction would be infinitely extended, k would vary over the real axis and the eigenvalues of $H_\alpha(k)$ would form spectral branches $\varepsilon_\nu^\alpha(\hat{k})$, $\hat{k} \in \mathbb{R}$ labelled by the Landau level index ν. These spectral branches are strictly monotone, entire functions with the properties $\varepsilon_\nu^\ell(-\infty) = +\infty$, $\varepsilon_\nu^\ell(+\infty) = (\nu + \frac{1}{2})B$ and $\varepsilon_\nu^r(-\infty) = (\nu + \frac{1}{2})B$, $\varepsilon_\nu^r(+\infty) = +\infty$.

Here because of the periodic boundary conditions the set of k values is discrete so that the spectrum of H_α

$$\sigma(H_\alpha) = \left\{E^\alpha_{\nu k}; \nu \in \mathbb{N}, k \in \tfrac{2\pi}{L}\mathbb{Z}\right\} \tag{6.13}$$

consists of isolated points on the spectral branches $E^\alpha_{\nu k} = \varepsilon^\alpha_\nu(k)$, $k \in \frac{2\pi}{L}\mathbb{Z}$. The corresponding eigenfunctions $\psi^\alpha_{\nu k}$ have the form

$$\psi^\alpha_{\nu k}(x, y) = \frac{1}{\sqrt{L}} e^{iky} \varphi^\alpha_{\nu k}(x) \tag{6.14}$$

with $\varphi^\alpha_{\nu k}$ the normalized eigenfunctions of the one-dimensional Hamiltonian $H_\alpha(k)$. By definition, the current of the state $\psi^\alpha_{\nu k}$ in the $y-$direction is given by the expectation value of the velocity $v_y = p_y - Bx$,

$$J^\alpha_{\nu k} = (\psi^\alpha_{\nu k}, v_y \psi^\alpha_{\nu k}) = \int_{\mathbb{R}} |\varphi^\alpha_{\nu k}(x)|^2 (k - Bx)\, \mathrm{d}x = \partial_{\hat{k}} \varepsilon^\alpha_\nu(\hat{k}) \Big|_{\hat{k} = \frac{2\pi m}{L}} \tag{6.15}$$

where the last equality follows from the Feynman-Hellman theorem. From (6.15) we notice that for any $\varepsilon > 0$, one can find $j(\varepsilon) > 0$ and $L(\varepsilon)$ such that for $L > L(\varepsilon)$ the states of the two branches $\nu = 0$, $\alpha = \ell, r$ with energies $E^\alpha_{0k} \geq \frac{1}{2}B + \varepsilon$ satisfy

$$J^\ell_{0k} \leq -j(\varepsilon) < 0 \qquad J^r_{0k} \geq j(\varepsilon) > 0\,. \tag{6.16}$$

In other words the eigenstates of the edge Hamiltonians carry an appreciable current. The spacing of two consecutive eigenvalues greater than $\frac{1}{2}B + \varepsilon$ satisfies

$$\left| E^\alpha_{0\frac{2\pi(m+1)}{L}} - E^\alpha_{0\frac{2\pi m}{L}} \right| > \frac{j(\varepsilon)}{L} \qquad \alpha = \ell, r\,. \tag{6.17}$$

Note that these observations extend to other branches but $j(\varepsilon)$ and $L(\varepsilon)$ are not uniform with respect to the index ν. In the rest of the paper we limit ourselves to $\nu = 0$ for simplicity. On the other hand the spacing between the energies of $\sigma(H_\ell)$ and $\sigma(H_r)$ is a priori arbitrary. We assume that the confining potentials U_ℓ and U_r are such that the following hypothesis is fulfilled.

Hypothesis 6.1. *Fix any $\varepsilon > 0$ and let $\Delta_\varepsilon = \left[\frac{1}{2}B + \varepsilon, \frac{1}{2}B + V_0\right]$. There exist $L(\varepsilon)$ and $d(\varepsilon) > 0$ such that for all $L > L(\varepsilon)$*

$$\operatorname{dist}\left(\sigma(H_\ell) \cap \Delta_\varepsilon, \sigma(H_r) \cap \Delta_\varepsilon\right) \geq \frac{d(\varepsilon)}{L}\,. \tag{6.18}$$

This hypothesis is important because a minimal amount of non-degeneracy between the spectra of the two edge systems is needed in order to control backscattering effects induced by the random potential. Indeed in a system with two boundaries backscattering favors localization and has a tendency to destroy currents. This hypothesis can easily be realized by taking non-symmetric confining potentials U_ℓ and U_r. In a more realistic model with impurities close to the edges one expects that it is automatically satisfied with a large probability.

Now we describe the spectral properties of the bulk random Hamiltonian (6.10). From the bound (6.83) on the kernel of $R_0(z)$ and the fact that V_ω is bounded with compact support we can see that V_ω is relatively compact w.r.t. H_0, thus $\sigma_{ess}(H_b) = \{(\nu + \frac{1}{2})B; \nu \in \mathbb{N}\}$. Since $|V_\omega(x,y)| \leq V_0 < B$ the eigenvalues E_β^b of H_b are contained in Landau bands $\bigcup_{\nu \geq 0} \left[(\nu + \frac{1}{2})B - V_0, (\nu + \frac{1}{2})B + V_0\right]$. We will assume

Hypothesis 6.2. *Fix any $\varepsilon > 0$. There exist $\mu(\varepsilon)$ a strictly positive constant and $L(\varepsilon)$ such that for all $L > L(\varepsilon)$ one can find a set of realizations of the random potential Ω'_Λ with $\mathbb{P}_\Lambda(\Omega'_\Lambda) \geq 1 - L^{-\theta}$, $\theta > 0$, with the property that if $\omega \in \Omega'_\Lambda$ the eigenstates corresponding to $E_\beta^b \in \sigma(H_b) \cap \Delta_\varepsilon$ satisfy*

$$|\psi_\beta^b(x, \bar{y}_\beta)| \leq e^{-\mu(\varepsilon)L} \qquad , \qquad |\partial_y \psi_\beta^b(x, \bar{y}_\beta)| \leq e^{-\mu(\varepsilon)L} \tag{6.19}$$

for some $\bar{y}_\beta$ depending on ω and L.

Since V_ω is random we expect that wavefunctions with energies in Δ_ε (not too close to the Landau levels where the localization length diverges) are exponentially localized on a scale $\mathcal{O}(1)$ with respect to L. Inequalities (6.19) are a weaker version of this statement, and have been checked for the special case where the random potential is a sum of rank one perturbations [FM01] using the methods of Aizenman and Molchanov [AM93] (see for example [DMP99] where the case of point impurities is treated by these methods). Presumably one could adapt existing techniques for multiplicative potentials to our geometry, to prove hypothesis $(H2)$ at least for energies close to the band tail $\frac{B}{2} + V_0$. One also expects that $\mu(\varepsilon) \to 0$ as $\varepsilon \to 0$. The main physical consequence of $(H2)$ (as shown in Section 6.5) is that a state satisfying (6.19) does not carry any appreciable current (contrary to the eigenstates of H_α) in the sense that $J_\beta^b = (\psi_\beta^b, v_y \psi_\beta^b) = \mathcal{O}(e^{-\mu(\varepsilon)L})$.

We now state our main result.

Theorem 6.1. *Fix $\varepsilon > 0$ and assume that $(H1)$ and $(H2)$ are fulfilled. Assume $B > 4V_0$. Let $p \geq 7$ and $s = \min(\theta, p - 6)$. Then there exists a numerical constant $\gamma > 0$ and an $L(\varepsilon, p, B, V_0)$ such that for all for all $L > L(\varepsilon, p, B, V_0)$ one can find a set $\hat{\Omega}_\Lambda$ of realizations of the random potential with $\mathbb{P}_\Lambda(\hat{\Omega}_\Lambda) \geq 1 - 3L^{-s}$ such that for any $\omega \in \hat{\Omega}_\Lambda$, $\sigma(H_\omega) \cap \Delta_\varepsilon$ is the union of three sets $\Sigma_\ell \cup \Sigma_b \cup \Sigma_r$, each depending on ω and L, and characterized by the following properties:*

a) $E_k^\alpha \in \Sigma_\alpha$ $(\alpha = \ell, r)$ are a small perturbation of $E_{0k}^\alpha \in \sigma(H_\alpha) \cap \Delta_\varepsilon$ with

$$|E_k^\alpha - E_{0k}^\alpha| \leq e^{-\gamma B (\log L)^2}, \qquad \alpha = \ell, r \,. \tag{6.20}$$

b) For $E_k^\alpha \in \Sigma_\alpha$ the current J_k^α of the associated eigenstate satisfies

$$|J_k^\alpha - J_{0k}^\alpha| \leq e^{-\gamma B (\log L)^2}, \qquad \alpha = \ell, r \,. \tag{6.21}$$

c) Σ_b *contains the same number of energy levels as* $\sigma(H_b) \cap \Delta_\varepsilon$ *and*

$$\mathrm{dist}(\Sigma_b, \Sigma_\alpha) \geq L^{-p+1}, \qquad \alpha = \ell, r \; . \tag{6.22}$$

d) The current associated to each level $E_\beta \in \Sigma_b$ *satisfies*

$$|J_\beta| \leq e^{-\gamma B (\log L)^2} \; . \tag{6.23}$$

The proof of the theorem is organized as follows. In Section 6.3 we set up a decoupling scheme by which we express the resolvent of H_ω as an approximate sum of those of the edge and bulk systems. Parts *a*) and *c*) of Theorem 6.1 are proven in Section 6.4. First we compute approximations for the spectral projections of H_ω in terms of the projectors $P(E^\alpha_{0k})$ of H_α and $P_b(\bar{\Delta})$ of H_b (Proposition 6.1). This is done for realizations of the disorder such that the levels of H_b are not "too close" to those of H_α. We then show that these realizations are typical (have large probability) thanks to a Wegner estimate (Proposition 6.2). Parts *b*) and *d*) are proven in Section 6.5 by estimating currents in term of norms of differences between projectors. The appendices contain some technical estimates.

6.3 Decoupling of the Bulk and the Edge Systems

The resolvent $R(z) = (z - H_\omega)^{-1}$ can be expressed, up to a small term, as a sum of the resolvents of the bulk system $R_b(z) = (z - H_b)^{-1}$ and the two edge systems $R_\alpha(z) = (z - H_\alpha)^{-1}$ $(\alpha = \ell, r)$. Here this will be achieved by a *decoupling formula* developed in other contexts [BCD89], [BG91]. We set $D = \log L$ and introduce the characteristic functions

$$\begin{aligned} \tilde{J}_\ell(x) &= \chi_{]-\infty, -\frac{L}{2} + \frac{D}{2}]}(x) \qquad \tilde{J}_b(x) = \chi_{[-\frac{L}{2} + \frac{D}{2}, \frac{L}{2} - \frac{D}{2}]}(x) \\ \tilde{J}_r(x) &= \chi_{[\frac{L}{2} - \frac{D}{2}, +\infty[}(x) \; . \end{aligned} \tag{6.24}$$

We will also use three bounded $C^\infty(\mathbb{R})$ functions $|J_i(x)| \leq 1$, $i \in \mathcal{I} \equiv \{\ell, b, r\}$, with bounded first and second derivatives $\sup_x |\partial_x^n J_i(x)| \leq 2$, $n = 1, 2$, and such that

$$J_\ell(x) = \begin{cases} 1 & \text{if } x \leq -\frac{L}{2} + \frac{3D}{4} \\ 0 & \text{if } x \geq -\frac{L}{2} + \frac{3D}{4} + 1 \end{cases} \qquad J_b(x) = \begin{cases} 1 & \text{if } |x| \leq \frac{L}{2} - \frac{D}{4} \\ 0 & \text{if } |x| \geq \frac{L}{2} - \frac{D}{4} + 1 \end{cases}$$

$$J_r(x) = \begin{cases} 1 & \text{if } x \geq \frac{L}{2} - \frac{3D}{4} \\ 0 & \text{if } x \leq \frac{L}{2} - \frac{3D}{4} - 1 \end{cases} \; . \tag{6.25}$$

For $i \in \mathcal{I}$ we have $H_\omega J_i = H_i J_i$ thus

$$(z - H_\omega) \sum_{i \in \mathcal{I}} J_i R_i(z) \tilde{J}_i = \sum_{i \in \mathcal{I}} (z - H_i) J_i R_i(z) \tilde{J}_i = 1 - \mathcal{K}(z) \tag{6.26}$$

where

$$\mathcal{K}(z) = \sum_{i \in \mathcal{I}} K_i(z) = \sum_{i \in \mathcal{I}} \tfrac{1}{2}[p_x^2, J_i] R_i(z) \tilde{J}_i \ . \tag{6.27}$$

To obtain the second equality one commutes $(z - H_i)$ and J_i and then uses the identity $\sum_{i\in\mathcal{I}} J_i \tilde{J}_i = \sum_{i\in\mathcal{I}} \tilde{J}_i = 1$. From (6.26) we deduce the decoupling formula

$$R(z) = \left(\sum_{i\in\mathcal{I}} J_i R_i(z) \tilde{J}_i \right) (1 - \mathcal{K}(z))^{-1} \ . \tag{6.28}$$

The main result of this section is an estimate of the operator norm of $\mathcal{K}(z)$. In particular it will assure $\|\mathcal{K}(z)\| < 1$.

Lemma 6.1. *Let* $\mathcal{R}e\, z \in \Delta_\varepsilon$ *such that* $\mathrm{dist}(z, \sigma(H_\ell)\cup\sigma(H_r)\cup\sigma(H_b)) \geq e^{-\frac{B}{512}(\log L)^2}$. *Then for* L *large enough there exists a constant* $C(B, V_0) > 0$ *independent of* L *such that*

$$\|\mathcal{K}(z)\| \leq \varepsilon^{-1} C(B, V_0) L e^{-\frac{B}{512}(\log L)^2} \ . \tag{6.29}$$

Proof. Computing the commutator in the definition of $K_i(z)$ and applying the second resolvent formula we have

$$\begin{aligned} K_i(z) &= -\tfrac{1}{2}(\partial_x^2 J_i) R_i(z) \tilde{J}_i - (\partial_x J_i)\partial_x R_i(z) \tilde{J}_i \\ &= -\tfrac{1}{2}(\partial_x^2 J_i) R_0(z) \tilde{J}_i - \tfrac{1}{2}(\partial_x^2 J_i) R_0(z) W_i R_i(z) \tilde{J}_i \\ &- (\partial_x J_i)\partial_x R_0(z) \tilde{J}_i - (\partial_x J_i)\partial_x R_0(z) W_i R_i(z) \tilde{J}_i \end{aligned} \tag{6.30}$$

where we have set $W_\ell = U_\ell$, $W_b = V_\omega$ and $W_r = U_r$. From the triangle inequality and $\|R_i(z)\| = \mathrm{dist}(z, \sigma(H_i))^{-1}$ we obtain

$$\begin{aligned} \|K_i(z)\| &\leq \tfrac{1}{2}\|(\partial_x^2 J_i) R_0(z) \tilde{J}_i\| + \tfrac{1}{2}\|(\partial_x^2 J_i) R_0(z) W_i\| \ \mathrm{dist}(z, \sigma(H_i))^{-1} \\ &+ \|(\partial_x J_i)\partial_x R_0(z) \tilde{J}_i\| + \|(\partial_x J_i)\partial_x R_0(z) W_i\| \ \mathrm{dist}(z, \sigma(H_i))^{-1} \ . \end{aligned} \tag{6.31}$$

To estimate the operator norms on the right hand side it is sufficient to bound them by the Hilbert-Schmidt norms $\|.\|_2$. Using bounds (6.83) on the kernels of $\partial_x^n R_0(z)$ for $n = 0, 1$, and the properties of the functions J_i, $\tilde{J}_i$ we obtain

$$\begin{aligned} \|(\partial_x^{2-n} J_i)\partial_x^n R_0(z) \tilde{J}_i\|_2^2 &= \int_{\mathrm{supp}\,\partial_x^{2-n} J_i} \mathrm{d}\mathbf{x} |\partial_x^{2-n} J_i(x)|^2 \int_{\mathrm{supp}\,\tilde{J}_i} \mathrm{d}\mathbf{x}' |\partial_x^n R_0(\mathbf{x}, \mathbf{x}'; z)|^2 \\ &\leq 4 C_n^2(z, B) \int_{\mathrm{supp}\,\partial_x^{2-n} J_i} \mathrm{d}\mathbf{x} \int_{\mathrm{supp}\,\tilde{J}_i} \mathrm{d}\mathbf{x}' e^{-\frac{B}{4}(x-x')^2} \\ &\leq 4 C_n^2(z, B) e^{-\frac{B}{8}\left(\frac{D}{4}+1\right)^2} \int_{\mathrm{supp}\,\partial_x^{2-n} J_i} \mathrm{d}\mathbf{x} \int_{\mathbb{R}\times[-\frac{L}{2}, \frac{L}{2}]} \mathrm{d}\mathbf{x}' e^{-\frac{B}{8}(x-x')^2} \\ &\leq 16 \sqrt{\tfrac{\pi}{B}} C_n^2(z, B) L^2 e^{-\frac{B}{128} D^2} \ . \end{aligned} \tag{6.32}$$

For the norms involving the potentials W_i we obtain in a similar way

$$\begin{aligned} & \|\partial_x^{2-n} J_i \partial_x^n R_0(z) W_i\|_2^2 \\ = & \int_{\operatorname{supp} \partial_x^{2-n} J_i} \mathrm{d}\mathbf{x} |\partial_x^{2-n} J_i(x)|^2 \int_{\operatorname{supp} W_i} \mathrm{d}\mathbf{x}' |\partial_x^\alpha R_0(\mathbf{x}, \mathbf{x}'; z)|^2 |W_i(\mathbf{x}')|^2 \\ \leq & \; 4C_n^2(z,B) e^{-\frac{B}{128} D^2} \int_{\operatorname{supp} \partial_x^{2-n} J_i} \mathrm{d}\mathbf{x} \int_{\operatorname{supp} W_i} \mathrm{d}\mathbf{x}' e^{-\frac{B}{8}(x-x')^2} |W_i(\mathbf{x}')|^2 \,. \end{aligned} \tag{6.33}$$

It is clear that since V_ω is bounded, and U_ℓ, U_r do not grow faster than polynomials, the double integral in the right hand side of the last inequality is bounded above by L^2 times a constant depending only on B and V_0. From this result, (6.31), (6.32) and $\operatorname{dist}(z, \sigma(H_\ell) \cup \sigma(H_r) \cup \sigma(H_b)) \geq e^{-\frac{B}{512}(\log L)^2}$ we obtain ($\tilde{C}(B, V_0)$ a constant independent of L)

$$\|K_i(z)\| \leq \tilde{C}(B, V_0) \varepsilon^{-1} L e^{-\frac{B}{512}(\log L)^2} \,, \tag{6.34}$$

where we used the expression for $C_n(z, B)$ in Appendix 6.A and the fact that $\mathcal{R}e\, z \in \Delta_\varepsilon$.

□

6.4 Estimates of Eigenprojectors of H_ω

In this section we use the decoupling formula (6.28) to give deterministic estimates for the difference between projectors of H_ω and H_b, H_ℓ and H_r. We then combine this information with a probabilistic estimate (Wegner estimate) to deduce that the spectrum of H_ω is the union of the three sets Σ_ℓ, Σ_r and Σ_b satisfying the parts *a*) and *c*) of Theorem 6.1.

Proposition 6.1. *Assume that* $(H1)$ *holds. Take* $p \geq 7$ *and any* $e^{-\frac{B}{512}(\log L)^2} < \rho < \frac{d(\varepsilon)}{2} L^{-p}$. *For* $L > L(\varepsilon)$ *let* Ω''_Λ *be the set of realizations of the random potential such that for each* $\omega \in \Omega''_\Lambda$ $\operatorname{dist}(\sigma(H_b) \cap \Delta_\varepsilon, E_{0k}^\alpha) \geq d(\varepsilon) L^{-p}$ *for all* $E_{0k}^\alpha \in \Delta_\varepsilon$, $\alpha = \ell, r$. *Then*

i) If $P(E_{0k}^\alpha)$ *is the eigenprojector of* H_α *associated to the eigenvalue* $E_{0k}^\alpha \in \Delta_\varepsilon$ *and* P_k^α *the eigenprojector of* H_ω *for the intervals* $I_k^\alpha = [E_{0k}^\alpha - \rho, E_{0k}^\alpha + \rho]$ *we have*

$$\|P_k^\alpha - P(E_{0k}^\alpha)\| \leq \varepsilon^{-1} C'(B, V_0) L e^{-\frac{B}{512}(\log L)^2} \,. \tag{6.35}$$

ii) Let $\bar{\Delta} \subset \Delta_\varepsilon$ *be an interval such that* $\operatorname{dist}(\bar{\Delta}, \sigma(H_\ell) \cup \sigma(H_r)) = \frac{d(\varepsilon)}{2} L^{-p}$. *If* $P_b(\bar{\Delta})$ *is the eigenprojector of* H_b *for the interval* $\bar{\Delta}$ *and* $P(\bar{\Delta})$ *the eigenprojector of* H_ω *for the interval* $\bar{\Delta}$ *we have*

$$\|P(\bar{\Delta}) - P_b(\bar{\Delta})\| \leq \varepsilon^{-3} C'(B, V_0) L^p e^{-\frac{B}{512}(\log L)^2} \,. \tag{6.36}$$

Proof. We start by proving (6.35) for $\alpha = r$. The case $\alpha = \ell$ is identical. From the decoupling formula we have

$$\begin{aligned} R(z) - R_r(z) &= \left(\sum_{i\in\mathcal{I}} J_i R_i(z)\tilde{J}_i\right)\left(\sum_{n=1}^{\infty}\mathcal{K}(z)^n\right) - (1-J_r)R_r(z) \\ &- J_r R_r(z)(1-\tilde{J}_r) + J_\ell R_\ell(z)\tilde{J}_\ell + J_b R_b(z)\tilde{J}_b\,. \end{aligned} \tag{6.37}$$

Let Γ be a circle of radius ρ in the complex plane, centered at E^r_{0k}. Because of $(H1)$ and $\operatorname{dist}(\sigma(H_b)\cap\Delta_\varepsilon, E^r_{0k}) \geq d(\varepsilon)L^{-p}$, $R_b(z)$ and $R_\ell(z)$ have no poles in Γ. Moreover the only pole of $R_r(z)$ is precisely E^r_{0k}. Thus integrating (6.37) along the circle Γ

$$\begin{aligned} P^r_k - P(E^r_{0k}) &= \frac{1}{2\pi i}\oint_\Gamma \left(\sum_{i\in\mathcal{I}} J_i R_i(z)\tilde{J}_i\right)\sum_{n=1}^{\infty}\mathcal{K}(z)^n\,\mathrm{d}z \\ &- (1-J_r)P(E^r_{0k}) - J_r P(E^r_{0k})(1-\tilde{J}_r)\,. \end{aligned} \tag{6.38}$$

We proceed to estimate the norms of the three contributions on the right hand side of (6.38). The norm of the first term is smaller than

$$\rho\left(\sum_{i\in\mathcal{I}}\sup_{z\in\Gamma}\|R_i(z)\|\right)\frac{\sup_{z\in\Gamma}\|\mathcal{K}(z)\|}{1-\sup_{z\in\Gamma}\|\mathcal{K}(z)\|} \leq 6\varepsilon^{-1}C(B,V_0)Le^{-\frac{B}{512}(\log L)^2}\,. \tag{6.39}$$

Indeed, for $i = r$ we have $\sup_{z\in\Gamma}\|R_r(z)\| = \rho^{-1}$ by construction. For $i = \ell, b$ we have $\sup_{z\in\Gamma}\|R_i(z)\| < \frac{2}{d(\varepsilon)}L^p$. Since $\rho < \frac{d(\varepsilon)}{2}L^{-p}$ we note that in all three cases ($i \in \mathcal{I}$) $\rho\sup_{z\in\Gamma}\|R_i(z)\| \leq 1$. Furthermore, since $\rho > e^{-\frac{B}{512}(\log L)^2}$, using Lemma 6.1 we get (6.39). To estimate the second term in (6.38) we note that by the second resolvent formula

$$\frac{P(E^r_{0k})}{(z-E^r_{0k})} = (z-H_0)^{-1}P_r(E^r_{0k}) + (z-H_0)^{-1}U_r\frac{P(E^r_{0k})}{(z-E^r_{0k})}\,. \tag{6.40}$$

Integrating (6.40) along Γ we obtain the identity

$$P(E^r_{0k}) = (E^r_{0k}-H_0)^{-1}U_r P(E^r_{0k}) \tag{6.41}$$

this implies

$$\begin{aligned} \|(1-J_r)P(E^r_{0k})\| &\leq \|(1-J_r)R_0(E^r_{0k})U_r\| \leq \|(1-J_r)R_0(E^r_{0k})U_r\|_2 \\ &= \left\{\int \mathrm{d}\mathbf{x}|1-J_r(x)|^2\int \mathrm{d}\mathbf{x}'|R_0(\mathbf{x},\mathbf{x}';E^r_{0k})U_r(x')|^2\right\}^{1/2} \end{aligned} \tag{6.42}$$

since the distance (in the $x-$direction) between the supports of $(1-J_r)$ and U_r is greater than $\frac{D}{2}+1$ we can proceed in a similar way as in the estimate of (6.33) to obtain

$$\|(1-J_r)P(E^r_{0k})\| \leq \varepsilon^{-1}\bar{C}(B)Le^{-\frac{B}{64}(\log L)^2} \tag{6.43}$$

where $\bar{C}(B)$ is a constant depending only on B. For the third term in (6.38) we use the adjoint of (6.41)

$$P(E_{0k}^r) = P(E_{0k}^r)U_r(E_{0k}^r - H_0)^{-1} \tag{6.44}$$

to get

$$\|J_r P(E_{0k}^r)(1-\tilde{J}_r)\| \leq \|U_r R_0(E_{0k}^r)(1-\tilde{J}_r)\| \tag{6.45}$$

from which we obtain the same bound as in (6.43). Combining this result with (6.38), (6.39), (6.43) we obtain (6.35) in the proposition.

Let us now sketch the proof of (6.36). From the decoupling formula we have

$$\begin{aligned} R(z) - R_b(z) &= \left(\sum_{i\in\mathcal{I}} J_i R_i(z)\tilde{J}_i\right)\left(\sum_{n=1}^{\infty}\mathcal{K}(z)^n\right) - (1-J_b)R_b(z) \\ &- J_b R_b(z)(1-\tilde{J}_b) + J_\ell R_\ell(z)\tilde{J}_\ell + J_r R_r(z)\tilde{J}_r \,. \end{aligned} \tag{6.46}$$

Given an interval $\bar{\Delta} \subset \Delta_\varepsilon$ such that $\mathrm{dist}(\bar{\Delta}, \sigma(H_\ell)\cup\sigma(H_r)) = \frac{d(\varepsilon)}{2}L^{-p}$, we choose a circle $\bar{\Gamma}$ in the complex plane with diameter equal to $|\bar{\Delta}|$. Then if we integrate over $\bar{\Gamma}$ the last two terms on the right hand side do not contribute while the second and third ones give $(1-J_b)P_b(\bar{\Delta})$ and $J_b P_b(\bar{\Delta})(1-J_b)$. Therefore

$$\begin{aligned} \|P - P_b(\bar{\Delta})\| &\leq |\bar{\Delta}|\left(\sum_{i\in\mathcal{I}}\sup_{z\in\bar{\Gamma}}\|R_i(z)\|\right)\frac{\sup_{z\in\bar{\Gamma}}\|\mathcal{K}(z)\|}{1-\sup_{z\in\bar{\Gamma}}\|\mathcal{K}(z)\|} \\ &+ \|(1-J_b)P_b(\bar{\Delta})\| + \|J_b P_b(\bar{\Delta})(1-\tilde{J}_b)\| \,. \end{aligned} \tag{6.47}$$

From Lemma 6.1, $|\bar{\Delta}| < d(\varepsilon)L^{-1}$ and $\sup_{z\in\bar{\Gamma}}\|R_i(z)\| < \frac{2}{d(\varepsilon)}L^p$ the first term is bounded above by

$$12\varepsilon^{-1}C(B,V_0)L^p e^{-\frac{B}{512}(\log L)^2} \,. \tag{6.48}$$

In order to estimate the second norm in (6.47) we notice that (in the same way as in (6.40), (6.41))

$$P_b(\bar{\Delta}) = \sum_{E_\beta^b\in\bar{\Delta}} R_0(E_\beta^b)V_\omega P_b(E_\beta^b) \tag{6.49}$$

thus

$$\|(1-J_b)P_b(\bar{\Delta})\| \leq \sum_{E_\beta^b\in\bar{\Delta}} \|(1-J_b)R_0(E_\beta^b)V_\omega\|_2 \,. \tag{6.50}$$

Each term of the sum can be bounded in a way similar to (6.33), and since the number of terms in the sum is equal to $\mathrm{Tr}\, P_b(\bar{\Delta})$ we get

$$\begin{aligned} \|(1-J_b)P_b(\bar{\Delta})\| &\leq \varepsilon^{-1}C(B,V_0)Le^{-\frac{B}{64}(\log L)^2}\,\mathrm{Tr}\,P_b(\bar{\Delta}) \\ &\leq 2\varepsilon^{-3}c(B)^2C(B,V_0)V_0^2L^5e^{-\frac{B}{64}(\log L)^2} \,. \end{aligned} \tag{6.51}$$

The second inequality follows from Lemma 6.4 in Appendix 6.B (where we need $B > 4V_0$). For $\|J_b P_b(\bar{\Delta})(1-\tilde{J}_b)\|$ one uses the adjoint of identity (6.49) to obtain the same result. The result (6.36) of the proposition then follows by combining (6.47), (6.48) and (6.51). □

In Appendix 6.B we adapt the method of [CH96] to our geometry to get the following Wegner estimate.

Proposition 6.2. *Let* $B \geq 4V_0$ *and* $E \in \Delta_\varepsilon$

$$\mathbb{P}_\Lambda \{\mathrm{dist}(\sigma(H_b), E) < \delta\} \leq 4c(B)\|h\|_\infty \delta\varepsilon^{-2} V_0 L^4 \ . \tag{6.52}$$

Proof of Theorem 6.1, part a) and c). Let $\omega \in \Omega''_\Lambda$ where Ω''_Λ is the set given in Proposition 6.1. Since for L large enough the right hand side of (6.35) is strictly smaller than one the two projectors necessarily have the same dimension. Therefore $\sigma(H_\omega) \cap I_k^\alpha$ contains a unique energy level E_k^α for each I_k^α of radius ρ. In particular by taking the smallest value $\rho = e^{-\frac{B}{512}(\log L)^2}$ we get (6.20). The number of such levels is $\mathcal{O}(L)$ since they are in one to one correspondence with the energy levels of H_α. The sets Σ_α of Theorem 6.1 are precisely

$$\Sigma_\alpha = \bigcup_k (\sigma(H_\omega) \cap I_k^\alpha \cap \Delta_\varepsilon) , \qquad \alpha = \ell, r \ . \tag{6.53}$$

The set of all other eigenvalues in $\sigma(H_\omega) \cap \Delta_\varepsilon$, defines Σ_b, and is necessarily contained in intervals $\bar{\Delta}$ such that $\mathrm{dist}(\bar{\Delta}, \sigma(H_\ell) \cup \sigma(H_r)) = \frac{d(\varepsilon)}{2} L^{-p}$. In view of (6.20) this implies (6.22). Since the two projectors in (6.36) necessarily have the same dimension, the number of eigenstates in Σ_b is the same than that of $\sigma(H_b) \cap \Delta_\varepsilon$. It remains to estimate the probability of the set Ω''_Λ. The realizations of the complementary set are such that for at least one $E_{0k}^\alpha \in \Delta_\varepsilon$

$$\mathrm{dist}(\sigma(H_b), E_{0k}^\alpha) < d(\varepsilon) L^{-p} \tag{6.54}$$

but from Proposition 6.2 this has a probability smaller than

$$4c(B)\|h\|_\infty d(\varepsilon) L^{-p} \varepsilon^{-2} V_0 L^4 \cdot \mathcal{O}(L) \tag{6.55}$$

where $\mathcal{O}(L)$ comes from the number of levels in $[\sigma(H_\ell) \cup \sigma(H_r)] \cap \Delta_\varepsilon$. Thus for L large enough

$$\mathbb{P}_\Lambda(\Omega''_\Lambda) \geq 1 - L^{6-p} \ . \tag{6.56}$$

We recall that $p \geq 7$. □

6.5 Estimates of Currents

In this section we characterize the eigenvalues of H_ω in terms of the current carried by the corresponding eigenstates. This will yield parts b) and d) of Theorem 6.1.

Proof of Theorem 6.1, part b). Let $E_k^\alpha \in \Sigma_\alpha$. The associated current is by definition

$$J_k^\alpha = \mathrm{Tr}\, v_y P_k^\alpha \tag{6.57}$$

and will be compared to that of ψ^α_{0k}

$$J^\alpha_{0k} = \operatorname{Tr} v_y P(E^\alpha_{0k}) \ . \tag{6.58}$$

The difference between these two currents will be estimated by $\|P^\alpha_k - P(E^\alpha_{0k})\|$. First we observe that $v_y P^\alpha_k$ is trace class. Indeed, $v_y P^\alpha_k = v_y P^\alpha_k P^\alpha_k$ with $v_y P^\alpha_k$ bounded and $\|P^\alpha_k\|_1 = \operatorname{Tr} P^\alpha_k = 1$

$$\|v_y P^\alpha_k\|_1^2 \le \|v_y P^\alpha_k\|^2 \le \|P^\alpha_k v_y^2 P^\alpha_k\| \le 2\|P^\alpha_k (H_\omega - V_\omega) P^\alpha_k\| \le 2E^\alpha_k + V_0 \tag{6.59}$$

to get the second inequality one has simply added positive terms to v_y^2. Similarly

$$\begin{aligned} \|v_y P(E^\alpha_{0k})\|_1^2 &\le \|v_y P(E^\alpha_{0k})\|^2 \le \|P(E^\alpha_{0k}) v_y^2 P(E^\alpha_{0k})\| \\ &\le 2\|P(E^\alpha_{0k}) H_\alpha P(E^\alpha_{0k})\| \le 2E^\alpha_{0k} \ . \end{aligned} \tag{6.60}$$

The identity

$$\begin{aligned} P^\alpha_k - P(E^\alpha_{0k}) &= [P^\alpha_k - P(E^\alpha_{0k})]^2 + [P^\alpha_k - P(E^\alpha_{0k})] P(E^\alpha_{0k}) \\ &+ P(E^\alpha_{0k}) [P^\alpha_k - P(E^\alpha_{0k})] \end{aligned} \tag{6.61}$$

implies

$$\begin{aligned} |J^\alpha_k - J^\alpha_{0k}| &= \left|\operatorname{Tr} v_y [P^k_\alpha - P(E^\alpha_{0k})]\right| \le \left|\operatorname{Tr} v_y [P^\alpha_k - P(E^\alpha_{0k})]^2\right| \\ &+ |\operatorname{Tr} v_y [P^\alpha_k - P(E^\alpha_{0k})] P(E^\alpha_{0k})| \\ &+ |\operatorname{Tr} v_y P(E^\alpha_{0k}) [P^\alpha_k - P(E^\alpha_{0k})]| \ . \end{aligned} \tag{6.62}$$

From (6.62), (6.59) and (6.60) we get

$$\begin{aligned} |J^\alpha_k - J^\alpha_{0k}| &\le 2\left(\|v_y P^\alpha_k\|_1 + \|v_y P(E^\alpha_{0k})\|_1\right) \|P^\alpha_k - P(E^\alpha_{0k})\| \\ &\le 2\left((B+3V_0)^{1/2} + (B+2V_0)^{1/2}\right) \|P^\alpha_k - P(E^\alpha_{0k})\| \ . \end{aligned} \tag{6.63}$$

Combining this last inequality with (6.35) we get the result (6.21) of Theorem 6.1. □

In order to prove part *d*) of Theorem 6.1 we need the following lemma.

Lemma 6.2. *Fix $\omega \in \Omega'_\Lambda$ the set of realizations in (H2). Let ψ^b_1, ψ^b_2 be two eigenstates of H_b with eigenvalues E^b_1 and E^b_2. Then*

$$|(\psi^b_1, v_y \psi^b_2)| \le 2|E^b_1 - E^b_2| L + e^{-\frac{\mu(\varepsilon)}{4} L} \ . \tag{6.64}$$

For $\psi^b_1 = \psi^b_2$, $E^b_1 = E^b_2$ this shows that eigenstates of H_b do not carry any appreciable current. The main idea of the proof sketched below is that v_y is equal to the commutator $[-iy, H_b]$ up to a small boundary term.

