A TOY MORE DIFFICULT TO CONTROL THAN THE REAL THING

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Abstract

Standard helicopters use the propeller angle to alter the aerodynamic force and have been shown to be feedback linearizable with the pitch, yaw and roll angles being the flat outputs. In this work, a-two-degree-of-freedom, laboratory-scale helicopter-like system (termed the *toycopter*), where the aerodynamic force is manipulated using the propeller speed, is considered. For such a system the yaw and pitch angles are not flat outputs contrarily of the real helicopters. Moreover, the system does not seem to be flat and is shown to admit at most a one dimensional defect. Various properties of this system are investigated to illustrate the difficulties that arise in the control of the *toycopter*.

1 Introduction

Laboratory-scale experimental setups have always been used to illustrate certain aspects of important industrial and technological problems. However, they sometimes, in their own potential, throw challenging control issues unseen in the real-world setups that they try to imitate. The classical example is that of an inverted pendulum, which was proposed to illustrate the problems encountered in rocket control [8]. However, the problem of swinging the pendulum to its upright position has occupied many researchers, ever since its conception. Swinging is completely foreign to the original rocket control problem; nevertheless, it has been the testground for many an interesting idea [1].

A similar situation arises in the helicopter problem as well. The elements to be illustrated are the multi-input multi-output scenario that naturally arises and the gyroscopic effects that come into picture due to a moving propeller. In the standard helicopter [7], the aerodynamic force is controlled using the propeller blade angle. In small setups however, it is much easier to have a propeller with fixed blade angle and to adjust the aerodynamic force by manipulating the propeller speed. Such a setup, termed a *toycopter*, will be considered in this work. A modification of this sort does not remove any of the essential couplings that need to be illustrated, but adds an extra coupling caused by the reaction of the force necessary to change the propeller speed.

The presence of this coupling has drastic effects on the control problem at large. If the blade angle had been used for control, the system would have been feedback linearizable [6] and flat. For the sake of simplicity, we restrict ourselves to two-degrees-of-freedom (2-DOF) systems where there is no roll movement. The flat outputs are then the pitch and yaw angles. However, it will be shown that if the propeller speed is varied, the system is no longer flat due to the presence of this extra coupling. Flat systems are those in which the states and the inputs can be reconstructed from the outputs and their time derivatives [4]. Such systems are easy to control and the control strategies are well studied. However, if a system falls out of this category, control becomes more involved, and sophisticated techniques need to be employed.

In this paper, the *toycopter* is compared and contrasted with a 2-DOF helicopter, a system where the blade angle is used for control. The modeling of the two systems is performed in an unified Newtonian framework. Various properties of these two systems, such as linearizations, flat outputs, defects and the residual dynamics are studied to illustrate the need for more sophisticated control schemes for the control of a *toycopter*.

Section 2 is devoted to the modeling of the 2-DOF helicopter and the *toycopter*. The linearized systems are analyzed and the their flat outputs derived in Section 3. A discussion of flat outputs and defects of the nonlinear systems is undertaken in Section 4, and Section 5 proposes a mechanical solution to overcome the defect. The paper concludes with Section 6 where possible control alternatives are discussed.

2 Modeling of a 2-DOF helicopter and a *toycopter*

The aerodynamic force of a propeller can be varied by either changing the propeller speed or the propeller blade angle. In this section, we will develop a model of a helicopter-like system where the aerodynamic force of the main propeller is controlled using both the blade angle and propeller speed. The rear axis however has a fixed blade angle. The model of a 2-DOF helicopter and a *toycopter* will then be derived as special cases.

Figure 1 shows the descriptive diagram of such a helicopter system which, however, cannot fly. The main body is fixed to the ground through a rotational joint permitting a rotation around the ϕ axis. The second part, the arm, is attached to the main body through another rotational joint which permits rotation in the ψ direction. Two electrical drives, perpendicular to each other, that rotate the main and the rear propellers on each end of the arm are mounted. These drives are positioned in such a way that the aerodynamical force of the main motor generates a torque in the ψ direction while the rear propeller acts on the ϕ coordinate.

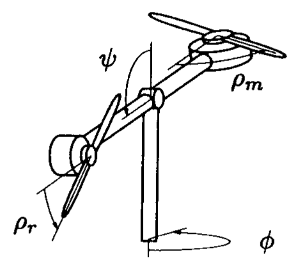

Figure 1: Helicopter model

The following hypotheses are considered in the mechanical modeling of the above mentioned system:

- The aerodynamical force is considered proportional to the propeller speed.
- The aerodynamic cross coupling is neglected.
- Frictional effects are neglected.
- The electrical drives are torque controlled.