Proof. The wavefunctions ψ_1^b and ψ_2^b are defined on $\mathbb{R}\times[-\frac{L}{2},\frac{L}{2}]$, are periodic along y and are twice differentiable in y. Here we will work with periodized versions of these functions where the $y-$direction is infinite (but we keep the same notation). This allows us to shift integrals over y from $[-\frac{L}{2},\frac{L}{2}]$ to $[\bar{y}_2,\bar{y}_2+L]$. We have

$$(\psi_1^b, v_y\psi_2^b) = \int_{\mathbb{R}} \mathrm{d}x \int_{\bar{y}_2}^{\bar{y}_2+L} \mathrm{d}y[\psi_1^b(\mathbf{x})]^*(-i\partial_y - Bx)\psi_2^b(\mathbf{x}) \ . \tag{6.65}$$

An integration by parts yields

$$\begin{aligned} i(\psi_1^b, v_y\psi_2^b) &= \frac{1}{2}\int_{\mathbb{R}} \mathrm{d}x \int_{\bar{y}_2}^{\bar{y}_2+L} \mathrm{d}y[\psi_1^b(\mathbf{x})]^* y(-i\partial_y - Bx)^2\psi_2^b(\mathbf{x}) \\ &- \frac{1}{2}\int_{\mathbb{R}} \mathrm{d}x \int_{\bar{y}_2}^{\bar{y}_2+L} \mathrm{d}y[(-i\partial_y - Bx)^2\psi_1^b(\mathbf{x})]^* y\psi_2^b(\mathbf{x}) + \mathcal{B} \end{aligned} \tag{6.66}$$

where $\mathcal{B}$ is a boundary term given by

$$\begin{aligned} \mathcal{B} &= i\frac{L}{2}\int_{\mathbb{R}} \mathrm{d}x[(-i\partial_y - Bx)\psi_1^b(x,\bar{y}_2)]^*\psi_2^b(x,\bar{y}_2) \\ &+ [\psi_1^b(x,\bar{y}_2)]^*(-i\partial_y - Bx)\psi_2^b(x,\bar{y}_2) \ . \end{aligned} \tag{6.67}$$

We can add a periodized version of V_ω and $\frac{1}{2}p_x^2$ to the kinetic energy operator in both terms on the right hand side of (6.66) and use that ψ_1^b and ψ_2^b are eigenfunctions of H_b to obtain

$$i(\psi_1^b, v_y\psi_2^b) = (E_2^b - E_1^b)\int_{\mathbb{R}} \mathrm{d}x \int_{\bar{y}_2}^{\bar{y}_2+L} \mathrm{d}y y[\psi_1^b(\mathbf{x})]^*\psi_2^b(\mathbf{x}) + \mathcal{B} \ . \tag{6.68}$$

From $|y| \leq |\bar{y}_2| + L \leq 2L$ and the Schwarz inequality we obtain

$$|(\psi_1^b, v_y\psi_2^b)| \leq 2L|E_2^b - E_1^b| + |\mathcal{B}| \ . \tag{6.69}$$

With the help of (6.112), (6.113) in Appendix 6.C we get

$$|\mathcal{B}| \leq e^{-\frac{\mu(\varepsilon)}{4}L} \tag{6.70}$$

this concludes the proof of (6.64). □

Proof of Theorem 6.1, part d). Let $\bar{\Delta}$ an interval like in part *ii*) of Proposition 6.1. We consider the maximal set of intervals $\mathcal{F}_k \subset \bar{\Delta}$ such that $|\mathcal{F}_k| = e^{-\frac{B}{1024}(\log L)^2}$ and $\mathrm{dist}(\mathcal{F}_k,\mathcal{F}_\lambda) \geq 4e^{-\frac{B}{512}(\log L)^2}$, $k \neq \lambda$. Since the number of gaps between the $\mathcal{F}_k$ in $\bar{\Delta}$ is less than $e^{\frac{B}{1024}(\log L)^2}|\bar{\Delta}|$ and $|\bar{\Delta}| < \frac{d(\varepsilon)}{L}$, it follows from Proposition 6.2 that

$$\begin{aligned} \mathbb{P}_\Lambda(\Omega_\Lambda{'''}) &\equiv \mathbb{P}_\Lambda\left(\omega \in \Omega_\Lambda \ : \ \sigma(H_b)\cap\bar{\Delta} \subset \bigcup_k \mathcal{F}_k\right) \\ &\geq 1 - 16c(B)\|h\|_\infty\varepsilon^{-2}V_0L^4e^{-\frac{B}{512}(\log L)^2}e^{\frac{B}{1024}(\log L)^2}\frac{d(\varepsilon)}{L} \\ &= 1 - 16c(B)\|h\|_\infty\varepsilon^{-2}V_0d(\varepsilon)L^3e^{-\frac{B}{1024}(\log L)^2} \ . \end{aligned} \tag{6.71}$$

Now suppose that ψ_β is an eigenstate of H_ω corresponding to $E_\beta \in \bar{\Delta}$. For a given $\omega \in \Omega_\Lambda'''$ one can show that E_β is necessarily included in one of the fattened intervals $\tilde{\mathcal{F}}_k \equiv \mathcal{F}_k + e^{-\frac{B}{512}(\log L)^2}$. In order to check this it is sufficient to adapt the estimates (6.47) to (6.51) to the difference of projectors $\|P(\tilde{\mathcal{F}}_k) - P_b(\tilde{\mathcal{F}}_k)\|$. The main point is to check that with our choice of intervals one is allowed to replace the circle $\bar{\Gamma}$ by circles $\bar{\Gamma}_k$ centered at the midpoint of $\mathcal{F}_k$ and of diameter $e^{-\frac{B}{1024}(\log L)^2} + 2e^{-\frac{B}{512}(\log L)^2}$. We do not give the details here. One finds

$$\|P(\tilde{\mathcal{F}}_k) - P_b(\tilde{\mathcal{F}}_k)\| \leq \varepsilon^{-3} C''(B, V_0) L e^{-\frac{B}{1024}(\log L)^2} \; . \tag{6.72}$$

Therefore $P(\tilde{\mathcal{F}}_k)\psi_\beta = \psi_\beta$ for some k and we have

$$\begin{aligned} J_\beta &= (\psi_\beta, v_y \psi_\beta) = (\psi_\beta, v_y P(\tilde{\mathcal{F}}_k)\psi_\beta) = (P_b(\tilde{\mathcal{F}}_k)\psi_\beta, v_y P_b(\tilde{\mathcal{F}}_k)\psi_\beta) \\ &+ ([P(\tilde{\mathcal{F}}_k) - P_b(\tilde{\mathcal{F}}_k)]\psi_\beta, v_y P_b(\tilde{\mathcal{F}}_k)\psi_\beta) + (\psi_\beta, v_y [P(\tilde{\mathcal{F}}_k) - P_b(\tilde{\mathcal{F}}_k)]\psi_\beta) \; . \end{aligned} \tag{6.73}$$

To estimate the first term on the right hand side of (6.73) we use the spectral decomposition in terms of eigenstates of H_b,

$$P_b(\tilde{\mathcal{F}}_k)\psi_\beta = \sum_{E_\tau^b \in \tilde{\mathcal{F}}_k} (\psi_\tau^b, \psi_\beta)\psi_\tau^b \; . \tag{6.74}$$

We have

$$(P_b(\tilde{\mathcal{F}}_k)\psi_\beta, v_y P_b(\tilde{\mathcal{F}}_k)\psi_\beta) = \sum_{E_\tau^b, E_\sigma^b \in \tilde{\mathcal{F}}_k} (\psi_\beta, \psi_\tau^b)(\psi_\sigma^b, \psi_\beta)(\psi_\tau^b, v_y \psi_\sigma^b). \tag{6.75}$$

From Lemma 6.2 and Lemma 6.4 in Appendix 6.B we get

$$\begin{aligned} |(P_b(\tilde{\mathcal{F}}_k)\psi_\beta, v_y P_b(\tilde{\mathcal{F}}_k)\psi_\beta)| &\leq (\mathrm{Tr}\, P_b(\mathcal{F}_k))^2 4 L e^{-\frac{B}{1024}(\log L)^2} \\ &\leq 16 c(B)^4 \varepsilon^{-4} V_0^4 L^9 e^{-\frac{B}{1024}(\log L)^2} \; . \end{aligned} \tag{6.76}$$

The second term on the right hand side of (6.73) is estimated by the Schwarz inequality

$$\begin{aligned} & ([P(\tilde{\mathcal{F}}_k) - P_b(\tilde{\mathcal{F}}_k)]\psi_\beta, v_y P_b(\tilde{\mathcal{F}}_k)\psi_\beta)^2 \leq \|v_y P_b(\tilde{\mathcal{F}}_k)\psi_\beta\|^2 \|P(\tilde{\mathcal{F}}_k) - P_b(\tilde{\mathcal{F}}_k)\|^2 \\ \leq\ & 2(P_b(\tilde{\mathcal{F}}_k)\psi_\beta, (H_b - V_\omega)P_b(\tilde{\mathcal{F}}_k)\psi_\beta)\|P(\tilde{\mathcal{F}}_k) - P_b(\tilde{\mathcal{F}}_k)\|^2 \\ \leq\ & (B + 3V_0)\|P(\tilde{\mathcal{F}}_k) - P_b(\tilde{\mathcal{F}}_k)\|^2 \; . \end{aligned} \tag{6.77}$$

The third term is treated in a similar way

$$\begin{aligned} (\psi_\beta, v_y[P(\tilde{\mathcal{F}}_k) - P_b(\tilde{\mathcal{F}}_k)]\psi_\beta)^2 &\leq \|v_y \psi_\beta\|^2 \|P(\tilde{\mathcal{F}}_k) - P_b(\tilde{\mathcal{F}}_k)\|^2 \\ &\leq 2(\psi_\beta, (H_\omega - V_\omega)\psi_\beta)\|P(\tilde{\mathcal{F}}_k) - P_b(\tilde{\mathcal{F}}_k)\|^2 \\ &\leq (B + 3V_0)\|P(\tilde{\mathcal{F}}_k) - P_b(\tilde{\mathcal{F}}_k)\|^2 \; . \end{aligned} \tag{6.78}$$

The last estimate (6.23) of Theorem 6.1 then follows from (6.72), (6.76), (6.77) and (6.78). □

Remark. The set $\hat{\Omega}_\Lambda$ in Theorem 6.1 may be taken equal to $\Omega_\Lambda' \cap \Omega_\Lambda'' \cap \Omega_\Lambda'''$. This set has a probability larger than $1 - 3L^{-s}$ with $s = \min(\theta, p - 6)$.

6.A Resolvent of the Landau Hamiltonian

The kernel $R_0(\mathbf{x},\mathbf{x}';z)$ of the resolvent $R_0(z)=(z-H_0)^{-1}$ with periodic boundary conditions along y can be expressed in term of the kernel $R_0^\infty(\mathbf{x},\mathbf{x}';z)$ of the resolvent of the two dimensional Landau Hamiltonian defined on the whole plane $\mathbb{R}^2$. Since the spectrum and the eigenfunctions of H_0 are exactly known, by writing down the spectral decomposition of $R_0(\mathbf{x},\mathbf{x}';z)$ and applying the Poisson summation formula we get for $z\in\rho(H_0)$

$$R_0(\mathbf{x},\mathbf{x}';z)=\sum_{m\in\mathbb{Z}} R_0^\infty(x\,y-mL,x'\,y';z)\ . \tag{6.79}$$

The formula for $R_0^\infty(\mathbf{x},\mathbf{x}';z)$ is (see for example [DMP99])

$$R_0^\infty(\mathbf{x},\mathbf{x}';z)=\frac{B}{2\pi}\Gamma(\alpha_z)U\left(\alpha_z,1;\tfrac{B}{2}|\mathbf{x}-\mathbf{x}'|^2\right)e^{-\frac{B}{4}|\mathbf{x}-\mathbf{x}'|^2}M(\mathbf{x},\mathbf{x}') \tag{6.80}$$

where $\alpha_z=(\frac{1}{2}-\frac{z}{B})$ and

$$M(\mathbf{x},\mathbf{x}')=\exp\left(\tfrac{i}{2}B(x+x')(y-y')\right) \tag{6.81}$$

is the phase factor in the Landau gauge. In (6.80) the Landau levels appear as simple poles of the Euler Γ function and $U(-\lambda,b;\rho)$ is the logarithmic solution of the Kummer equation (see eqns. (13.1.1) and (13.1.6) of [AS70])

$$\rho\frac{d^2U}{d\rho^2}+(b-\rho)\frac{dU}{d\rho}+\lambda\rho=0\ . \tag{6.82}$$

Lemma 6.3. *If* $|\mathcal{I}m\,z|\le 1$, $\mathcal{R}e\,z\in\ \left]\frac{1}{2}B,\frac{3}{2}B\right[$ *and* $\frac{B}{2}|x-x'|^2>1$ *then, for* L *large enough, there exists* $C_n(z,B)$, $n=0,1$ *independent of* L *such that*

$$|\partial_x^n R_0(\mathbf{x},\mathbf{x}';z)|\ \le\ C_n(z,B)e^{-\frac{B}{8}(x-x')^2} \tag{6.83}$$

where $C_n(z,B)=C_nB^{1+\frac{n}{2}}\,\mathrm{dist}(z,\sigma(H_0))^{-1}$ *with* C_n *a numerical positive constant.*

For our purposes we need only decay in the $x-$direction as provided by the lemma but in fact there is also a Gaussian decay in the $y-$direction as long as $|y-y'|<\frac{L}{2}$. One can also prove similar estimates when $\mathcal{R}e\,z$ is between higher Landau levels but the constant is not uniform with respect to ν. Finally we point out that this estimate does not hold for $\frac{B}{2}|\mathbf{x}-\mathbf{x}'|^2<1$ because of the logarithmic singularity in the Kummer function for $\rho\to 0$ (see also Appendix 6.C).

Proof. The proof relies on the estimate (6.10) of [DMP99] which we state here for convenience. For $\lambda=x+iy$, $N-1<x<N$ $(N\ge 1)$, $b\in\mathbb{N}$ and $\rho>1$

$$\begin{aligned}|U(-\lambda,b;\rho)|\ &\le\ 2^{b+N-1}\rho^x(b+N+|y|)^N\frac{|\Gamma(-x)|}{|\Gamma(-\lambda)|}\\ &+\ e^{-(\rho-2)}(\rho+1+|y|)^N\frac{(b+N)!}{|\Gamma(N-\lambda)|}\ .\end{aligned} \tag{6.84}$$

Using this estimate for $N = 1$, $|y| < 1$ and $b = n$ together with $\Gamma(1-\lambda) = -\lambda\Gamma(-\lambda)$ we have (C_n' a numerical constant)

$$|\Gamma(-\lambda)||U(-\lambda, n+1; \rho)| \leq C_n' \rho \left\{ \Gamma(-x) + |\lambda|^{-1} \right\} . \tag{6.85}$$

From (6.85) if $|\mathcal{I}m\, z| \leq 1$, $\mathcal{R}e\, z \in]\frac{1}{2}B, \frac{3}{2}B[$ and $\frac{B}{2}|\mathbf{x}-\mathbf{x}'|^2 > 1$ we deduce the estimate (C_n'' a numerical constant)

$$|\Gamma(\alpha_z) U\left(\alpha_z, n+1; \frac{B}{2}|\mathbf{x}-\mathbf{x}'|^2\right)| \leq B C_n'' \operatorname{dist}(z, \sigma(H_0))^{-1} |\mathbf{x}-\mathbf{x}'|^2 . \tag{6.86}$$

From (6.86) for $n = 0$ and (6.79) we get

$$|R_0(\mathbf{x}, \mathbf{x}'; z)| \leq 2BC_0'' \operatorname{dist}(z, \sigma(H_0))^{-1} e^{-\frac{B}{8}(x-x')^2} \sum_{m\in\mathbb{Z}} e^{-\frac{B}{8}(y-y'-mL)^2} \tag{6.87}$$

since $|y - y'| < L$ the last sum can be bounded by a constant, which yields (6.83) for $n = 0$.

To estimate the first derivative it is convenient to use the relation [AS70]

$$\frac{\mathrm{d}U(-\lambda, 1; \rho)}{\mathrm{d}\rho} = U(-\lambda, 1; \rho) - U(-\lambda, 2; \rho) \tag{6.88}$$

which yields

$$\begin{aligned} \partial_x R_0^\infty(\mathbf{x}, \mathbf{x}'; z) &= \frac{B}{2} \left[(x-x') + i(y-y')\right] R_0^\infty(\mathbf{x}, \mathbf{x}'; z) \\ &- B(x-x') \frac{B}{2\pi} \Gamma(\alpha_z) U\left(\alpha_z, 2; \tfrac{B}{2}|\mathbf{x}-\mathbf{x}'|^2\right) e^{-\frac{B}{4}|\mathbf{x}-\mathbf{x}'|^2} M(\mathbf{x}, \mathbf{x}') . \end{aligned} \tag{6.89}$$

Using (6.86) to bound the two terms on the right hand side of (6.89) we get

$$|\partial_x R_0^\infty(x\, y, x'\, y' - mL; z)| \leq B^{\frac{3}{2}} C_1'' \operatorname{dist}(z, \sigma(H_0))^{-1} e^{-\frac{B}{8}\left[(x-x')^2 + (y-y'-mL)^2\right]} \tag{6.90}$$

the result (6.83) for $n = 1$ then follows from (6.90) and (6.79). □

6.B Bounds on the Number of Eigenvalues in Small Intervals

We first prove a deterministic Lemma on the maximal number of eigenvalues of H_b belonging to energy intervals I contained in Δ_ε. Then we sketch the proof of Proposition 6.2. The ideas in this appendix come from the method used by Combes and Hislop to obtain the Wegner estimate which gives the expected number of eigenvalues in I. Since Lemma 6.4 does not appear in [CH96] and we need to adapt the technique to our geometry we give some details below.

We begin with some preliminary observations on the kernel $P_0(\mathbf{x},\mathbf{x}')$ of the projector onto the first Landau level with periodic boundary conditions along y. Using the spectral decomposition and the Poisson summation formula one gets

$$P_0(x\,y,x'\,y') = \sum_{m\in\mathbb{Z}} P_0^\infty(x\,y-mL,x'\,y') \tag{6.91}$$

where

$$P_0^\infty(\mathbf{x},\mathbf{x}') = \frac{B}{2\pi}e^{-\frac{B}{4}|\mathbf{x}-\mathbf{x}'|^2}e^{i\frac{B}{2}(x+x')(y-y')} \tag{6.92}$$

is the projector on the first Landau level for the infinite plane. The above formula can also be obtained by computing the residues of the poles of the Γ function. We observe that $V_{\mathbf{i}}^{1/2}P_0V_{\mathbf{j}}^{1/2}$ is trace class. Indeed it is the product of two Hilbert-Schmidt operators $V_{\mathbf{i}}^{1/2}P_0$ and $P_0V_{\mathbf{j}}^{1/2}$ and from the expression of the kernel (6.91) it is easily seen that ($c(B)$ a constant independent of L)

$$\|V_{\mathbf{i}}^{1/2}P_0V_{\mathbf{j}}^{1/2}\|_1 \le \|V_{\mathbf{i}}^{1/2}P_0\|_2\|P_0V_{\mathbf{j}}^{1/2}\|_2 \le c(B)V_0\ . \tag{6.93}$$

Lemma 6.4. *Let I be any interval contained in Δ_ε and $P_b(I)$ the eigenprojector associated to H_b. Then*

$$\operatorname{Tr} P_b(I) \le 2\varepsilon^{-2}c(B)^2V_0^2L^4\ . \tag{6.94}$$

Proof. Let $Q_0 = 1-P_0$ and E the middle point of I. Using $Q_0(H_0-E)Q_0 \ge 0$ and $Q_0R_0(E)Q_0 \le (B-V_0)^{-1}Q_0$ we can write

$$\begin{aligned} P_b(I)Q_0P_b(I) &= P_b(I)Q_0(H_0-E)^{1/2}R_0(E)(H_0-E)^{1/2}Q_0P_b(I) \\ &\le (B-V_0)^{-1}P_b(I)(H_0-E)Q_0P_b(I) \\ &\le (B-V_0)^{-1}\left[P_b(I)(H_b-E)Q_0P_b(I)-P_b(I)V_\omega Q_0P_b(I)\right] \end{aligned} \tag{6.95}$$

and thus from $\|P_b(I)(H_b-E)\| \le \frac{|I|}{2}$, we get

$$\|P_b(I)Q_0P_b(I)\| \le (B-V_0)^{-1}\left(\frac{|I|}{2}+V_0\right) \le \frac{3V_0}{2(B-V_0)} \le \frac{1}{2}\ . \tag{6.96}$$

In the last inequality we have assumed that $B \ge 4V_0$. Using $\operatorname{Tr} P_b(I) = \operatorname{Tr} P_b(I)P_0P_b(I) + \operatorname{Tr} P_b(I)Q_0P_b(I)$, $\operatorname{Tr} P_b(I)Q_0P_b(I) \le \|P_b(I)Q_0P_b(I)\| \operatorname{Tr} P_b(I)$, and (6.96) we obtain

$$\operatorname{Tr} P_b(I) \le 2\operatorname{Tr} P_b(I)P_0P_b(I) = 2\operatorname{Tr} P_0P_b(I)P_0\ . \tag{6.97}$$

Now, from

$$\operatorname{dist}(I,\frac{B}{2})^2P_b(I)^2 \le \left(P_b(I)(H_b-\frac{B}{2})P_b(I)\right)^2 \tag{6.98}$$

it follows that

$$\begin{aligned} \operatorname{Tr} P_0P_b(I)P_0 &\le \varepsilon^{-2}\operatorname{Tr}(P_0P_b(I)(H_b-\frac{B}{2})P_b(I)(H_b-\frac{B}{2})P_b(I)P_0) \\ &= \varepsilon^{-2}\operatorname{Tr}(P_0V_\omega P_b(I)V_\omega P_0) \le \varepsilon^{-2}\|P_0V_\omega\|_2\|V_\omega P_0\|_2 \end{aligned} \tag{6.99}$$

each Hilbert-Schmidt norm in (6.99) is bounded by $c(B)V_0L^2$. This observation together with (6.97) gives the result of the lemma. □

Let us now sketch the proof of Proposition 6.2.

Proof of Proposition 6.2. Let $E \in \Delta_\varepsilon$ and $I = [E-\delta, E+\delta]$ for δ small enough (we require that I is contained in Δ_ε). By the Chebyshev inequality we have

$$\mathbb{P}_\Lambda\left\{\mathrm{dist}(\sigma(H_b), E) < \delta\right\} = \mathbb{P}_\Lambda\left\{\mathrm{Tr}\, P_b(I) \geq 1\right\} \leq \mathbb{E}_\Lambda\{\mathrm{Tr}\, P_b(I)\} \tag{6.100}$$

where $\mathbb{E}_\Lambda$ is the expectation with respect to the random variables in Λ. To estimate it we use an intermediate inequality of the previous proof

$$\mathbb{E}_\Lambda\{\mathrm{Tr}\, P_b(I)\} \leq 2\varepsilon^{-2}\mathbb{E}_\Lambda\{\mathrm{Tr}\, P_0 V_\omega P_b(I) V_\omega P_0\}\ . \tag{6.101}$$

Writing $V_{\omega,\Lambda} = \sum_{\mathbf{i}\in\Lambda} X_{\mathbf{i}}(\omega) V_{\mathbf{i}}$

$$\begin{aligned} \mathrm{Tr}\, P_0 V_\omega P_b(I) V_\omega P_0 &= \sum_{\mathbf{i},\mathbf{j}\in\Lambda^2} X_{\mathbf{i}}(\omega) X_{\mathbf{j}}(\omega)\, \mathrm{Tr}\, P_0 V_{\mathbf{i}} P_b(I) V_{\mathbf{j}} P_0 \\ &= \sum_{\mathbf{i},\mathbf{j}\in\Lambda^2} X_{\mathbf{i}}(\omega) X_{\mathbf{j}}(\omega)\, \mathrm{Tr}\, V_{\mathbf{j}}^{1/2} P_0 V_{\mathbf{i}}^{1/2} V_{\mathbf{i}}^{1/2} P_b(I) V_{\mathbf{j}}^{1/2}\ . \end{aligned} \tag{6.102}$$

Since $V_{\mathbf{j}}^{1/2} P_0 V_{\mathbf{i}}^{1/2}$ is trace class we can introduce the singular value decomposition

$$V_{\mathbf{j}}^{1/2} P_0 V_{\mathbf{i}}^{1/2} = \sum_{n=0}^{\infty} \mu_n (\psi_n, .)\phi_n \tag{6.103}$$

where $\sum_{n=0}^{\infty} \mu_n = \|V_{\mathbf{j}}^{1/2} P_0 V_{\mathbf{i}}^{1/2}\|_1$. Then

$$\begin{aligned} \mathrm{Tr}\, V_{\mathbf{j}}^{1/2} P_0 V_{\mathbf{i}}^{1/2} V_{\mathbf{i}}^{1/2} P_b(I) V_{\mathbf{j}}^{1/2} &= \sum_{n=0}^{\infty} \mu_n (\phi_n, V_{\mathbf{i}}^{1/2} P_b(I) V_{\mathbf{j}}^{1/2} \psi_n) \\ \leq &\ \sum_{n=0}^{\infty} \mu_n (\phi_n, V_{\mathbf{i}}^{1/2} P_b(I) V_{\mathbf{i}}^{1/2} \phi_n)^{1/2} (\psi_n, V_{\mathbf{j}}^{1/2} P_b(I) V_{\mathbf{j}}^{1/2} \psi_n)^{1/2} \\ \leq &\ \frac{1}{2} \sum_{n=0}^{\infty} \mu_n \left\{ (\phi_n, V_{\mathbf{i}}^{1/2} P_b(I) V_{\mathbf{i}}^{1/2} \phi_n) + (\psi_n, V_{\mathbf{j}}^{1/2} P_b(I) V_{\mathbf{j}}^{1/2} \psi_n) \right\}\ . \end{aligned} \tag{6.104}$$

An application of the spectral averaging theorem of [CH96] shows that

$$\mathbb{E}_\Lambda\{(\psi_n, V_{\mathbf{j}}^{1/2} P_b(I) V_{\mathbf{j}}^{1/2} \psi_n)\} \leq \|h\|_\infty 2\delta \tag{6.105}$$

as well as for the term with $\mathbf{i}$ replacing $\mathbf{j}$ and ϕ_n replacing ψ_n. Combining (6.101), (6.104), (6.105) and (6.102) we get

$$\mathbb{E}_\Lambda\{\mathrm{Tr}\, P_b(I)\} \leq 4\|h\|_\infty \delta\varepsilon^{-2} \sum_{\mathbf{i},\mathbf{j}\in\Lambda^2} \|V_{\mathbf{j}}^{1/2} P_0 V_{\mathbf{i}}^{1/2}\|_1 \leq 4\|h\|_\infty \delta\varepsilon^{-2} c(B) V_0 L^4\ . \tag{6.106}$$

□

6.C Estimate on the Eigenfunction of H_b

In this section we prove Gaussian decay of the eigenfunction ψ_β^b and its $y-$derivative outside the support of the random potential V_ω. From the eigenvalue equation $(H_0 + V_\omega)\psi_\beta^b = E_\beta^b \psi_\beta^b$ we get

$$\psi_\beta^b = R_0(E_\beta^b) V_\omega \psi_\beta^b \,. \tag{6.107}$$

Thus

$$\begin{aligned} |\psi_\beta^b(\mathbf{x})| &\leq \int_{\mathbb{R}\times I_p} |R_0(\mathbf{x},\mathbf{x}'; E_\beta^b) V_\omega(\mathbf{x}') \psi_\beta^b(\mathbf{x}')| \,\mathrm{d}\mathbf{x}' \\ &\leq V_0 \left\{ \int_{\operatorname{supp} V_\omega} |R_0(\mathbf{x},\mathbf{x}'; E_\beta^b)|^2 \,\mathrm{d}\mathbf{x}' \right\}^{1/2} , \end{aligned} \tag{6.108}$$

and

$$|\partial_y \psi_\beta^b(\mathbf{x})| \leq V_0 \sup_{\mathbf{x}} |\psi_\beta^b(\mathbf{x})| \int_{\operatorname{supp} V_\omega} |\partial_y R_0(\mathbf{x},\mathbf{x}'; E_\beta^b)| \,\mathrm{d}\mathbf{x}' \,. \tag{6.109}$$

We need bounds on the integral kernel R_0 and its $y-$derivative to get an estimate of the eigenfunctions and their $y-$derivative. From [DMP99] we have ($E \in \Delta_\varepsilon$)

$$\begin{aligned} |R_0^\infty(\mathbf{x},\mathbf{x}'; E)| &\leq C(B) |\Gamma(\alpha_E)| e^{-\frac{B}{8}|\mathbf{x}-\mathbf{x}'|^2} \times \\ &\times \begin{cases} 1 & \text{if } \frac{B}{2}|\mathbf{x}-\mathbf{x}'|^2 > 1 \\ 1 + \left|\ln(\frac{B}{2}|\mathbf{x}-\mathbf{x}'|^2)\right| & \text{if } \frac{B}{2}|\mathbf{x}-\mathbf{x}'|^2 \leq 1 \,. \end{cases} \end{aligned} \tag{6.110}$$

Calculating the $y-$derivative thanks to (6.88), and using bounds (6.16) of [DMP99] we have

$$\begin{aligned} &|\partial_y R_0^\infty(\mathbf{x},\mathbf{x}'; E)| \leq C'(B) |\Gamma(\alpha_E)| e^{-\frac{B}{8}|\mathbf{x}-\mathbf{x}'|^2} \times \\ \times &\begin{cases} 1 + |x| & \text{if } \frac{B}{2}|\mathbf{x}-\mathbf{x}'|^2 > 1 \\ \left(1 + \left|\ln(\frac{B}{2}|\mathbf{x}-\mathbf{x}'|^2)\right|\right)(1 + |x| + |\mathbf{x}-\mathbf{x}'|^{-1}) & \text{if } \frac{B}{2}|\mathbf{x}-\mathbf{x}'|^2 \leq 1 \,. \end{cases} \end{aligned} \tag{6.111}$$

With the help of (6.110) and (6.111) we can see that for L large enough

$$|\psi_\beta^b(\mathbf{x})| \leq C(B)\varepsilon^{-1} V_0 L \times \begin{cases} e^{-\frac{B}{8}(x-\frac{L}{2}+\log L)^2} & \text{if } x \notin \left[-\frac{L}{2}, \frac{L}{2}\right] \\ \ln(BL^2) & \text{if } x \in \left[-\frac{L}{2}, \frac{L}{2}\right] \,. \end{cases} \tag{6.112}$$

and

$$|\partial_y \psi_\beta^b(\mathbf{x})| \leq C'(B)\varepsilon^{-2} V_0^2 L^2 \times \begin{cases} e^{-\frac{B}{8}(x-\frac{L}{2}+\log L)^2}(1+|x|) & \text{if } x \notin \left[-\frac{L}{2}, \frac{L}{2}\right] \\ L(\ln(BL^2)^2(1+|x|) & \text{if } x \in \left[-\frac{L}{2}, \frac{L}{2}\right] \,. \end{cases} \tag{6.113}$$

Indeed, for $|m| > 1$ $\frac{B}{2}[(x-x')^2 + (y-y'-mL)^2] > 1$ thus we have

$$|R_0(\mathbf{x},\mathbf{x}'; E_\beta^b)| \leq \tilde{C}(B)\varepsilon^{-1} e^{-\frac{B}{8}(x-x')^2} + \sum_{|m|\leq 1} |R_0^\infty(x\,y, x'\,y' - mL; E_\beta^b)| \,. \tag{6.114}$$

If $x \notin \left[-\frac{L}{2}, \frac{L}{2}\right]$ since $\mathbf{x}' \in \operatorname{supp} V_\omega$ the terms $|m| \leq 1$ have also a Gaussian bound and

$$|R_0(\mathbf{x}, \mathbf{x}'; E^b_\beta)| \leq \tilde{C}'(B)\varepsilon^{-1} e^{-\frac{B}{8}(x-x')^2} \ . \tag{6.115}$$

Replacing this bound in (6.108) we get the Gaussian decay in (6.112) On the other hand if $x \in \left[-\frac{L}{2}, \frac{L}{2}\right]$ we can use the logarithmic bounds for the terms $|m| \leq 1$ and we remark they are integrable and bounded by $L^2 \ln(BL^2)$. The same arguments hold for the $y-$derivative.

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Part II

Magnetic Stark Resonances

Chapter 7

Introduction to quantum resonances

In this chapter we introduce the notion of quantum resonance, but first we need to briefly expose the spectral deformation theory. After a first definition of quantum resonances for Schrödinger operators, we relate it to the spectral deformation theory. We then discuss an alternative definition of resonance related to the notion of time decay, finally we give some hints about the physical notion of resonance. The main reference for this chapter is [HS96].

7.1 Spectral deformation theory

In this section we briefly present the spectral deformation theory, that is one of the main tools used for the study of quantum resonances. We will present this technique in a simple way, and directly related to our study of quantum resonances. We remark that a complementary and important tool is the geometric perturbation theory, namely the geometric resolvent equation already discussed in Section 4.5.

The basic idea of the spectral deformation theory is to consider a one parameter family of transformation of the Euclidian space $\mathbb{R}^d$, represented by a family of unitary operators U_θ, $\theta \in \mathbb{R}$, acting on the Hilbert space $L^2(\mathbb{R}^d)$. Then, given a self-adjoint operator H on $L^2(\mathbb{R}^d)$, we first consider the family of unitary equivalent operators $H(\theta) = U_\theta H U_\theta^{-1}$, $\theta \in \mathbb{R}$. In a second step, we let the parameter θ become complex, provided that the operators $H(\theta)$ satisfy analyticity properties (this constrains the complex parameter θ to be in a suitable open domain $\mathcal{D}_\theta$ in the complex plane having a non empty intersection with the real line). The knowledge of the spectral properties of the non self-adjoint operators $H(\theta)$, $\theta \in \mathcal{D}_\theta$, is of great importance for the study of the quantum resonances. Let us now explain the spectral deformation theory and the main spectral properties of the family of non self-adjoint operators $H(\theta)$, $\theta \in \mathcal{D}_\theta$.

Consider a one parameter Lie group G acting on the Euclidian space $\mathbb{R}^d$. From the group representation theory we know that there exists a strongly continuous unitary representation of G on an Hilbert space $\mathcal{H}$, that will be chosen as $\mathcal{H} = L^2(\mathbb{R}^d)$. Let T the self-adjoint operator generating this one parameter unitary group, that is, G is represented as (Stone Theorem [RS72])

$$\{U_\theta : \theta \in \mathbb{R}\} \quad \text{with} \quad U_\theta = \exp(iT\theta) \ . \tag{7.1}$$

With this we have introduced the first element of the spectral deformation, namely the family of unitary operators U_θ, $\theta \in \mathbb{R}$. Associated to it there is the notion of *analytic vectors for the generator* T. A vector $\psi \in L^2(\mathbb{R}^d)$ is said to be analytic for T if the power series

$$\sum_{n=0}^{\infty} \frac{\theta^n}{n!} \|T^n \psi\| \tag{7.2}$$

has a non zero radius of convergence. We denote $\psi(\theta)$ the corresponding vector valued analytic function. Note that since T is self-adjoint its domain $\mathcal{D}(T)$ contains a dense set $\mathcal{A}$ of analytic vectors [RS75, Cor. 1, p. 203]. On the vectors $\psi \in \mathcal{A}$ the function $\mathbb{R} \ni \theta \mapsto \psi(\theta)$ can be analytically continued to a small complex neighborhood of the origin.

Let now H be a self-adjoint operator acting in the Hilbert space $L^2(\mathbb{R}^d)$ with domain $\mathcal{D}(H)$, we define a *spectral deformation family for* H as a set of linear operators on $L^2(\mathbb{R}^d)$ $\mathcal{U} = \{U_\theta : \theta \in \mathcal{D}_\theta\}$ such that

($H1$) U_θ is unitary for $\theta \in \mathcal{D}_\theta \cap \mathbb{R}$, $U_\theta \mathcal{D}(H) = \mathcal{D}(H)$ for all $\theta \in \mathcal{D}_\theta$ and $U_0 = 1$.

($H2$) There exists a dense set of vectors $\mathcal{A}$ in $L^2(\mathbb{R}^d)$ such that

- the map $\mathcal{A} \times \mathcal{D}_\theta \ni (\psi, \theta) \mapsto U_\theta \psi$ is analytic on $\mathcal{D}_\theta$ with values in $L^2(\mathbb{R}^d)$,
- for $\theta \in \mathcal{D}_\theta$, $U(\theta)\mathcal{A}$ is dense in $\mathcal{A}$.

($H3$) The family of operators $H(\theta) = U_\theta H U_\theta^{-1}$, that are unitary equivalent for $\theta \in \mathcal{D}_\theta \cap \mathbb{R}$, is analytic of type A for $\theta \in \mathcal{D}_\theta$.

We now consider the case where is given an Hamiltonian $H = H_0 + V$, acting in the Hilbert space $L^2(\mathbb{R}^d)$, and a spectral deformation family $\mathcal{U}$. We then say that the real valued function V on $\mathbb{R}^d$ is an *admissible potential* for the spectral deformation family $\mathcal{U}$ if V is an H_0-compact perturbation and $V(\theta) = U_\theta V U_\theta^{-1}$, $\theta \in \mathbb{R}$, has an H_0-compact analytic continuation in an open connected domain of the complex plane with a non empty intersection with the real line.

Let $H = H_0 + V$ be a self-adjoint operator in $L^2(\mathbb{R}^d)$. In what follows we assume that $\mathcal{U} = \{U_\theta : \theta \in \mathcal{D}_\theta\}$ is a spectral deformation family for H, and that V is an admissible potential for $\mathcal{U}$. We then are interested in the study of some general properties of the spectrum of the spectrally deformed operators $H(\theta)$ for $\theta \in \mathcal{D}_\theta$. We will not give a lot of results but just some properties that can characterize "geometrically" the spectrum.

The main properties of the discrete spectrum of $H(\theta)$ are contained in the following

Proposition 7.1.

1. *$\sigma_d(H(\theta))$ is locally independent of θ, that means that the discrete spectrum does not change as long as it is not covered by the essential spectrum (i.e. as long as, varying θ, the eigenvalues in $\sigma_d(H(\theta))$ remain isolated eigenvalues).*

2. *The location of the discrete spectrum with respect to the essential one can be discovered using the following argument: If γ_t, $t \in [0,1]$, is a curve in $\mathcal{D}_\theta$ and $\lambda \notin \sigma_{ess}(H(\gamma_t))$ for any t, then if $\lambda \in \sigma_d(H(\gamma_0))$ also $\lambda \in \sigma_d(H(\gamma_1))$.*

Proof. $H(\theta)$, $\theta \in \mathcal{D}_\theta$ is analytic family of type A. Remark that analyticity of type A implies analyticity in the sense of Kato (see [RS78]).
Point 1. Fix $\theta_0 \in \mathcal{D}_\theta$ and suppose $E \in \sigma_d(H(\theta_0))$. Since $H(\theta)$ is analytic in the sense of Kato, the Kato-Rellich Theorem [RS78, Thm. XII.13] implies that for θ near θ_0 $H(\theta)$ as eigenvalues $E_k(\theta)$ $(1 \leq k \leq m_{alg}(E))$ with $E_k(\theta_0) = E$, and that the branches $E_k(\theta)$ are analytic functions near θ_0.
Now, for $\varphi \in \mathbb{R}$ we have the unitary equivalence $H(\theta_0) \simeq H(\theta_0 + \varphi) = U_\varphi H(\theta_0) U_\varphi^{-1}$. This implies that E remains an isolated eigenvalue of $H(\theta_0 + \varphi)$ and there are no other eigenvalues near E, and thus for $\theta - \theta_0 \in \mathbb{R}$ sufficiently small $E_k(\theta) = E$. The analyticity of $E_k(\theta)$ implies $E_k(\theta) = E$ for all $\theta \in \mathcal{D}_\theta$ where the functions $E_k(\theta)$ are defined, that is where $H(\theta)$ has only point spectrum around $E \in \sigma_d(H(\theta_0))$.
Point 2. First define a sequence θ_n with points on the net defined by the path γ_t as follows $\theta_n = \gamma_{1-\frac{1}{n}}$, $n \geq 1$ $(\theta_1 = \gamma_0,\ \theta_\infty = \gamma_1)$. The argument of Point 1. above, together with $\lambda \notin \sigma_{ess}(H(\theta_n))$ for all n, imply that if $\lambda \in \sigma_d(H(\gamma_0)) \equiv \sigma_d(H(\theta_1))$ then $\lambda \in \sigma_d(H(\theta_n))$ for all n. Then observe that if $\theta_n \to \theta_\infty$, by analyticity, $H(\theta_n) \to H(\theta_\infty)$ in the norm resolvent sense. Therefore, since $\lambda \in \sigma_d(H(\theta_n))$ for all n, one has $\lambda \in \sigma_d(H(\theta_\infty)) \equiv \sigma_d(H(\gamma_1))$ [RS78, pag. 187]. □

To characterize the essential spectrum of $H(\theta)$ we use the result following from the Weyl's theorem on the stability of the essential spectrum under relatively compact perturbations.

Lemma 7.1. [RS78, Cor. 2, p. 113] *Let A be a self-adjoint operator and let B be a relatively compact perturbation of A. Then $\mathcal{D}(A+B) = \mathcal{D}(A)$ and $\sigma_{ess}(A+B) = \sigma_{ess}(A)$.*

This result, together with the fact that $V(\theta)$ is H_0-compact, is useful to characterize $\sigma_{ess}(H(\theta))$ in many situations.

7.2 Aguilar–Combes–Balslev–Simon theory of quantum resonances

This chapter is based on the so called Aguilar–Balslev–Combes–Simon theory of quantum resonances as presented in [HS96]. The main idea of this theory is to define quantum resonances as poles of the meromorphic continuation of certain matrix elements of the resolvent. Then the poles will be identified as the eigenvalues of certain non self-adjoint operators constructed from H. We want to emphasize that the resonances of H do not correspond directly to any spectral data for the self-adjoint operator H.

Let $R_H(z)$ the resolvent of the Hamiltonian H, we define the resonances of H as follows.

Definition 7.1. *The quantum resonances of a Schrödinger operator H associated with a dense set of vectors $\mathcal{A}$ in the Hilbert space $\mathcal{H}$ are the poles of the meromorphic continuations of all matrix elements $(\psi, R_H(z)\varphi)$, $\psi, \varphi \in \mathcal{A}$, from $\{z \in \mathbb{C} : \Im z > 0\}$ to $\{z \in \mathbb{C} : \Im z \leq 0\}$.*

The existence of such meromorphic continuations, the association of the poles of these continuations with the eigenvalues of certain non self-adjoint operators related to H, and the identification of these eigenvalues as resonances, are the main results of the Aguilar–Balslev–Combes–Simon theory. We report here these fundamental results. But before stating them, we need to introduce two supplementary assumptions about the spectra of H and its spectrally deformed $H(\theta)$. These hypothesis are introduced in a slightly different form than in [HS96], this to take in account the specific problem discussed in the next chapter.

($H4$) $\sigma_{ess}(H) = \mathbb{R}$.

($H5$) There exists an open, connected set $\Omega \subset \mathbb{C}$, such that $\Omega^+ \equiv \Omega \cap \mathbb{C}^+ \neq \emptyset$, and $\Omega^- \equiv \Omega \cap \mathbb{C}^- \neq \emptyset$, and for all $\theta \in \mathcal{D}_\theta^+ \equiv \mathcal{D}_\theta \cap \mathbb{C}^+$ one has $\sigma(H(\theta)) \cap \Omega^+ = \emptyset$. Moreover, for each $\varepsilon > 0$, there exists a subset Ω_ε^- (with non empty intersection with $\mathbb{R}$) in the closure $\overline{\Omega^-}$ of Ω^- such that for some $\theta \in \mathcal{D}_{\theta,\varepsilon}^+ \equiv \{\vartheta \in \mathcal{D}_\theta : \Im\vartheta > \varepsilon\}$, we have $\sigma_{ess}(H(\theta)) \cap \Omega_\varepsilon^- = \emptyset$.

where the symbol $\mathbb{C}^+$ means $\mathbb{C} \cap \{z : \Im z > 0\}$ and $\mathbb{C}^-$ means $\mathbb{C} \cap \{z : \Im z < 0\}$.

For the theorem below we suppose that $\mathcal{U}$ is a spectral deformation family that satisfies the assumptions ($H1$) to ($H5$).

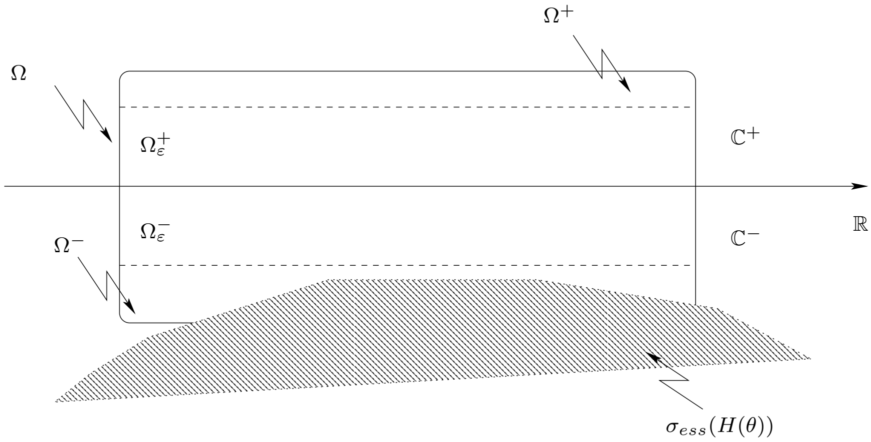

Figure 7.1: The domains in hypothesis $(H5)$ for $\theta \in \mathcal{D}^+_{\theta,\varepsilon}$.

Theorem 7.1. *Let H be a self-adjoint Schrödinger operator with spectral deformation family $\mathcal{U}$ and analytic vectors $\mathcal{A}$. Then*

1. *For $\psi, \varphi \in \mathcal{A}$, the function*

$$F_{\psi,\varphi}(z) = (\psi, R_H(z)\varphi)$$

defined for $\Im z > 0$, has a meromorphic continuation across $\sigma_{ess}(H) = \mathbb{R}$ into Ω_ε^-, for any $\varepsilon > 0$.

2. *The poles of the continuation of $F_{\psi,\varphi}(z)$ into Ω_ε^- are eigenvalues of all the operators $H(\theta)$, $\theta \in \mathcal{D}^+_{\theta,\varepsilon}$, such that $\sigma_{ess}(H(\theta)) \cap \Omega_\varepsilon^- = \emptyset$.*

3. *These poles are independent of $\mathcal{U}$ in the following sense. If $\mathcal{V}$ is another spectral deformation family for H with a set of analytic vectors $\mathcal{B}$ such that the assumptions $(H1)$ to $(H5)$ are satisfied and $\mathcal{A} \cap \mathcal{B}$ is dense, then the eigenvalues of $\hat{H}(\theta) = V_\theta H V_\theta^{-1}$, $\theta \in \mathcal{D}^+_{\theta,\varepsilon}$, in $\overline{\Omega_\varepsilon^-}$ are the same as those of $H(\theta)$ in this region.*

We then have an identification of the quantum resonances as defined in Definition 7.1 with the eigenvalues of the spectrally deformed Hamiltonians $H(\theta)$ in the lower half-plane. More precisely, together with Proposition 7.1, from Theorem 7.1 follows that the resonances of H, denoted $\mathcal{R}(H)$, in the sector $\Omega_\varepsilon^- \subset \mathbb{C}^-$ can be given as

$$\mathcal{R}(H) \cap \Omega_\varepsilon^- = \bigcup_{\theta \in \mathcal{D}^+_{\theta,\varepsilon}} \sigma_d(H(\theta)) \,. \tag{7.3}$$

Clearly we take ε as large as possible, but il may be that the spectral deformation theory does not give all resonances of H, that is $\bigcup_{\theta \in \mathcal{D}^+_{\theta,\varepsilon}} \sigma_d(H(\theta)) \subset \mathcal{R}(H)$.

We now sketch the proof of Parts 1. and 2. of the above theorem, we follow [HS96].

Proof of Theorem 7.1. Denote $F_\pm(z) \equiv F_{\psi,\varphi}(z)$ for $\psi, \varphi \in \mathcal{A}$ and $\Im z \gtrless 0$, and define the set $\Omega_\varepsilon^+ = \{z \in \mathbb{C} : \bar{z} \in \Omega_\varepsilon^-\}$. See also Figure 7.1 for the sets involved in the proof.

Part 1: $F_\pm(z)$ are clearly analytic for $z \in \mathbb{C}\backslash\mathbb{R}$, $(H4)$. In what follows we deal only with $F_+(z)$ and prove that it has a meromorphic continuation across $\sigma_{ess}(H)$. Fix $z \in \mathbb{C}^+$. By $(H1)$ for $\theta \in \mathcal{D}_\theta \cap \mathbb{R}$, U_θ is invertible, thus

$$F_+(z) = (\psi, R_H(z)\varphi) = (\psi, U_\theta^{-1}U_\theta R_H(z)U_\theta^{-1}U_\theta\varphi) = (U_\theta\psi, R_{H(\theta)}(z)U_\theta\varphi) \tag{7.4}$$

where we used $U_\theta^{-1} = U_\theta^*$ and $U_\theta R_H(z)U_\theta^{-1} = R_{H(\theta)}(z)$ since by $(H1)$ $\mathcal{D}(H)$ is invariant under U_θ.
For $\theta \in \mathcal{D}_\theta$ define the function

$$F_+(z;\theta) = (U_{\bar{\theta}}\psi, R_{H(\theta)}(z)U_\theta\varphi)\,. \tag{7.5}$$

Since for $\theta \in \mathbb{R}$, $F_+(z;\theta) = (U_\theta\psi, R_{H(\theta)}(z)U_\theta\varphi)$ and for $\theta \in \mathcal{D}_\theta$, by $(H2)$ and $(H3)$

$$\theta \longmapsto U_\theta\psi \quad , \quad \theta \longmapsto U_\theta\varphi \quad , \quad \theta \longmapsto R_{H(\theta)}(z)\,, \quad z \notin \sigma(H(\theta)) \tag{7.6}$$

are analytic maps[1], $F_+(z;\theta)$ is an analytic map for $z \notin \sigma(H(\theta))$ and is the analytic continuation of $(U_\theta\psi, R_{H(\theta)}(z)U_\theta\varphi)$ for $\theta \in \mathcal{D}_\theta$.
Take now $\varepsilon > 0$ and *fix* $z \in \Omega_\varepsilon^+ \subset \Omega^+$. By $(H3)$ and $(H5)$ the function $F_+(z;\theta)$ defined for $\theta \in \mathcal{D}_{\theta,\varepsilon}^+$ is analytic since there is no spectrum in Ω_ε^+. Furthermore $F_+(z;\theta)$ is constant in θ for $\theta \in \mathbb{R}$, then it is constant for any $\theta \in \mathcal{D}_{\theta,\varepsilon}^+ \subset \mathcal{D}_\theta$, and according to (7.4)

$$F_+(z;\theta) = F_+(z)\,, \qquad z \in \Omega_\varepsilon^+\,. \tag{7.7}$$

Fix now $\theta \in \mathcal{D}_{\theta,\varepsilon}^+$, and let $\Omega_\varepsilon = \Omega_\varepsilon^+ \cup \Omega_\varepsilon^-$ (as required in $(H5)$), by $(H5)$ we have $\sigma_{ess}(H(\theta)) \cap \Omega_\varepsilon = \emptyset$. Therefore $F_+(z;\theta)$ can be meromorphically continued in z from Ω_ε^+ into Ω_ε^-, denote $\tilde{F}_+(z;\theta)$ this continuation. Now by (7.7) $\tilde{F}_+(z;\theta) = F_+(z;\theta) = F_+(z)$, $z \in \Omega_\varepsilon^+$ and by the identity principle for meromorphic function [Rem91] there exist a meromorphic function $\tilde{F}_+(z)$ on Ω_ε equal to $F_+(z)$ for $z \in \Omega_\varepsilon^+$ ($\tilde{F}_+(z) \neq F_-(z)$ for $z \in \Omega_\varepsilon^-$). This function provides the meromorphic continuation of $F_+(z)$ into Ω_ε^- and is given by $\tilde{F}_+(z;\theta)$ for any $\theta \in \mathcal{D}_{\theta,\varepsilon}^+$.

Part 2: The meromorphic continuation of $F_+(z)$ into Ω_ε^- is given by

$$(\psi_\theta, R_{H(\theta)}(z)\varphi_\theta) \tag{7.8}$$

where ψ_θ is the continuation of $U_\theta\psi$ and φ_θ that of $U_\theta\varphi$. By $(H2)$, for $\theta \in \mathcal{D}_\theta$, the set of vectors in $U_\theta\mathcal{A}$ is dense.

[1]This also imply that $\theta \longmapsto U_\theta\bar{\psi}$ is analytic.

Suppose that $H(\theta)$ has an eigenvalue $E(\theta) \in \Omega_\varepsilon^-$, then $(\psi_\theta, R_{H(\theta)}(z)\varphi_\theta)$ will have a (simple) pole at $z = E(\theta)$. Indeed,

$$\lim_{z \to E(\theta)} (z - E(\theta))(\psi_\theta, R_{H(\theta)}(z)\varphi_\theta) = \frac{1}{2\pi i} \oint_\Gamma (\psi_\theta, R_{H(\theta)}(z)\varphi_\theta)\, \mathrm{d}z = (\psi_\theta, P_{H(\theta)}(E(\theta))\varphi_\theta) \ .$$

Now, by density, $(\psi_\theta, P_{H(\theta)}(E(\theta))\varphi_\theta)$ cannot vanish for all $\psi_\theta, \varphi_\theta$ unless $P_{H(\theta)}(E(\theta)) = 0$, that contradict the hypothesis $E(\theta) \in \sigma_d(H(\theta))$. This implies that the meromorphic continuation of $F_+(z)$ have a pole at $z = E(\theta)$ for some $\psi, \varphi \in \mathcal{A}$. Remark that, $E(\theta)$ is independent of θ as long as $E(\theta)$ remains away from the essential spectrum of $H(\theta)$.
On the other hand, if the meromorphic continuation of $F_+(z)$ has a pole at $E(\theta) \in \Omega_\varepsilon^-$, then $E(\theta)$ is an eigenvalue of $H(\theta)$. Indeed, since $E(\theta)$ is a pole the residue associate to it is non vanishing, and given by

$$\frac{1}{2\pi i} \oint_\Gamma (\psi_\theta, R_{H(\theta)}(z)\varphi_\theta)\, \mathrm{d}z = (\psi_\theta, P_{H(\theta)}(E(\theta))\varphi_\theta) \ . \tag{7.9}$$

Thus $P_{H(\theta)}(E(\theta)) \neq 0$. □

7.3 Exponential law and lifetime

The Definition 7.1 is the mathematical definition of quantum resonance in relation with the meromorphic continuation of the resolvent. However there are other possible definition of resonances (see for example [Sim78]). Here we discuss various possible definitions of the so called *time decay* and we shorty explain a formal connection with Definition 7.1. The connection to the spectral deformation theory, at least for the specific model studied in the next chapter, is given. A first tentative definition of time decay is:
Let H be a self-adjoint operator acting in the Hilbert space $\mathcal{H}$. A state $\psi \in \mathcal{H}$ in a resonant state of H with width Γ, if

$$|(\psi, e^{-itH}\psi)|^2 = e^{-\Gamma t} \qquad \textit{for all } t > 0. \tag{7.10}$$

This definition of a quantum resonant state if subjected to several criticisms, indeed it cannot be a good definition for all times. The above definition, for an Hamiltonian bounded from below (or with spectral gaps), can be true only for times neither too small nor too large.

First we look at short times. Let $F(t) \equiv |(\psi, e^{-itH}\psi)|^2$. If the state $\psi \in \mathcal{H}$ has finite energy, $(\psi, H\psi) \leq C < \infty$, then $F(t)$ is differentiable at $t = 0$. Since $F(t) \leq F(0)$ for all $t \in \mathbb{R}$ we have $\frac{dF(t)}{dt}|_{t=0} = 0$. Finally, since $F(t) = F(-t)$ the form $F(t) = e^{-\Gamma|t|}$ is impossible for short times ($t \to 0$).

We now look at long times. Suppose H bounded from below and that $F(t)$ has a upper bound $0 \leq F(t) \leq Ce^{-A|t|}$ $(A, C > 0)$ for $|t| \to \infty$. By the spectral theorem

$$(\psi, e^{-itH}\psi) = \int_{\sigma(H)} e^{-i\lambda t} \mathrm{d}\mu_\psi(\lambda) \; . \tag{7.11}$$

On the other hand, in view of the upper bound on $F(t)^{1/2} = |(\psi, e^{-itH}\psi)|$ for $|t| \to \infty$, the Paley-Wiener theorem [PW34, Thm. I, p. 3] tell us that

$$(\psi, e^{-itH}\psi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-it\lambda}\phi(\lambda)\mathrm{d}\lambda \tag{7.12}$$

with ϕ analytic in the strip $|\Im z| < \frac{1}{2}A$. Thus the spectral measure has the form $\mathrm{d}\mu_\psi(\lambda) = g(\lambda)\mathrm{d}\lambda$ with g an analytic function in the strip $\{z \in \mathbb{C} : |\Im z| < \frac{1}{2}A\}$. But clearly $g(\lambda) = 0$ if $\lambda < \inf \sigma(H)$ (or $\lambda \in \rho(H) \cap \mathbb{R}$ – spectral gap), and by analyticity $g = 0$, that implies $F(t) = 0$.

The short time behavior of (7.10) is always wrong, while the long time one is only valid for unbounded from below Hamiltonians with absolutely continuous spectrum covering the real line. If $\sigma_{ac}(H) = \mathbb{R}$, we have the following meaningfull definition

Definition 7.2. *Let H be a self-adjoint operator acting in the Hilbert space $\mathcal{H}$. A state $\psi \in \mathcal{H}$ in a resonant state of H with width Γ, if there exists some $\varepsilon > 0$, such that*

$$|(\psi, e^{-itH}\psi)|^2 = e^{-\Gamma t}(1 + R(t)) \tag{7.13}$$

for all $t > 0$, where

$$|R(t)| = \mathcal{O}\left(e^{-t\varepsilon}\right), \qquad \textit{as} \qquad t \to \infty.$$

This definition is that used in the next chapter, where we deal with an Hamiltonian that is supposed to have $\sigma_{ac}(H) = \mathbb{R}$.

There is a formal method to get the relationship between the poles of the meromorphic continuation of the resolvent and time decay. Suppose that the matrix element of the resolvent $(\psi, R_H(E + i\epsilon)\psi)$ $(\epsilon > 0)$ has an analytic continuation to the lower half-plane with a pole at $E_r - i\Gamma/2$ $(\Gamma > 0)$. Write the evolution group as the Fourier transform of the spectral density

$$(\psi, e^{-itH}\psi) = \frac{1}{\pi} \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} e^{-itE}\, \Im(\psi, R_H(E + i\epsilon)\psi)\, \mathrm{d}E \; , \tag{7.14}$$

where we used Stone formula for the spectral density

$$Q(E) = \frac{1}{\pi} \lim_{\epsilon \to 0} \Im(\psi, R_H(E + i\epsilon)\psi) \; . \tag{7.15}$$

Then, by residue theorem, we get that the coefficient related to the pole $E_r - i\Gamma/2$ in the integral is ($t \geq 0$)

$$e^{-itE_r - t\Gamma/2} \tag{7.16}$$

which decays exponentially at the rate given by the pole energy imaginary part Γ.
From Definition 7.2 on can introduce the notion of *lifetime* of a resonant state, it is defined as the inverse on the resonance width, namely

$$\tau = \frac{1}{\Gamma} \, . \tag{7.17}$$

The connection between Definition 7.2 and the spectral deformation theory, namely the eigenvalues (discrete spectrum) of the deformed Hamiltonians $H(\theta)$ can be done in a precise mathematical way for some models (i.e. Hamiltonians). Indeed, on can relate the imaginary part of the eigenvalues of $H(\theta)$ with the resonance width Γ associated to a resonant state. Of course, this connection makes sense only if the Hamiltonian has an absolutely continuous spectrum covering the real line.
The idea behind this connection is that, if the spectrum is absolutely continuous and cover the real line, one can write $(\psi, e^{-itH}\psi)$ as the Fourier transform of the spectral density, see (7.14). This has been done for a class of Hamiltonians $H = H_0 + V$ in $L^2(\mathbb{R}^3)$ with $H_0 = -\Delta - Fx$ and V in a class of admissible potential [Her80]. This kind of connection is generalized in presence of a constant magnetic field, for a two dimensional system, with unperturbed Hamiltonian $H_0 = (p - A)^2 - Fx$, see next chapter [FK03b].

7.4 Physical notion of resonances

In theoretical physics, resonances are used to describe quantum states which are almost-bound states[2]. Almost-bound states means the following. At time $t = 0$, consider a state ψ_0 almost localized in a compact set Ω of $\mathbb{R}^d$. This state evolves under the evolution group $U_t = e^{-itH}$, that is $\psi_t = U_t\psi_0$. An almost-bound state is a state that remain concentred for a long time in Ω: such a state is characterized by the fact that it has a finite lifetime and it is called a resonant state or *quantum resonance*.
In many situations an almost-bound quantum state appear as follows. In a first step we have a system with well defined bound states, supported for example in the neighborhood of a local potential V. Then, in a second step, as a perturbation is switched on, these states disappear, due for example to the quantum tunnelling effect. However, we expect that there is a memory of these bound states.
Suppose that the unperturbed Hamiltonian H_0 has some bound states at energies $\{e_k\}$. Their memory, once the perturbation V is added, will be reflected in the following way

[2]Remark that in some domain of physics resonances appear as unstable particles and this is reflected as a bump in the scattering cross section.

on the spectral density $Q(E)$ of $H = H_0 + V$ (associated to a dense set of vectors $\mathcal{A}$). Suppose that there is a resonance at energy $E_k = E_{r,k} - i\Gamma_k/2$ then $Q(E)$ as a sharp peak at energy equal to $E_{r,k} \sim e_k$ (if the perturbation is small in suitable sense) and the width of this peak is Γ_k.
A formal argument for this is the following. From (7.14) and (7.16) we have that ($t \geq 0$)

$$\int_{-\infty}^{\infty} e^{-itE} Q(E)\,\mathrm{d}E = \sum_k C_k \exp\left(-it(E_{r,k} - i\Gamma_k/2)\right) \tag{7.18}$$

and thus by inverse Fourier transform

$$Q(E) = \sum_k \tilde{C}_k \frac{\frac{1}{2}\Gamma_k}{\pi\left[(E - E_{r,k})^2 + \frac{1}{4}\Gamma_k^2\right]} \tag{7.19}$$

that is a sum of Lorentzian functions. For E close of a given resonance energy $E_k = E_{r,k} - i\Gamma_k/2$ the spectral density $Q(E)$ has a peak whose width at half maximum is equal to Γ_k.

Chapter 8

Resonances in crossed fields

In this chapter we present the articles reported in the two next chapters. First we shortly review the work of Martin and Gyger [GM99] that inspired the analysis of resonances in crossed electric and magnetic field, work done in [FK03a] and [FK03b]. Then we present in detail the model studied and we report the main results with some comments.

8.1 The case of a delta interaction

In this section we report the study of quantum resonances for crossed electric and magnetic field in a two dimensional model where the impurity potential is a delta-like interaction at the origin, that is a point impurity.

In [GM99] the authors consider the following model. Denote by H_0 the crossed fields Hamiltonian (electric field F, magnetic field B) in the Landau gauge where the vector potential is given by $A(x,y) = (0, Bx)$

$$H_0 = p_x^2 + (p_y - Bx)^2 - Fx\,. \tag{8.1}$$

Since $H_0 \simeq \int_{\mathbb{R}}^{\oplus} [p_x^2 + (k - Bx)^2 - Fx]\, \mathrm{d}k$, by standard arguments, $\sigma(H_0) = \mathbb{R}$. The total Hamiltonian is obtained by formally adding to H_0 the singular (attractive) potential $V(x,y) = \lambda\delta(x,y)$, where $\delta(x,y)$ is the two-dimensional Dirac distribution, $\lambda < 0$. Remark that, since the delta interaction is a too strong singularity, the model needs to be renormalized.

For the case without electric field ($F = 0$) the essential spectrum is given by $\{(2n+1)B : n \in \mathbb{N}\}$ (the Landau levels), and the discrete spectrum consists of non degenerate eigenvalues $\{E_j : j \in \mathbb{N}\}$ in between the Landau levels (one for each "gap"), with $E_0 < B$. The eigenfunctions associated to the energy levels E_j are denoted by ψ_j.

When the electric field is switched on all localized states created by the impurity are turned into resonances. The resonance lifetimes are characterized as follows.

The time dependent decay amplitude $(\psi_j, e^{-iHt}\psi_j)$ of the j^{th} impurity state under a weak electric field is given by the Fourier transform of the spectral density, namely

$$Q_j(E) = \frac{1}{2\pi i} \lim_{\eta \to 0^+} (\psi_j, [R(E+i\eta) - R(E-i\eta)]\psi_j) \tag{8.2}$$

where $R(z) = (H-z)^{-1}$. In [GM99], the authors show that $Q_j(E)$, in a neighborhood of E_j, for F small, behaves as a Lorentzian plus a correction,

$$Q_j(E) \sim \frac{\frac{1}{2}\Gamma_j}{\pi[(E-E_j)^2 + \frac{1}{4}\Gamma_j^2]} + Q_0(E) \tag{8.3}$$

where $Q_0(E)$ is bounded (for E in a neighborhood of E_j). The factor Γ_j has the form

$$\Gamma_j = C_j \frac{\sqrt{B^3}}{F} \left(\frac{F}{\sqrt{B}\Delta_j} \right)^{-2j} \exp\left(-\frac{B\Delta_j^2}{F^2} \right) \tag{8.4}$$

where Δ_j is the distance between E_j and the closest Landau level. The lifetime of the j^{th} resonance is given by the inverse of Γ_j: $\tau_j = \Gamma_j^{-1}$.

Theorem 8.1. [GM99] *For the model above all the localized states created by the impurity are turned into resonances, and the lifetime of the j^{th} resonance is*

$$\tau_j \simeq \mathcal{O}\left(\exp\left[\frac{\Delta_j^2 B}{F^2} \right] \right) \tag{8.5}$$

for $F \to 0$.

We remark that, in the presence of the magnetic field, the lifetimes are no more exponential in $1/F$ as in the usual Stark effect, but gaussian in $1/F$. We will see below that such a behavior is a lower bound for the lifetime in the more general case studied in this thesis.

8.2 The case of a multiplicative potential

Here we discuss the model and present the main results obtained in the works [FK03a] (see Chapter 9) and [FK03b] (see Chapter 10).

In [FK03a] and [FK03b] we are interested in the study of resonances for the following physical model. Consider, in a first step, a spinless quantum particle (an electron) on the configuration space $\mathbb{R}^2$ that is submitted to a perpendicular homogeneous magnetic field B. The associated vector potential is denoted by $A = (A_x, A_y)$, and satisfies $B = \partial_x A_y - \partial_y A_x$. The particle is also submitted to a potential V that satisfies certain localization conditions. The self-adjoint operator $H(0) = (p_x - A_x)^2 + (p_y - A_y)^2 + V$ has typically a pure point spectrum. The essential spectrum consists of the Landau levels, while the discrete spectrum consists of eigenvalues in between the Landau levels, the latter correspond to the so called impurity states and are created by the potential V. The main question that we address is what happens with these localized states when a constant electric field F is switched on. In particular one would like to know, whether the eigenvalues of $H(0)$ may survive in the presence of a nonzero electric field and if not, what is the characteristic time in which they dissolve.

8.2.1 The model and the spectral deformation family

We now describe the basic properties of the model, in what follows we will work in the system of units, where $m = 1/2$, $e = 1$, $\hbar = 1$. The Hilbert space for the model is clearly

$$\mathcal{H} = L^2(\mathbb{R}^2, \mathrm{d}x\,\mathrm{d}y). \tag{8.6}$$

A first useful Hamiltonian is the crossed fields Hamiltonian

$$H_1(F) = H_L - Fx = (-i\partial_x + By)^2 - \partial_y^2 - Fx\,. \tag{8.7}$$

Here we use the Landau gauge with $A(x,y) = (-By, 0)$. This choice of the gauge is different from that used in (8.1) and turn out to be the "right" gauge to use when we deal with complex translations along the $x-$direction (see below).
A straightforward application of [RS75, Thm. X.37] shows that $H_1(F)$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$, see also [RS75, Prob. X.38]. Moreover, one can easily check that

$$\sigma(H_1(F)) = \sigma_{ac}(H_1(F)) = \mathbb{R}\,. \tag{8.8}$$

The second useful Hamiltonian is the impurity Hamiltonian

$$H(0) = H_L + V = (-i\partial_x + By)^2 - \partial_y^2 + V \tag{8.9}$$

where V is an H_L-compact bounded symmetric perturbation that satisfies the *assumptions* given below. $H(0)$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$ [RS75, Thm. X.34], and its spectrum is given by

$$\sigma_{ess} = \{(2n+1)B : n \in \mathbb{N}\} \qquad \sigma_d = \{e_\alpha\}\,.$$

The full Hamiltonian, for which we want to study the resonances, is

$$H(F) = H_L + V - Fx = (-i\partial_x + By)^2 - \partial_y^2 + V - Fx\,, \tag{8.10}$$

it is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$.

We now introduce the *spectral deformation family* for $H(F)$. We consider the one parameter family of translations of $\mathbb{R}^2$ in the $x-$direction. To each $\theta \in \mathbb{R}$ corresponds a translation

$$(x, y) \longmapsto (x - \theta, y)$$

that is represented in the Hilbert space $L^2(\mathbb{R}^2)$ by a unitary operator U_θ. The generator of the one parameter group is the self-adjoint operator p_x. We have

$$U_\theta = \exp(ip_x\theta) \tag{8.11}$$

and its action on a function in $L^2(\mathbb{R}^2)$ is

$$(U_\theta f)(x, y) = (\exp(ip_x\theta)f)(x, y) = f(x + \theta, y) \,. \tag{8.12}$$

We now let the parameter θ become complex, that is $\theta = a + ib$, moreover we impose $\theta \in \mathcal{D}_\theta$ where $\mathcal{D}_\theta$ is the following strip in the complex plane

$$\mathcal{D}_\theta = \{\theta \in \mathbb{C} : |\Im\theta| < C_F\}$$

where $C_F \to 0$ for $F \to 0$[1]. Moreover, without loss of generality we take $a = 0$ and $b \in \mathbb{R}_+$. This sets up the spectral deformation family $\mathcal{U}$. Before commenting about the hypothesis on $\mathcal{U}$, we look at its action on the Hamiltonians defined above and we define the class of $H_1(F)-$*translation analytic potentials.*

An elementary calculation shows that the translated operator $H_1(F, \theta)$ is given by

$$H_1(F, \theta) = U(\theta)H_1(F)U^{-1}(\theta) = H_1(F) - F\theta \tag{8.13}$$

that is clearly an analytic family of type A (for θ).

Definition 8.1. *Suppose that $V(z, y)$ is analytic in the strip $|\Im z| < \beta$, $\beta > 0$ independent of y. We then say that V is $H_1(F)-$translation analytic if $V(x + z, y)(H_1(F) + i)^{-1}$ is a compact analytic operator valued function of z in the given strip.*

We are now ready to give the *Assumptions on the potential* V:

(*a*) $V(x, y)$ is $H_1(F)-$translation analytic in the strip $|\Im z| < \beta$.

(*b*) There exists $\beta_0 \leq \beta$ such that for $|\Im z| \leq \beta_0$ the function $V(x + z, y)$ is uniformly bounded and

$$\lim_{x,y\to\pm\infty} |V(x + z, y)| = 0 \,.$$

(*c*) The operator $H(F) = H_1(F) + V$ has purely absolutely continuous spectrum.

[1]To understand this choice see Chapter 10.

In order to characterise the potential class for which the above conditions are fulfilled let us assume for the moment, that the integral kernel of $(H_1(F) + i)^{-1}$ has at most a local logarithmic singularity at the origin. This is a very plausible hypothesis (see Chapter 9), it then follows that any $L^2(\mathbb{R}^2)$ function which tends to zero at infinity and can be analytically continued in a given strip $|\Im z| < \beta$ satisfies the conditions (a) and (b). We can take a Gaussian as an elementary example. The condition (c) is more delicate. For a fixed value of F one can specify the corresponding potential class satisfying (c) with the help of the Mourre commutator method, see [Mou81]. Roughly speaking, the spectrum of $H(F)$ will be purely absolutely continuous whenever $\|\partial_x V(x,y)\|_\infty < F$. This gives us the sought criteria in the situation, when F is fixed.

From assumption (b) and the analyticity of V it follows that

$$H(F,\theta) = U(\theta)H(F)\,U^{-1}(\theta) = H_1(F,\theta) + V\,(x+\theta, y) \tag{8.14}$$

forms an analytic family of type A. Indeed, $V(x+\theta, y)$ and clearly $-F\theta$ are bounded operators for $\theta \in \mathcal{D}_\theta$, thus the domain $\mathcal{D}(H(F,\theta))$ is independent of θ and given by $\mathcal{D}(H_1(F)) \equiv \mathcal{D}$. Moreover, for each $\psi \in \mathcal{D}$, $H(F,\theta)\psi$ is a vector valued analytic function of θ. This can be seen using the fact that $(\varphi, H(F,\theta)\psi)$ is a complex valued analytic function for each $\varphi \in L^2(\mathbb{R}^2)$, and that weak analyticity implies strong analyticity [RS72, Thm. VI.4].
Type A analytic family property implies that the hypothesis $(H1)$ and $(H3)$ given in Section 7.1 for a spectral deformation family are fulfilled. We now look at hypothesis $(H2)$. Since we are dealing with a one parameter group, we have immediately a dense set of analytic vectors $\mathcal{A}$ contained in the domain $\mathcal{D}(p_x)$ of p_x, the generator of the one parameter group. [RS75, Cor. 1, p. 203].

Given such set $\mathcal{A}$, we have the following required properties [HS96, Prop. 17.10]

- the map $\mathcal{A} \times \mathcal{D}_\theta \ni (\psi, \theta) \mapsto U_\theta \psi$ is an analytic $L^2(\mathbb{R}^2)-$valued function,
- for any $\theta \in \mathcal{D}_\theta$, $U(\theta)\mathcal{A}$ is dense in $L^2(\mathbb{R}^2)$.

Therefore $\mathcal{U}$ is a spectral deformation family for $H(F)$.

We now briefly look at the spectral properties of the deformed Hamiltonians $H_1(F,\theta)$ and $H(F)$. We have

$$\sigma(H_1(F,\theta)) = \mathbb{R} - ibF$$

and, since $V(x+\theta, y)(H_1(F) + i)^{-1}$ is compact by (a), we have [RS78, Cor. 2, p. 113]

$$\sigma_{ess}(H(F,\theta) + ibF) = \sigma_{ess}(H_1(F)) = \mathbb{R} \implies \sigma_{ess}(H(F,\theta)) = \mathbb{R} - ibF \tag{8.15}$$

where $\theta = ib$, $b \in \mathbb{R}$. Moreover, all the eigenvalues of $H(F, ib)$ lie in the strip $-bF < \Im z \leq 0$ and are independent of b as long as they are not covered by the essential spectrum. By the way, remark that hypothesis $(H5)$ in Section 7.2 is fulfilled. In Figure 8.1 we represent schematically the spectrum of $H(F, ib)$.

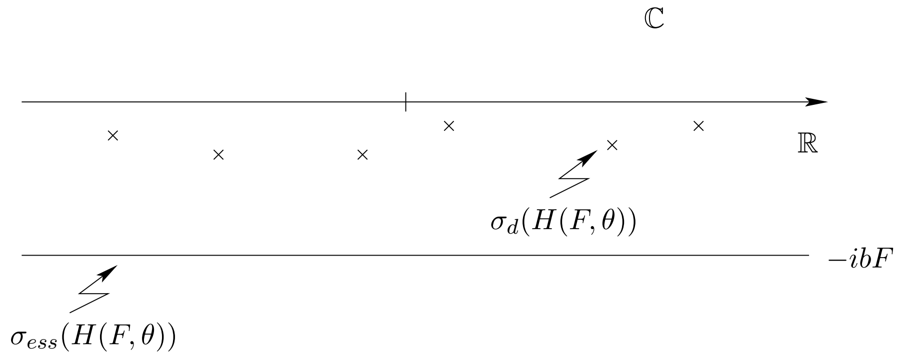

Figure 8.1: A schematic representation of the spectrum of $H(F, \theta)$.

Finally, we remark that (a) implies $\sigma_{ess}(H) = \sigma_{ess}(H_1) = \mathbb{R}$, thus also the hypothesis $(H4)$ given in Section 7.2 is satisfied. Thus via Theorem 7.1 we can identify the resonances of $H(F)$, defined as the poles of the meromorphic continuation across the real axis of the matrix elements of the resolvent, with the eigenvalues of $H(F, \theta)$ in the lower complex plane. We will see that this identification can also be proven, for our model, starting form the time decay definition of quantum resonances (Section 7.3).

8.2.2 Main results and comments

We now report the main Theorems contained in [FK03a] and [FK03b]. The first one makes the connection between the time decay definition of a quantum resonance and the eigenvalues of the spectrally deformed Hamiltonian $H(F, \theta)$. On the other hand, in the second paper we prove some upper bound on the resonance width, or equivalently a lower bound for the lifetime associated to the corresponding resonant state.

Before giving the first theorem we need the following definition.

Definition 8.2. *Let A be any open complex domain having non-empty intersection with $\mathbb{R}$. Then we denote by $\mathcal{D}(A)$ the set of those vectors f, for which $f_\theta = U(\theta)f$, $\theta \in \mathbb{R}$ can be analytically continued to A.*

Theorem 8.2. [FK03a] *Take $\alpha := \alpha_0 F > 0$ sufficiently small such that the conditions (a), (b) and (c) are satisfied for $\beta_0 > \alpha$. Assume moreover that $bF > \alpha$ and let*

ψ, ϕ, $H_1(F)\psi$, $H_1(F)\phi \in \mathcal{D}(\{z \in \mathbb{C} : |\Im z| < 2bF\})$. *Then for any* $t \geq 0$

$$(\psi, e^{-itH(F)}\phi) = \sum_{-\Im E_j \leq \alpha} (\psi_{-ib}, P_j(ib)\phi_{ib})\, e^{-itE_j} + R(t) \tag{8.16}$$

where

$$R(t) \leq \mathcal{C}\, e^{-t(\alpha+\epsilon)}$$

for some $\epsilon > 0$. *Here* $P_j(ib)$ *is the spectral projector of* $H(F, ib)$ *associated with the eigenvalue* E_j. *Moreover, the sum in* (8.16) *is finite and* $f_j(z) = (\psi_{\bar{z}}, P_j(z)\phi_z)$ *is independent of* z *as long as* $-F\Im z < \Im E_j$.

We thus know that the resonance widths are given by the imaginary parts of the eigenvalues of $H(F, ib)$, we now give the results concerning these imaginary parts. For the theorems that follow we need a stronger hypothesis on the decay at the infinity for the potential V (deformed):

(d)

$$|V(x+ib, y)| \leq \begin{cases} V_0 & \text{if } x \in [-a_0, a_0], y \in [-a_1, a_1] \\ V_0\, e^{-\nu x^2},\ \nu > 0 & \text{if } x \notin [-a_0, a_0] \end{cases}$$

and

$$|V(x+ib, y)| = 0, \quad y \notin [-a_1, a_1]$$

for given positive constants a_0, a_1, independent of F. We remark that we could replace the localization of V w.r.t. y by a Gaussian decay, we choose (d) in order to keep the computations as simple as possible.

The first result concerns the behavior of the eigenvalues of $H(F, ib)$ as $F \to 0$.

Theorem 8.3. [FK03b] *Assume* V *satisfies* (a), (d) *and let* e_α *be an eigenvalue of* $H(0)$ *of multiplicity* $r_\alpha < \infty$ *at finite distance from the Landau levels. Then near* e_α *there are eigenvalues* $E_{\alpha,i}$ *of* $H(F, ib), (1 \leq i \leq r_\alpha)$, *repeated according to their multiplicity, and*

$$E_{\alpha,i} \to e_\alpha \quad \text{as} \quad F \to 0.$$

Our main result concerns the imaginary part of the above eigenvalues.

Theorem 8.4. [FK03b] *Assume* V *satisfies* (a) *and* (d). *Let* e_α *and* $E_{\alpha,i}$ *be the eigenvalues defined in Theorem 8.3. Then there exist some positive constants* $\mathcal{C}$ *and* $R_\alpha(B)$, *such that for* F *small enough the following inequality holds true*

$$|\Im E_{\alpha,i}| \leq \mathcal{C}\, e^{-\frac{R_\alpha(B)}{F^{2(1-\varepsilon)}}}, \quad \varepsilon > 0$$

ε *can be made arbitrarily small and* $R_\alpha(B) = B\tilde{R}_\alpha$. *The result in not uniform in* α *since* $\tilde{R}_\alpha \to 0$ *as* $e_\alpha \to \infty$.

Finally for the lifetime we have

Corollary 8.1. *The life-times of the resonant states satisfy:*

$$\tau_\alpha = \tfrac{1}{2} \sup_{\varepsilon>0} |\Im E_{\alpha,i}|^{-1} \geq 1/\mathcal{C} \, \exp\left(\frac{B\tilde{R}_\alpha}{F^2}\right) .$$

We now give the idea of the proof of the two last theorems. To show that the eigenvalues of $H(F,\theta)$ are located in a Gaussian small vicinity of real axis as $F \to 0$ we employ a geometric resolvent equation to separate the configuration space in many pieces. The idea of our proof is based on the fact that the eigenfunctions of $H(0)$ have a Gaussian-like decay at infinity and therefore "feel" the electric field only locally. That leads us to a construction of the reference Hamiltonian $H_2(F)$, which describes the system where the electric field is localized in the vicinity of impurity potential V by a suitable cut-off function. When $F \to 0$ we let the cut-off function tend to 1 at the rate proportional to $F^{-1+\varepsilon}$ ($\varepsilon > 0$), which assures the convergence of spectra of $H_2(F)$ to that of $H(0)$. Moreover $\sigma(H_2(F,\theta))$ remains real even when θ becomes complex. The geometric resolvent equation then allows us to deduce that for F small enough the resolvent $R(z;\theta) = (z - H(F,\theta))^{-1}$ is bounded except in a small neighborhood of the eigenvalues of $H_2(F,\theta)$. More precisely, we show that the norm of $R(z;\theta)$ remains bounded as long as the distance between z and $\sigma(H_2(F,\theta))$ is at least of order

$$e^{-\frac{B\,C}{F^{2(1-\varepsilon)}}}, \quad \varepsilon > 0, \tag{8.17}$$

where C is a strictly positive constant and ε can be taken arbitrarily small. Moreover, we prove that on the energy intervals well separated from Landau levels the spectral projector of $H(F,\theta)$ converges uniformly to that of $H_2(F,\theta)$ as $F \to 0$. These results give us the existence of eigenvalues of $H(F,\theta)$ and an upper bound on their imaginary parts.