2.1 Modeling using the Newtonian approach

The classical Newtonian approach using Euler formula will be used for modeling [5]. The Newtonian modeling consists of the following steps: (a) decomposition of the system into smaller units (subsystems) (b) modeling the dynamics of these subsystems and (c) the computation of the external torques, which include (i) gravity, (ii) aerodynamical forces and (iii) forces from the other subsystems.

The system of Figure 1 is decomposed into two subsystems: (i) the arm, and (ii) the two rotating propellers. The arm dynamics will be derived in detail in the next subsection while the dynamics of the propellers are just integrators. The coupling between the propeller subsystem and the arm is caused by the gyroscopical torques resulting from the change in kinetic momentum of the propellers. However, the coupling between the arm and the propellers will be neglected, since its speed is far less than that of the propellers.

2.2 Arm dynamics

Let ϵ_1, ϵ_2, ϵ_3 be a frame (ϵ-frame) attached to the arm at its center of rotation, as shown in Figure 2. The arm speed, $\boldsymbol{\omega}$, with respect to an inertial frame, expressed in the ϵ-frame is given by,

$$\boldsymbol{\omega} = \dot{\psi}\epsilon_1 - \dot{\phi}\cos(\psi)\epsilon_2 + \dot{\phi}\sin(\psi)\epsilon_3 \quad (1)$$

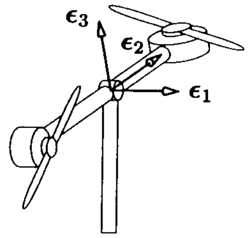

Figure 2: Epsilon frame attached to the arm

The momentum of the arm is then $\boldsymbol{L} = \sum_{i=1}^{3} I_i \omega_i \epsilon_i$, where I_i is the inertia along the ith axis. The dynamics is obtained by inserting the momentum into the fundamental equation $\frac{d}{dt}\boldsymbol{L} = \boldsymbol{M}$, with M_i being the external torque along the ϵ_i axis. Noting that $\dot{\epsilon}_i = \sum_{j=1}^{3} \omega_j \epsilon_j \wedge \epsilon_i$, the dynamics, after some simplification, can be obtained as,

$$I_\psi \ddot{\psi} - \frac{1}{2} I_c \dot{\phi}^2 \sin(2\psi) = M_1 \quad (2)$$

$$\left[I_\phi + I_c \sin^2(\psi)\right] \ddot{\phi} + I_c \dot{\phi}\dot{\psi} \sin(2\psi) = M_3 \sin(\psi) - M_2 \cos(\psi) \quad (3)$$

where $I_c = I_3 - I_2$, $I_\phi = I_2$ and $I_\psi = I_1$.

2.3 External torques

Three types of torques are applied to the arm:

1. Aerodynamical torques (M_i^a): These are due to the propeller angular speed and blade angle. The aerodynamical forces are proportional to the angular speed,

and the ratio is an invertible function of the blade angle of attack.

2. Gravitational torque (M_i^G): This torque is due to gravity.

3. Gyroscopical torques (M_i^g): These are due to the gyroscopic effects caused by the change in kinetic momenta of the propellers.

The first two types of torques can be computed as:

$$\begin{aligned} \boldsymbol{M}^a &= C_m(\eta)\omega_m\epsilon_1 + C_r\omega_r\epsilon_3 & (4)\\ \boldsymbol{M}^G &= (G_s\sin(\psi) + G_c\cos(\psi))\epsilon_1 & (5)\end{aligned}$$

where $C_m(\eta)$ is a monotonic function of η and C_r, G_s, G_c are appropriately-defined constants. The gyroscopic torque has two components, $\boldsymbol{M}_m^g$ and $\boldsymbol{M}_r^g$ given by:

$$\boldsymbol{M}_m^g = I_m\omega_m\dot{\phi}\cos(\psi)\epsilon_1 + I_m\omega_m\dot{\psi}\epsilon_2 - I_m\dot{\omega}_m\epsilon_3 \quad (6)$$
$$\boldsymbol{M}_r^g = -I_r\dot{\omega}_r\epsilon_1 - I_r\omega_r\dot{\phi}\sin(\psi)\epsilon_2 - I_r\omega_r\dot{\phi}\cos(\psi)\epsilon_3 \quad (7)$$