Finally we make some comments. First note that our result doesn't exclude the existence of point spectrum of $H(F)$. In other words, we do not answer the question, whether all impurity states become unstable once the electric field with finite intensity is switched on. Although the quantum tunnelling phenomenon leads us to believe that it is indeed the case, a rigorous proof is missing and the question remains open.

However, if we assume that the spectrum of $H(F)$ is purely absolutely continuous (assumption (c)), we get a lower bound on the life-times of the corresponding resonances in the form

$$\tau(B,F) \geq \mathcal{C}\, e^{\frac{BC}{F^2}},$$

which is to be compared with the exponential law for the life-times of purely electric Stark resonances. The fact that the lower bound on the resonance life-times is Gaussian in F^{-1} and not exponential is due to the presence of the magnetic field. However, further comparison with the purely electric Stark effect shows much larger restriction on the class of admissible potentials, in particular the condition on the Gaussian decay of $V(x, y)$. As follows from the analysis of the Stark resonances, [Opp28] [HS80] [Sig88], the exponential law for the resonant states is in that case directly connected with the exponential decay of the eigenfunctions of a "free" Hamiltonian, i.e. without electric field. If we suppose that the same connection exists also in the magnetic case, then our result should hold whenever the eigenfunctions of $H(0) = H_L + V$, associated with the discrete spectrum, fall off as a Gaussian. Sufficient condition for the latter is the Gaussian decay of $V(x, y)$, see [CN98], which is compatible with our assumption (d). Up to now, the optimal condition is known only for the ground state, in which case a sort of exponential decay of $V(x, y)$ is shown to be sufficient and necessary for Gaussian behavior of the corresponding eigenfunctions at infinity, [Erd96].

Such a restriction is in contrast with the non magnetic Schrödinger operator, whose eigenfunctions decrease exponentially in the classically forbidden region independently on the rate at which $V(x, y)$ tends to zero at infinity. This might indicate a principal difference between the behavior of resonant states in the presence respectively absence of magnetic field.

In the next two chapters we report the articles [FK03a] (Chapter 9) and [FK03b] (Chapter 10) without the references that are included in the bibliography of this thesis.

Chapter 9

Exponential decay

In this chapter we report the article [FK03a].

On the Exponential Decay of Magnetic Stark Resonances

Christian Ferrari and Hynek Kovarik

Abstract

We study the time decay of magnetic Stark resonant states. As our main result we prove that for sufficiently large time these states decay exponentially with the rate given by the imaginary parts of eigenvalues of certain non-selfadjoint operator. The proof is based on the method of complex translations.

9.1 Introduction

The purpose of this paper is to study the decay properties of resonances in two dimensions in the presence of crossed magnetic and electric fields and a potential type perturbation. We assume that the magnetic field acts in the direction perpendicular to the electron plane with a constant intensity B and that the electric field of constant intensity F points in the $x-$direction. The perturbation $V(x,y)$ is supposed to satisfy certain localisation conditions. The corresponding quantum Hamiltonian reads as follows

$$H(F) = H(0) - Fx = H_L + V - Fx,$$

where H_L is the Landau Hamiltonian of an electron in a homogeneous magnetic field of intensity B.

We begin with the definition of a resonance in terms of an exponential time decay of the corresponding resonant states. In Section 9.3 we show the connection between these time decaying states and the usual spectral deformation notion of resonance. The basic mathematical tool we use is the method of complex translations for Stark Hamiltonians, which was introduced in [AH77] as a modification of the original theory of complex scaling [AC71], [BC71]. Following [AH77] we consider the transformation $U(\theta)$, which acts as a translation in $x-$direction; $(U(\theta)\psi)(x) = \psi(x+\theta)$. For non real θ the translated operator $H(F,\theta) = U(\theta)H(F)U^{-1}(\theta)$ is non-selfadjoint and therefore can have some complex eigenvalues. The main result of Section 9.3, Theorem 9.1, tells us that if ϕ is an eigenfunction of $H(0)$, then $(\phi, e^{-itH(F)}\,\phi)$ decays exponentially at the rate given by the imaginary parts of the eigenvalues of $H(F,\theta)$. Theorem 9.1 thus can be regarded as a generalisation of the result obtained in [Her80], where the exponential decay was proved for the Stark Hamiltonians without magnetic field.

Of course on would like to know how the resonance widths behave as functions of F. This question is discussed in [FK03b] in which we prove that for $F \to 0$ the resonance widths decay as $\exp[-\frac{B}{F^2}]$ in contrast with the usual Stark resonances, where the behaviour is exponential. However, the technique used in our next paper requires some specific properties of the Green's function $G_1(\mathbf{x},\mathbf{x}';z)$ of the operator

$$H_1(F) = H_L - Fx,$$

in the limit $F \to 0$. In particular, on need to know that $G_1(\mathbf{x},\mathbf{x}';z)$ is exponentially decaying with respect to $(x'-x)^2$ and $|y'-y|$. While similar behaviour is well known in case of purely magnetic Hamiltonian, where the Green's function is given explicitly, to the best of our knowledge there is no explicit formula for the Green's function of the crossed fields Hamiltonian $H_1(F)$. The direct application of these results on the crossed fields Green's function motivates us to include them as a second part of the present paper. However, the estimations of $G_1(\mathbf{x},\mathbf{x}';z)$ could be of general interest for other problems dealing with simultaneous electric and magnetic fields.

9.2 The Model

We work in the system of units, where $m = 1/2$, $e = 1$, $\hbar = 1$. The crossed fields Hamiltonian is then given by

$$H_1(F) = H_L - Fx = (-i\partial_x + By)^2 - \partial_y^2 - Fx, \quad \text{on} \quad L^2(\mathbb{R}^2). \tag{9.1}$$

Here we use the Landau gauge with $\mathbf{A}(x,y) = (-By, 0)$. A straightforward application of [RS75, Thm. X.37] shows that $H_1(F)$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$, see also

[RS75, Prob. X.38]. Moreover, one can easily check that

$$\sigma(H_1(F)) = \sigma_{ac}(H_1(F)) = \mathbb{R} \tag{9.2}$$

As mentioned in the Introduction we employ the translational analytic method developed in [AH77]. We introduce the translated operator $H_1(F,\theta)$ as follows:

$$H_1(F,\theta) = U(\theta)H_1(F)U^{-1}(\theta) \tag{9.3}$$

where

$$(U(\theta)f)(x,y) := \left(e^{ip_x\theta}f\right)(x,y) = f(x+\theta,y) \tag{9.4}$$

An elementary calculation shows that

$$H_1(F,\theta) = H_1(F) - F\theta \tag{9.5}$$

The operator $H_1(F,\theta)$ is clearly analytic in θ. Following [AH77] we define the class of $H_1(F)-$translation analytic potentials.

Definition 9.1. *Suppose that $V(z,y)$ is analytic in the strip $|\Im z| < \beta$, $\beta > 0$ independent of y. We then say that V is $H_1(F)-$translation analytic if $V(x+z,y)(H_1(F)+i)^{-1}$ is a compact analytic operator valued function of z in the given strip.*

We can thus formulate the conditions to be imposed on V:

(a) $V(x,y)$ is $H_1(F)-$translation analytic in the strip $|\Im z| < \beta$.

(b) There exists $\beta_0 \leq \beta$ such that for $|\Im z| \leq \beta_0$ the function $V(x+z,y)$ is uniformly bounded and
$$\lim_{x,y\to\pm\infty} |V(x+z,y)| = 0$$

(c) The operator $H(F) = H_1(F) + V$ has purely absolutely continuous spectrum.

In order to characterise the potential class for which the above conditions are fulfilled, let us assume for the moment that the integral kernel of $(H_1(F)+i)^{-1}$ has at most a local logarithmic singularity at the origin. This is a very plausible hypothesis, see Lemma 9.3. It then follows that any $L^2(\mathbb{R}^2)$ function which tends to zero at infinity and can be analytically continued in a given strip $|\Im z| < \beta$ satisfies the conditions (a) and (b). We can take a Gaussian as an elementary example.

The condition (c) is more delicate. For the fixed value of F one can specify the correspnding potential class satisfying (c) with the help of the Mourre commutator method, see [Mou81]. The central point of the latter is to find a suitable conjugate operator A such that the expectation value of the commutator $[H(F), iA]$ will have a definite sign in certain energy states. The Mourre theorem then says, under some additional conditions on A, that these states belong to the absolutely continuous spectrum of $H(F)$. Since $H(F)$ is unitarily equivalent to

$$\tilde{H}(F) = -\partial_x^2 + (-i\partial_y + Bx)^2 - Fx + V(x,y),$$

we can follow [MMP99] and take as A the generator of magnetic translations, $A = -i\partial_x - By$, so that

$$[\tilde{H}(F), iA] = F - \partial_x V(x,y).$$

Thus the spectrum of $\tilde{H}(F)$, which coincides with the spectrum of $H(F)$, will be purely absolutely continuous whenever $\|\partial_x V(x,y)\|_\infty < F$. This gives us the sought criteria in the situation when F is fixed.

From the well known perturbation argument, [Kat66], we see that under assumption (b)

$$H(F,\theta) = U(\theta)H(F)\,U^{-1}(\theta) = H_1(F,\theta) + V\,(x+\theta,y) \tag{9.6}$$

forms an analytic family of type A.
Furthermore, since $V(x+\theta,y)(H_1(F)+i)^{-1}$ is compact by (a), we have [RS78, Cor. 2, p. 113]

$$\sigma_{ess}(H(F,\theta)+ibF) = \sigma_{ess}(H_1(F)) = \mathbb{R} \implies \sigma_{ess}(H(F,\theta)) = \mathbb{R} - ibF \tag{9.7}$$

where $\theta = ib$, $b \in \mathbb{R}$. By standard arguments [RS78, Prob. XIII.76], all eigenvalues of $H(F,ib)$ lie in the strip $-bF < \Im z \le 0$ and are independent of b as long as they are not covered by the essential spectrum.

9.3 Exponential decay

The resonant states for our model are defined in the following way:

Definition 9.2. *We say that φ is a resonant state of $H(F)$ with width Γ, if there exists some $\epsilon > 0$, such that*

$$|(\varphi, e^{-itH(F)}\,\varphi)|^2 = e^{-t\,\Gamma}(1+R(t)),$$

where

$$|R(t)| = \mathcal{O}(e^{-t\,\epsilon}), \quad \text{as} \quad t \to \infty.$$

We remark that for a bounded below Hamiltonian the decay law can be exponential only for times neither too small nor too large, [Exn84]. However, in our case, due to the fact that $H(F)$ is unbounded from below, the above definition makes sense. For a detailed discussion of the problem of definition of resonance see also [Sim78]. The goal of this section is to prove that the resonance width Γ is given by an imaginary part of the associated complex eigenvalue of $H(F,\theta)$. We will borrow the ideas from [Her80] where a similar problem in three dimensions was treated in the absence of magnetic field. The main ingredient of our analysis is the proof of the fact that $H(F,\theta)$ can have only a finite number of eigenvalues in a given strip. We will need the following claim.

Proposition 9.1. *Let f, g be bounded functions with compact support in $\mathbb{R}^2$. Then*

$$\lim_{\lambda\to\pm\infty} \|f(H_1(F)-\lambda-i\,\gamma)^{-1}g\| = 0$$

for $F \geq 0$ and uniformly for γ in the compacts of $\mathbb{R}\setminus\{0\}$.

Proof. We take $\gamma < 0$ and write[1] ($\epsilon < \frac{\pi}{2B}$)

$$\begin{aligned} f(H_1(F)-\lambda-i\,\gamma)^{-1}g &= -i\int_0^\infty (fe^{itH_1(F)}g)e^{\gamma t}e^{-i\lambda t}\,\mathrm{d}t := \int_0^\infty G(t)e^{-i\lambda t}\,\mathrm{d}t \\ &= \int_0^\epsilon G(t)e^{-i\lambda t}\,\mathrm{d}t + \sum_{n\in\mathbb{N}}\int_{n\pi/B-\epsilon}^{n\pi/B+\epsilon} G(t)e^{-i\lambda t}\,\mathrm{d}t \\ &+ \sum_{n\in\mathbb{N}_0}\int_{n\pi/B+\epsilon}^{(n+1)\pi/B-\epsilon} G(t)e^{-i\lambda t}\,\mathrm{d}t \end{aligned} \tag{9.8}$$

The first term on the right hand side is bounded from above by $\|f\|_\infty\,\|g\|_\infty\,\epsilon$. For the second we have

$$\left\|\sum_{n\in\mathbb{N}}\int_{n\pi/B-\epsilon}^{n\pi/B+\epsilon} G(t)e^{-i\lambda t}\,\mathrm{d}t\right\| \leq 2\epsilon\,\|f\|_\infty\,\|g\|_\infty \sum_{n\in\mathbb{N}} e^{\gamma(n\pi/B-\epsilon)}$$

which implies

$$\begin{aligned} \|f(H_1(F)-\lambda-i\,\gamma)^{-1}g\| &\leq \epsilon\,\|f\|_\infty\,\|g\|_\infty\left(\frac{2e^{-\gamma\epsilon}}{1-e^{\gamma\pi/B}}+1\right) \\ &+ \left\|\sum_{n\in\mathbb{N}_0}\int_{n\pi/B+\epsilon}^{(n+1)\pi/B-\epsilon} G(t)e^{-i\lambda t}\,\mathrm{d}t\right\| \end{aligned} \tag{9.9}$$

All terms in the sum on the r.h.s. of (9.9) can be integrated by parts to give

$$\begin{aligned} \int_{n\pi/B+\epsilon}^{(n+1)\pi/B-\epsilon} G(t)e^{-i\lambda t}\,\mathrm{d}t &= \frac{1}{i\,\lambda}\int_{n\pi/B+\epsilon}^{(n+1)\pi/B-\epsilon} G'(t)e^{-i\lambda t}\,\mathrm{d}t \\ &- \left[\frac{1}{i\,\lambda}\,G(t)e^{-i\,\lambda t}\right]_{n\pi/B+\epsilon}^{(n+1)\pi/B-\epsilon} \end{aligned} \tag{9.10}$$

where the second term on the r.h.s. is bounded above by $2\|f\|_\infty\,\|g\|_\infty|\lambda|^{-1}$. In order to estimate the first term we use the integral kernel of the evolution operator $e^{-i\,tH_1(F)}$ in the

[1] here $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$

gauge where $H_L = p_x^2 + (p_y - Bx)^2$ (keeping in mind that the norm is gauge-invariant). From the formula (9.118) given in Appendix 9.A we then deduce the integral kernel of $G'(t)$

$$\begin{aligned}(x, y|G'(t)|x_0, y_0) &= \frac{1}{2\pi i}\sqrt{\frac{B}{2}}e^{\gamma t} f(x,y) g(x_0,y_0) e^{iS_{-t}[w_{cl}(\cdot)]}\frac{1}{\sin(Bt)} \times \\ \times \quad &\left\{\gamma + B\cot(Bt) + \frac{i}{4}\Big(u^2 - 2F(x+x_0) - \frac{B^2}{\sin^2(Bt)}[(x-x_0)^2 + (y - y_0 + ut)^2]\right. \\ + \quad &\left. 2F\cot(Bt)(y - y_0 + ut)\Big)\right\} \end{aligned} \tag{9.11}$$

with $u = \frac{F}{B}$. After some manipulations we find an upper bound on the Hilbert-Schmidt norm of $G'(t)$

$$\|G'(t)\|_{HS} \le \frac{C\, e^{\delta\, t}}{|\sin^3(Bt)|}$$

where $\gamma < \delta < 0$ and the constant C is uniform in t and depends on f, g, F, B. The last inequality yields the following estimate

$$\left\| \sum_{n\in\mathbb{N}_0} \int_{n\pi/B+\epsilon}^{(n+1)\pi/B-\epsilon} G(t)e^{-i\lambda t}\,\mathrm{d}t \right\| \le |\lambda|^{-1}\left[2\,\|f\|_\infty\,\|g\|_\infty + C(\delta)\int_\epsilon^{\pi/B-\epsilon}\frac{1}{|\sin^3(Bt)|}\,\mathrm{d}t\right].$$

Here we have put

$$C(\delta) = \frac{C\, e^{\delta\epsilon/2}}{1 - e^{\delta\pi/2B}}\,, \qquad (\delta < 0)$$

Finally, we can sum up all the contributions on the r.h.s. of (9.8) to write

$$\begin{aligned}\|f(H_1(F) - \lambda - i\,\gamma)^{-1}g\| \quad &\le \quad \|f\|_\infty\,\|g\|_\infty\left\{\left(1 + \frac{2e^{-\gamma\epsilon}}{1 - e^{\gamma\pi/B}}\right)\epsilon + 2|\lambda|^{-1}\right\} \\ &+ \quad C(\delta)|\lambda|^{-1}\int_\epsilon^{\pi/B-\epsilon}\frac{1}{|\sin^3(Bt)|}\,\mathrm{d}t \end{aligned} \tag{9.12}$$

Sending ϵ to zero in a suitable way, for example as $|\lambda|^{-\alpha}$ with $\alpha > 0$ and sufficiently small, we can make sure that the last term in (9.12) tends to zero as $\lambda \to \pm\infty$ and the claim of the Proposition then follows. The case $\gamma > 0$ can be proved in a similar way. □

Armed with Proposition 9.1 we can prove the promised result about the finite number of eigenvalues in the vicinity of real axis.

Proposition 9.2. *Suppose that assumptions (b) and (c) hold true. Then for any $aF < bF < \beta_0$ there exists some $M(a)$ such that $H(F, ib)$ has no eigenvalues in the strip $S_a := \{0 \ge \Im z \ge -aF,\ |\Re z| \ge M(a)\}$.*

Proof. We write $V_1 := |V(x+ib,y)|^{1/2}$, $\quad V_2 := |V(x+ib,y)|^{1/2}\,\mathrm{phase}V(x+ib,y)$ and, for $z \in S_a$, $R_1(z) = (z - H_1(F,ib))^{-1}$, $\quad R(z) = (z - H(F,ib))^{-1}$. Then, by an approximation argument and Proposition 9.1

$$\lim_{\lambda\to\pm\infty} \|V_1(H_1(F,ib) - \lambda - i\gamma)^{-1}V_2\| = 0, \quad \gamma > F(b-a) > 0, \tag{9.13}$$

which means that we can take $M(a)$ large enough, so that

$$\|V_1(H_1(F,ib) - \lambda - i\gamma)^{-1}V_2\| < 1 \quad \forall\, z \in S_a.$$

The Neumann series

$$R(z) = \sum_{n=0}^{\infty} R_1(z)(VR_1(z))^n = R_1(z) + R_1(z)V_1\left(\sum_{n=0}^{\infty}(V_2R_1(z)V_1)^n\right)V_2R_1(z)$$

thus converges in norm for $z \in S_a$. Moreover, since $\|R_1(z)\| \leq ((b-a)F)^{-1}$ and V_1, V_2 are in S_a uniformly bounded by assumption, we can conclude that

$$\sup_{z\in S_a} \|(z - H(F,ib))^{-1}\| < \infty$$

□

The following definition is a "translational version" of the notion of analytic vectors for dilatation group introduced in [AC71].

Definition 9.3. *Let A be any open complex domain having non-empty intersection with $\mathbb{R}$. Then we denote by $\mathcal{D}(A)$ the set of those vectors f, for which $f_\theta = U(\theta)f$, $\theta \in \mathbb{R}$ can be analytically continued to A.*

We are now able to state the main theorem of this section. Since a similar analysis was made in [Her80] for a non magnetic case, we skip some details of the proof referring to the latter.

Theorem 9.1. *Take $\alpha := \alpha_0 F > 0$ sufficiently small such that the conditions (a), (b) and (c) are satisfied for $\beta_0 > \alpha$. Assume moreover that $bF > \alpha$ and let ψ, ϕ, $H_1(F)\psi$, $H_1(F)\phi \in \mathcal{D}(\{z \in \mathbb{C} : |\Im z| < 2bF\})$. Then for any $t \geq 0$*

$$(\psi, e^{-itH(F)}\phi) = \sum_{-\Im E_j \leq \alpha} (\psi_{-ib}, P_j(ib)\phi_{ib})\, e^{-itE_j} + R(t)$$

where

$$R(t) \leq \mathcal{C}\, e^{-t(\alpha+\epsilon)}$$

for some $\epsilon > 0$. Here $P_j(ib)$ is the spectral projector of $H(F,ib)$ associated with the eigenvalue E_j.

Proof. Following [Her80] we put $K_1(z) = (\psi, (z - H(F))^{-1}\phi)$ for $\Im z > 0$ and note that $K_1(z)$ has a meromorphic continuation to $\mathbb{C}$, which is for $\Im z > -bF$ given by $K_1(z) = (\psi_{-ib}, (z - H(F, ib))^{-1}\phi_{ib})$. Similarly $K_2(z) = (\psi, (z - H(F))^{-1}\phi)$, $\Im z < 0$ has for $\Im z < bF$ a meromorphic continuation given by $K_2(z) = (\psi_{ib}, (z - H(F, -ib))^{-1}\phi_{-ib})$.

From the spectral theorem it follows that

$$(\psi, e^{-itH(F)}\phi) = \int_{-\infty}^{\infty} Q(\lambda)\, e^{-it\lambda}\, \mathrm{d}\lambda \tag{9.14}$$

where $Q(\lambda)$ is the spectral density. We have

$$\begin{aligned} Q(\lambda) &= \lim_{\delta \to 0} \frac{i}{2\pi}(\psi, [\lambda + i\delta - H(F))^{-1} - (\lambda - i\delta - H(F))^{-1}]\phi) \\ &= -(2\pi i)^{-1}(K_1(\lambda) - K_2(\lambda)), \quad \lambda \in \mathbb{R} \end{aligned} \tag{9.15}$$

Let us now take a such that $\alpha < aF < bF$. By Proposition 9.2 and assumption (c), the meromorphic continuation of $Q(\lambda)$ to $\mathbb{C}$, which is given by

$$Q(z) = -(2\pi i)^{-1}(K_1(z) - K_2(z))$$

is then analytic in the strip S_a and on the real axis. In addition, the argument of [Her80] shows that for $0 < \gamma < aF$ and $|E|$ large enough

$$Q(E - i\gamma) = \mathcal{O}(|E|^{-2}) \tag{9.16}$$

This allows us to shift the integration in (9.14) from the real axis downwards to the lower complex half-plane by

$$\lambda \to \lambda - i\,(\alpha + \epsilon) \qquad \alpha + \epsilon < aF$$

so that

$$\begin{aligned} (\psi, e^{-itH(F)}\phi) &= 2\pi i \sum_{-\Im E_j \le \alpha} \operatorname{Res} K_1(z)|_{z=E_j}\, e^{-itE_j} \\ &+ e^{-t(\alpha+\epsilon)} \int_{-\infty}^{\infty} Q(\lambda - i(\alpha + \epsilon))\, e^{-it\lambda}\, \mathrm{d}\lambda \end{aligned} \tag{9.17}$$

For the residues of $K_1(z)$ we have

$$\operatorname{Res} K_1(z)|_{z=E_j} = \frac{1}{2\pi i} \int_{|z-E_j|=\varepsilon} \mathrm{d}z(\psi_{-ib}, (z - H(F, ib))^{-1}\phi_{ib}) = (\psi_{-ib}, P_j(ib)\phi_{ib})$$

However, $f_j(z) = (\psi_{\bar{z}}, P_j(z)\phi_z)$ is by assumption an analytic function of z for $-F\Im z < \Im E_j$. Since $f_j(z)$ is constant for z real, we can conclude that $f_j(z)$ is independent of z as long as $-F\Im z < \Im E_j$. □

Theorem 9.1 can be applied with $\psi = \phi = \varphi$ where φ is and eigenvector of the Hamiltonian without electric field $H(0)$. In this case for large t we get the announced exponential decay of the matrix element $(\varphi, e^{-itH(F)}\varphi)$ at a rate proportional to imaginary part of the complex eigenvalues of $H(F, ib)$. Thus, φ is a resonant state whose resonance width is given in term of the imaginary part of the complex eigenvalues of $H(F, ib)$.

9.4 Green's function of $H_1(F, ib)$

As already announced, we now proceed to the estimations of the Green's function of the crossed fields Hamiltonian $H_1(F, ib)$. Results of this Section have a technical character and will be used in the announced forthcoming paper, in which we prove an upper bound on the resonance widths.

9.4.1 General solution

We want to find an upper bound on the Green's function (and its first derivatives) of

$$H_1(ib) := H_1(F, ib) = -\partial_x^2 + (-i\partial_y - Bx)^2 - Fx - Fib \tag{9.18}$$

Since $H_1(ib)$ is translationally invariant in $y-$direction, it can be written as

$$H_1(ib) \simeq \int_{\mathbb{R}}^{\oplus} H_1(ib, k)\, \mathrm{d}k \tag{9.19}$$

where

$$H_1(ib, k) = -\partial_x^2 + (k - Bx)^2 - Fx - Fib \tag{9.20}$$

is the corresponding fiber Hamiltonian on $L^2(\mathbb{R}, \mathrm{d}x)$. Its spectral equation

$$H_1(ib, k)\psi(x, k) = z\psi(x, k) \tag{9.21}$$

can be solved explicitly to give two linearly independent solutions. Namely, with the notation

$$x(k) := x - \frac{k}{B} - \frac{F}{2B^2}, \quad z(k) := z + ibF + \frac{F}{B}k + \frac{F^2}{4B^2} \tag{9.22}$$

we get for $x(k) > 0$:

$$\psi_1(x, k) = e^{-Bx^2(k)/2}\, U\left(\frac{B - z(k)}{4B}, \frac{1}{2}, B\,x^2(k)\right) \tag{9.23}$$

$$\begin{aligned}\psi_2(x, k) &= e^{-Bx^2(k)/2}\, V\left(\frac{B - z(k)}{4B}, \frac{1}{2}, B\,x^2(k)\right) \qquad (9.24)\\ &= e^{-Bx^2(k)/2}\sqrt{\pi}\left[\frac{M\left(\frac{B-z(k)}{4B}, \frac{1}{2}, B\,x^2(k)\right)}{\Gamma\left(\frac{3B-z(k)}{4B}\right)} + 2\sqrt{B}\,x(k)\,\frac{M\left(\frac{3B-z(k)}{4B}, \frac{3}{2}, B\,x^2(k)\right)}{\Gamma\left(\frac{B-z(k)}{4B}\right)}\right]\end{aligned}$$

and for $x(k) \le 0$:

$$\psi_1(x, k) = e^{-Bx^2(k)/2}\, V\left(\frac{B - z(k)}{4B}, \frac{1}{2}, B\,x^2(k)\right) \tag{9.25}$$

$$\psi_2(x, k) = e^{-Bx^2(k)/2}\, U\left(\frac{B - z(k)}{4B}, \frac{1}{2}, B\,x^2(k)\right) \tag{9.26}$$

where U and M are solutions to Kummer's equation, see [AS70, chap. 13]. Here we have followed the analysis made in [EJK99] for purely magnetic Hamiltonian. Clearly, $V((B-z(k))/4B, 1/2, B\,x^2(k))$ is analytical continuation of $U((B-z(k))/4B, 1/2, B\,x^2(k))$ for $x(k)<0$. We note that $\psi_1(x,k)\in L^2([0,\infty))$ and $\psi_2(x,k)\in L^2((-\infty,0])$. The Green's function of $H_1(ib,k)$ is thus given by

$$G(x,x';z,k)=\frac{\psi_1(x_>,k)\,\psi_2(x_<,k)}{W(\psi_1,\psi_2)} \tag{9.27}$$

with

$$x_> = \max(x,x'),\quad x_< = \min(x,x') \tag{9.28}$$

With the help of [AS70, p. 505] one can calculate the Wronskian

$$W(\psi_1,\psi_2)=\sqrt{\pi B}\,2^{\frac{3}{2}-\frac{z(k)}{2B}}\,\Gamma^{-1}\left(\frac{B-z(k)}{2B}\right) \tag{9.29}$$

The Green's function of $H_1(ib)$ then reads

$$G_1(\mathbf{x},\mathbf{x}';z)=(\pi B)^{-1/2}\int_{\mathbb{R}} 2^{-\frac{3}{2}+\frac{z(k)}{2B}}\,\psi_1(x_>,k)\,\psi_2(x_<,k)\,\Gamma\left(\frac{B-z(k)}{2B}\right)\,e^{ik(y-y')}\,\mathrm{d}k \tag{9.30}$$

To discuss the convergence of the integral in the definition of $G_1(\mathbf{x},\mathbf{x}';z)$ we recall the behaviour of the hypergeometric functions U and M, see [AS70, p. 504]. The latter gives the asymptotic of the integrand in (9.30) in the form:

$$e^{-k[|x'-x|-i(y'-y)]}\left(\frac{x-kB^{-1}}{x'-kB^{-1}}\right)^{\frac{z(k)}{2B}}\frac{1}{\sqrt{(x-kB^{-1})(x'-kB^{-1})}}\,[1+\mathcal{O}(k^{-2})]$$

as $k\to\infty$, and

$$e^{k[|x'-x|-i(y'-y)]}\left(\frac{x'-kB^{-1}}{x-kB^{-1}}\right)^{\frac{z(k)}{2B}}\frac{1}{\sqrt{(x-kB^{-1})(x'-kB^{-1})}}\,[1+\mathcal{O}(k^{-2})]$$

as $k\to-\infty$. Thus, for $x'\neq x$ the integral converges independently on the value of y',y, for in that case the asymptotic is given by

$$e^{-|k||x'-x|}\,\alpha(k)^k\,k^{-1},\quad |k|\to\infty \tag{9.31}$$

with $\lim_{|k|\to+\infty}\alpha(k)=1$. Similarly, when $y'\neq y$ the integral converges even for $x'=x$, since the asymptotic then reads

$$e^{-ik(y'-y)}\,\frac{1}{\sqrt{(x-kB^{-1})(x-kB^{-1})}}\,[1+\mathcal{O}(k^{-2})],\quad |k|\to\infty, \tag{9.32}$$

and simple integration by parts shows that $G_1(\mathbf{x},\mathbf{x}';z)$ converges pointwise for any $y'\neq y$. From the definition of hypergeometric functions and the construction of ψ_1 and ψ_2 it follows, that the product $\psi_1(x,k)\,\psi_2(x,k)$ is analytic w.r.t. k. The integrand of (9.30) is thus a meromorphic function with poles at

$$k_2 = -BF^{-1}(z_2 + bF), \quad k_1(n) = BF^{-1}\left[(2n+1)B - z_1 - F^2/(4B)\right], n \geq 0 \tag{9.33}$$

where we write $k = k_1 + ik_2$ and $z = z_1 + iz_2$. Moreover the integrand vanishes in the limit $|k_1| \to \infty$, see (9.31), (9.32). Therefore we can shift the integration to the lower complex half-plane by substituting

$$p := -\frac{k}{B} - \frac{F}{2B^2} - i\,\frac{z_2 + bF}{2F}\,\delta \quad , \quad \delta = \frac{y - y'}{|y - y'|}, \tag{9.34}$$

so that

$$x(p) = x + p + i\,\Delta, \quad x'(p) = x' + p + i\,\Delta, \quad \Delta = \frac{z_2 + bF}{2F}\,\delta \tag{9.35}$$

Since $U(a,b,t)$ is a many-valued function with a principal branch $-\pi < \arg t \leq \pi$, we have to consider its analytical continuation, see [AS70, p. 504]. The fundamental solutions $\psi_1(x_>, p)$ and $\psi_2(x_<, p)$ will be given by different combinations of hypergeometric functions corresponding to different values of quasimomentum p;

1. For $p < -x' < -x$:

$$\psi_1(x', p) = e^{-Bx'^2(p)/2}\, V\left(\frac{B - z(p)}{4B}, \frac{1}{2}, B\, x'^2(p)\right) \tag{9.36}$$

$$\psi_2(x, p) = e^{-Bx^2(p)/2}\, U\left(\frac{B - z(p)}{4B}, \frac{1}{2}, B\, x^2(p)\right) \tag{9.37}$$

2. For $-x' < p < -x$:

$$\psi_1(x', p) = e^{-Bx'^2(p)/2}\, U\left(\frac{B - z(p)}{4B}, \frac{1}{2}, B\, x'^2(p)\right) \tag{9.38}$$

$$\psi_2(x, p) = e^{-Bx^2(p)/2}\, U\left(\frac{B - z(p)}{4B}, \frac{1}{2}, B\, x^2(p)\right) \tag{9.39}$$

3. For $-x' < -x < p$:

$$\psi_1(x', p) = e^{-Bx'^2(p)/2}\, U\left(\frac{B - z(p)}{4B}, \frac{1}{2}, B\, x'^2(p)\right) \tag{9.40}$$

$$\psi_2(x, p) = e^{-Bx^2(p)/2}\, V\left(\frac{B - z(p)}{4B}, \frac{1}{2}, B\, x^2(p)\right) \tag{9.41}$$

The Cauchy theorem now yields

$$\begin{aligned} G_1(\mathbf{x}, \mathbf{x}'; z) &= (\pi B)^{-1/2} e^{-\frac{z_2 + bF}{2F}\,|y - y'|}\, e^{-iF(y-y')/2 + B(z_2 + bF)^2/(4F)} \\ &\times \int_{\mathbb{R}} 2^{-\frac{3}{2} + \frac{z(k(p))}{2B}}\, \psi_1(x', k(p))\, \psi_2(x, k(p))\, \Gamma\left(\frac{B - z(k(p))}{2B}\right) e^{ipB(y'-y)}\, \mathrm{d}p \end{aligned} \tag{9.42}$$

with $k(p)$ defined through (9.34).

9.4.2 Long distances: $G_1(\mathbf{x}, \mathbf{x}'; z)$

Let us suppose, for definiteness, that $x' > x$ and examine the case where $|x' - x| > 1$. For x and x' we have to consider the following three cases: $x' > x > 0$, $x' > 0 > x$ and $0 > x' > x$. In each case we perform the integral (9.42) by dividing it in several pieces depending on the value of p. Before doing so we give some general estimates on the hypergeometric functions which will be used throughout the text.

Remark 9.1. *The symbol C below denotes a positive real number, which depends on the energy z, but not on the size of the electric field F.*

For the product $U(a,b,t)\,M(a,b,t)$ we use the asymptotic expressions, [AS70, p. 504], and the corresponding estimate of the error term to get

$$\left|2^{-\frac{3}{2}+\frac{z(p)}{2B}}\, V\left(\frac{B-z(p)}{4B}, \frac{1}{2}, B\,x'^2(p)\right)\, U\left(\frac{B-z(p)}{4B}, \frac{1}{2}, B\,x^2(p)\right)\Gamma\left(\frac{B-z(p)}{2B}\right)\right| \leq$$
$$C\, e^{Bx'^2(p)} \left|\frac{p+x+i\Delta}{p+x'+i\Delta}\right|^{z(p)/2B} B^{-1/2}|(x+p+i\Delta)(x'+p+i\Delta)|^{-1/2}\,[1+C\Delta^{-2}] \quad (9.43)$$

where we have used the doubling formula for the gamma function, [AS70, p. 256]

$$\Gamma(2z) = \pi^{-\frac{1}{2}}\, 2^{2z-1}\,\Gamma(z)\,\Gamma(z+\tfrac{1}{2}) \quad (9.44)$$

Henceforth we will work only with the leading term and drop the factor $[1+C\Delta^{-2}]$. Moreover, as the asymptotic behaviour of both summands in the definition of V is identical, we will consider only the first one.

The following bound can be easily found

$$|(x+p+i\Delta)(x'+p+i\Delta)|^{-1/2} \leq \Delta^{-1}\,. \quad (9.45)$$

We have

$$\left|\frac{p+x+i\Delta}{p+x'+i\Delta}\right|^{z(p)/2B} = \left(1 + \frac{(x-x')^2}{(p+x')^2+\Delta^2} + \frac{2(x'-x)(p+x')}{(p+x')^2+\Delta^2}\right)^{\frac{\tilde{z}_1 - Fp}{4B}} \quad (9.46)$$

with $\tilde{z}_1 = z_1 - F^2/4B^2$. Remark that $|\cdots| > 1$, thus for $\tilde{z}_1 \leq 0$ and $p \geq 0$ this term can be neglected. For $\tilde{z}_1 > 0$ we can apply the following inequality

$$1 + \frac{(x-x')^2}{(p+x')^2+\Delta^2} + \frac{2(x'-x)(p+x')}{(p+x')^2+\Delta^2} \leq 1 + \frac{2(x-x')^2}{\Delta^2}\,. \quad (9.47)$$

For $p < 0$ we write $|\cdots|^{-\frac{Fp}{2B}} = e^{-\frac{Fp}{2B}\ln|\cdots|}$. Finally, note that the same result holds true if we interchange x and x', which correspond to interchange the functions U and V.

Let $x' > x > 0$

We divide the interval of integration in five parts as follows

$$\mathbb{R} = (-\infty, -2x'] \cup (-2x', -x'] \cup (-x', -x] \cup (-x, -x/2] \cup (-x/2, \infty)$$

For $p \in (-\infty, -2x']$:
Keeping in mind that $F \to 0$ one gets from (9.43)

$$\begin{aligned}
&\int_{-\infty}^{-2x'} \left| 2^{-\frac{3}{2}+\frac{z(p)}{2B}} \, \psi_1(x', x, p) \, \psi_2(x', x, p) \, \Gamma\left(\frac{B - z(p)}{2B}\right) \right| \mathrm{d}p \\
\leq\ & \frac{C}{\sqrt{B}} \Delta^{-1} \left[1 + \frac{2(x'-x)^2}{\Delta^2}\right]^{\frac{z_1}{4B}} e^{\frac{B}{2}\,(x'^2 - x^2)} \int_{-\infty}^{-2x'} e^{p\,B(x'-x)} \, e^{\frac{-Fp}{4B}\,\ln|\cdots|} \, \mathrm{d}p \\
\leq\ & \frac{C}{\sqrt{B}} \Delta^{-1} \left[1 + \frac{2(x'-x)^2}{\Delta^2}\right]^{\frac{z_1}{4B}} e^{\frac{B}{2}\,(x'^2 - x^2)} \int_{-\infty}^{-2x'} e^{pB(x'-x)/2} \, \mathrm{d}p \\
\leq\ & \frac{C}{B^{3/2}} \Delta^{-1} \left[1 + \frac{2(x'-x)^2}{\Delta^2}\right]^{\frac{z_1}{4B}} e^{-\frac{B}{2}\,(x'-x)^2}
\end{aligned} \tag{9.48}$$

For $p \in (-x/2, \infty)$:
(9.43) (with x and x' interchanged) and the bounds given before lead to

$$\begin{aligned}
&\int_{-x/2}^{\infty} \left| 2^{-\frac{3}{2}+\frac{z(p)}{2B}} \, \psi_1(x', x, p) \, \psi_2(x', x, p) \Gamma\left(\frac{B - z(p)}{2B}\right) \right| \mathrm{d}p \\
\leq\ & \frac{C}{\sqrt{B}} \Delta^{-1} \left[1 + \frac{2(x'-x)^2}{\Delta^2}\right]^{\frac{z_1}{4B}} e^{\frac{B}{2}(x^2 - x'^2)} \times \\
\times\ & \left\{ \int_{-x/2}^{0} e^{-Bp(x'-x)} e^{-\frac{Fp}{2B}\ln|\ldots|} \, \mathrm{d}p + \int_0^\infty e^{-Bp(x'-x)} \, \mathrm{d}p \right\} \\
\leq\ & \frac{C}{\sqrt{B}} \Delta^{-1} \left[1 + \frac{2(x'-x)^2}{\Delta^2}\right]^{\frac{z_1}{4B}} e^{\frac{B}{2}(x^2 - x'^2)} \left\{ \int_{-x/2}^{0} e^{-2Bp(x'-x)} \, \mathrm{d}p + \int_0^\infty e^{-Bp(x'-x)} \, \mathrm{d}p \right\} \\
\leq\ & \frac{C}{B^{3/2}} \Delta^{-1} \left[1 + \frac{2(x'-x)^2}{\Delta^2}\right]^{\frac{z_1}{4B}} 2e^{-\frac{B}{2}\,(x'-x)^2}
\end{aligned} \tag{9.49}$$

For $p \in (-2x', -x']$:
Here the estimate (9.43) does not give us the sought result. Instead we will rewrite the corresponding part of the integration in (9.42) in the following way,

$$\begin{aligned}
&\int_{-2x'}^{-x'} \left| 2^{-\frac{3}{2}+\frac{z(p)}{2B}} \, \psi_1(x', x, p) \, \psi_2(x', x, p) \, \Gamma\left(\frac{B - z(p)}{2B}\right) \right| \mathrm{d}p \\
\equiv\ & \Delta^{-1} \, x'^{-1} \, (x'-x)^{\frac{z_1}{2B}} e^{-\frac{B}{4}\,(x'-x)^2} \int_{-2x'}^{-x'} \Phi(x', x, p) \, \mathrm{d}p
\end{aligned} \tag{9.50}$$

and look at the maximum of the function $\Phi(x', x, p)$ in the interval $[-2x', -x']$. We denote the maximum value by $\Phi_0(x', x)$. In particular we want to show that Φ_0 is bounded above by certain function of F, which does not grow faster than a power function of F^{-1} as $F \to 0$. To be more precise, we want to show, that there exist some positive constants Θ_0, θ_1, such that

$$|\Phi(x', x, p)| \leq \Theta_0 \, F^{-\theta_1}$$

holds uniformly for $p \in (-2x', -x']$ and F small enough. This procedure will used below also for other values of p.

We recall the asymptotic properties of the gamma function, see [AS70, p. 257]

$$\Gamma(az+b) \sim \sqrt{2\pi}\, e^{-az}\, (az)^{az+b-\frac{1}{2}}, \quad |z| \to \infty, \ |\arg z| < \pi, \ a > 0 \tag{9.51}$$

It is then easy to see, that $\Phi(x', x, p)$ is bounded at the endpoints of the interval $[-2x', -x']$. We can thus confine ourselves to the case when Φ acquires its maximum inside the considered interval. Let us denote the corresponding extremal point by

$$p_0(x') = -x' - j(x')$$

First of all we note that if $j(x')$ is bounded, one can show the boundedness of $\Phi(x', x, p_0(x'))$ in the same way as that of $\Phi(x', x, -x')$. Without loss we may thus assume that $j(x')$ is unbounded. We shall distinguish two different situations according to different behaviour of the function $j(x')$.

1. $j^2(x')/x'$ bounded as $x' \to \infty$. In this case the first parameter of

$$M\left(\frac{B - z(p_0(x'))}{4B}, \frac{1}{2}, B\, x'^2(p_0(x'))\right) \tag{9.52}$$

   does not grow more slowly than its argument, for

$$\begin{aligned} z(p_0(x')) &= z_1 + F(x' + j(x')) - \frac{F^2}{4B^2} + \frac{i}{2}\,(z_2 + bF)(2-\delta) && (9.53)\\ B\, x'^2(p_0(x')) &= B\,(j(x') + i\Delta)^2. && (9.54) \end{aligned}$$

   We observe that in our case real parts of $z(p_0(x'))$ and $x'^2(p_0(x'))$ increase faster than their imaginary parts in the limit $x' \to \infty$. It then follows from the definition of function M, [AS70, p. 504], that the behaviour of (9.52) at infinity will be governed by

$$M\left(\frac{B - \Re z(p_0(x'))}{4B}, \frac{1}{2}, \Re B\, x'^2(p_0(x'))\right) \tag{9.55}$$

   The application of a suitable asymptotic expansion, [Buc53, p. 105], also [AS70, p. 509, 13.5.21], thus gives us the following inequality for $x' \to \infty$

$$\left| M\left(\frac{B - \Re z(p_0(x'))}{4B}, \frac{1}{2}, \Re B\, x'^2(p_0(x'))\right)\right| \leq C\, F^{-1}\, e^{\frac{j^2(x')}{2}} \tag{9.56}$$

Recalling (9.51) we can conclude that

$$\Phi(x', x, p_0(x')) \leq C\,\Delta\,x' \,\exp\left[-\frac{B}{4}\,\left((x'-x)^2 + 2j^2(x') + 4j(x')(x'-x)\right)\right]$$
$$|\,B(x'-x+j(x'))|^{\frac{F(x'+j(x'))}{2B}}\,\left|\Gamma\left(\frac{B - z(p_0(x'))}{4B}\right)\right| \tag{9.57}$$

is bounded above by a constant times $\Delta\,F^{-1}$.

2. $j^2(x')/x'$ unbounded. Here we can use again (9.43) and the boundedness of $\Phi(x', x, p_0(x'))$ then follows after some elementary manipulations.

To sum up we have

$$\int_{-2x'}^{-x'} \left| 2^{-\frac{3}{2}+\frac{z(p)}{2B}}\,\psi_1(x', x, p)\,\psi_2(x', x, p)\,\Gamma\left(\frac{B - z(p)}{2B}\right)\right|\,\mathrm{d}p$$
$$\leq C\,(F^{-1} + \Delta^{-1})\,(x'-x)^{\frac{z_1}{2B}}\,e^{-\frac{B}{4}\,(x'-x)^2} \tag{9.58}$$

For $p \in (-x, -x/2]$:
Same estimations as for $p \in (-2x', -x']$.

For $p \in (-x', -x]$:
We show that the function to be integrated is bounded by some constant uniform in x, x' times $e^{-\frac{B}{4}(x-x')^2}$. At the boundary it has been shown above that the function is bounded, we suppose that there is an extremal point $p_0 = p_0(x, x') \in (-x', x]$. Denote

$$d(x, x') = |p_0 + x| \qquad \text{and} \qquad d'(x, x') = |p_0 + x'|$$

the distances between the end points and the extremum p_0.
We have to consider the following cases, which correspond to the different behaviours of the argument of U: $d(x, x')$ unbounded, $d(x, x') < C$ and the same for $d'(x, x')$.

1) $d(x, x')$, $d'(x, x')$ unbounded: we have for $p = p_0$

$$\begin{aligned}
\mathcal{A}_1(x, x') &:= e^{\frac{B}{4}(x+p_0+i\Delta)^2}\,\sqrt{|W^{-1}\,(\psi_1, \psi_2)|}\,|\psi_1(x, p)| \qquad (9.59)\\
&= \left|2^{\frac{z(p_0)}{4B}}\,e^{-\frac{B}{4}(x+p_0+i\Delta)^2}\,B(x+p_0+i\Delta)^{\frac{z(p)-B}{2B}}\right|\left|\Gamma\left(\frac{B - z(p_0)}{2B}\right)\right|^{1/2}\\
&\leq 2^{\frac{\tilde{z}_1 - Fp_0}{4B}}\,e^{-\frac{B}{4}(x+p_0)^2}\,(B\,|x+p_0+i\Delta|)^{\frac{\tilde{z}_1 - B}{2B}}\,\left|\Gamma\left(\frac{B - \tilde{z}_1 - Fp_0}{2B} + i\eta\right)\right|^{1/2}
\end{aligned}$$

where η denote the imaginary part of the argument in the gamma function. $\mathcal{A}_2(x, x')$ is defined in the same way where ψ_1 is replaced with ψ_2 and x, x' are interchanged. In the

limit $x', x \to \infty$ we consider the following cases.
a)

$$B(d^2(x,x') + \Delta^2),\ B(d'^2(x,x') + \Delta^2) > \nu_0 \frac{\tilde{z}_1 - Fp_0}{4B} : \tag{9.60}$$

where $\nu_0 = 4(1 + \ln 2) f_0^{-1} > 1$ and $f_0 > 0$ is the global minimum of $(1 - t\ln(2/t))$ for $t \geq 0$. Using the asymptotic properties of the gamma function we get for the leading term of (9.59):

$$\exp\left\{ -\frac{B}{4}(x + p_0)^2 \left[1 + f(x,x') \ln\left(-2f^{-1}(x,x')\right)\right] + (1 + \ln 2)\, \frac{\tilde{z}_1 - Fp_0}{4B} \right\} \tag{9.61}$$

where

$$f(x,x') = \frac{F\, p_0(x,x')}{B^2\,(x + p_0(x,x'))^2} < 0 \tag{9.62}$$

The boundedness of $\mathcal{A}_1(x,x')$ follows from (9.60). The same analysis for $\mathcal{A}_2(x,x')$ then gives

$$\begin{aligned} &\left|\, \psi_1(x',x,p)\, \psi_2(x',x,p)\, W^{-1}(\psi_1,\psi_2) \right| \leq e^{-\frac{B}{4}(x+p_0)^2} e^{-\frac{B}{4}(x'+p_0)^2} \mathcal{A}_1 \mathcal{A}_2 \\ &\leq C\, e^{-\frac{B}{8}\,(x'-x)^2} \end{aligned} \tag{9.63}$$

To continue we recall again the asymptotic behaviour of $U(a,b,z)$, see [AS70, p. 504], to assure that

$$\begin{aligned} &\left| U\left(\frac{B - z(p_0(x',x))}{4B}, \frac{1}{2}, B\,(x' + p + i\Delta)^2) \right) \right| \leq \\ C &\left| U\left(\frac{B - \Re z(p_0(x',x))}{4B}, \frac{1}{2}, B\,((x' + p)^2 + \Delta^2) \right) \right| \,[1 + C\Delta^{-2}] \end{aligned} \tag{9.64}$$

Let us now consider
b)

$$B(d^2(x,x') + \Delta^2) > \nu_0\, \frac{\tilde{z}_1 - Fp_0}{4B}, \quad B(d'^2(x,x') + \Delta^2) = \nu\, \frac{\tilde{z}_1 - Fp_0}{4B},\ \nu \in [1, \nu_0],$$

in which case the part corresponding to $\mathcal{A}_1(x,x')$ can be treated as above and for the rest of the integrand we use [AS70, p. 509, 13.5.20] to get

$$\left| e^{-\frac{B}{2}(x'+p_0)^2} U\left(\frac{B - \Re z(p_0(x'))}{4B}, \frac{1}{2}, B\,((x' + p)^2 + \Delta^2) \right) \right| \leq C\, e^{-\frac{B}{4\nu}\,(x'+p_0)^2} \tag{9.65}$$

and consequently

$$\begin{aligned} &\left|\, \psi_1(x',x,p)\, \psi_2(x',x,p)\, W^{-1}(\psi_1,\psi_2) \right| \leq e^{-\frac{B}{4}(x+p_0)^2} e^{-\frac{B}{4\nu}(x'+p_0)^2} \mathcal{A}_1 \\ &\leq C\, e^{-\frac{B}{8\nu}\,(x'-x)^2} \end{aligned} \tag{9.66}$$

c)

$$B(d^2(x,x') + \Delta^2) \geq \frac{\tilde{z}_1 - Fp_0}{4B}, \quad B(d'^2(x,x') + \Delta^2) < \frac{\tilde{z}_1 - Fp_0}{4B}, \tag{9.67}$$

The part which includes $\psi_1(x,p)$ can be controlled by one of the estimates given above. For the second part we observe that, [AS70, p. 509, 13.5.22], $|\psi_2(x',p)|$ is uniformly bounded for p in $(-x', -x]$. The properties of gamma function then lead to the following inequality for the Wronskian

$$\begin{aligned} |W^{-1/2}(\psi_1,\psi_2)| \quad &\leq C \, \exp\left[\frac{Fp_0}{4B}(\ln(\sqrt{-Fp_0/2B}) - 1 - \ln 2)\right] e^{\frac{Fp_0}{4B}\, \ln(\sqrt{-Fp_0/2B})} \left|\frac{Fp_0}{2B}\right|^{-\frac{\tilde{z}_1}{4B}} \\ &\leq C \, \exp[-B((x'+p_0)^2 + \Delta^2)(\ln(\sqrt{(x'+p_0)^2 + \Delta^2}) - 1 - \ln 2)] \\ &\leq C \, e^{-B(x'+p_0)^2}, \end{aligned} \tag{9.68}$$

so that

$$\begin{aligned} &\big|\, \psi_1(x',x,p)\, \psi_2(x',x,p)\, W^{-1}(\psi_1,\psi_2)\big| \leq e^{-\frac{B}{4\nu}(x+p_0)^2}\, |W^{-1/2}(\psi_1,\psi_2)| \\ &\leq C\, e^{-\frac{B}{8\nu}\,(x'-x)^2} \end{aligned} \tag{9.69}$$

d)

$$B(d^2(x,x') + \Delta^2) < \frac{\tilde{z}_1 - Fp_0}{4B}, \quad B(d'^2(x,x') + \Delta^2) < \frac{\tilde{z}_1 - Fp_0}{4B}$$

Here both the functions $|\psi_2(x',p)|$ and $|\psi_1(x,p)|$ are uniformly bounded and the exponential decay then comes from the Wronskian in the same way as in the case *c*).

2) One of $d(x,x')$, $d'(x,x')$ bounded.

Let us suppose for definiteness, that $d(x,x')$ is bounded. At the point $p = p_0(x,x')$ we apply again (9.64) and [AS70, p. 508, 13.5.16] to find that

$$|\psi_1(x,p)| \leq C \left|\Gamma\left(\frac{1}{2} - \frac{B - z(p_0(x',x))}{4B}\right)\right| \tag{9.70}$$

For the function $\psi_2(x',p)$ and for the Wronskian we use the suitable estimate given above in one of the cases *a*), *b*), *c*), *d*), which gives the desired result.

In all these cases the same analysis can be made when $d(x,x')$ and $d'(x,x')$ interchange their roles.

3) Both $d(x,x')$ and $d'(x,x')$ bounded.

Since this can only happen when $|x'-x| \leq C$, it suffices to show that the integrand is bounded. The latter however follows immediately from (9.70) and

$$\left|\Gamma^2\left(\frac{1}{2} - \frac{B - z(p_0(x',x))}{4B}\right) W^{-1}(\psi_1,\psi_2)\right| \leq C, \quad \forall\, p \in (-x', -x]$$

Finally we conclude that there exists certain constant $\omega > 0$, which depends on B but not on F, such that

$$\int_{-x'}^{-x} \left| \psi_1(x',x,p)\, \psi_2(x',x,p)\, W^{-1}(\psi_1,\psi_2) \right| \,\mathrm{d}p \le \int_{-x'}^{-x} e^{-\frac{B}{4}(x+p_0)^2} e^{-\frac{B}{4}(x'+p_0)^2} \mathcal{A}_1 \mathcal{A}_2 \,\mathrm{d}p$$
$$\le C\,\Delta^{-1}\,(x'-x)\,e^{-\omega\,(x'-x)^2} \tag{9.71}$$

Remark 9.2. *We do not present the analysis of all the possible combinations, because the in the remaining cases one can proceed in a completely analogous way as above.*

Let $x' > 0 > x$

In this case we divide the interval of integration in four parts as

$$\mathbb{R} = (-\infty, -2x'] \cup (-2x', -x'] \cup (-x', -x] \cup (-x, \infty)$$

The intervals $(-\infty, -2x']$, $(-2x', -x']$ can be treated exactly as in the previous case. For $p \in (-x, \infty)$ we proceed in the same way as for $p \in (-x/2, \infty)$ in the previous case, keeping in mind that since $x < 0$ one has $p > 0$.

For $p \in (-x', -x]$ we separate the analysis of the integrand in two pieces.
(1) $p \in (-x', 0]$: Same argument as for the interval $(-x', -x]$ when x', x are both positive.
(2) $p \in (0, -x]$: We divide the interval in $(0, p_c+1] \cup (p_c+1, -x]$, where $p_c = \frac{\tilde{z}_1 - B}{F}$. For $p > p_c$ we have $\Re a(p) > 0$ with $a(p)$ the first parameter of the function U. In this case we can use the integral representation of U to get [DMP99]

$$|U(a(p), \tfrac{1}{2}, \rho(p))| \le \frac{C}{\Re a(p)} |\Gamma(a(p))|^{-1} \qquad \text{for } \Re\rho(p) > 0,\ \Re a(p) > 0 \tag{9.72}$$

In $(0, p_c+1]$ the analysis of the maximum of

$$|x' + p + i\Delta|^2 |x + p + i\Delta|^2$$

shows that it is a power function in $(x'-x)$. Thus, since the Γ function remains in this interval bounded, we get the bound $e^{-\frac{B}{2}(x'-x)^2}$ times a polynomial in $(x'-x)$.
In $(p_c+1, -x-|\Delta|]$ we use the bounds (9.72) and the asymptotic behaviour of the gamma function to get a uniform upper bound. In $(-x-|\Delta|, -x]$ we use (9.72) for the function U depending on x' while for the other U we use its expression in term of a sum of function M. In this case we get a uniform estimate since the argument of M is bounded.

Let $0 > x' > x$

We divide the interval of integration in four parts as follows

$$\mathbb{R} = (-\infty, 0] \cup (0, -x'] \cup (-x', -x] \cup (-x, \infty)$$

For the interval $(-x, \infty)$ the remarks above hold. When $p \in (-x', -x]$ a slight modification of the analysis done in $(0, -x]$ above leads to the desired bound.
For $p \in (-\infty, 0]$:

$$\begin{aligned}
&\int_{-\infty}^{0} \left| 2^{-\frac{3}{2}+\frac{z(p)}{2B}} \, \psi_1(x', x, p) \, \psi_2(x', x, p) \, \Gamma\left(\frac{B - z(p)}{2B}\right) \right| \, \mathrm{d}p \\
\leq\ & \frac{C}{\sqrt{B}} \, \Delta^{-1} \left[1 + \frac{2(x'-x)^2}{\Delta^2}\right]^{\frac{z_1}{4B}} e^{\frac{B}{2}\,(x'^2 - x^2)} \int_{-\infty}^{0} e^{p\,B(x'-x)} \, e^{\frac{-Fp}{4B}\,\ln|\cdots|} \, \mathrm{d}p \\
\leq\ & \frac{C}{\sqrt{B}} \, \Delta^{-1} \left[1 + \frac{2(x'-x)^2}{\Delta^2}\right]^{\frac{z_1}{4B}} e^{\frac{B}{2}\,(x'^2 - x^2)} \int_{-\infty}^{0} e^{pB(x'-x)/2} \, \mathrm{d}p \\
\leq\ & \frac{C}{B^{3/2}} \Delta^{-1} \left[1 + \frac{2(x'-x)^2}{\Delta^2}\right]^{\frac{z_1}{4B}} e^{-\frac{B}{2}\,(x'-x)^2}
\end{aligned} \tag{9.73}$$

For $p \in (0, -x']$:

$$\begin{aligned}
&\int_{0}^{-x'} \left| 2^{-\frac{3}{2}+\frac{z(p)}{2B}} \, \psi_1(x', x, p) \, \psi_2(x', x, p) \, \Gamma\left(\frac{B - z(p)}{2B}\right) \right| \, \mathrm{d}p \\
\leq\ & \frac{C}{\sqrt{B}} \, \Delta^{-1} \left[1 + \frac{2(x'-x)^2}{\Delta^2}\right]^{\frac{z_1}{4B}} e^{\frac{B}{2}\,(x'^2 - x^2)} \int_{0}^{-x'} e^{p\,B(x'-x)} \, \mathrm{d}p \\
\leq\ & 2\frac{C}{B^{3/2}} \Delta^{-1} \left[1 + \frac{2(x'-x)^2}{\Delta^2}\right]^{\frac{z_1}{4B}} e^{-\frac{B}{2}\,(x'-x)^2}
\end{aligned} \tag{9.74}$$

Let us finally formulate the results in

Lemma 9.1. *For F small enough and $|x' - x| \geq 1$ there exist some strictly positive constants C_1, C_2, $\tilde{\omega}$, which depend on B and z, such that the following inequality holds true*

$$|G_1(\mathbf{x}, \mathbf{x}; z)| \leq C_1 \, \Delta^{-1} e^{-\Delta\,|y-y'|} \, e^{-\tilde{\omega}\,(x'-x)^2} \left[1 + \frac{2(x'-x)^2}{\Delta^2}\right]^{\frac{z_1}{4B}} \left[1 + C_2 \Delta^{-2}\right] \tag{9.75}$$

with $\Delta = \frac{z_2 + bF}{2F}$.

9.4.3 Long distances: $\partial_{x,y} \, G_1(\mathbf{x}, \mathbf{x}'; z)$

In this section we want to prove similar result to that one described in Lemma 9.1 also for the derivatives of the Green's function w.r.t. x and y. We suppose again that $x' > x$ and $|x' - x| > 1$. As we have already seen the most general and complicated case is the one where $x', x > 0$ and the all the others can be regarded as its simplification. Therefore here we confine ourselves to the situation when both x', x are positive.

We start with the derivative w.r.t. x. For $|x' - x| > 1$ the integral

$$\int_{\mathbb{R}} \left| 2^{-\frac{3}{2}+\frac{z(p)}{2B}} \psi_1(x',p)\, \partial_x \psi_2(x,p)\, \Gamma\left(\frac{B-z(p)}{2B}\right) \right| \, \mathrm{d}p$$

converges uniformly with respect to x, see (9.31). We can thus interchange the differentiation and integration in (9.42) to get the following inequality for the derivative of $G_1(\mathbf{x},\mathbf{x};z)$:

$$\begin{aligned} &|\partial_x G_1(\mathbf{x},\mathbf{x};z)| \leq \\ &C\, e^{-\Delta|y'-y|} \int_{\mathbb{R}} \left| 2^{-\frac{3}{2}+\frac{z(p)}{2B}} \psi_1(x',p)\, \partial_x \psi_2(x,p)\, \Gamma\left(\frac{B-z(p)}{2B}\right) \right| \, \mathrm{d}p \end{aligned} \tag{9.76}$$

We split again the integration in (9.42) into five intervals:

$$\mathbb{R} = (-\infty, -2x'] \cup (-2x', -x'] \cup (-x', -x] \cup (-x, -x/2] \cup (-x/2, \infty)$$

and use [AS70, p. 507, 13.4.8/21] to calculate the derivatives of hypergeometric functions. When $p \in (-x/2, \infty)$ we get for the corresponding integrand in (9.76)

$$\begin{aligned} &-B(x+p+i\Delta)\, \psi_1(x',p)\, \psi_2(x,p)\, W^{-1}(\psi_1,\psi_2) + 2B(x+p+i\Delta) e^{-B(x+p+i\Delta)^2/2}\, a(p)\sqrt{\pi} \\ &\times \left[\frac{M(a(p)+1, \frac{3}{2}, B(x+p+i\Delta)^2)}{\frac{1}{2}\Gamma(a(p)+1/2)} + 2\sqrt{B}(x+p+i\Delta)\, \frac{M(a(p)+\frac{3}{2}, \frac{5}{2}, B(x+p+i\Delta)^2)}{\frac{3}{2}\Gamma(a(p))} \right. \\ &\left. +2\sqrt{B}\, \frac{M(a(p)+1, \frac{3}{2}, B(x+p+i\Delta)^2)}{a(p)\Gamma(a(p))} \right] \psi_1(x',p)\, W^{-1}(\psi_1,\psi_2) \end{aligned} \tag{9.77}$$

where

$$a(p) = \frac{B - z(p)}{4B} \,. \tag{9.78}$$

The first term can be controlled in the same way as the Green's function itself due to (9.43) and the fact that

$$\left| \frac{x+p+i\Delta}{x'+p+i\Delta} \right|^2 \leq 1 + \frac{2(x-x')^2}{\Delta^2} \tag{9.79}$$

As for the term which includes the derivative of the function M, using [AS70, p. 504] and $\Gamma(a+1) = a\Gamma(a)$, we note that the asymptotic behaviour of

$$\frac{a(p)\, M(a(p)+1, \frac{3}{2}, B(x+p+i\Delta)^2)}{\Gamma(a(p)+1/2)}\, W^{-1}(\psi_1,\psi_2) \tag{9.80}$$

is the same as that of

$$\frac{M(a(p), \frac{1}{2}, B(x+p+i\Delta)^2)}{\Gamma(a(p)+1/2)}\, W^{-1}(\psi_1,\psi_2) \tag{9.81}$$

The rest of the analysis is then identical with the case of $G_1(\mathbf{x},\mathbf{x}';z)$ itself.

For $p < -x'$ are x, x' interchanged and we have to differentiate the function U:

$$\partial_x U\left(a(p), \frac{1}{2}, B(x+p+i\Delta)^2\right) = -2B(x+p+i\Delta)\, a(p)\, U\left(a(p)+1, \frac{3}{2}, B(x+p+i\Delta)^2\right) \tag{9.82}$$

The pre-factor $(x+p+i\Delta)$ is again well controlled due to (9.79). In addition we observe that for the product

$$a(p)\, U\left(a(p)+1, \frac{3}{2}, B(x+p+i\Delta)^2\right) V\left(a(p), \frac{1}{2}, B(x+p+i\Delta)^2\right) \tag{9.83}$$

we get the upper bound (9.43) multiplied by

$$\left|\frac{a(p)}{(x+p+i\Delta)^2}\right| \tag{9.84}$$

and that for $p < -2x'$ is the latter uniformly bounded w.r.t. to x, x'. Thus, for $x \in (-\infty, -2x']$ we can use the same estimations as for $G_1(\mathbf{x}, \mathbf{x}'; z)$.

For $p \in (-2x', -x'] \cup (-x, -x/2]$ we multiply the function $\Phi(x', x, p)$ introduced in (9.50) by $a(p)$, which leads to an additional factor F^{-1} in the estimate (9.58).

Similarly is for $p \in (-x', -x]$ the factor (9.84), coming from the derivative of U, controlled by the decay of the upper bounds that we have found above. More exactly, for the case 1a) we see from the inequality (9.60) that (9.84) is uniformly bounded in the interval $(-x', -x]$. The case 1b) is treated in an analogous way. As for 1c), we note that

$$a(p_0)\, e^{-\frac{B}{8}(x+p_0)^2}$$

is bounded due to (9.67). The result then follows from (9.63). When the inequalities of the case 1d) hold, then following (9.68) we get

$$|W^{-1}(\psi_1, \psi_2)\, a(p_0)| \le C e^{-\frac{B}{2}(x'+p_0)^2}\, e^{-\frac{B}{2}(x+p_0)^2},$$

which gives again the exponential decay of the integrand. In the cases 2) and 3) we proceed in the same way as for the Green's function itself noting that both

$$\left|a(p_0)\Gamma(1/2 - a(p_0))W^{-1/2}(\psi_1, \psi_2)\right|, \quad \left|a(p_0)\Gamma^2(1/2 - a(p_0))W^{-1}(\psi_1, \psi_2)\right|$$

are uniformly bounded. We thus conclude that

$$\left|\partial_x\, \psi_1(x', x, p)\, \psi_2(x', x, p)\, W^{-1}(\psi_1, \psi_2)\right| \le C\, e^{-\frac{B}{16\nu}(x'-x)^2} \tag{9.85}$$

for $p \in (-x', -x]$.

The same arguments can be then used for $\partial_y\, G_1(\mathbf{x}, \mathbf{x}'; z)$. Since the substitution $k \to p$ is not analytic in y, the differentiation w.r.t. y has to be done before this substitution is made. In other words, we have to differentiate the formula (9.30) and then substitute

p for k through (9.34). This leads to a multiplication of the integrand in (9.76) by the factor Bp, which is well controlled by the previously given arguments, noting that

$$\left| \frac{p}{\sqrt{(x+p+i\Delta)(x'+p+i\Delta)}} \right|$$

is uniformly bounded on $(-\infty, -2x'] \cup (-x/2, \infty)$.

Finally we get

Lemma 9.2. *For F small enough and $|x'-x| \geq 1$ there exist some strictly positive constants C_3, C_4, $\tilde{\omega}$, which depend on B and z, such that the following inequality holds true*

$$|\partial_{x,y} G_1(\mathbf{x},\mathbf{x}';z)| \leq C_3\, F^{-2}\, \Delta^{-1} e^{-\Delta\,|y-y'|}\, e^{-\tilde{\omega}\,(x'-x)^2} \left[1 + \frac{2(x'-x)^2}{\Delta^2}\right]^{\frac{z_1}{4B}+\frac{1}{4}} [1 + C_4 \Delta^{-2}] \tag{9.86}$$

with $\Delta = \frac{z_2+bF}{2F}$.

9.4.4 Short distances

Up to now we have considered that $|x'-x| \geq 1$ and $|y'-y|$ was arbitrary. Here we want to investigate the case where $|x'-x| < 1$ for any value of $|y'-y|$. Since our system is two-dimensional, we expect the Green's function $G_1(\mathbf{x},\mathbf{x}';z)$ to have a logarithmic singularity as $x \to x'$ and $y \to y'$ of the following type:

$$G_1(\mathbf{x},\mathbf{x}';z) \sim \ln(|\mathbf{x}'-\mathbf{x}|)$$

Our goal in this section is to show that

$$\int_{\mathbb{R}} \int_{|x'-x|\leq 1} |\partial^n_{x,y} G_1(\mathbf{x},\mathbf{x}';z)| e^{\frac{\Delta}{2}|y-y'|}\, \mathrm{d}x'\, \mathrm{d}y' \quad n = 0, 1 \tag{9.87}$$

is bounded as a function of x and y. We will work only with the derivatives of $G_1(\mathbf{x},\mathbf{x}';z)$, noting that same arguments then apply also to $G_1(\mathbf{x},\mathbf{x}';z)$ itself.

We divide the real axis as above and present again only the case $x', x > 0$.

<u>$\partial_x G_1(\mathbf{x},\mathbf{x}';z)$</u>

From the asymptotic expansion for the integrand of $G_1(\mathbf{x},\mathbf{x};z)$, see (9.31), (9.32), it follows that

$$\int_{\mathbb{R}} |\partial_x\, \psi_1(x',x,p)\, \psi_2(x',x,p)\, W^{-1}(\psi_1,\psi_2)|\, \mathrm{d}p$$

converges only if $x' \neq x$. This reflects the usual behaviour of the Green's function, i.e. the discontinuity of the derivative for $x' = x$. We will thus investigate $\partial_x G_1(\mathbf{x},\mathbf{x}';z)$ separately for $(x'-x)$ in the compacts of $(0,1)$ and $(-1,0)$.

Assume first that $(x' - x) \in (0,1)$. For the derivative w.r.t. x we write

$$|\partial_x G_1(\mathbf{x}, \mathbf{x}'; z)| = C\, e^{-\Delta|y'-y|} \left| \int_{\mathbb{R}} g(x', x, p)\, e^{ipB(y'-y)}\, \mathrm{d}p \right| \tag{9.88}$$

where for $p > -x$

$$g(x', x, p) = \psi_1(x', p)\, \partial_x \psi_2(x, p)\, W^{-1}(\psi_1, \psi_2) \tag{9.89}$$

Let us perform first the integration in the interval $p \in (-x/2, \infty)$. We have

$$\begin{aligned}\partial_x \psi_2(x, p) &= -B(x + p + i\Delta)\, \psi_2(x, p) + e^{-\frac{B}{2}(x+p+i\Delta)^2}\, \partial_x V\left(a(p), \frac{1}{2}, B(x + p + i\Delta)^2\right) \\ &=: \phi_1(x, p) + \phi_2(x, p)\end{aligned} \tag{9.90}$$

Using the asymptotic expansions for M and U and integrating by parts we find

$$\begin{aligned}&\left| \int_{-x/2}^{\infty} \psi_1(x', p)\, \phi_1(x, p)\, W^{-1}(\psi_1, \psi_2)\, e^{ipB(y'-y)}\, \mathrm{d}p \right| = C\, e^{-B(x'^2 - x^2)/2} \\ &\times \left| \int_{-x/2}^{\infty} e^{-pB[(x'-x) - i(y'-y)]} \left(\frac{p + x' + i\Delta}{p + x + i\Delta} \right)^{\frac{z(p)}{2B}} \frac{p + x + i\Delta}{\sqrt{(p + x + i\Delta)(p + x' + i\Delta)}} \right. \\ &\left. [1 + \mathcal{O}(|p + x + i\Delta|^{-2})][1 + \mathcal{O}(|p + x' + i\Delta|^{-2})]]\, \mathrm{d}p \right| \\ &\le \frac{C}{B|(x' - x) - i(y' - y)|} \left[\Delta^{-1} + e^{-B(x'^2 - x^2)/2} \int_{-x/2}^{\infty} e^{-pB(x'-x)}\, w(x', x, p)\, \mathrm{d}p \right] [1 + C\, \Delta^{-2}]\end{aligned} \tag{9.91}$$

where

$$w(x', x, p) = \partial_p \left\{ \left(\frac{p + x' + i\Delta}{p + x + i\Delta} \right)^{\frac{z(p)}{2B}} \frac{p + x + i\Delta}{\sqrt{(p + x + i\Delta)(p + x' + i\Delta)}} \right\} \tag{9.92}$$

Here we have used the fact that the integrand of (9.91) is an analytic function of p and therefore we can differentiate the term

$$[1 + \mathcal{O}(|p + x + i\Delta|^{-2})][1 + \mathcal{O}(|p + x' + i\Delta|^{-2})]]$$

w.r.t. p. It then follows from the Cauchy formula that the derivative is an $L^1[(-x/2, \infty)]$ function with the corresponding norm smaller than a constant times Δ^{-1}. The first term on the last line of (9.91) gives the expected result. The point is now that, as one can easily verify, the function $w(x', x, p)$ is proportional to $(x' - x)$ in the sense that

$$\frac{w(x', x, p)}{x' - x}$$

is uniformly bounded. In other words

$$\left| e^{-B(x'^2 - x^2)/2} \int_{-x/2}^{\infty} e^{-pB(x'-x)}\, w(x', x, p)\, \mathrm{d}p \right| \le C \tag{9.93}$$

and

$$\left|\int_{-x/2}^{\infty} \psi_1(x',p)\,\phi_1(x,p)\,W^{-1}(\psi_1,\psi_2)\,e^{ipB(y'-y)}\,\mathrm{d}p\right| \le \frac{C\,\Delta^{-1}}{|(x'-x)-i(y'-y)|}\,[1+C\,\Delta^{-2}] \tag{9.94}$$

All constants in the latter inequality are uniform for $(x'-x)$ in the compacts of $(0,1)$. Same analysis can be made also for the term $\phi_2(x,p)$, which includes the derivative of the function M, see the remarks below (9.79).

For p in the interval $(-\infty,-2x']$ are x' and x interchanged and we have

$$g(x',x,p) = \psi_2(x',p)\,\partial_x\psi_1(x,p)\,W^{-1}(\psi_1,\psi_2) \tag{9.95}$$

so that $\phi_1(x,p)$ is unchanged and instead of $\phi_2(x,p)$ we get

$$\tilde\phi_2(x,p) = e^{\frac{B}{2}(x+p+i\Delta)^2}\,\partial_x U\left(a(p),\frac{1}{2},B(x+p+i\Delta)^2\right) \tag{9.96}$$

Using (9.82) and (9.84) we can proceed as above replacing $w(x',x,p)$ with

$$\begin{aligned}\tilde w(x',x,p) &= w(x',x,p)\,\frac{a(p)}{(x+p+i\Delta)^2} \\ &+\left(\partial_p\,\frac{a(p)}{(x+p+i\Delta)^2}\right)\left(\frac{p+x+i\Delta}{p+x'+i\Delta}\right)^{\frac{z(p)}{2B}}\frac{p+x+i\Delta}{\sqrt{(p+x+i\Delta)(p+x'+i\Delta)}}\end{aligned} \tag{9.97}$$

It is now sufficient to realize that

$$\partial_p\left(\frac{a(p)}{(x+p+i\Delta)^2}\right) \in L^1((-\infty,-2x']) \tag{9.98}$$

with the corresponding L^1 norm being uniformly bounded from above by a constant times Δ^{-1}, and that

$$e^{\frac{pB}{2}(x'-x)}\left(\frac{p+x+i\Delta}{p+x'+i\Delta}\right)^{\frac{z(p)}{2B}}\frac{p+x+i\Delta}{\sqrt{(p+x+i\Delta)(p+x'+i\Delta)}} \tag{9.99}$$

is uniformly bounded for $p\in(-\infty,-2x']$ provided F is small enough. This follows from

$$\ln\left|\frac{p+x+i\Delta}{p+x'+i\Delta}\right| \le C, \quad \forall\,p\in(-\infty,-2x'] \tag{9.100}$$

Then

$$\begin{aligned}&\left|\int_{-\infty}^{-2x'} \psi_1(x,p)\,\tilde\phi_2(x',p)\,W^{-1}(\psi_1,\psi_2)\,e^{ipB(y'-y)}\,\mathrm{d}p\right| \\ &\le \frac{C}{B|(x'-x)-i(y'-y)|}\left[\Delta^{-1}+e^{B(x'^2-x^2)/2}\int_{-\infty}^{-2x'} e^{pB(x'-x)}\,\tilde w(x',x,p)\,\mathrm{d}p\right][1+C\,\Delta^{-2}] \\ &\le \frac{C\,\Delta^{-1}}{|(x'-x)-i(y'-y)|}\,[1+C\,\Delta^{-2}]\end{aligned} \tag{9.101}$$

uniformly for $(x' - x)$ in the compacts of $(0, 1)$, since both

$$e^{B(x'^2-x^2)/2} \int_{-\infty}^{-2x'} e^{pB(x'-x)} |w(x', x, p)| \, \mathrm{d}p, \quad e^{B(x'^2-x^2)/2} e^{-Bx'(x'-x)} \tag{9.102}$$

are bounded. Same bounds on $|\partial_x G_1(\mathbf{x}, \mathbf{x}; z)|$ can be found for $(x' - x) \in (-1, 0)$.

$\underline{\partial_y G_1(\mathbf{x}, \mathbf{x}'; z)}$

As it was already noticed, differentiation w.r.t. y leads to a multiplication of the corresponding integrand by the factor iBp:

$$|\partial_y \, G_1(\mathbf{x}, \mathbf{x}'; z)| = C \, e^{-\Delta|y'-y|} \left| \int_{\mathbb{R}} h(x', x, p) \, e^{ipB(y'-y)} \, \mathrm{d}p \right| \tag{9.103}$$

where for $p > -x$

$$h(x', x, p) = iBp \, \psi_1(x', p) \, \psi_2(x, p) \, W^{-1}(\psi_1, \psi_2) \tag{9.104}$$

and for $p < -x'$

$$h(x', x, p) = iBp \, \psi_1(x, p) \, \psi_2(x', p) \, W^{-1}(\psi_1, \psi_2). \tag{9.105}$$

We can thus proceed in the same way as for $\partial_x \, G_1(\mathbf{x}, \mathbf{x}'; z)$. The only new ingredient which we need is the fact that that

$$\left(\partial_p \, \frac{p}{\sqrt{(p + x + i\Delta)(p + x' + i\Delta)}} \right) \in L^1 \left((-\infty, -2x'] \cup (-x/2, \infty) \right), \tag{9.106}$$

where the L^1 norm is again bounded by a constant times Δ^{-1}.

For $p \in (-2x', -x/2]$ we apply to both $\partial_x G_1(\mathbf{x}, \mathbf{x}'; z)$ and $\partial_y G_1(\mathbf{x}, \mathbf{x}'; z)$ the same arguments as for $|x' - x| \geq 1$ noting that these are independent on the value of $(x' - x)$. We have thus proved

Lemma 9.3. *For F small enough there exists some strictly positive constant G_0' such that the following inequality holds true*

$$\int_{\mathbb{R}} \int_{|x'-x|<1} |\partial_{x,y}^m G_1(\mathbf{x}, \mathbf{x}'; z)| e^{\frac{\Delta}{2}|y-y'|} \, \mathrm{d}x' \mathrm{d}y' \leq G_0' \, \Delta^{-3}, \tag{9.107}$$

where $m = 0, 1$.