These are generated by the reaction torque to the one needed to change the kinetic momentum of each propeller (Set $\boldsymbol{L}_m = I_m\omega_m\epsilon_3$ for the main propeller and $\boldsymbol{L}_r = I_r\omega_r\epsilon_1$ for the rear one and noticing that $\boldsymbol{M}_i = \frac{d\boldsymbol{L}_i}{dt} = -\boldsymbol{M}_i^g$ with $i = m, r$ gives the previous formulas).

Note that the external torques do not contain a $\omega_r\dot{\phi}$ term due to the combination $M_3\sin(\psi) - M_2\cos(\psi)$. This phenomenon can be explained by the fact that the change of orientation of the rear kinetic momentum creates a torque only on the roll coordinate whose movement is prevented by the structure.

2.4 The model

Putting together the results of the previous subsections gives the final model:

$$\begin{aligned} I_\psi\ddot{\psi} + I_r\dot{\omega}_r &= C_m(\eta)\omega_m + G_s\sin(\psi) + G_c\cos(\psi)\\ &\quad + I_m\omega_m\dot{\phi}\cos(\psi) + \frac{1}{2}I_c\sin(2\psi)\dot{\phi}^2 \quad (8)\end{aligned}$$

$$\begin{aligned}\left[I_\phi + I_c\sin^2(\psi)\right]\ddot{\phi} + I_m\dot{\omega}_m\sin(\psi) &= C_r\omega_r\sin(\psi)\\ -I_m\omega_m\cos(\psi)\dot{\psi} - I_c\sin(2\psi)\dot{\psi}\dot{\phi} & \quad (9)\end{aligned}$$

$$\begin{aligned} I_m\dot{\omega}_m &= K_m u_m - C_{mm}(\eta)\omega_m & (10)\\ I_r\dot{\omega}_r &= K_r u_r - C_{rr}\omega_r & (11)\end{aligned}$$

Where η is the blade angle of attack and the terms $C_{mm}(\eta)\omega_m$ and $C_{rr}\omega_r$ are aerodynamic reaction torques as seen by the different motors, with the constants defined appropriately.

The model of the 2-DOF helicopter is obtained by setting ω_m to be a constant, $\dot{\omega}_m = 0$ and dropping (10). The states are $\{\psi, \dot{\psi}, \phi, \dot{\phi}, \omega_r\}$ and inputs $\{\eta, u_r\}$. The input u_m is utilised by another controller (outside the realm of this study) to maintain ω_m at a constant value and hence is unavailable to us. On the other hand, the model of the *toycopter* can be derived by having η constant. The states of the *toycopter* are $\{\psi, \dot{\psi}, \phi, \dot{\phi}, \omega_m, \omega_r\}$ and inputs $\{u_m, u_r\}$. For both systems the outputs are $\{\psi, \phi\}$.

As far as the dynamics are concerned, the most important change is that the term $I_m\dot{\omega}_m\sin(\psi)$ of (9) drops out in the model of the 2-DOF-helicopter since $\dot{\omega}_m = 0$. To understand the source of this term, note that the gyroscopic torque can be generated two ways : (i) a change in orientation of the kinetic momentum and (ii) a change in propeller speed. $I_m\dot{\omega}_m\sin(\psi)$ is typically a term of latter type. The complexity that such a term generates is illustrated in the next section by analyzing the linearized models of these two systems.

3 Linearized systems and their corresponding flat outputs

Flat outputs of a system are those combinations of the states which have the following properties: (i) Given the flat outputs and their derivatives, all states and inputs can be reconstructed. (ii) The number of flat outputs is equal to the number of inputs. (iii) If the system outputs are flat then the residual dynamics are trivial (no residual dynamics). Flat outputs of a linear system are the standard Brunowsky outputs discussed in literature [2]. In this section, the models of the two systems under consideration are first linearized. The flat outputs and the nature of the residual dynamics of the linearized systems are then analyzed.

3.1 Linearization of the 2-DOF helicopter

The linearization of the 2-DOF helicopter has the form,

$$\begin{aligned} \Delta\ddot{\psi} + c_1\Delta\dot{\omega}_r &= c_2\Delta\psi + c_3\Delta\dot{\phi} + c_4\Delta\eta & (12)\\ \Delta\ddot{\phi} &= c_5\Delta\omega_r + c_6\Delta\psi \\ &\quad + c_7\Delta\dot{\psi} + c_8\Delta\dot{\phi} & (13)\\ \Delta\dot{\omega}_r &= c_9\Delta\omega_r + c_{10}\Delta u_r & (14)\end{aligned}$$

where $c_i, i = 1, 2, \cdots, 10$ are coefficients of the linearized system expressible as a function of the states and model constants. It can be easily seen that the flat outputs for this system are variations of the original outputs, $\Delta\psi$ and $\Delta\phi$ themselves. Let $\tau = \Delta\psi$ and $\kappa = \Delta\phi$ be used to represent the flat outputs. The variations of the various states and outputs can be derived as follows: $\Delta\omega_r$ can be obtained from (13) since the other quantities are assumed to be known. Differentiating and back substituting $\Delta\omega_r$ and $\Delta\dot{\omega}_r$ in (12) gives $\Delta\eta$. Since the outputs of the system are flat the residual dynamics are trivial.

3.2 Linearization of the toycopter

The linearization of the *toycopter* has the form,

$$\begin{aligned}\Delta\ddot{\psi} + C_1\Delta\dot{\omega}_r &= C_2\Delta\psi + C_3\Delta\dot{\phi} + C_4\Delta\omega_m \quad (15)\\ \Delta\ddot{\phi} + C_5\Delta\dot{\omega}_m &= C_6\Delta\omega_r + C_7\Delta\omega_m\\ &\quad +C_8\Delta\psi + C_9\Delta\dot{\psi} + C_{10}\Delta\dot{\phi} \quad (16)\\ \Delta\dot{\omega}_m &= C_{11}\Delta\omega_m + C_{12}\Delta u_\eta \quad (17)\\ \Delta\dot{\omega}_r &= C_{13}\Delta\omega_r + C_{14}\Delta u_r \quad (18)\end{aligned}$$

where $C_i, i = 1, 2, \cdots, 14$ are the coefficients of this linearized system.

Clearly we see that $\Delta\psi$ and $\Delta\phi$ are not the flat outputs. To find the flat outputs of a differentially cross-coupled system, we transform the same so that the Brunovsky outputs can be readily obtained. The state transformation that needs to be employed and the transformed system should look like :

$$\begin{aligned}\Delta\tilde{\omega}_m &= \Delta\omega_m + \bar{C}_1\Delta\psi + \bar{C}_2\Delta\dot{\psi} + \bar{C}_3\Delta\phi + \bar{C}_4\Delta\dot{\phi},\\ \Delta\tilde{\omega}_r &= \Delta\omega_r + \bar{C}_5\Delta\psi + \bar{C}_6\Delta\dot{\psi} + \bar{C}_7\Delta\phi + \bar{C}_8\Delta\dot{\phi}(19)\end{aligned}$$

$$\begin{aligned}\tilde{C}_1\Delta\ddot{\psi} + \tilde{C}_2\Delta\ddot{\phi} + \tilde{C}_3\Delta\dot{\tilde{\omega}}_m + \tilde{C}_4\Delta\dot{\tilde{\omega}}_r &= \Delta\tilde{\omega}_m\\ \tilde{C}_5\Delta\ddot{\psi} + \tilde{C}_6\Delta\ddot{\phi} + \tilde{C}_7\Delta\dot{\tilde{\omega}}_m + \tilde{C}_8\Delta\dot{\tilde{\omega}}_r &= \Delta\tilde{\omega}_r \quad (20)\end{aligned}$$

where $\bar{C}_i$s and $\tilde{C}_i$s can be expressed in terms of coefficients C_is. It is easy to verify that $\tau = \int\int \Delta\tilde{\omega}_m\, dt\, dt$ and $\kappa = \int\int \Delta\tilde{\omega}_m\, dt\, dt$ expressed as a function of the original states are the Brunowsky outputs of (21) and hence of the system (16)-(19). If the constants in equations (16)-(19) are such that they are already in the form (21), then the physical interpretation of the flat outputs is that they are the integral of the two propeller angles.

Looking at the development, it is clear that the cross-coupling term has increased the complexity of finding a flat output. The expressions are lenghty and mathematically complicated even when the linearized system is considered and it is not clear how the flat outputs can be found for the complete nonlinear model.