9.A Integral kernel of e^{-itH_1}

Here we sketch the calculation of the integral kernel of evolution operator e^{-itH_1} in the gauge $H_L = p_x^2 + (p_y - Bx)^2$. We employ the functional integration to write

$$(x, y|e^{-itH_1}|x_0, y_0) = \int_{x_0,y_0;0}^{x,y;t} \mathrm{d}[w(\cdot)] \exp\left\{ i \int_0^t \mathrm{d}\, s\, L[w(s), \dot{w}(s)] \right\} \tag{9.108}$$

where

$$L[w(s), \dot{w}(s)] = \frac{1}{4}\, |\dot{w}(s)|^2 + F w_x(s) - \dot{w}_y(s) B w_x(s)$$

is the Lagrangian and

$$S_t[w(\cdot)] = \int_0^t \mathrm{d}\, s\, L[w(s), \dot{w}(s)] \tag{9.109}$$

the corresponding action. The integral in (9.108) is then taken over all trajectories $w(s)$ which satisfy the boundary conditions

$$w(0) = (x_0, y_0), \quad w(t) = (x, y) \tag{9.110}$$

We will write w as a sum of a classical trajectory plus certain fluctuation:

$$w(s) = w_{cl}(s) + \xi(s)$$

and evaluate $S_t[w(\cdot)]$ in the vicinity of the classical action $S_t[w_{cl}(\cdot)]$. As $L[w(s), \dot{w}(s)]$ is a quadratic function of canonical variables, all higher variations of $S_t[w_{cl}(\cdot)]$ are identically zero and

$$S_t[w(\cdot)] = S_t[w_{cl}(\cdot)] + \delta^{(1)} S_t[w_{cl}(\cdot)] + \delta^{(2)} S_t[w_{cl}(\cdot)] \tag{9.111}$$

Moreover, since $w_{cl}(s)$ minimises the classical action, the second term on the r.h.s. of (9.111) vanishes and for the last term we have

$$\delta^{(2)} S_t[w_{cl}(\cdot)] = \int_0^t \mathrm{d}\, s \left\{ \frac{1}{4}\, |\dot{\xi}(s)|^2 - \dot{\xi}_y(s) B \xi_x(s) \right\}$$

From the Van Vleck formula it then follows that the kernel (9.108) can be expressed in terms of the classical action only:

$$(x, y|e^{-itH_1}|x_0, y_0) = \frac{1}{2\pi i}\, e^{i\, S_t[w_{cl}(\cdot)]} \left[\det \left\{ -\frac{\partial^2 S_t[w_{cl}(\cdot)]}{\partial\alpha \partial\beta_0} \right\}_{\alpha,\beta} \right]^{1/2} \tag{9.112}$$

with $\alpha, \beta \in \{x, y\}$.

To compute $S_t[w_{cl}(\cdot)]$ we have to find the solution of the classical equations of motion

$$\begin{aligned} \frac{1}{2}\, \ddot{w}_x^{cl} &= -B \dot{w}_y^{cl} + F \\ \frac{1}{2}\, \ddot{w}_y^{cl} &= B \dot{w}_x^{cl} \end{aligned} \tag{9.113}$$

It is not difficult to verify that the general solution of (9.113) reads

$$\begin{aligned} w_x^{cl}(s) &= C_1(t)\cos(2Bs) + C_2(t)\sin(2Bs) + C_3(t) \\ w_y^{cl}(s) &= -C_2(t)\cos(2Bs) + C_1(t)\sin(2Bs) + u\,s + B^{-1}\,C_4(t) \end{aligned} \tag{9.114}$$

where $u = \frac{F}{B}$ is the drift velocity in $y-$direction and the "constants" $\{C_i(t), i = 1,2,3,4\}$ depend on t through the boundary conditions (9.110). A straightforward calculation gives

$$\begin{aligned} w_x^{cl}(s) &= \tfrac{1}{2}\,[(y-y_0-ut) + (x-x_0)\cot(Bt)]\,\sin(2Bs) \\ &- \tfrac{1}{2}\,[(x-x_0) - (y-y_0-ut)\cot(Bt)]\,\cos(2Bs) \\ &+ \tfrac{1}{2}\,[(x+x_0) - (y-y_0-ut)\cot(Bt)] \end{aligned} \tag{9.115}$$

and similarly

$$\begin{aligned} w_y^{cl}(s) &= -\tfrac{1}{2}\,[(x-x_0) - (y-y_0-ut)\cot(Bt)]\,\sin(2Bs) \\ &- \tfrac{1}{2}\,[(y-y_0-ut) + (x-x_0)\cot(Bt)]\,\cos(2Bs) \\ &+ \tfrac{1}{2}\,[(y+y_0-ut) + (x-x_0)\cot(Bt)] + u\,s \end{aligned} \tag{9.116}$$

The action then takes the form

$$\begin{aligned} S_t[w_{cl}(\cdot)] &= \tfrac{1}{4}\tfrac{F^2}{B^2}\,t + \tfrac{1}{2}\tfrac{F}{B}\,\left(y-y_0-\tfrac{F}{B}t\right) - \tfrac{1}{2}\,B(x+x_0)\left(y-y_0-\tfrac{F}{B}t\right) \\ &+ \tfrac{1}{4}\,B\cot(Bt)\left[\left(y-y_0-\tfrac{F}{B}t\right)^2 + (x-x_0)^2\right] \end{aligned} \tag{9.117}$$

and Van Vleck's determinant is thus easily calculated to give the integral kernel of e^{-itH_1}

$$(x,y|e^{-itH_1}|x_0,y_0) = \frac{1}{2\pi i}\sqrt{\tfrac{B}{2}}\,e^{i\,S_t[w_{cl}(\cdot)]}\,\frac{1}{\sin(Bt)} \tag{9.118}$$

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Chapter 10

Resonance width

In this chapter we report the article [FK03b].

Resonances Width in Crossed Electric and Magnetic Fields

Christian Ferrari and Hynek Kovarik

Abstract

We study the spectral properties of a charged particle confined to a two-dimensional plane and submitted to homogeneous magnetic and electric fields and an impurity potential V. We use the method of complex translations to prove that the life-times of resonances induced by the presence of electric field are at least Gaussian long as the electric field tends to zero.

10.1 Introduction

The purpose of this paper is to study the dynamics of an electron in two dimensions in the presence of crossed magnetic and electric fields and a potential type perturbation. We assume that the magnetic field acts in the direction perpendicular to the electron plane with a constant intensity B and that the electric field of constant intensity F points in the $x-$direction. The perturbation $V(x, y)$ is supposed to satisfy certain localisation conditions. The corresponding quantum Hamiltonian reads as follows

$$H(F) = H(0) - Fx = H_L + V - Fx,$$

where H_L is the Landau Hamiltonian of an electron in a homogeneous magnetic field of intensity B. Its spectrum is given by the infinitely degenerate eigenvalues (Landau levels) $(2n+1)B$, $n \in \mathbb{N}$.

When $F = 0$, the impurity potential V creates generically an infinite number of eigenvalues of $H(0)$ in between the Landau levels. These eigenvalues, which correspond to the so-called impurity states, then accumulate at Landau levels. This holds for any sign definite, bounded V, which tends to zero at infinity, see [Rai90], [MR03]. Classically, such impurity states represent the electron motion on localised trajectories. The main question that we address is what happens with these localised states when a constant electric field is switched on. In particular one would like to know, whether the eigenvalues of $H(0)$ may survive in the presence of a nonzero electric field and if not, what is the characteristic time in which they dissolve.

Answer to this question is well known for the hydrogen atom in a homogeneous electric field, in which case the corresponding Schrödinger operator has no eigenvalues, [Tit58]. The localised states turn into so-called Stark resonances, whose life-times are exponentially long as $F \to 0$. This was first computed by Oppenheimer in [Opp28] and later rigorously proved in [HS80]. The Oppenheimer formula was then partially generalised also for many body and non Coulombic potentials, see [Sig88] and references therein.

On the other hand, results concerning systems with simultaneous constant magnetic and electric fields are scarce. Such a model is considered in [GM99] where the impurity V is supposed to act as a $\delta-$potential. Using the special properties of a two-dimensional $\delta-$interaction, the authors of [GM99] compute the spectral density of $H(F)$ in the neighbourhood of the discrete spectrum of $H(0)$ and prove that all impurity states are unstable. Their life-times are then shown to be of order $\exp[\frac{B}{F^2}]$ as $F \to 0$ and it is conjectured that such a behaviour holds in general. It is our motivation to extend this result for continuous impurity potentials when the method of [GM99] is no longer applicable. In particular, we will prove under some assumptions on V that the life-times of magnetic Stark resonances are for F small enough at least Gaussian long, i.e. we find a lower bound compatible with the asymptotics obtained in [GM99].

Let us now describe the content of our paper more in detail. The basic mathematical tool we use is the method of complex translations for Stark Hamiltonians, which was introduced in [AH77] as a modification of the original theory of complex scaling [AC71], [BC71]. Following [AH77] we consider the transformation $U(\theta)$, which acts as a translation in $x-$direction; $(U(\theta)\psi)(x) = \psi(x+\theta)$. For non real θ the translated operator $H(F,\theta) = U(\theta)H(F)U^{-1}(\theta)$ is non-selfadjoint and therefore can have some complex eigenvalues. The complex eigenvalues of $H(F,\theta)$ with $\Im\theta > 0$ are called the spectral resonances of $H(F)$, see e.g. [HS96], and the corresponding resonance widths are given by their imaginary parts. Moreover, the result of [FK03a] tells us that if ϕ is an eigenfunction of $H(0)$, then $(\phi, e^{-itH(F)}\,\phi)$ decays exponentially at the rate given by the imaginary parts of the eigenvalues of $H(F,\theta)$.

In Section 10.5 we show that the eigenvalues of $H(F,\theta)$ are located in the Gaussian small vicinity of real axis as $F \to 0$, see Theorem 10.2. In order to prove this we employ a geometric resolvent equation in the form developed in [BG91] for the study of Stark Wannier Ladders. The idea of our proof is based on the fact that the eigenfunctions of $H(0)$ have a Gaussian-like decay at infinity and therefore "feel" the electric field only locally. That leads us to a construction of the reference Hamiltonian $H_2(F)$, which describes the system where the electric field is localised in the vicinity of impurity potential V by a suitable cut-off function. For a precise definition of $H_2(F)$ see Section 10.3. When $F \to 0$ we let the cut-off function tend to 1 at the rate proportional to $F^{-1+\varepsilon}$ ($\varepsilon > 0$), which assures the convergence of spectra of $H_2(F)$ to that of $H(0)$. It follows from the general theory of complex deformations that the discrete spectrum of $H_2(F)$ is not affected by the transformation $U(\theta)$. Moreover, for $H_2(F)$ also the essential spectrum does not change under $U(\theta)$. Therefore $\sigma(H_2(F,\theta))$ remains real even when θ becomes complex. The geometric resolvent equation, (10.22), then allows us to deduce that for F small enough the resolvent $R(z;\theta) = (z - H(F,\theta))^{-1}$ is bounded except in a small neighbourhood of the eigenvalues of $H_2(F,\theta)$. More precisely, we show that the norm of $R(z;\theta)$ remains bounded as long as the distance between z and $\sigma(H_2(F,\theta))$ is at least of order

$$e^{-\frac{B\,C}{F^{2(1-\varepsilon)}}}, \quad \varepsilon > 0, \tag{10.1}$$

where C is a strictly positive constant and ε can be taken arbitrarily small. Moreover, we prove that on the energy intervals well separated from Landau levels the spectral projector of $H(F,\theta)$ converges uniformly to that of $H_2(F,\theta)$ as $F \to 0$. These results give us the existence of eigenvalues of $H(F,\theta)$ and an upper bound on their imaginary parts. Let us note, that our result does not exclude the existence of point spectrum of $H(F)$. In other words, we do not answer the question whether all impurity states become unstable once the electric field with finite intensity is switched on. Although the quantum tunnelling phenomenon leads us to believe that it is indeed the case, a rigorous proof is missing and the question remains open.

10.2 The Model

We work in the system of units, where $m = 1/2$, $e = 1$, $\hbar = 1$. The crossed fields Hamiltonian is then given by

$$H_1(F) = H_L - Fx = (-i\partial_x + By)^2 - \partial_y^2 - Fx, \quad \text{on} \quad L^2(\mathbb{R}^2). \tag{10.2}$$

Here we use the Landau gauge with $\mathbf{A}(x,y) = (-By, 0)$. A straightforward application of [RS75, Thm. X.37] shows that $H_1(F)$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$, see also [RS75, Prob. X.38]. Moreover, one can easily check that

$$\sigma(H_1(F)) = \sigma_{ac}(H_1(F)) = \mathbb{R} \tag{10.3}$$

As mentioned in the Introduction we employ the translational analytic method developed in [AH77]. We introduce the translated operator $H_1(F,\theta)$ as follows:

$$H_1(F,\theta) = U(\theta)H_1(F)U^{-1}(\theta) \tag{10.4}$$

where

$$\left(U(\theta)f\right)(x,y) := \left(e^{ip_x\theta} f\right)(x,y) = f(x+\theta,y) \tag{10.5}$$

An elementary calculation shows that

$$H_1(F,\theta) = H_1(F) - F\theta \tag{10.6}$$

Operator $H_1(F,\theta)$ is clearly analytic in θ. Following [AH77] we define the class of $H_1(F)-$translation analytic potentials.

Definition 10.1. *Suppose that $V(z,y)$ is analytic in the strip $|\Im z| < \beta$, $\beta > 0$ independent of y. We then say that V is $H_1(F)-$translation analytic if $V(x+z,y)(H_1(F)+i)^{-1}$ is a compact analytic operator valued function of z in the given strip.*

We can thus formulate the conditions to be imposed on V:

(a) $V(x,y)$ is $H_1(F)-$translation analytic in the strip $|\Im z| < \beta$.

(b) There exists $\beta_0 \le \beta$ such that for $|\Im z| \le \beta_0$ the function $V(x+z,y)$ satisfies

$$|V(x+z,y)| \le \begin{cases} V_0 & \text{if } x \in [-a_0 - \Re z, a_0 - \Re z], y \in [-a_1, a_1] \\ V_0\, e^{-\nu\,(x+\Re z)^2},\ \nu > 0 & \text{if } x \notin [-a_0 - \Re z, a_0 - \Re z] \end{cases}$$

and

$$|V(x+z,y)| = 0, \quad y \notin [-a_1, a_1]$$

for given positive constants a_0, a_1, independent of F.

In order to characterise the potential class for which the above conditions are fulfilled let us assume for the moment, that the integral kernel of $(H_1(F)+i)^{-1}$ has at most a local logarithmic singularity at the origin. This is a very plausible hypothesis, see Lemma 4.3 in [FK03a], it then follows that any $L^2(\mathbb{R}^2)$ function that can be analytically continued in a strip $|\Im z| < \beta$ satisfies the condition (a). If in addition the analytic continuation satisfy (b), both assumptions are satisfied.

Remark 10.1. *It follows from the proof of our main result, given below, that the localisation of V w.r.t. y could be replaced by a Gaussian decay. However, we use the assumption (b) in order to keep the computations as simple as possible. Note that this assumption is of crucial importance to get the Gaussian upper bound, in $1/F$, on the imaginary part of the eigenvalues of $H(F,ib)$. See in particular Remark 10.4 in Appendix 10.A.*

From the well known perturbation argument, [Kat66], we see that under assumption (b) and the analyticity of V

$$H(F,\theta) = U(\theta)H(F)\,U^{-1}(\theta) = H_1(F,\theta) + V\,(x+\theta,y) \qquad (10.7)$$

forms an analytic family of type A.

Furthermore, since $V(x+\theta,y)(H_1(F)+i)^{-1}$ is compact by (a), we have [RS78, Cor. 2, p. 113]

$$\sigma_{ess}(H(F,\theta)+ibF) = \sigma_{ess}(H_1(F)) = \mathbb{R} \implies \sigma_{ess}(H(F,\theta)) = \mathbb{R} - i\Im\theta F \qquad (10.8)$$

From now on we take $\theta = ib$, $b \in \mathbb{R}_+$. By standard arguments [RS78, Prob. XIII.76], all eigenvalues of $H(F,ib)$ lie in the strip $-bF < \Im z \le 0$ and are independent of b as long as they are not covered by the essential spectrum.
The complex eigenvalues of $H(F,\theta)$ with $\Im\theta > 0$, in $\{z \in \mathbb{C} : -\Im\theta F < \Im z < 0\}$ are called the spectral *resonances* of $H(F)$, and are intrinsic to $H(F)$, see [HS96, Chap. 16]. The corresponding *resonance widths* are given by the imaginary parts of the eigenvalues E_α of $H(F,\theta)$: $\Gamma_\alpha = -2\Im E_\alpha$, and the *lifetimes* by $\tau_\alpha = \Gamma_\alpha^{-1}$.

Next we will show that, for sufficiently weak electric field F, the eigenvalues E_α of $H(F,ib)$ exist and are located in Gaussian small neighborhood of the real axis. In particular, we will prove that

$$|\Im E_\alpha| \le e^{-\frac{B\tilde{R}_\alpha}{F^{2(1-\varepsilon)}}}$$

where the positive constant $\tilde{R}_\alpha$ depends on the real part of E_α and ε can be made arbitrarily small. The method we employ is based on the decoupling formula developed in [BG91], see also [FM02].

10.3 Auxiliary Hamiltonian

The reference Hamiltonian reads

$$H_2(F) = H_L + V - Fxh_F(x)\chi_A(y) \equiv H_L + V + W_F$$

with χ_A being characteristic function of the set $A = [-\bar{y},\bar{y}]$ ($\bar{y} = y_1 + \frac{1}{F^\tau}$, with y_1 and τ defined in Section 10.4 below) and

$$h_F(x) = \tfrac{1}{2}\left\{\tanh(\gamma_F(x+\bar{x})) - \tanh(\gamma_F(x-\bar{x}))\right\}$$

where[1] $\gamma_F = \frac{\gamma_0}{F^{1-\varepsilon}} > 0$ and $\bar{x} > 0$ must satisfy

$$F\bar{x} \to 0 \quad \text{as} \quad F \to 0\,. \qquad (10.9)$$

[1]We will often drop the subscript F.

This is required because we don't want the local electric field to modify significatively the impurity potential V. We can thus expect that the spectrum of $H_2(F)$ is "close" to that of $H(0)$. We will chose $\bar{x} = \frac{\bar{C}}{F^{1-\varepsilon}} > 0$, for $\varepsilon > 0$.
In Figure 10.1 we sketch the $x-$section of $V(x,y) - xh_F(x)\chi_A(y)$ for the case of impurity potential given by $V(x,y) = -V_0 e^{-x^2} f(y)$ (f being any locally supported positive bounded function).

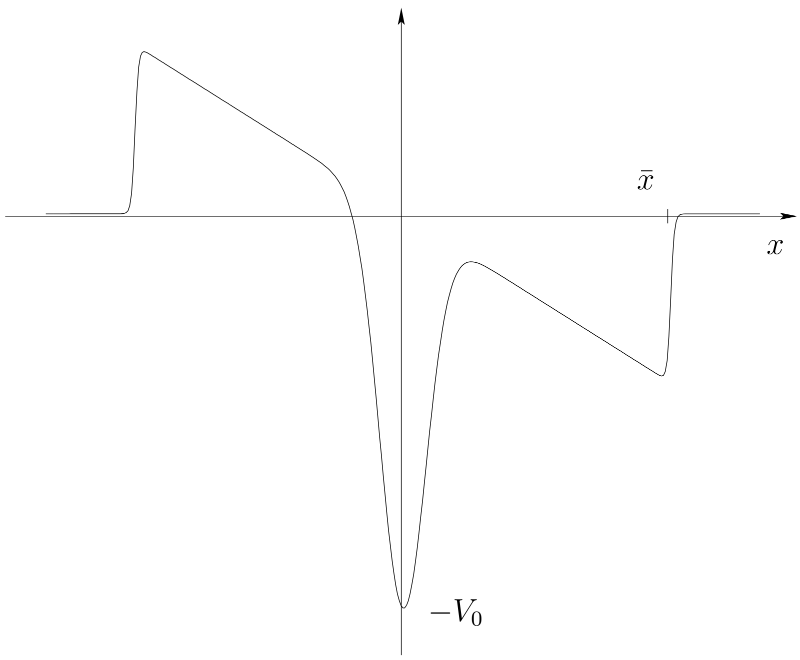

Figure 10.1: The $x-$section for the potential of $H_2(F)$ satisfying condition (10.9) *for a negative Gaussian potential.*

Before giving the results on the spectral properties of $H_2(F)$ and its translated correspondent $H_2(F, ib)$ we define the set of $\theta = ib$ for which W_F can be analytically continued in the x variable. Since $\tanh(z)$ has an analytic continuation for $|\Im z| < \frac{\pi}{2}$ we have $\gamma_F|b| < \frac{\pi}{2}$. For our purpose we will consider the family of operator $U(\theta) \equiv U(ib)$ defined in Section 10.2, with $\theta \in \mathcal{D}_\theta$ where

$$\mathcal{D}_\theta = \{\theta \in \mathbb{C} : \gamma_F|\Im\theta| < \tfrac{\pi}{4}\}$$

Since $\gamma_F = \frac{\gamma_0}{F^{1-\varepsilon}}$ we take

$$b = b_0 F^\alpha, \alpha > 2 \tag{10.10}$$

that insure $\gamma_F|\Im\theta| < \frac{\pi}{4}$ for $F \to 0$.

Proposition 10.1. *Assume V satisfies (a) and (b). Then*

1. *For each $e_\alpha \in \sigma(H(0))$ there is a family of $\lambda_\alpha(F) \in \sigma(H_2(F))$ such that $\lambda_\alpha(F) \to e_\alpha$ for $F \to 0$.*

2. *Let $P_\Delta(F)$ respectively $P_\Delta(0)$ be the eigenprojector of $H_2(F)$ respectively $H(0)$ on the open interval Δ. Then $\|P_\Delta(F) - P_\Delta(0)\| \to 0$ as $F \to 0$.*

3. *$\sigma_{ess}(H_2(F)) = \sigma_{ess}(H_L) = \{(2n+1)B; n \in \mathbb{N}\}$*

4. *For each $e_\alpha \in \sigma_d(H(0))$ there exists a constant c such that*

$$\lambda_\alpha(F) \in [e_\alpha - cF^\varepsilon, e_\alpha + cF^\varepsilon]$$

Proof. We have

$$\begin{aligned} \|(H(0)-z)^{-1} - (H_2-z)^{-1}\| &= \|(H_2-z)^{-1}[H_2 - H(0)](H(0)-z)^{-1}\| \\ &\leq \|(H_2-z)^{-1}\|\|(H_2 - H(0))\|\|(H(0)-z)^{-1}\| \\ &\leq \frac{1}{|\Im z|^2}\|Fxh_F(x)\chi_A(y)\| \to 0 \end{aligned} \tag{10.11}$$

as $F \to 0$ due to the choice of h_F. Thus $H_2(F) \to H(0)$ in the norm resolvent sense. The Statement 1. and 2. of the Lemma now follows from [Kat66, Thm. VIII.1.14] and [RS72, Thm. VIII.23]. Statement 3. follows from the fact that W_F and V are H_L-compact, see proof of Lemma 10.1 below. Finally the estimate

$$\|Fxh_F(x)\chi_A(y)\| \leq F\|xh_F(x)\|_\infty \leq cF^\varepsilon \tag{10.12}$$

yields Statement 4. □

We now show that the spectrum of $H_2(F)$ is not affected by the transformation $U(ib)$:

Lemma 10.1. *Under the assumptions of Proposition 10.1 $\{H_2(F,\theta) : \theta \in \mathcal{D}_\theta\}$ forms a self-adjoint holomorphic family of type A. Moreover, for each $ib \in \mathcal{D}_\theta$ one has*

$$\begin{aligned} \sigma_{ess}(H_2(F,ib)) &= \sigma_{ess}(H_2(F)) \\ \sigma_d(H_2(F,ib)) &= \sigma_d(H_2(F)) \end{aligned}$$

Proof. To prove that $\{H_2(F,\theta) : \theta \in \mathcal{D}_\theta\}$ forms a self-adjoint holomorphic family we have show that $H_2(F,\theta)$ is holomorphic w.r.t. $\theta \in \mathcal{D}_\theta$ and that its domain is independent of θ, see [Kat66, pp. 375, 385]. First claim follows from the assumptions on V and from the explicit form of W_F. The boundedness of V, W_F then implies the $\theta-$independence of the domain. For the the stability of essential spectrum we recall [HS96, Thm. 18.8], which tells us that it is enough to prove that $W_F(x+ib,y)(H_L+i)^{-1}$ and $V(x+ib,y)(H_L+i)^{-1}$ are compact. We first observe that

$$h_F(x+ib) = \frac{e^{2\gamma_F\bar{x}} - e^{-2\gamma_F\bar{x}}}{e^{2\gamma_F\bar{x}} + e^{-2\gamma_F\bar{x}} + e^{2\gamma_F(x+ib)} + e^{-2\gamma_F(x+ib)}}.$$

Thus

$$|h_F(x+ib)| \leq \frac{e^{2\gamma_F\bar{x}}}{[e^{2\gamma_F x} + e^{-2\gamma_F x}]\cos(2\gamma_F b) + [e^{2\gamma_F\bar{x}} + e^{-2\gamma_F\bar{x}}]}$$

From the latter estimate we deduce that $\lim_{x\to\pm\infty}|W_F(x+ib,y)| = 0$ and that $|W_F(x+ib,y)|$ is uniformly bounded. Since χ_A has compact support, $W_F(ib) \in L^2(\mathbb{R}^2)$. Then

$$\begin{aligned}\|W_F(ib)(H_L+i)^{-1}\|_{HS}^2 &= \int_{\mathbb{R}^2} d\mathbf{x}|W_F(x+ib,y)|^2 \int_{\mathbb{R}^2} d\mathbf{x}'|G_L(\mathbf{x},\mathbf{x}';i)|^2 \\ &= \int_{\mathbb{R}^2} d\mathbf{x}|W_F(x+ib,y)|^2 \int_{\mathbb{R}^2} d\mathbf{u}|G_L(\mathbf{u};i)|^2 < \infty \quad (10.13)\end{aligned}$$

where $|G_L(\mathbf{x},\mathbf{x}';i)| = |G_L(\mathbf{x}-\mathbf{x}';i)| = |G_L(\mathbf{u};i)| \in L^2(\mathbb{R}^2)$ is the integral kernel of $(H_L+i)^{-1}$, see for example [CN98]. Hence $W_F(ib)(H_L+i)^{-1}$ is compact. The same argument shows that also $V(ib)(H_L+i)^{-1}$ is compact.

Finally the stability of the discrete spectrum follows from a standard analyticity argument [RS78, Prob. XIII.76]. □

We now give a result on the norm of $R_2(z;ib)$, which will be used later in the proof of our main theorem. [2]

Lemma 10.2. *Let $z \in \mathbb{C}$ such that $(2q-1)B+\delta < \Re z < (2q+1)B-\delta$ $(\delta > 0)$ for some $q \in \mathbb{N}$. Then there exists a natural number $0 < s < \infty$, such that*

$$\|R_2(z;ib)\| \leq \mathcal{C}\,|\Im z|^{-s},$$

holds true provided F is small enough.

Proof. We introduce the operator $A(ib)$ by

$$A(ib) = H_2(ib) - H_2 \qquad (10.14)$$

(here we note $H_2(ib) \equiv H_2(F,ib)$ and $H_2 \equiv H_2(F)$). ¿From the definition of $H_2(ib)$ it easily follows that there exists certain constant A_0 such that for $b = b_0F^\alpha$

$$\|A(ib)\| \leq A_0F^{\alpha-1+\varepsilon}(1+\mathcal{O}(F^\alpha))$$

We need a preliminary result. A standard perturbation argument now shows that if

$$\operatorname{dist}(\sigma(H_2(F)),\xi) = d_0F^\varepsilon$$

[2]Henceforth the symbol $\mathcal{C}$ denotes a strictly positive real number independent of F.

then

$$\|R_2(\xi;ib)\| \leq \frac{\|R_2(\xi;0)\|}{1-\|A(ib)R_2(\xi;0)\|} = F^{-\varepsilon}\,\frac{1}{d_0 - F^{\alpha-1}A_0} \tag{10.15}$$

whenever $d_0 > F^{\alpha-1}A_0$, i.e. whenever F is small enough. To continue let e_α be the eigenvalue of $H(0)$ which minimises $|z-(e_\alpha \pm cF^\varepsilon)|$. We define a circle $\tilde{\Gamma} \equiv \{\xi \in \mathbb{C} : |\xi - e_\alpha| = \Gamma_0 F^\varepsilon\}$ enclosing only the eigenvalues of $H_2(F)$ converging to e_α for given e_α. Let $P_2^{\tilde{\Gamma}}(ib)$ the projector onto $\overline{\operatorname{Int}\tilde{\Gamma}}$ associated to $H_2(ib)$

$$P_2^{\tilde{\Gamma}}(ib) \equiv P_2(ib) = \frac{1}{2\pi i}\oint_{\tilde{\Gamma}} R_2(z;ib)\,\mathrm{d}z$$

Since $P_2(ib)$ is a projector, applying [Kat66, Thm.III.6.17], the resolvent of $H_2(ib)$ decomposes as follows

$$R_2(z;ib) = R_2'(z;ib) + R_2''(z;ib)$$

where

$$R_2'(z;ib) = P_2(ib)R_2'(z;ib) = R_2'(z;ib)P_2(ib) \tag{10.16}$$

$$R_2''(z;ib) = [1-P_2(ib)]R_2'(z;ib) = R_2'(z;ib)[1-P_2(ib)] \tag{10.17}$$

Let H' be the restriction of $H_2(ib)$ on $M' \equiv \operatorname{Ran} P_2(ib)$ and H'' the restriction of $H_2(ib)$ on $M'' \equiv \operatorname{Ran}[1-P_2(ib)]$. From [Kat66, Thm.III.6.17] it follows that $R_2'(z;ib)$ coincides with $(z-H')^{-1}$ on M' and vanishes on M''. Similarly $R_2''(z;ib)$ coincides with $(z-H'')^{-1}$ on M'' and vanishes on M'. Since $\operatorname{dist}(\sigma(H''),z)$ is bounded from below by a constant we can use (10.15) to get

$$\|R_2''(z;ib)\| \leq \mathcal{C}$$

Let us denote $r_0 = \dim P_2(ib)$. We can then write

$$R_2'(z;ib) = \sum_{h=1}^{r_0}\left[(z-\zeta_h)^{-1}P_h + (z-\zeta_h)^{-1}\sum_{n=1}^{m_h-1}(z-\zeta_h)^{-n}D_h^n\right]$$

where $\zeta_h \equiv \lambda_{\alpha,h} \in \mathbb{R}$ are the eigenvalues of H', P_h the corresponding projectors, $m_h = \dim P_h$ and D_h denotes the nilpotent associated to ζ_h, see [Kat66, Chap.I]. So we can always find some $s \in \mathbb{N}$ $(1 \leq s \leq \max_h m_h \leq r_0)$, such that

$$\|R_2'(z;ib)\| \leq \mathcal{C}\operatorname{dist}(z,\sigma(H'))^{-s} \leq \mathcal{C}\,|\Im z|^{-s},$$

which concludes the proof. □

10.4 Setup of a decoupling scheme

As already mentioned in the Introduction, the eigenfunctions of $H(0)$ “feel” the electric field only locally and the properties of the Hamiltonian $H(F)$ can be derived on the basis of those of the "local field" Hamiltonian $H_2(F)$ described above. To make this idea work we use the geometric resolvent perturbation theory in the form developed in [BG91] (see also [BCD89], [HS96]). It consists of dividing the configuration space $\mathbb{R}^2$ in different regions and study of Hamiltonians H_i with associated potentials V_i which are in the considered regions close to that of the full Hamiltonian $H(F)$.

We introduce the following functions that give a decoupling along the $x-$axis.

$$\begin{aligned}
J_-(x) &= \tfrac{1}{2}\left[1+\tanh(\gamma_F(x-x_2))\right] \\
\tilde{J}_-(x) &= \tfrac{1}{2}\left[1+\tanh(\gamma_F(x-x_0))\right] \\
J_0(x) &= \tfrac{1}{2}\left[\tanh(\gamma_F(x+x_1))-\tanh(\gamma_F(x-x_1))\right] \\
\tilde{J}_0(x) &= \tfrac{1}{2}\left[\tanh(\gamma_F(x+x_0))-\tanh(\gamma_F(x-x_0))\right] \\
J_+(x) &= \tfrac{1}{2}\left[1-\tanh(\gamma_F(x+x_2))\right] \\
\tilde{J}_+(x) &= \tfrac{1}{2}\left[1-\tanh(\gamma_F(x+x_0))\right]
\end{aligned} \tag{10.18}$$

where $0 < x_2 = \frac{C_2}{F^{1-\varepsilon}} < x_0 = \frac{C_0}{F^{1-\varepsilon}} < x_1 = \frac{C_1}{F^{1-\varepsilon}} < \bar{x}$. Along the $y-$axis we use three bounded $C^\infty(\mathbb{R})$ functions

$$J_<(y) = \begin{cases} 1 & \text{if } y \le -y_0 + \frac{1}{F^\tau} \\ 0 & \text{if } y \ge -y_2 \end{cases} \qquad J_c(y) = \begin{cases} 1 & \text{if } |y| \le y_0 + \frac{1}{F^\tau} \\ 0 & \text{if } |y| \ge y_1 \end{cases}$$

$$J_>(y) = \begin{cases} 1 & \text{if } y \ge y_0 - \frac{1}{F^\tau} \\ 0 & \text{if } y \le y_2 \end{cases} \tag{10.19}$$

where $0 < y_2 = a_1 + 1$, $y_0 = y_2 + \frac{1}{F^\tau} + 1$, $y_1 = y_0 + \frac{1}{F^\tau} + 1$, where $\tau > \alpha + 2$. We will also assume that $\|J_i'\|_\infty$, $\|J_i''\|_\infty < \infty$, $i \in \{<, >, c\}$.
Note that for the $x-$cut the dependence on F of x_0, x_1, x_2 is the optimal choice to get the desired results, while in the $y-$cut the dependence on F, i.e. the factor $F^{-\tau}$, is such that τ can be chosen as large as we need.
The system is then cut in five parts according to the following "full" decoupling functions (see Figure 2):

$$\begin{cases} J_1(x,y) &= J_-(x)J_c(y) \\ \tilde{J}_1(x,y) &= \tilde{J}_-(x)\tilde{J}_c(y) \end{cases} \qquad \begin{cases} J_2(x,y) &= J_0(x)J_c(y) \\ \tilde{J}_2(x,y) &= \tilde{J}_0(x)\tilde{J}_c(y) \end{cases}$$

$$\begin{cases} J_3(x,y) &= J_>(y) \\ \tilde{J}_3(x,y) &= \tilde{J}_>(y) \end{cases} \qquad \begin{cases} J_4(x,y) &= J_<(y) \\ \tilde{J}_4(x,y) &= \tilde{J}_<(y) \end{cases} \qquad \begin{cases} J_5(x,y) &= J_+(x)J_c(y) \\ \tilde{J}_5(x,y) &= \tilde{J}_+(x)\tilde{J}_c(y) \end{cases}$$

with

$$\tilde{J}_<(y) = \chi_{(-\infty,-y_0]}(y), \quad \tilde{J}_c(y) = \chi_{[-y_0,y_0]}(y), \quad \tilde{J}_>(y) = \chi_{[y_0,\infty)}(y)$$

We remark that all these functions have an analytic continuation in the x variable $(x \to x + ib)$ if $ib \in \mathcal{D}_\theta$.

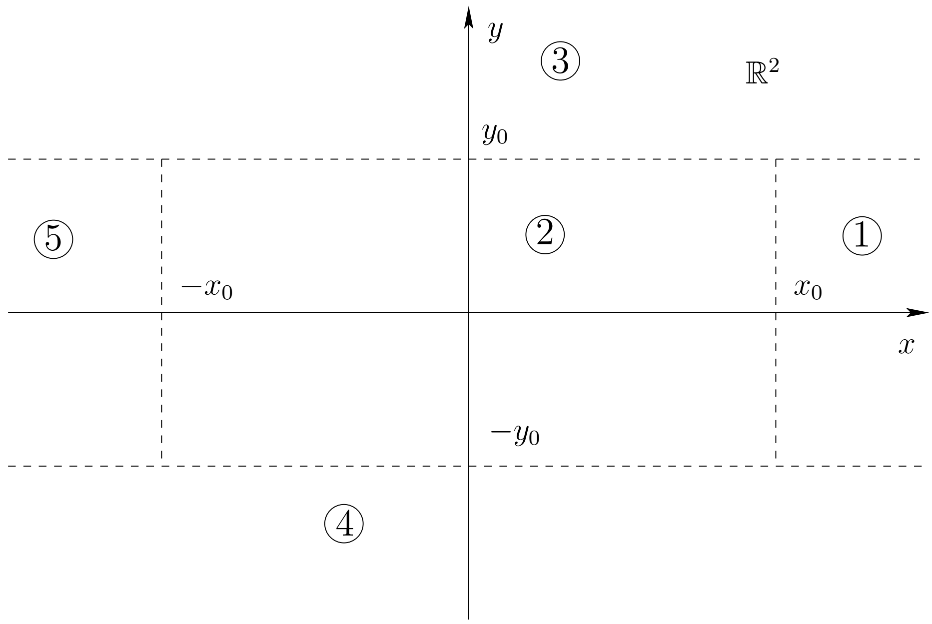

Figure 10.2: Schematic representation of the decoupling scheme. In region 2 the total potential $V(x,y) - Fx$ is close to the local potential of the auxiliary Hamiltonian $H_2(F)$, while in the others it is close to the electric potential $-Fx$.

We are now ready to establish the decoupling scheme. We introduce the following auxiliary Hamiltonians: $H_3 = H_4 = H_5 = H_1 = H_L - Fx$ and $H_2(F) \equiv H_2$ treated in the previous paragraph. For simplicity we write H for $H(F)$.
Note that

$$HJ_1 = H_1J_1 + VJ_1, \quad HJ_5 = H_5J_5 + VJ_5, \quad HJ_3 = H_3J_3, \quad HJ_4 = H_4J_4$$

and, using $\chi_A(y)J_c(y) = J_c(y)$,

$$HJ_2 = H_2J_2 - Fx(1 - h_F)(x)J_2$$

thus

$$(z - H)\sum_{i=1}^{5} J_iR_i(z)\tilde{J}_i = \sum_{i=1}^{5}(z - H_i)J_iR_i(z)\tilde{J}_i + A_1 + A_5 + A_2 = 1 - K(z) \qquad (10.20)$$

where $A_1 = VJ_1R_1(z)\tilde{J}_1$, $A_5 = VJ_5R_5(z)\tilde{J}_5$, $A_2 = -Fx(1 - h_F)(x)J_2R_2(z)\tilde{J}_2$ and

$$K(z) = \sum_{i=1}^{5}[H_L, J_i]R_i(z)\tilde{J}_i + \left(\sum_{i=1}^{5} J_i\tilde{J}_i - 1\right) - A_1 - A_5 - A_2$$

From (10.20) we deduce the decoupling formula

$$R(z) = \left(\sum_{i=1}^{5} J_iR_i(z)\tilde{J}_i\right)(1 - K(z))^{-1} . \qquad (10.21)$$

which is now to be transformed by the translation group $U(ib)$:

$$R(z;ib) = \left(\sum_{i=1}^{5} J_i(ib)R_i(z;ib)\tilde{J}_i(ib)\right)(1-K(z;ib))^{-1} \tag{10.22}$$

To prove that the eigenvalues of $H(F,ib)$ are at distance $\mathcal{O}\left(\exp\left(-1/F^{2(1-\varepsilon)}\right)\right)$ from those of $H_2(F,ib)$, we have to show that the norm of $K(z;ib)$ becomes smaller than 1 as $\mathrm{dist}(\sigma(H_2(F)),z)$ becomes Gaussian small. We will write $K(z;ib)$ as

$$K(z;ib) = \sum_{j=1}^{5} K_j(z;ib) + M(z;ib) \tag{10.23}$$

where

$$K_j(z;ib) = [H_L, J_j(ib)]R_j(z;ib)\tilde{J}_j(ib)$$

and

$$M(z;ib) = \left(\sum_{j=1}^{5} J_j(ib)\tilde{J}_j(ib) - 1\right) - A_1(ib) - A_5(ib) - A_2(ib)$$

In Appendix 10.A we estimate the norm of each term in the definition of $K(z;ib)$ separately. Our strategy is the following. Each of $K_j(z;ib)$ can be viewed as an integral operator with the corresponding kernel of the form $f(\mathbf{x})G(\mathbf{x},\mathbf{x}';z)h(\mathbf{x}')$, where $G(\mathbf{x},\mathbf{x}';z)$ is the Green function of H_1. Typically, the overlap of the functions $f(x)$ and $h(x')$ decreases as $F \to 0$. Fact, which together with the Gaussian decay of $G(\mathbf{x},\mathbf{x}';z)$ at large distances, see Appendix 10.A, assures that the norm of each of $K_j(z;ib)$ will tend to zero in the limit $F \to 0$. As for the operator $M(z;ib)$, we will see that for small values of F its norm can be made arbitrarily small by a proper choice of the parameters of the decoupling functions.

The results of Appendix 10.A yield the following estimate on the norm of $K(z;ib)$

$$\begin{aligned} \|K(z;ib)\| \;&\leq\; \mathcal{C}\, F^{-\mathcal{C}}\, \beta(z)^{-\sigma(\Re z)} \left(e^{-\frac{\beta(z)}{F^{\tau}}} + e^{-B\frac{\mathcal{C}'(B,\Re z)}{F^{2(1-\varepsilon)}}}\right)(1+\|R_2(z;ib)\|) \\ &+\; \mathcal{C}e^{-\frac{\tilde{\mathcal{C}}}{F^{2(1-\varepsilon)}}}\left(\|R_1(z;ib)\| + \|R_2(z;ib)\| + 1\right) \end{aligned} \tag{10.24}$$

with $\mathcal{C}'(B,\Re z) = Bc(\Re z) \to 0$ as $\Re z \to \infty$, $\tilde{\mathcal{C}}$ depending on the decoupling scheme (in particular we can set $\tilde{\mathcal{C}} = B\tilde{c}$), $\beta(z) = \frac{\Im z + bF}{2F}$ and $\sigma(\Re z) \geq 1$ ($\sigma(\Re z) \to \infty$ as $\Re z \to \infty$). We remark that for $F < 1$ we have $\beta(z) \leq \mathrm{dist}(\sigma(H_1(ib)), z)$. Using the inequality

$$\|R_1(z;ib)\| \leq \frac{1}{\mathrm{dist}(z, \overline{\Theta(H_1(ib))})} = \frac{1}{\mathrm{dist}(z, \mathbb{R} - ibF)}, \tag{10.25}$$

where $\Theta(H_1(ib))$ is the numerical range of $H_1(ib)$, see [HS96, Prop. 19.7], we can rewrite (10.24) as in the following Lemma:

Lemma 10.3. *Let F be small enough. Then for a given $z \in \mathbb{C}$ there exist positive numbers $\mathcal{C}_1$, $\mathcal{C}_2$, $\sigma(\Re z) \geq 1$ and $\mathcal{C}(B, \Re z) > 0$, with $\mathcal{C}(B, \Re z) = Bc(\Re z) \to 0$ as $\Re z \to \infty$, such that*

$$\begin{aligned} \|K(z;ib)\| \; &\leq \; \mathcal{C}_1 \, F^{-\mathcal{C}_2} \, \mathrm{dist}(\sigma(H_1(ib)), z)^{-\sigma(\Re z)} \left(e^{-\frac{\mathrm{dist}(\sigma(H_1(ib)),z)}{F^{\tau}}} + e^{-\frac{\mathcal{C}(B,\Re z)}{F^{2(1-\varepsilon)}}} \right) \\ & \quad \times (1 + \|R_2(z;ib)\|) \, . \end{aligned} \tag{10.26}$$

10.5 Main result

Armed with Lemma 10.3 we are ready to prove an estimate on the difference between the spectral projectors of $H(F, ib)$ and $H_2(F, ib)$.

Let $\Gamma(e_\alpha)$ the path in the complex plane enclosing the eigenvalue $e_\alpha \in \sigma(H(0))$ at finite distance to the Landau levels (see Figure 3). More precisely

$$\begin{aligned} \Gamma(e_\alpha) &:= \Gamma_1(e_\alpha) \cup \Gamma_2(e_\alpha) \cup \Gamma_3(e_\alpha) \cup \Gamma_4(e_\alpha) \\ \Gamma_1(e_\alpha) &:= \{\xi \in \mathbb{C} : \, \Re\xi = e_\alpha - cF^{\varepsilon/2}, \, |\Im\xi| \leq \rho\} \\ \Gamma_2(e_\alpha) &:= \{\xi \in \mathbb{C} : \, \Re\xi = e_\alpha + cF^{\varepsilon/2}, \, |\Im\xi| \leq \rho\} \\ \Gamma_3(e_\alpha) &:= \{\xi \in \mathbb{C} : \, e_\alpha - cF^{\varepsilon/2} \leq \Re\xi \leq e_\alpha + cF^{\varepsilon/2}, \, \Im\xi = \rho\} \\ \Gamma_4(e_\alpha) &:= \{\xi \in \mathbb{C} : \, e_\alpha - cF^{\varepsilon/2} \leq \Re\xi \leq e_\alpha + cF^{\varepsilon/2}, \, \Im\xi = -\rho\} \, . \end{aligned} \tag{10.27}$$

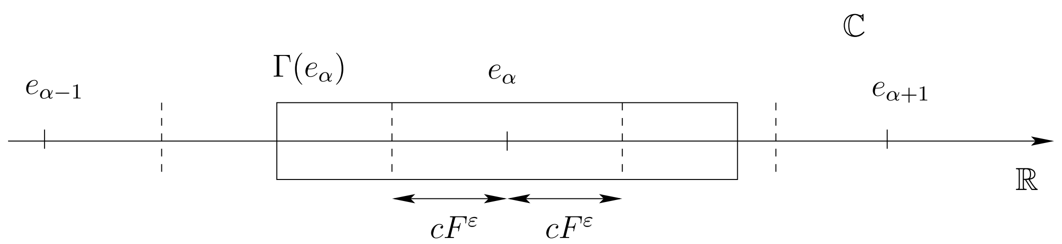

Figure 10.3: The path $\Gamma(e_\alpha)$ in the complex plane. The spectrum of $H_2(F, ib)$ is localised in the vicinity of e_α, represented by the dashed vertical lines. (Proposition 10.1).

For F sufficiently small this construction can be made in such a way that the spectrum of $H_2(F, ib)$ enclosed by $\Gamma(e_\alpha)$ consists only of the eigenvalues $\lambda_{\alpha,i}(F) \to e_\alpha$, where i denote the degeneracy index of the eigenvalue e_α $(1 \leq i \leq r_\alpha)$, see Proposition 10.1. Moreover for $z \in \Gamma(e_\alpha)$ holds by Lemma 10.2

$$\|R_2(z;ib)\| \leq \mathcal{C}\rho^{-s} \, . \tag{10.28}$$

To control the inverse $(1 - K(z, ib))^{-1}$ we need $\|K(z;ib)\| < 1$ for $z \in \Gamma(e_\alpha)$. In particular we want $\|K(z;ib)\| \to 0$ as $F \to 0$. Looking at Lemma 10.3, together with (10.28) we see that the above requirement on the norm of $K(z;ib)$ is satisfied at best by taking

$$\rho = e^{-\frac{\rho_0}{F^{2(1-\varepsilon)}}} \quad \text{with } s\rho_0 < \mathcal{C}(B,\Re z) \tag{10.29}$$

We point out that the Gaussian smallness of ρ is the optimal choice to get the eigenprojectors convergence. From the decoupling formula (10.22) we have

$$\begin{aligned} R(z;ib) - R_2(z;ib) &= \left(\sum_{i=1}^{5} J_i(ib)R_i(z;ib)\tilde{J}_i(ib)\right)\sum_{n=1}^{\infty} K(z;ib)^n - (1-J_2(ib))R_2(z;ib) \\ &- J_2(ib)R_2(z;ib)(1-\tilde{J}_2(ib)) + \sum_{i\in\{1,3,4,5\}} J_i(ib)R_i(z;ib)\tilde{J}_i(ib)\ . \end{aligned} \tag{10.30}$$

Because of $\sigma(H_i(ib)) = \mathbb{R} - ibF$ (see (10.3)), $R_i(z;ib)$, $i \neq 2$, have no poles in $\Gamma(e_\alpha)$. Moreover the only poles of $R_2(z;ib)$ are precisely $\lambda_{\alpha,i}(F)$ $(1 \leq i \leq r_\alpha)$. Thus integrating (10.30) along the path $\Gamma(e_\alpha) \equiv \Gamma$

$$\begin{aligned} P^\Gamma(ib) - P_2^\Gamma(ib) &= \frac{1}{2\pi i}\oint_\Gamma \left(\sum_{i=1}^{5} J_i(ib)R_i(z;ib)\tilde{J}_i(ib)\right)\sum_{n=1}^{\infty} K(z;ib)^n \,\mathrm{d}z \\ &- J_2(ib)P_2^\Gamma(ib)(1-\tilde{J}_2(ib)) - (1-J_2(ib))P_2^\Gamma(ib)\ . \end{aligned} \tag{10.31}$$

where $P_2^\Gamma(ib)$ is the spectral projector of $H_2(ib)$ onto $\overline{\mathrm{Int}\,\Gamma}$ and

$$P^\Gamma(ib) = \frac{1}{2\pi i}\oint_\Gamma (z - H(ib))^{-1}\,\mathrm{d}z$$

We estimate the norms of the three contributions on the r.h.s. of (10.31). If ρ_0 in the definition of $\Gamma(e_\alpha)$ satisfies a bit stronger condition than the bound in (10.29), the norm of the first term is smaller than

$$\mathcal{C}\left(\sum_{i=1}^{5}\sup_{z\in\Gamma}\|R_i(z;ib)\|\right)\frac{\sup_{z\in\Gamma}\|K(z;ib)\|}{1-\sup_{z\in\Gamma}\|K(z;ib)\|} \leq g(F) \to 0 \quad \text{as } F \to 0\ . \tag{10.32}$$

Indeed, for $i = 2$, by (10.28) and (10.29) there exists a smooth function $g(F)$ such that

$$\|R_2(z;ib)\|\|K(z;ib)\| \leq \mathcal{C}g(F)$$

for each $z \in \Gamma(e_\alpha)$ and $\lim_{F\to 0} g(F) = 0$ provided $2s\rho_0 < \mathcal{C}(B,\Re z)$. For $i \neq 2$ remembering that $b = b_0F^\alpha$, by (10.25) we have $\sup_{z\in\Gamma}\|R_i(z;ib)\| \leq \frac{\mathcal{C}}{F^{\alpha+1}}$, and the result follows.

To estimate the second term in (10.31) we write

$$\begin{aligned} \|J_2(ib)P_2^\Gamma(ib)(1-\tilde{J}_2(ib))\| &\leq \|J_2(ib)\|_\infty\|P_2^\Gamma(ib)(1-\tilde{J}_2(ib))\| \\ &\leq \|[P_2^\Gamma(ib) - P_2^\Gamma(0)](1-\tilde{J}_2(ib))\| \\ &+ \|[P_2^\Gamma(0) - P^\Gamma](1-\tilde{J}_2(ib))\| + \|P^\Gamma(1-\tilde{J}_2(ib))\| \\ &\leq \left(\|P_2^\Gamma(ib) - P_2^\Gamma(0)\| + \|P_2^\Gamma(0) - P^\Gamma\|\right)\|(1-\tilde{J}_2(ib))\|_\infty \\ &+ \sum_{i=1}^{r_\alpha} |(1-\tilde{J}_2(ib),\phi_0^i)| \end{aligned} \tag{10.33}$$

where P^Γ is the spectral projector of $H(0)$ onto the eigenfunctions ϕ_0^i $(i = 1, \ldots, r_\alpha)$ corresponding to the eigenvalue e_α. In order to control the term $\|P_2^\Gamma(ib) - P_2^\Gamma(0)\|$ we define a circle $\tilde\Gamma \equiv \{\xi \in \mathbb{C} : |\xi - e_\alpha| = \Gamma_0 F^\varepsilon\}$. Then for F small enough holds

$$\begin{aligned} \|P_2^\Gamma(ib) - P_2^\Gamma(0)\| &\leq (2\pi)^{-1} \oint_{\tilde\Gamma} \|R_2(\xi; ib) A(ib) R_2(\xi; 0)\| \, |\mathrm{d}\xi| \\ &\leq \mathcal{C}\, F^{\alpha-1} \end{aligned} \tag{10.34}$$

where $A(ib)$ is defined in (10.14) and the second inequality follows form (10.15). By Proposition 10.1 $\|P_2^\Gamma(0) - P^\Gamma\| \to 0$ as $F \to 0$. Thus for $F \to 0$ the two terms converge to 0. The last term can be easily estimated using the result of [CN98, Thm. 4.2], which says that for any at least gaussian decaying potential one has the estimate

$$|\phi(\mathbf{x})| \leq \mathcal{C} e^{-\mu|\mathbf{x}|^2},$$

where ϕ is associated to a discrete eigenvalue of $H(0)$. Using this result and a bound on $|1 - \tilde J_2(ib)|$ similar to that of (10.39) we get

$$\|J_2(ib) P_2^\Gamma(ib)(1 - \tilde J_2(ib))\| \to 0 \qquad \text{as } F \to 0 \tag{10.35}$$

For the third term in (10.31) we obtain the same estimate, since for any bounded operator A, $\|A^*\| = \|A\|$. In conclusion we arrive at

Proposition 10.2. *Let* $\Gamma(e_\alpha)$ *be as in* (10.27)*, then*

$$\|P^\Gamma(ib) - P_2^\Gamma(ib)\| \to 0, \qquad F \to 0$$

In other words, $\dim \operatorname{Ran} P^\Gamma(ib) = \dim \operatorname{Ran} P_2^\Gamma(ib)$ *for* F *sufficiently small.*

Propositions 10.2 and 10.1 yield

Theorem 10.1. *Assume* V *satisfies* (a), (b) *and let* e_α *be an eigenvalue of* $H(0)$ *of multiplicity* $r_\alpha < \infty$*. Then near* e_α *there are eigenvalues* $E_{\alpha,i}$ *of* $H(F, ib)$, $(1 \leq i \leq r_\alpha)$*, repeated according to their multiplicity, and*

$$E_{\alpha,i} \to e_\alpha \quad \text{as} \quad F \to 0.$$

Now we can formulate our main result.

Theorem 10.2. *Assume* V *satisfies* (a) *and* (b)*. Let* e_α *and* $E_{\alpha,i}$ *be the eigenvalues defined in Theorem 10.1. Then there exist some positive constants* $\mathcal{C}$ *and* $R_\alpha(B)$*, such that for* F *small enough the following inequality holds true*

$$|\Im E_{\alpha,i}| \leq \mathcal{C}\, e^{-\frac{R_\alpha(B)}{F^{2(1-\varepsilon)}}}, \quad \varepsilon > 0,$$

where ε *can be made arbitrarily small and* $R_\alpha(B) = B\tilde R_\alpha$*.*

Proof. Consider the path $\Gamma(e_\alpha)$ defined through (10.27), with $\rho_0 = R_\alpha(B)$. We have proved in Proposition 10.2 that if

$$2s\, R_\alpha(B) < \mathcal{C}(B, e_\alpha), \tag{10.36}$$

with $\mathcal{C}(B, e_\alpha)$ defined in Lemma 10.3, then $\dim \operatorname{Ran} P^\Gamma(ib) = \dim \operatorname{Ran} P_2^\Gamma(ib)$ and the only eigenvalues of $H(F, ib)$ in $\overline{\operatorname{Int}\Gamma}$ are the eigenvalues $E_{\alpha,i}$. By construction their imaginary parts satisfy the announced upper bound. The linear dependence on B follows from the linear dependence of $\mathcal{C}(B, e_\alpha)$ on B. □

Remark 10.2. *The behaviour of $\tilde{R}_\alpha$ w.r.t. α is not uniform. Indeed $\tilde{R}_\alpha \to 0$ as $e_\alpha \to \infty$, because $\mathcal{C}(B, \Re z) \to 0$ as $\Re z \to \infty$.*

As already mentioned at the end of Section 10.2 the resonance widths are given by the imaginary parts of the eigenvalues of $H(F, ib)$, and the lifetime by the inverse of the resonance width. Since ε is arbitrarily small, we thus get a lower bound on the life-times:

Corollary 10.1. *The life-times of the resonant states satisfy:*

$$\tau_\alpha = \tfrac{1}{2} \sup_{\varepsilon > 0} |\Im E_{\alpha,i}|^{-1} \geq 1/\mathcal{C}\, \exp\left(\frac{B\tilde{R}_\alpha}{F^2}\right) .$$

Conclusion

Theorem 10.2 gives a partial generalisation of the result obtained in [GM99]. As expected, the fact that the lower bound on the resonance life-times is Gaussian in F^{-1} and not exponential is due to the presence of the magnetic field. However, further comparison with the purely electric Stark effect shows much larger restriction on the class of admissible potentials, in particular the condition on the Gaussian decay of $V(x, y)$. Let us now briefly discuss the issue of Gaussian versus exponential behaviour. As follows from the analysis of the Stark resonances, [Opp28] [HS80] [Sig88], the exponential law for the resonant states is in that case directly connected with the exponential decay of the eigenfunctions of a "free" Hamiltonian, i.e. without electric field. If we suppose that the same connection exists also in the magnetic case, then our result should hold whenever the eigenfunctions of $H(0) = H_L + V$, associated with the discrete spectrum, fall off as a Gaussian. Sufficient condition for the latter is the Gaussian decay of $V(x, y)$, see [CN98], which is compatible with our assumption (b). Up to now, the optimal condition is known only for the ground state, in which case a sort of exponential decay of $V(x, y)$ is shown to be sufficient and necessary for Gaussian behaviour of the corresponding eigenfunctions at infinity, [Erd96].

Such a restriction is in contrast with the non magnetic Schrödinger operator, whose eigenfunctions decrease exponentially in the classically forbidden region independently on the rate at which $V(x, y)$ tends to zero at infinity. This might indicate a principal difference between the behaviour of resonant states in the presence respectively absence of magnetic field.

10.A Estimate of $\|K(z;ib)\|$

Here we estimate the norm of each term in the definition of $K(z;ib)$ separately. Since the calculations are often analogous, we skip the details in many places.

Norm of $M(z;ib)$

Terms $\|A_1(ib)\|$ and $\|A_5(ib)\|$:

$$\begin{aligned}\|A_1(ib)\| &\leq \|V(ib)J_1(ib)\|_\infty \|R_1(z;ib)\| \|\tilde{J}_1(ib)\| \\ &\leq \mathcal{C}\|V(ib)J_1(ib)\|_\infty \|R_1(z;ib)\| \end{aligned} \tag{10.37}$$

and for F sufficiently small

$$\begin{aligned}\|V(ib)J_1(ib)\|_\infty &= \sup_{(x,y)} |V(x+ib,y)||J_-(x+ib)||J_c(y)| \\ &\leq \sup_x |V(x+ib,\hat{y})| \frac{e^{2\gamma(x-x_2)}}{(e^{4\gamma(x-x_2)}+1)^{1/2}}\end{aligned}$$

We estimate this term as $\max\{a,b,c\}$ where a, b, c are

$$\begin{aligned} a &= \sup_{|x|<a_0} |V(x+ib,\hat{y})|e^{2\gamma(x-x_2)} \leq V_0 e^{2\gamma(a_0-x_2)} \leq V_0 e^{\frac{2\gamma_0 a_0}{F^{1-\varepsilon}}} e^{-\frac{2\gamma_0 C_2}{F^{2(1-\varepsilon)}}} \\ b &= \sup_{a_0\leq|x|\leq a_0+\delta} V_0 e^{-\nu x^2} e^{2\gamma(x-x_2)} \leq V_0 e^{-\nu a_0^2} e^{2\gamma(a_0+\delta-x_2)} \leq V_0 e^{\frac{2\gamma_0 a_0}{F^{1-\varepsilon}}} e^{-\frac{2\gamma_0(C_2-\delta_0)}{F^{2(1-\varepsilon)}}} \\ c &= \sup_{|x|>a_0+\delta} V_0 e^{-\nu x^2} \leq V_0 e^{-\frac{\delta_0{}^2}{F^{2(1-\varepsilon)}}} \end{aligned}$$

and $\delta = \delta_0 F^{-(1-\varepsilon)} < x_2$. This leads to

$$\|A_1(ib)\| \leq \mathcal{C} e^{-\frac{\mathcal{C}}{F^{2(1-\varepsilon)}}} \|R_1(z;ib)\|$$

In the same way we prove the estimate for $\|A_5(ib)\|$.

Term $\|A_2(ib)\|$:

$$\begin{aligned}\|A_2(ib)\| &\leq F\|(x+ib)(1-h_F(x+ib))J_2(ib)\|_\infty \|R_2(z;ib)\| \|\tilde{J}_2(ib)\| \\ &\leq \mathcal{C}F\|(x+ib)(1-h_F(x+ib))J_0(x+ib)\|_\infty \|R_2(z;ib)\| \end{aligned} \tag{10.38}$$

We can easily found the following bounds

$$|J_0(x+ib)| \leq \frac{1}{\cos(2\gamma b)} \begin{cases} e^{2\gamma(x+x_1)} & \text{if } x<0 \\ e^{-2\gamma(x-x_1)} & \text{if } x>0 \end{cases} \tag{10.39}$$

and

$$|1 - h_F(x+ib)| \le \left(e^{-4\gamma(x-\bar{x})} + 1\right)^{-1/2} + \left(e^{4\gamma(x+\bar{x})} + 1\right)^{-1/2} \equiv h_1 + h_2 \tag{10.40}$$

For $x > \frac{\bar{x}+x_1}{2} > 0$

$$|h_1|^2 |J_0(x+ib)|^2 \le \mathcal{C}\frac{e^{-4\gamma(x-x_1)}}{e^{-4\gamma(x-\bar{x})}+1} \le \mathcal{C}\frac{e^{-4\gamma(x-\frac{\bar{x}+x_1}{2})}}{e^{-2\gamma(x_1-\bar{x})}}$$

the last inequality follows after multiplication by $(e^{2\gamma(\bar{x}-x_1)})/(e^{2\gamma(\bar{x}-x_1)})$. Now, $y = x - (\bar{x}+x_1)/2$, yields

$$\begin{aligned} \sup_{x>\frac{\bar{x}+x_1}{2}} F|x||h_1 J_0(x+ib)| &\le \mathcal{C}F\sup_y \left(|y| + |\bar{x}+x_1|/2\right) e^{-\gamma(\bar{x}-x_1)} e^{-2\gamma|y|} \\ &\le \mathcal{C}(F+F^{\varepsilon}) e^{-\frac{\mathcal{C}}{F^{2(1-\varepsilon)}}} \end{aligned} \tag{10.41}$$

For $x < -\frac{\bar{x}+x_1}{2} < 0$ we get in the same way the upper bound (10.41). Finally, for $|x| \le \frac{\bar{x}+x_1}{2}$ obviously $\sup_x |x| = \frac{\bar{x}+x_1}{2}$ and

$$|h_1 J_0(x+ib)| \le e^{-2\gamma(\bar{x}-x_1)}$$

which gives a similar estimate as (10.41).
A similar argument holds for $|h_2 J_0(x+ib)|$ that leads to

$$\|A_2(ib)\| \le \mathcal{C}e^{-\frac{\mathcal{C}}{F^{2(1-\varepsilon)}}} \|R_2(z;ib)\| \tag{10.42}$$

Term $\|\sum_{j=1}^{5} J_j(ib)\tilde{J}_j(ib) - 1\|$:
First we remark that we can write $1 = \tilde{J}_c(y) + (1 - \tilde{J}_c(y))$ and that $\sum_{i=3}^{4} J_i(ib)\tilde{J}_i(ib) - (1-\tilde{J}_c) = 0$, thus it remains to estimate $\sum_{i\in\{1,2,5\}} J_i(ib)\tilde{J}_i(ib) - \tilde{J}_c$. We have

$$\begin{aligned} \sum_{i\in\{1,2,5\}} J_i(ib)\tilde{J}_i(ib) - \tilde{J}_c &= \Big[J_-(x+ib)\tilde{J}_-(x+ib) + J_0(x+ib)\tilde{J}_0(x+ib) \\ &+ J_+(x+ib)\tilde{J}_+(x+ib) - 1\Big]\tilde{J}_c(y) := \mathcal{X}(ib)\tilde{J}_c(y) \end{aligned}$$

Now $\|\tilde{J}_c(y)\|_\infty = 1$, and it remain to estimate

$$\|\mathcal{X}(ib)\|_\infty = \left\|\sum_{\alpha\in\{\pm,0\}} J_\alpha(x)\tilde{J}_\alpha(x) - 1\right\|_\infty \tag{10.43}$$

This can be done by developing explicitly the functions in term of the exponentials and by writing the sum as fraction (denote by $\mathcal{K}$ the denominator). After a tedious straightforward computation we find out that each term in the sum

$$\sum_{\alpha\in\{\pm,0\}} J_\alpha(x+ib)\tilde{J}_\alpha(x+ib) - 1$$

can be bounded from above uniformly w.r.t. x by $\mathcal{C}e^{-\mathcal{C}F^{-(2-\varepsilon)}}$. For example

$$\left|\frac{e^{-2\gamma(2x+x_0+x_2)}}{\mathcal{K}}\right| \le \frac{e^{-2\gamma(2x+x_0+x_2)}}{\cos(4\gamma b)e^{4\gamma x}} = \frac{e^{-2\gamma(x_0+x_2)}}{\cos(4\gamma b)} \le \mathcal{C}e^{-\frac{\mathcal{C}}{F^{2(1-\varepsilon)}}}$$

for $F \to 0$ due to (10.10) and similarly in other cases. Therefore

$$\left\|\sum_{i=1}^{5} J_i(ib)\tilde{J}_i(ib) - 1\right\|_\infty \le \mathcal{C}e^{-\frac{\mathcal{C}}{F^{2(1-\varepsilon)}}}$$

Finally,

$$\|M(z;ib)\| \le \mathcal{C}e^{-\frac{\mathcal{C}}{F^{2(1-\varepsilon)}}}\left(\|R_1(z;ib)\| + \|R_2(z;ib)\| + 1\right)$$

Norm of $K_3(z;ib)$ and $K_4(z;ib)$

To control the operator norm we will use alternatively the Hilbert-Schmidt norm and the following inequality for the norm of an integral operator which can be found in [Kat66, p. 144]

$$\|A\| \le \max\left\{\sup_{\mathbf{x}}\int |A(\mathbf{x},\mathbf{x}')|\,\mathrm{d}\mathbf{x}'; \sup_{\mathbf{x}'}\int |A(\mathbf{x},\mathbf{x}')|\,\mathrm{d}\mathbf{x}\right\} \tag{10.44}$$

Each integration that we need to evaluate is split in two parts according to $|x-x'| \ge 1$ and $|x-x'| < 1$:

Let φ such that $\|\varphi\| = 1$, and A an operator with integral kernel $A(\mathbf{x},\mathbf{x}')$, then

$$\|A\varphi\|^2 = \int_{\mathbb{R}^2}\left|\int_{\mathbb{R}^2} A(\mathbf{x},\mathbf{x}')\varphi(\mathbf{x}')\,\mathrm{d}\mathbf{x}'\right|^2 \mathrm{d}\mathbf{x} \tag{10.45}$$

$$\le 2\int_{\mathbb{R}^2}\left|\int_{\mathbb{R}^2:|x-x'|\ge 1} A(\mathbf{x},\mathbf{x}')\varphi(\mathbf{x}')\,\mathrm{d}\mathbf{x}'\right|^2 \mathrm{d}\mathbf{x}$$

$$+2\int_{\mathbb{R}^2}\left|\int_{\mathbb{R}^2:|x-x'|< 1} A(\mathbf{x},\mathbf{x}')\varphi(\mathbf{x}')\,\mathrm{d}\mathbf{x}'\right|^2 \mathrm{d}\mathbf{x} =: 2(a+b)\,. \tag{10.46}$$

We now treat the two terms separately. By the Schwarz inequality we have

$$a \le \int_{\mathbb{R}^2}\int_{\mathbb{R}^2:|x-x'|\ge 1} |A(\mathbf{x},\mathbf{x}')|^2\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x}\|\varphi\|^2 \le \|A\|_{HS}^2\|\varphi\|^2$$

For b we proceed as follows, let

$$\psi(\mathbf{x}) \equiv \int_{\mathbb{R}^2:|x-x'|<1} A(\mathbf{x},\mathbf{x}')\varphi(\mathbf{x}')\,\mathrm{d}\mathbf{x}'$$

and

$$A(\mathbf{x}) = \int_{\mathbb{R}^2:|x-x'|<1} |A(\mathbf{x},\mathbf{x}')|\,\mathrm{d}\mathbf{x}' \qquad A'(\mathbf{x}') = \int_{\mathbb{R}^2:|x-x'|<1} |A(\mathbf{x},\mathbf{x}')|\,\mathrm{d}\mathbf{x}$$

we first remark that $\int_{\mathbb{R}^2:|x-x'|<1} |A(\mathbf{x},\mathbf{x}')|/A(\mathbf{x})\,\mathrm{d}\mathbf{x}' = 1$, this implies by convexity, that

$$\left(\frac{|\psi(\mathbf{x})|}{A(\mathbf{x})}\right)^2 \leq \int_{\mathbb{R}^2:|x-x'|<1} \frac{|A(\mathbf{x},\mathbf{x}')|}{A(\mathbf{x})}|\varphi(\mathbf{x}')|^2\,\mathrm{d}\mathbf{x}'$$

and thus

$$\begin{aligned} b = \int_{\mathbb{R}^2} |\psi(\mathbf{x})|^2\,\mathrm{d}\mathbf{x} &\leq \sup_{\mathbf{x}} A(\mathbf{x}) \int_{\mathbb{R}^2}\int_{\mathbb{R}^2:|x-x'|<1} |A(\mathbf{x},\mathbf{x}')||\varphi(\mathbf{x}')|^2\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x} \\ &= \sup_{\mathbf{x}} A(\mathbf{x}) \int_{\mathbb{R}^2}\int_{\mathbb{R}^2:|x-x'|<1} |A(\mathbf{x},\mathbf{x}')||\varphi(\mathbf{x}')|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{x}' \\ &\leq \sup_{\mathbf{x}} A(\mathbf{x}) \sup_{\mathbf{x}'} A'(\mathbf{x}')\|\varphi\|^2 \\ &\leq \max\left\{\sup_{\mathbf{x}} A(\mathbf{x}), \sup_{\mathbf{x}'} A'(\mathbf{x}')\right\}^2 \|\varphi\|^2 \qquad (10.47) \end{aligned}$$

Therefore, for $|x-x'| \geq 1$ we can use a Hilbert-Schmidt-like norm, while for $|x-x'| < 1$ we can use a (10.44) norm. We will need results on the behaviour of the Green function $G_1(\mathbf{x},\mathbf{x}';z)$ of $H_1(ib)$. We expect that at points $\mathbf{x},\mathbf{x}'$ with $|x-x'|$ large the Green function decay in the $x-$direction as a Gaussian due to the magnetic field, while in the $y-$direction (the drift direction of the classical particle) we expect only exponential decay. On the other we also expect integrable singularity at the origin. These properties are contained in the following two lemmas which are obtained in [FK03a].

Lemma 10.4. *Let $|x-x'| \geq 1$ and let F be small enough. Then there exist some strictly positive constants G_0, $\omega(z)$ and $\sigma(z) \geq 1$ such that*

$$|\partial^n_{x,y} G_1(\mathbf{x},\mathbf{x}';z)| \leq G_0\,\beta(z)^{-\sigma(z)}\,e^{-\beta(z)|y'-y|}\,e^{-\omega(z)(x'-x)^2},$$

where $n = 0,1$ and $\beta(z) = \frac{\Im z + bF}{2F}$.

Lemma 10.5. *For F small enough there exists some strictly positive constants G_0' and $\sigma(z)$, such that the following inequality holds true*

$$\int_{\mathbb{R}}\int_{|x'-x|<1} |\partial^n_{x,y} G_1(\mathbf{x},\mathbf{x}';z)|e^{\frac{\beta(z)}{2}|y-y'|}\,\mathrm{d}x'\mathrm{d}y' \leq G_0'\,\beta(z)^{-\sigma(z)}, \qquad (10.48)$$

where $n = 0,1$ and $\beta(z) = \frac{\Im z + bF}{2F}$.

Since the integrands are positive functions, for $|x-x'| \geq 1$ we first substitute the integral kernels by their upper bounds and then integrate without any restriction.

Remark 10.3. *In the Lemmas above the coefficient $\omega(z)$ depends only in $\Re z$ and decreases as $\Re z$ increases. Moreover, $\omega(z)$ is linear in B: $\omega(z) \sim B$. $\sigma(z) \geq 1$, and also depends only on $\Re z$ and diverges for $\Re z \to \infty$. For the sake of brevity we do not write z in the arguments of σ and ω.*

We now evaluate the norm of $K_3(z;ib)$. The terms in the commutator are

$$[p_y^2, J_3(ib)]R_3(z;ib)\tilde{J}_3(ib) = -2\partial_x J_3(ib)\partial_x R_3(z;ib)\tilde{J}_3(ib) - \partial_x^2 J_3(ib)R_3(z;ib)\tilde{J}_3(ib)$$

We use again inequality (10.44). Due to the upper bound on the Green function and its derivatives when $|x-x'| \geq 1$ the integration can be separated in two parts, which for F small enough gives us (for $n=1,2$)

$$\begin{aligned} &\sup_{\mathbf{x}} \int \mathrm{d}\mathbf{x}' |\partial_y^n J_3(x+ib,y)||\partial_y^{2-n} G_3(\mathbf{x},\mathbf{x}';z)||\tilde{J}_3(x'+ib,y')| \\ \leq\ & \mathcal{C} \sup_y \int dy' |\partial_y^n J_>(y)|\beta(z)^{-\sigma} e^{-\beta(z)|y-y'|}|\tilde{J}_>(y')| \\ \leq\ & \mathcal{C}\beta(z)^{-\sigma} \sup_{y\in\operatorname{supp}\partial_y^n J_>} \ \sup_{y'\in\operatorname{supp}\tilde{J}_>} e^{-\frac{\beta(z)}{2}|y-y'|} = \mathcal{C}\beta(z)^{-\sigma} e^{-\frac{\beta(z)}{2F^\tau}} \end{aligned}$$

and similarly for the second term. We now consider the situation $|x-x'|<1$, let be the set $D = \{x' \in \mathbb{R} : |x-x'|<1\} \times \mathbb{R}$

$$\begin{aligned} &\sup_{\mathbf{x}} \int_D \mathrm{d}\mathbf{x}' |\partial_y^n J_3(x+ib,y)||\partial_y^{2-n} G_3(\mathbf{x},\mathbf{x}';z)||\tilde{J}_3(x'+ib,y')| \\ \leq\ & \sup_{\mathbf{x}} \int_D \mathrm{d}\mathbf{x}' |\partial_y^n J_3(x+ib,y)|e^{-\frac{\beta(z)}{2}|y-y'|}|\tilde{J}_3(x'+ib,y')||\partial_y^{2-n} G_3(\mathbf{x},\mathbf{x}';z)|e^{\frac{\beta(z)}{2}|y-y'|} \\ \leq\ & \sup_{y\in\operatorname{supp}\partial_y^n J_>} \ \sup_{y'\in\operatorname{supp}\tilde{J}_>} e^{-\frac{\beta(z)}{2}|y-y'|} \sup_{\mathbf{x}} \int_D \mathrm{d}\mathbf{x}' |\partial_y^{2-n} G_3(\mathbf{x},\mathbf{x}';z)|e^{\frac{\beta(z)}{2}|y-y'|} \\ \leq\ & \mathcal{C}\beta(z)^{-\sigma} e^{-\frac{\beta(z)}{2F^\tau}} \end{aligned}$$

Thus we can conclude that

$$\|K_3(z;ib)\| \leq \mathcal{C}\beta(z)^{-\sigma} e^{-\frac{\beta(z)}{2F^\tau}}$$

In the same way we prove the estimate for $\|K_4(z;ib)\|$.

Norm of $K_1(z;ib)$ and $K_5(z;ib)$

Here below when we write $\|\cdot\|_{HS}$ for $|x-x'| \geq 1$ it is understood that the Hilbert-Schmidt corresponds to the integration over $\mathbb{R}^2$ with the restriction $|x-x'| \geq 1$. For the integral kernel of $R_1(z;ib)$ and $\partial_{x,y} R_1(z;ib)$ we then use the upper bounds of Lemma 10.4.