Since the original system outputs are not flat, the system possesses residual dynamics. Since the system is linear, the residual dynamics can be studied by setting $\Delta\psi = \Delta\phi = \Delta\dot{\psi} = \Delta\dot{\phi} = \Delta\ddot{\psi} = \Delta\ddot{\phi} = 0$ in (16)-(19). This leads to the homogeneous system

$$\begin{bmatrix}\dot{\omega}_m\\ \dot{\omega}_r\end{bmatrix} = \begin{bmatrix}C_7 & C_6\\ C_4 & 0\end{bmatrix}\begin{bmatrix}\omega_m\\ \omega_r\end{bmatrix} \quad (21)$$

The eigenvalues of the homogeneous system described in (22) are $\frac{C_7 \pm \sqrt{C_7^2 + 4C_4C_6}}{2}$. Since $C_4, C_6 > 0$, one of the eigenvalues is always positive and the other negative. This clearly shows that the system outputs exhibit nonminumum-phase characteristics.

4 Flat outputs and Defect of the nonlinear systems

Having obtained the flat outputs of the linearized systems in the previous section, we proceed with the flatness analysis of the nonlinear original models. As would be expected, the flat outputs of the 2-DOF helicopter are the system outputs themselves, while it is not possible to get a set of flat outputs for the *toycopter*. With the system outputs, the residual dynamics have dimension 2, though it can be reduced to 1 by a suitable choice of outputs. The dimension of the residual dynamics will be defined as the 'defect' of the system.

4.1 Flat outputs of the 2-DOF helicopter

From the dynamics of the 2-DOF helicopter, consider the flat outputs candidates $\tau = \psi$ and $\kappa = \phi$. We now check whether all states and initial inputs can be reconstructed from τ, κ and a finite number of their time derivatives. The states $\{\psi, \dot{\psi}, \phi, \dot{\psi}\}$ are trivially reconstructed. The procedure to reconstruct, ω_r and the inputs is given below. Using (9), ω_r can be expressed as:

$$\begin{aligned}\omega_r &= \frac{1}{C_r\sin(\tau)}\left\{\left[I_\phi + I_c\sin^2(\tau)\right]\ddot{\kappa} + I_m\omega_m\cos(\tau)\dot{\kappa}\right.\\ &\quad \left.+I_c\sin(2\tau)\dot{\tau}\dot{\kappa}\right\} = f_1(\tau, \dot{\tau}, \dot{\kappa}, \ddot{\kappa}) \quad (22)\end{aligned}$$

Then using (8) and noting that $C_m(\eta)$ is an invertible function of η, the input η can be computed as:

$$\begin{aligned}\eta &= C_m^{-1}\left(\frac{1}{\omega_m}\left[I_\psi\ddot{\tau} + I_r\dot{f}_1(\tau, \dot{\tau}, \ddot{\tau}, \dot{\kappa}, \ddot{\kappa}, \kappa^{(3)})\right.\right.\\ &\quad - G_s\sin(\tau) - G_c\cos(\tau)\\ &\quad \left.\left.-I_m\omega_m\dot{\kappa}\cos(\tau) - \frac{1}{2}I_c\sin(2\tau)\right]\right)\\ &= f_2(\tau, \dot{\tau}, \ddot{\tau}, \dot{\kappa}, \ddot{\kappa}, \kappa^{(3)}) \quad (23)\end{aligned}$$

Reconstruction of the other input is then straightforward by substituting (23) in (11). This leads to the expressions $u_r = f_3(\tau, \dot{\tau}, \ddot{\tau}, \dot{\kappa}, \ddot{\kappa}, \kappa^{(3)})$. By choosing $\ddot{\tau}$ and $\kappa^{(3)}$ as the new inputs, the 2-DOF helicopter can be transformed into two chains of integrators, one with 2 and another with 3 integrators. Note that the original system has 5 states, which is the same number as the number of states in the chain of integrators. Hence, the transformation can be realised using static feedback.