The first term in the commutator $[H_L, J_1(ib)]$ gives

$$[p_x^2, J_1(ib)]R_1(z;ib)\tilde{J}_1(ib) = -2\partial_x J_1(ib)\partial_x R_1(z;ib)\tilde{J}_1(ib) - \partial_x^2 J_1(ib)R_1(z;ib)\tilde{J}_1(ib) \quad (10.49)$$

In the case $|x-x'| \geq 1$ we estimate the "restricted" Hilbert-Schmidt norms term by term.

$$\begin{aligned} &\|\partial_x J_1(ib)\partial_x R_1(z;ib)\tilde{J}_1(ib)\|_{HS}^2 = \\ =\ & \int_{\mathbb{R}^4} |J'_-(x+ib)J_c(y)|^2 |\partial_x G_1(\mathbf{x},\mathbf{x}';z)|^2 |\tilde{J}_-(x'+ib)\tilde{J}_c(y')|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{x}' \end{aligned}$$

As before, due to the properties of the Green function for $|x - x'| \geq 1$, the integration can be separated into two parts. One can easily check that the integral with respect to y, y' gives the factor

$$\mathcal{C}\, F^{-2\tau}$$

The second part is bounded above by

$$\beta(z)^{-\sigma} \int_{\mathbb{R}} |J'_-(x+ib)|^2 f(x,x_0)\, \mathrm{d}x$$

where

$$f(x,x_0) := \int_{\mathbb{R}} e^{-\omega(x-x')^2} \frac{1}{1+e^{-4\gamma(x'-x_0)}}\, \mathrm{d}x'$$

Here we have used the fact that for F sufficiently small (see (10.10))

$$\begin{aligned} |\tilde{J}_-(x'+ib)|^2 &= \left(1 + e^{-4\gamma(x'-x_0)} + 2\cos(2\gamma b) e^{-2\gamma(x'-x_0)}\right)^{-1} \\ &\leq \frac{1}{1+e^{-4\gamma(x'-x_0)}} \end{aligned} \tag{10.50}$$

In the similar way we find out that

$$|J'_-(x+ib)|^2 \leq \mathcal{C}\, F^{-2} e^{-4\gamma|x-x_2|} \tag{10.51}$$

so that it suffices to look for an upper bound on the functional

$$\begin{aligned} \int_{\mathbb{R}} e^{-4\gamma|x-x_2|} f(x,x_0)\, \mathrm{d}x &= \int_{-\infty}^{x_2-\delta} e^{-4\gamma|x-x_2|} f(x,x_0)\, \mathrm{d}x \\ &+ \int_{x_2+\delta}^{\infty} e^{-4\gamma|x-x_2|} f(x,x_0)\, \mathrm{d}x + \int_{x_2-\delta}^{x_2+\delta} e^{-4\gamma|x-x_2|} f(x,x_0)\, \mathrm{d}x \\ &= I_1 + I_2 + I_3 \end{aligned} \tag{10.52}$$

where $\delta = \delta_0 F^{-1(1-\varepsilon)}$ such that $(x_2 + \delta) < x_0$. As $f(x,x_0)$ is by definition strictly positive and bounded, the first two integrals on the r.h.s. of (10.52) can be easily estimated as follows

$$\begin{aligned} I_1 + I_2 &\leq e^{-2\gamma\delta} \|f\|_\infty \left[\int_{-\infty}^{x_2-\delta} e^{2\gamma(x-x_2)}\, \mathrm{d}x + \int_{x_2+\delta}^{\infty} e^{-2\gamma(x-x_2)}\, \mathrm{d}x\right] \\ &\leq \gamma^{-1} \sqrt{\frac{\pi}{\omega}} e^{-2\gamma\delta} \end{aligned}$$

In order to control I_3 we have to look at the function $f(x,x_0)$ in more detail. First we note that

$$\begin{aligned} f(x,x_0) &= \int_{\mathbb{R}} e^{-\omega(x-x_0-t)^2} \frac{\mathrm{d}t}{1+e^{-4\gamma t}} \\ &\leq \int_0^{\infty} e^{-\omega(x-x_0-t)^2}\, \mathrm{d}t + \int_{-\infty}^{0} e^{-\omega(x-x_0-t)^2+4\gamma t}\, \mathrm{d}t \end{aligned} \tag{10.53}$$

From [GR80, p. 1064] (see also (10.72)) we then get the bound on $f(x,x_0)$ in the form

$$\begin{aligned} f(x,x_0) &\leq \sqrt{\frac{1}{2\omega}}\, e^{-\omega(x-x_0)^2}\Bigg[e^{\frac{\omega(x-x_0)^2}{2}} D_{-1}(\sqrt{2\omega}\,(x_0-x)) \\ &+ e^{\frac{(2\omega(x-x_0)+4\gamma)^2}{8\omega}} D_{-1}\left(\frac{2\omega(x-x_0)+4\gamma}{\sqrt{2\omega}}\right)\Bigg] \end{aligned}$$

where $D_{-1}(\cdot)$ denotes the parabolic cylinder function. Using its asymptotic expansion [GR80, p. 1065]

$$\begin{aligned} D_{-1}(z) &= e^{-z^2/4} z^{-1}(1-\mathcal{O}(z^{-2})), \quad z\to\infty \\ D_{-1}(z) &= e^{z^2/4}(1+\mathcal{O}(z^{-2})), \qquad z\to-\infty \end{aligned}$$

it is not difficult to verify that

$$f(x,x_0)\leq \mathcal{C}e^{-\mathcal{C}\,F^{-2(1-\varepsilon)}}, \quad F\to 0$$

uniformly for any $x\in[x_2-\delta,x_2+\delta]$. Now we employ the mean value theorem of the integral calculus which tells us that there exists some $\tilde{x}\in[x_2-\delta,x_2+\delta]$ for which

$$I_3 = f(\tilde{x})\int_{x_2-\delta}^{x_2+\delta} e^{-4\gamma|x-x_2|}\,\mathrm{d}x = \frac{1}{2\gamma}\left(1-e^{-4\gamma\delta}\right) f(\tilde{x})$$

Let us remark that the second term of the commutator (10.49) can be bounded in the same way, since

$$|J''_-(x+ib)|^2 \leq \mathcal{C}\,F^{-4}\,e^{-4\gamma|x-x_2|}, \quad F\to 0 \tag{10.54}$$

Moreover, due to the decoupling with respect to $y-$axis, the above procedure can be applied also to the second term in the commutator $[H_L, J_1(ib)]$, namely

$$[2Byp_x, J_1(ib)]R_1(z;ib)\tilde{J}_1(ib) = -2By\partial_x J_1(ib)R_1(z;ib)\tilde{J}_1(ib)$$

This allows us to find some $c_1(V,B)>0$ such that the following holds true for $|x-x'|\geq 1$:

$$\left\|[(p_x+By)^2, J_1(ib)]R_1(z;ib)\tilde{J}_1(ib)\right\|^2_{HS} \leq \mathcal{C}\,\beta(z)^{-\sigma}F^{-\mathcal{C}}\,e^{-c_1(B)F^{-2(1-\varepsilon)}} \tag{10.55}$$

where the constant $c_1(B)$ is proportional to B (since the factor ω is linear in B).
When $|x-x'|<1$ we use (10.47). As in the case $|x-x'|\geq 1$ all the term in the commutator $[H_L, J_1(ib)]$ involving $x-$derivatives are treated in the same way. For example for $\partial_x J_1(ib)\partial_x R_1(z;ib)\tilde{J}_1(ib)$ we have

$$\begin{aligned} &\sup_{\mathbf{x}}\int_{\mathbb{R}}\mathrm{d}y'\int_{x':|x-x'|<1}\mathrm{d}x'|J'_-(x+ib)J_c(y)||\partial_x G_1(\mathbf{x},\mathbf{x}';z)||\tilde{J}_-(x'+ib)\tilde{J}_c(y')| \\ \leq\ &\sup_{\mathbf{x}}\sup_{x':|x-x'|<1}|J'_-(x+ib)\tilde{J}_-(x'+ib)|\int_{\mathbb{R}}\mathrm{d}y'\int_{x':|x-x'|<1}\mathrm{d}x'|\partial_x G_1(\mathbf{x},\mathbf{x}';z)| \\ \leq\ &\mathcal{C}\beta(z)^{-\sigma}\sup_{x',x:|x-x'|<1}|J'_-(x+ib)\tilde{J}_-(x'+ib)| \end{aligned} \tag{10.56}$$

and similarly for $\mathbf{x}$ and $\mathbf{x}'$ interchanged. Now, using (10.50) and (10.51), we get

$$\sup_{x',x:|x-x'|<1} |J'_-(x+ib)\tilde{J}_-(x'+ib)| \le \mathcal{C}F^{-1} \sup_x \frac{e^{-4\gamma|x-x_2|}}{1+e^{-4\gamma(x-x_0)}} \le \mathcal{C}F^{-1}e^{-\mathcal{C}F^{-2(1-\varepsilon)}}$$

This with (10.55) leads to

$$\left\|[(p_x+By)^2, J_1(ib)]R_1(z;ib)\tilde{J}_1(ib)\right\|^2 \le \mathcal{C}\,\beta(z)^{-\sigma}F^{-\mathcal{C}}\,e^{-c_2(B)F^{-2(1-\varepsilon)}}$$

for $c_2(B) > 0$.
To control the operator norm of the last term in the commutator $[H_L, J_1(ib)]$, namely

$$[p_y^2, J_1(ib)]R_1(z;ib)\tilde{J}_1(ib)$$

we use again the inequality (10.44). When $|x-x'| \ge 1$, since both $f(x,x_0)$ and $f(x,x_2)$ are bounded as well as $J_-(x+ib)$, $\tilde{J}_-(x+ib)$, it suffices to estimate these parts in (10.44) which correspond to the integration w.r.t. y, y':

$$\begin{aligned}&\sup_y |J'_c(y)| \int_{\mathbb{R}} e^{-\beta(z)|y-y'|}|\tilde{J}_c(y')|\ \mathrm{d}y' \le \sup_y |J'_c(y)| \int_{-y_0}^{y_0} e^{-\beta(z)|y-y'|}\ \mathrm{d}y' \\ &\le 2y_0\|J'_c\|_\infty\, e^{-\beta(z)F^{-\tau}}\end{aligned} \tag{10.57}$$

On the other hand,

$$\begin{aligned}&\sup_{y'} |\tilde{J}_c(y')| \int_{\mathbb{R}} e^{-\beta(z)|y-y'|}|J'_c(y)|\ \mathrm{d}y \le \|\tilde{J}_c\|_\infty \sup_{y'\in[-y_0,y_0]} \int_{y_0+F^{-\tau}}^{y_0+F^{-\tau}+1} e^{-\beta(z)|y-y'|}\ \mathrm{d}y \\ &\le \|\tilde{J}_c\|_\infty\, e^{-\beta(z)F^{-\tau}}\end{aligned} \tag{10.58}$$

and similarly for the terms with $J''_c(y)$. When $|x-x'| < 1$ we proceed in a similar way as for the case $i = 3$ and we get the desired result.

Thus we can conclude that

$$\|[p_y^2, J_1(ib)]R_1(z;ib)\tilde{J}_1(ib)\| \le \mathcal{C}\,\beta(z)^{-\sigma}F^{-\mathcal{C}}e^{-\frac{\beta(z)}{F^\tau}} \tag{10.59}$$

Finally,

$$\|K_1(z;ib)\| \le \mathcal{C}F^{-\mathcal{C}}\beta(z)^{-\sigma}\left(e^{-\frac{\beta(z)}{F^\tau}} + e^{-\frac{\mathcal{C}}{F^{2(1-\varepsilon)}}}\right)$$

The upper bound on the term $\|K_5(z;ib)\|$ is found in the same way.

Norm of $K_2(z;ib)$

The operator $K_2(z;ib)$ includes the resolvent $R_2(z;ib)$, which can be evaluated with respect to $R_1(z;ib)$

$$R_2(z;ib) = R_1(z;ib) - R_1(z;ib)[F(x+ib)(\chi_A^c + h_F^c(ib)\chi_A) + V(ib)]R_2(z;ib) \tag{10.60}$$

Obviously, the first term coming from (10.60) is to be treated in the same way as above. The second term $R_1(z;ib)[\cdots]R_2(z;ib)$ is estimated using

$$\|[H_L, J_2(ib)]R_1(z;ib)[\cdots]R_2(z;ib)\tilde{J}_2(ib)\| \leq \|[H_L, J_2(ib)]R_1(z;ib)[\cdots]\|\|R_2(z;ib)\|\|\tilde{J}_2(ib)\|$$

Now, $\|\tilde{J}_2(ib)\|$ is bounded and for $\|R_2(z;ib)\|$ we use the result of Lemma 10.2. It then remains to estimate

$$\|[H_L, J_2(ib)]R_1(z;ib)[F(x+ib)(\chi_A^c + h_F^c(ib)\chi_A) + V(ib)]\| \tag{10.61}$$

Before we give the estimation of the different contribution to (10.61), we remind that

$$|J_0'(x+ib)| \leq \mathcal{C}\,F^{-1}\left\{e^{-2\gamma|x-x_1|} + e^{-2\gamma|x+x_1|}\right\} \tag{10.62}$$

$$|J_0''(x+ib)| \leq \mathcal{C}\,F^{-2}\left\{e^{-2\gamma|x-x_1|} + e^{-2\gamma|x+x_1|}\right\}, \tag{10.63}$$

where we have used the similar bounds as in (10.51). In the estimations we will separate the two contributions coming from $\bar{J}_+$ and $\bar{J}_-$.

Let us now look at the contribution to (10.61) which includes the potential $V(ib)$. We again begin with the Hilbert-Schmidt norm (case $|x-x'| \geq 1$) of the terms in the commutator involving the $x-$derivatives. After separation of variables we can write ($n = 1, 2$)

$$\begin{aligned}
&\|\partial_x^n \bar{J}_+(x+ib)J_c(y)\partial_x^{(2-n)}R_1(z;ib)V(ib)\|_{HS}^2 \\
&\leq \mathcal{C}\,F^{-2\tau}\beta(z)^{-\sigma}\int_{\mathbb{R}}|\partial_x^n\bar{J}_+(x+ib)|^2\,\mathrm{d}x\int_{\mathbb{R}}e^{-\omega(x-x')^2}\,|V(x'+ib,\hat{y})|^2\,\mathrm{d}x' \\
&\leq \mathcal{C}\,F^{-2-2\tau}\beta(z)^{-\sigma}\int_{\mathbb{R}}e^{-4\gamma|x-x_1|}\left[\int_{|x'|\leq a_0}e^{-\omega(x-x')^2}\,\mathrm{d}x' + \int_{|x'|>a_0}e^{-\omega(x-x')^2}e^{-\nu x'^2}\,\mathrm{d}x'\right]\mathrm{d}x \\
&\leq \mathcal{C}\,F^{-2-2\tau}\beta(z)^{-\sigma}\int_{\mathbb{R}}e^{-4\gamma|x-x_1|}\left[g(x,a_0) + \sqrt{\frac{\pi}{\omega+\nu}}\,e^{-\frac{\omega\nu}{\omega+\nu}x^2}\right]\mathrm{d}x
\end{aligned} \tag{10.64}$$

where we have defined

$$g(x,a_0) := \int_{|x'|\leq a_0}e^{-\omega(x-x')^2}\,\mathrm{d}x'$$

Now we can apply the same argument as in (10.52) and repeat it for $\|\partial_x^n\bar{J}_-(x+ib)J_c(y)\partial_x^{(2-n)}R_1(z;ib)V(ib)\|_{HS}^2$ to arrive at

$$\left\|[(p_x+By)^2, J_2(ib)]R_1(z;ib)V(ib)\right\|_{HS}^2 \leq \mathcal{C}\,\beta(z)^{-\sigma}F^{-\mathcal{C}}\,e^{-\mathcal{C}\,F^{-2(1-\varepsilon)}} \tag{10.65}$$

For $|x-x'|<1$ we proceed like in (10.56) evaluating separately the contributions coming from $\bar{J}_+$ and $\bar{J}_-$. For example, for $\partial_x^n\bar{J}_+(x+ib)J_c(y)\partial_x^{(2-n)}R_1(z;ib)V(ib)$ we get an upper bound of the form

$$\begin{aligned}
&\sup_{\mathbf{x}}\ \sup_{x':|x-x'|<1,y'}|\partial_x^n\bar{J}_+'(x)V(x'+ib,y')|\int_{\mathbb{R}}\mathrm{d}y'\int_{x':|x-x'|<1}\mathrm{d}x'|\partial_x^{2-n}G_1(\mathbf{x},\mathbf{x}';z)| \\
\leq\ & \mathcal{C}\beta(z)^{-\sigma}\sup_{x,x':|x-x'|<1,y'}|\partial_x^n\bar{J}_+'(x)V(x'+ib,y')| \leq \mathcal{C}\beta(z)^{-\sigma}F^{-\mathcal{C}}e^{-\mathcal{C}F^{-2(1-\varepsilon)}}
\end{aligned} \tag{10.66}$$

The last term in the commutator (10.61) which includes $V(ib)$ is the following

$$[p_y^2, J_2(ib)]R_1(z;ib)V(ib)$$

For $|x-x'|\geq 1$, since both

$$J_0(x+ib)\int_{\mathbb{R}} e^{-\omega(x-x')^2}\,\mathrm{d}x', \qquad \int_{\mathbb{R}} e^{-\omega(x-x')^2} J_0(x'+ib)\,\mathrm{d}x'$$

are bounded as functions of x, we apply again (10.44) to find out that

$$\sup_y |J_c'(y)|V_0\int_{-a_1}^{a_1} e^{-\beta(z)|y-y'|}\,\mathrm{d}y' \leq \|J_c'\|_\infty V_0\, 2a_1 \sup_{y\in \operatorname{supp} J_c'}\ \sup_{y'\in[-a_1,a_1]} e^{-\beta(z)|y-y'|}$$
$$\leq \|J_c'\|_\infty\, 2a_1 V_0\, e^{-\beta(z)F^{-\tau}} \tag{10.67}$$

and similarly the other way around

$$\sup_{y'} |V(x'+ib,y')|\int_{y_0+F^{-\tau}}^{y_0+F^{-\tau}+1} e^{-\beta(z)|y-y'|}|J_c'(y)|\,\mathrm{d}y \leq V_0\|J_c'\|_\infty\, e^{-\beta(z)F^{-\tau}}$$

For $|x-x'|<1$ we proceed as for $i=3$. Summing all the above given inequalities we obtain

$$\|[H_L,J_2(ib)]R_1(z;ib)V(ib)\| \leq \mathcal{C}\,\beta(z)^{-\sigma}F^{-\mathcal{C}}\left(e^{-\frac{\beta(z)}{F^\tau}} + e^{-\frac{\mathcal{C}}{F^{2(1-\varepsilon)}}}\right) \tag{10.68}$$

Remark 10.4. *Note that the hypothesis on the Gaussian-like decay of V w.r.t. x is necessary in order to obtain* (10.68) *as one can see from* (10.65) *and* (10.66).

Next we analyse those terms of (10.61), which include the potential $F(x+ib)h_F^c(ib)\chi_A$. We start again with the case $|x-x'|\geq 1$ looking at the Hilbert-Schmidt norm of

$$[(p_x+By)^2, J_2(ib)]R_1(z;ib)F(x+ib)h_F^c(ib)\chi_A \tag{10.69}$$

Note that since we have the same upper bounds on $J_c'(x+ib)$, $J_c''(x+ib)$ and also on $R_1(z;ib)$, $\partial_x R_1(z;ib)$, all terms in (10.69) can be estimated in the same way. As for the previous term we separate the contributions of $\bar{J}_\pm$, moreover $h_F^c = 1-h_F = h_+ + h_-$ with $h_\pm(x) = \frac{1}{2}\left[1\mp\tanh(\gamma_F(x\pm\bar{x}))\right]$, and thus we separate also the contributions of h_+ and h_-. We are left with four terms, each of them is estimated as follows ($n=1,2$):

$$\|\partial_x^n \bar{J}_+(x+ib)J_c(y)\partial_x^{(2-n)}R_1(z;ib)F(x+ib)h_-(ib)\chi_A\|_{HS}^2$$
$$\leq \mathcal{C}\,\beta(z)^{-\sigma}F^{-\mathcal{C}}\int_{\mathbb{R}} |\partial_x^n\bar{J}_+(x+ib)|^2\,\mathrm{d}x\int_{\mathbb{R}} e^{-\omega(x-x')^2}\,|F(x'+ib)h_-(x'+ib)|^2\,\mathrm{d}x'$$
$$\leq \mathcal{C}\,\beta(z)^{-\sigma}F^{-\mathcal{C}}\int_{\mathbb{R}} e^{-4\gamma|x-x_1|}\,\mathrm{d}x\int_{\mathbb{R}} e^{-\omega(x-\bar{x}-t)^2}|t+\bar{x}+ib|^2\,\frac{\mathrm{d}t}{1+e^{-4\gamma t}} \tag{10.70}$$

recalling that the integration w.r.t. y, y' gives again the factor of order $F^{-2\tau}$. To evaluate the integral with respect to t we write

$$\int_{\mathbb{R}} e^{-\omega(x-\bar{x}-t)^2}|t+\bar{x}+ib|^2 \frac{\mathrm{d}t}{1+e^{-4\gamma t}} \tag{10.71}$$
$$\leq \int_{-\infty}^{0} e^{-\omega(x-\bar{x}-t)^2+4\gamma t}(2t^2+2\bar{x}^2+b^2)\ \mathrm{d}t + \int_{0}^{\infty} e^{-\omega(x-\bar{x}-t)^2}(2t^2+2\bar{x}^2+b^2)\ \mathrm{d}t$$

and use the following general result which can be found in [GR80, p. 1064],

$$\int_0^\infty t^{\mu-1}e^{-bt^2-ct}\ \mathrm{d}t = (2b)^{-\mu/2}\Gamma(\mu)\exp(c^2/8b)D_{-\mu}(c/\sqrt{2b}) \tag{10.72}$$

Here $D_{-\mu}(\cdot)$ is the parabolic cylinder function of order $-\mu$. Its asymptotic behaviour is given by [GR80, p.1065]

$$\begin{array}{lcll} D_p(z) & \simeq & e^{-z^2/4}z^p(1-\mathcal{O}(z^{-2})), & z\to\infty \\ D_p(z) & \simeq & e^{z^2/4}z^{-p-1}(1+\mathcal{O}(z^{-2})), & z\to-\infty \end{array} \tag{10.73}$$

The asymptotic behaviour allows us to apply once more the argument used in (10.52). We can thus claim that

$$\left\|[(p_x+By)^2, J_2(ib)]R_1(z;ib)F(x+ib)h_F^c(ib)\chi_A\right\|_{HS}^2 \leq \mathcal{C}\,\beta(z)^{-\sigma}F^{-\mathcal{C}}\,e^{-\mathcal{C}\,F^{-2(1-\varepsilon)}}$$

Also for the case $|x-x'|<1$ all the terms are treated analogously. For example for $\partial_x^n\bar{J}_+(ib)J_c\partial_x^{2-n}R_1(z;ib)F(x+ib)h_-(ib)\chi_A$ we have

$$\begin{aligned} &\sup_{\mathbf{x}}\int_{\mathbb{R}}\mathrm{d}y'\int_{|x-x'|<1}\mathrm{d}x'|\partial_x^n\bar{J}_+(x+ib)J_c(y)||\partial_x^{2-n}G_1(\mathbf{x},\mathbf{x}';z)|F|x'+ib||h_-(x'+ib)\chi_A(y')| \\ \leq\ &\sup_{\mathbf{x}}\sup_{x':|x-x'|<1}|\partial_x^n\bar{J}_+(x+ib)h_-(x'+ib)|^{1/2}\times \\ \times\ &\int_{\mathbb{R}}\mathrm{d}y'\int_{|x-x'|<1}\mathrm{d}x'|\partial_x^{2-n}G_1(\mathbf{x},\mathbf{x}';z)||\partial_x^n\bar{J}_+(x)|^{1/2}F|x'+ib| \leq \mathcal{C}\beta(z)^{-\sigma}F^{-\mathcal{C}}e^{-\mathcal{C}F^{-2(1-\varepsilon)}} \end{aligned} \tag{10.74}$$

where we used the fact that $|x'|\leq|x|+1$ and $|\partial_x^n\bar{J}_+(x)|^{1/2}|x|\leq\mathcal{C}F^{-(1-\varepsilon)}$.
We are now left with the last term in the commutator:

$$\begin{aligned} &[p_y^2, J_2(ib)]R_1(z;ib)F(x+ib)h_F^c(ib)\chi_A = -2J_0(x+ib)J_c'(y)\partial_yR_1(z;ib)\times \\ &\times F(x+ib)h_F^c(ib)\chi_A - J_0(x+ib)J_c''(y)R_1(z;ib)F(x+ib)h_F^c(ib)\chi_A \end{aligned} \tag{10.75}$$

When $|x-x'|\geq 1$ the Hilbert-Schmidt norm of these terms can estimated separately for $h_\pm$. We do that for h_-, for the term coming from h_+ a similar argument holds.
For h_- the Hilbert-Schmidt norm is bounded above by a constant times $\beta(z)^{-\sigma}F^{-\tau}$ (coming from the integration w.r.t. y and y') times

$$\int_{\mathbb{R}}\mathrm{d}x|J_0(x+ib)|^2\int_{\mathbb{R}} e^{-\omega(x-x')^2}|x'|^2\frac{\mathrm{d}x'}{1+e^{-4\gamma(x'-\bar{x})}}$$
$$\leq\int_{\mathbb{R}}\mathrm{d}x|J_0(x+ib)|^2\int_{\mathbb{R}} e^{-\omega(x-\bar{x}-t)^2}(2t^2+2\bar{x}^2)\,\frac{\mathrm{d}t}{1+e^{-4\gamma t}} \tag{10.76}$$

The last integral can be again evaluated through (10.72) and (10.73) and estimated up to a constant from above by

$$F^{-\mathcal{C}}\, e^{-\mathcal{C}\, F^{-2(1-\varepsilon)}}, \tag{10.77}$$

To control the first term in (10.76), which is proportional to t^2, we proceed in the same way as in (10.53) to write

$$\int_{\mathbb{R}} e^{-\omega(x-\bar{x}-t)^2}\, t^2 \,\frac{\mathrm{d}t}{1+e^{-4\gamma t}} \le \mathcal{C}\, e^{-\omega(x-\bar{x})^2}\Bigg[e^{\frac{\omega(x-\bar{x})^2}{2}} D_{-3}(\sqrt{2\omega}\,(\bar{x}-x))$$
$$+ \quad e^{\frac{(2\omega(x-\bar{x})+4\gamma)^2}{8\omega}} D_{-3}\left(\frac{2\omega(x-\bar{x})+4\gamma}{\sqrt{2\omega}}\right)\Bigg] \tag{10.78}$$

We will split (10.76) in three parts:

$$(-\infty, x_1+\delta], \quad [x_1+\delta, \bar{x}], \quad [\bar{x}, \infty) \tag{10.79}$$

where $\delta = \delta_0\, F^{-(1-\varepsilon)}$ and $(x_1+\delta) < \bar{x}$. For the first part we get

$$\int_{\bar{x}}^{\infty} \mathrm{d}x e^{-4\gamma(x-x_1)}\, e^{-\omega(x-\bar{x})^2/2} D_{-3}(\sqrt{2\omega}(\bar{x}-x))$$
$$\le e^{-4\gamma(\bar{x}-x_1)} \int_0^{\infty} e^{-4\gamma t-\omega t^2/2} D_{-3}(-\sqrt{2\omega}\, t)\,\mathrm{d}t \le \mathcal{C}\, e^{-4\gamma(\bar{x}-x_1)} \tag{10.80}$$

since $e^{-4\gamma t-\omega t^2/2}D_{-3}(-\sqrt{2\omega}\, t)$ is clearly $L_1([0,\infty))$, see (10.73). The second part can be estimated as follows

$$\int_{x_1+\delta}^{\bar{x}} \mathrm{d}x e^{-4\gamma(x-x_1)}\, e^{-\omega(x-\bar{x})^2/2} D_{-3}(\sqrt{2\omega}(\bar{x}-x))\,\mathrm{d}x$$
$$\le e^{-4\gamma\delta} \int_{x_1+\delta}^{\bar{x}} e^{-\omega(x-\bar{x})^2/2} D_{-3}(\sqrt{2\omega}(\bar{x}-x))\,\mathrm{d}x$$
$$\le e^{-4\gamma\delta}(\bar{x}-x_1-\delta) \sup_{x\in[x_1+\delta,\bar{x}]} D_{-3}(\sqrt{2\omega}(\bar{x}-x)) \le \mathcal{C}\, F^{-(1-\varepsilon)}\, e^{-4\gamma\delta}, \quad F\to 0 \tag{10.81}$$

Finally, the third part is bounded above by

$$\int_{-\infty}^{x_1+\delta} e^{-\omega(x-\bar{x})^2/2} D_{-3}(\sqrt{2\omega}(\bar{x}-x))\,\mathrm{d}x \le e^{-\omega\bar{x}^2/2} \int_{-\infty}^{0} D_{-3}(\sqrt{2\omega}(\bar{x}-x))\,\mathrm{d}x$$
$$+e^{-\omega(\bar{x}-x_1-\delta)^2/2} \int_0^{x_1+\delta} D_{-3}(\sqrt{2\omega}(\bar{x}-x))\,\mathrm{d}x$$
$$\le \mathcal{C}\, e^{-\omega(\bar{x}-x_1-\delta)^2/2}, \quad F\to 0 \tag{10.82}$$

where we have employed the asymptotic expansion (10.73).

The estimate of the second part of (10.78), which contains the function

$$D_{-3}\left(\frac{2\omega(x-\bar{x})+4\gamma}{\sqrt{2\omega}}\right) \tag{10.83}$$

is a bit more subtle. After dividing the integration again in three parts according to (10.79) and substituting

$$t := \frac{2\omega(x-\bar{x})+4\gamma}{\sqrt{2\omega}} \tag{10.84}$$

one gets

$$\begin{aligned}
&\int_{\bar{x}}^{\infty} \mathrm{d}x e^{-4\gamma(x-x_1)} e^{-\omega(x-\bar{x})^2} e^{\frac{(2\omega(x-\bar{x})+4\gamma)^2}{8\omega}} D_{-3}\left(\frac{2\omega(x-\bar{x})+4\gamma}{\sqrt{2\omega}}\right) \\
&\leq e^{-4\gamma(\bar{x}-x_1)} \int_{4\gamma/\sqrt{2\omega}}^{\infty} \exp\left[-\frac{t^2}{4}+\frac{2\sqrt{2}\,\gamma}{\sqrt{\omega}}\,t-\frac{4\gamma^2}{\omega}\right] D_{-3}(t)\sqrt{2\omega}\,\mathrm{d}t \\
&\leq \mathcal{C}\, e^{-\mathcal{C}\,F^{-2(1-\varepsilon)}}, \quad F\to 0
\end{aligned} \tag{10.85}$$

provided

$$\omega(\bar{x}-x_1) > \gamma \tag{10.86}$$

this can be seen taking the maximum of the exponential function in the integral and the fact that $D_{-3}(t)\in L_1([0,\infty))$.

For $x\in(-\infty,x_1+\delta]$ we have similarly

$$\begin{aligned}
&\int_{-\infty}^{x_1+\delta} \mathrm{d}x e^{-\omega(x-\bar{x})^2} e^{\frac{(2\omega(x-\bar{x})+4\gamma)^2}{8\omega}} D_{-3}\left(\frac{2\omega(x-\bar{x})+4\gamma}{\sqrt{2\omega}}\right) \\
&\leq \int_{-\infty}^{\frac{2\omega(x_1+\delta-\bar{x})+4\gamma}{\sqrt{2\omega}}} \exp\left[-\frac{t^2}{4}+\frac{2\sqrt{2}\,\gamma}{\sqrt{\omega}}\,t-\frac{4\gamma^2}{\omega}\right] D_{-3}(t)\sqrt{2\omega}\,\mathrm{d}t
\end{aligned} \tag{10.87}$$

Since

$$\exp\left[-\frac{t^2}{4}+\frac{2\sqrt{2}\,\gamma}{\sqrt{\omega}}\,t\right] D_{-3}(t)\in L_1((-\infty,0]) \tag{10.88}$$

it suffices to estimate the integral for positive values of t. In this case we use the fact that

$$D_{-3}(z)e^{\xi z^2/4}\in L_1([0,\infty)),$$

for any $\xi<1$. Then

$$\begin{aligned}
&\int_{0}^{\frac{2\omega(x_1+\delta-\bar{x})+4\gamma}{\sqrt{2\omega}}} \exp\left[-\frac{t^2(1+\xi)}{4}+\frac{2\sqrt{2}\,\gamma}{\sqrt{\omega}}\,t-\frac{4\gamma^2}{\omega}\right] e^{\xi t^2/4}\, D_{-3}(t)\sqrt{2\omega}\,\mathrm{d}t \\
&\leq \mathcal{C}\, e^{-\mathcal{C}\,F^{-2(1-\varepsilon)}}, \quad F\to 0
\end{aligned} \tag{10.89}$$

whenever

$$1>\xi>\frac{4\gamma^2-\omega^2(x_1+\delta-\bar{x})^2}{4\gamma^2+\omega^2(x_1+\delta-\bar{x})^2}=\frac{4\gamma_0^2-\omega^2(C_1+\delta_0-\bar{C})^2}{4\gamma_0^2+\omega^2(C_1+\delta_0-\bar{C})^2}$$

We are thus left with

$$\int_{x_1+\delta}^{\bar{x}} \mathrm{d}x\, e^{-4\gamma(x-x_1)} e^{-\omega(x-\bar{x})^2} e^{\frac{(2\omega(x-\bar{x})+4\gamma)^2}{8\omega}} D_{-3}\left(\frac{2\omega(x-\bar{x})+4\gamma}{\sqrt{2\omega}}\right)$$
$$\leq e^{-4\gamma\delta} \int_{\frac{2\omega(x_1+\delta-\bar{x})+4\gamma}{\sqrt{2\omega}}}^{\frac{4\gamma}{\sqrt{2\omega}}} \exp\left[-\frac{t^2}{4}+\frac{2\sqrt{2}\,\gamma}{\sqrt{\omega}}\,t-\frac{4\gamma^2}{\omega}\right] D_{-3}(t)\sqrt{2\omega}\,\mathrm{d}t \qquad (10.90)$$

Due to (10.88) it is enough to show that

$$\int_0^{\frac{4\gamma}{\sqrt{2\omega}}} \exp\left[-\frac{t^2}{4}+\frac{2\sqrt{2}\,\gamma}{\sqrt{\omega}}\,t-\frac{4\gamma^2}{\omega}\right] D_{-3}(t)\sqrt{2\omega}\,\mathrm{d}t \leq \mathcal{C}\,F^{-(1-\varepsilon)} \qquad (10.91)$$

This is however easily seen since

$$-\frac{t^2}{2}+\frac{2\sqrt{2}\,\gamma}{\sqrt{\omega}}\,t-\frac{4\gamma^2}{\omega} \leq 0, \quad \forall\, t \in \left[0, \frac{4\gamma}{\sqrt{2\omega}}\right] \qquad (10.92)$$

and

$$\sup_{t\in[0,\frac{4\gamma}{\sqrt{2\omega}}]} e^{t^2/4} D_{-3}(t) \leq \sup_{t\in[0,\infty)} e^{t^2/4} D_{-3}(t) \leq \mathcal{C}$$

To conclude we remark that the second term of (10.76), which leads to

$$\int_{\mathbb{R}} e^{-\omega(x-\bar{x}-t)^2}\,\bar{x}^2\,\frac{\mathrm{d}t}{1+e^{-4\gamma t}} \leq \mathcal{C}\,F^{-2(1-\varepsilon)}\,e^{-\omega(x-\bar{x})^2}\left[e^{\frac{\omega(x-\bar{x})^2}{2}} D_{-1}(\sqrt{2\omega}\,(\bar{x}-x))\right.$$
$$\left. +\ e^{\frac{(2\omega(x-\bar{x})+4\gamma)^2}{8\omega}} D_{-1}\left(\frac{2\omega(x-\bar{x})+4\gamma}{\sqrt{2\omega}}\right)\right], \qquad (10.93)$$

can be control in the same way, because the asymptotic behaviour (10.73) is again governed by $\exp[\pm t^2/4]$.
Finally, for the case $|x-x'|<1$ we follows the same method as in (10.74) where the decay come from the "infinitesimally small" overlap of h_F^c with J_0 the latter also "localise" $|x'|$, i.e. $|J_0(x+ib)|^{1/2}|(x'+ib)| \leq \mathcal{C}F^{-(1-\varepsilon)}$. Summing up all the contributions we have

$$\|[H_L, J_2(ib)]R_1(z;ib)F(x+ib)h_F^c(ib)\chi_A\| \leq \mathcal{C}\,\beta(z)^{-\sigma}F^{-\mathcal{C}}e^{-\frac{\mathcal{C}}{F^{2(1-\varepsilon)}}} \qquad (10.94)$$

Let us next analyse the last term of (10.61), which includes the potential $F(x+ib)\chi_A^c$. When $|x-x'|\geq 1$, for the terms in the commutator involving the $x-$derivatives, the integration w.r.t. x and x' in the Hilbert-Schmidt norm gives a constant proportional to

$F^{-2(1-\varepsilon)}$. We then obtain the estimate on the Hilbert-Schmidt norm

$$\begin{aligned}
&\|\partial_x^n J_2(ib)\partial_x^{(2-n)} R_1(z;ib)F(x+ib)\chi_A^c\|_{HS}^2 \\
&\leq \mathcal{C}\,\beta(z)^{-\sigma} F^{-\mathcal{C}} \int_{-y_1}^{y_1} \mathrm{d}y \int_{|y'|\geq y_1+F^{-\tau}} e^{-2\beta(z)|y-y'|}\,\mathrm{d}y' \\
&\leq \mathcal{C}\,\beta(z)^{-\sigma} F^{-\mathcal{C}}\, e^{-\frac{\beta(z)}{F^\tau}} \int_{-\infty}^{\infty} e^{-\beta(z)\,|y-y'|}\,\mathrm{d}y' \\
&\leq \mathcal{C}\,\beta(z)^{-\sigma} F^{-\mathcal{C}}\, e^{-\frac{\beta(z)}{F^\tau}}
\end{aligned} \tag{10.95}$$

When $|x-x'|<1$ the $x-$derivative "localises" the term $|x'+ib|$ and the decay comes from the decay of the Green function along y as for the case $i=3$.
For the term of the commutator which corresponds to

$$\partial_y^n J_2(ib)\partial_y^{(2-n)} R_1(z;ib)F(x+ib)\chi_A^c, \quad |x-x'|\geq 1, \quad n=1,2$$

we recall (10.44) to find out that

$$\begin{aligned}
&\sup_{\mathbf{x}} \int_{\mathbb{R}^2} |J_0(x+ib)\partial_y^n J_c(y)\partial_y^{(2-n)} G_1(\mathbf{x},\mathbf{x}';z)F(x'+ib)\chi_A^c(y')|\,\mathrm{d}\mathbf{x}' \\
&\leq \frac{\mathcal{C}}{F^{1-\varepsilon}}\,\beta(z)^{-\sigma} \sup_{y\in \mathrm{supp}\,\partial_y^n J_c} \int_{|y'|\geq y_1+F^{-\tau}} e^{-\beta(z)|y-y'|}\,\mathrm{d}y' \\
&\leq \mathcal{C}\,\beta(z)^{-\sigma} F^{-\mathcal{C}}\, e^{-\frac{\beta(z)}{F^\tau}}
\end{aligned} \tag{10.96}$$

and similarly the other way around. Finally at short distances the same argument as in the previous case holds. Therefore

$$\|[H_L, J_2(ib)]R_1(z;ib)F(x+ib)\chi_A^c\| \leq \mathcal{C}\,\beta(z)^{-\sigma} F^{-\mathcal{C}} e^{-\frac{\beta(z)}{F^\tau}} \tag{10.97}$$

Taking into account all the estimates (10.68), (10.94), (10.97) made above, we can claim that for F small enough

$$\|K_2(z;ib)\| \leq \mathcal{C}\, F^{-\mathcal{C}}\,\beta(z)^{-\sigma(z)} \left(e^{-\frac{\beta(z)}{F^\tau}} + e^{-\frac{\mathcal{C}}{F^{2(1-\varepsilon)}}}\right)(1+\|R_2(z;ib)\|) \tag{10.98}$$

Inequality (10.98) plays an essential role in our estimates, because it tells us how close we can get to the spectrum of $H_2(F,ib)=H_2(F)$ and $H_1(F,ib)$ while keeping the resolvent of $H(F,ib)$ bounded.

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Chapter 11

Outlook

At the end of this thesis we would like to shortly expose some open problems directly related to the subject contained in the four papers reported in Part I and Part II.

In Part I we studied the spectral properties of the quantum Hall Hamiltonians defined on a configuration space given by a cylinder of circumference L. An open problem related to this study concerns the extension of the results of Chapter 6 for a system where the disordered potential can reach the edges. This corresponds to suppress the thin strip of size $\log L$ without disorder along the boundaries. Although is this done in Chapter 7, when the spectrum is analyzed in the spectral gaps of the bulk Hamiltonian, this suppression has not been yet studied when dealing with an energy interval in the Landau bands of the bulk Hamiltonian.
Another related question is the study of the spectrum in the Landau bands for the random Hamiltonian defined on the semi-infinite plane $\mathbb{R}_+ \times \mathbb{R}$, that is $H_\omega = H_L + V_\omega$ with Dirichlet boundary conditions at $x = 0$ (or a confining potential at $x = 0$ added to H_ω). What is the nature of the spectrum in this spectral interval ?

In Part II we studied the resonances in crossed electric and magnetic fields. In particular we have proved an upper bound on the resonance widths, but we did not answer the question whether the spectrum of the full Hamiltonian is purely absolutely continuous. Indeed, it could be that when the electric field is switched on not all the eigenvalues created by the impurity potential turn into resonances and remain embedded eigenvalues.
Another open problem consists in the proof of a lower bound on the resonance widths.
Finally, the analysis of resonances for crossed electric and magnetic fields with an other class of impurity potentials, characterized by a different decay properties, could be an interesting problem. For algebraically decaying potentials what is the behavior of the lifetime as a function of the electric field ?

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Seminars

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