4.2 Residual dynamics of the toycopter with system outputs

For the *toycopter*, one possibility is to choose $\tau = \psi$ and $\kappa = \phi$ as the flat output candidates. However, with such a choice, all the states cannot be reconstructed, and this leads to unstable residual dynamics. The residual dynamics, of dimension 2 (states $\{\omega_m, \omega_r\}$), are given by,

$$I_r\dot{\omega}_r - I_m\omega_m\dot{\kappa}\cos(\tau) - C_m\omega_m = \frac{1}{2}I_c\sin(2\tau)\dot{\kappa}^2 - I_\psi\ddot{\tau} + G_s\sin(\tau) + G_c\cos(\tau) \quad (24)$$

$$I_m\dot{\omega}_m\sin(\tau) + I_m\omega_m\cos(\tau)\dot{\tau} - C_r\omega_r\sin(\tau) = -\left[I_\phi + I_c\sin^2(\tau)\right]\ddot{\kappa} - I_c\sin(2\psi)\dot{\tau}\dot{\kappa} \quad (25)$$

The residual dynamics are linear in $\{\omega_m, \omega_r\}$ and the homogeneous part is given by,

$$\begin{bmatrix} \dot{\omega}_m \\ \dot{\omega}_r \end{bmatrix} = \begin{bmatrix} -I_m\dot{\tau}\tan(\tau) & C_r \\ C_m + I_m\dot{\kappa}\cos(\tau) & 0 \end{bmatrix} \begin{bmatrix} \omega_m \\ \omega_r \end{bmatrix} \quad (26)$$

Comparing (27) with the (22), it is clear that the residual dynamics are unstable.

4.3 Defect one outputs of the toycopter

Instead of taking ψ and ϕ, consider the following as flat output candidates:

$$\begin{aligned} \tau &= -I_\psi\cos(\psi) \\ &\quad + \frac{I_r}{C_r}(I_\phi\dot{\phi} + I_m\sin(\psi)\omega_m + I_c\dot{\phi}\sin^2(\psi)) \quad (27) \\ \kappa &= \phi \quad (28) \end{aligned}$$

The expressions have been so chosen that the input appears only after 3 differentiations for the τ output and after 2 for κ. Since the system has dimension 6, and since the input appears in $\ddot{\kappa}$ itself, we are left with an internal dynamics of one dimension. Thus we have reduced the dimension of the residual dynamics from 2 to 1. Hence the system is said to have a defect of *at most one*.

4.4 Reconstruction of the original states and the internal dynamics

To obtain the residual dynamics, we reconstruct from τ, κ and their derivatives, all possible original states. Reconstruction of $\phi = \kappa, \dot{\phi} = \dot{\kappa}$ are trivial. The states ω_m and ω_r can be obtained as expressions of ψ and $\dot{\psi}$:

$$\omega_m = \frac{\frac{C_r}{I_r}(\tau + I_\psi\cos(\psi)) - I_\phi\dot{\kappa} - I_c\dot{\kappa}\sin^2(\psi)}{I_m\sin(\psi)} \quad (29)$$

$$\omega_r = \frac{1}{I_r}\left\{\frac{\dot{\tau}}{\sin(\psi)} - I_\psi\dot{\psi}\right\} \quad (30)$$

Due to the one-dimensional defect, either of the variables ψ or $\dot{\psi}$ will be undetermined, and the relationship between them leads to the following internal dynamics. Using (30)-(31), we obtain,

$$\begin{aligned} \dot{\psi} = \frac{\sin(\psi)}{\dot{\tau}\cos(\psi)} \Big\{ & \ddot{\tau} - \frac{1}{2}I_c\sin(2\psi)\sin(\psi)\dot{\kappa}^2 - G_s\sin(\psi)^2 \\ & - G_c\sin(\psi)\cos(\psi) - \frac{(C_m + I_m\dot{\kappa}\cos(\psi))}{I_m I_r} \\ & \left[C_r(\tau + I_\psi\cos(\psi)) - I_r\dot{\kappa}(I_\phi + I_c\sin^2(\psi))\right] \Big\} \quad (31) \end{aligned}$$

If the internal dynamics are always stable for all $\{\tau, \kappa\}$ profiles, then one can design controller without taking these dynamics into account. However, for most trajectories these dynamics are unstable. To illustrate this, an evolution of $\{\psi, \phi\}$ is considered. The curves shown in Fig.3 is the step response of the closed loop system with a simple PD controller. For such an evolution, the defect one outputs $\{\tau, \kappa\}$ are calculated using the knowledge of the states $\{\omega_m, \omega_r\}$. From these defect one outputs, the internal dynamics (32) is simulated to give the curve ψ_{int} of Fig.3. It can be seen that even the numerical roundoff errors is sufficient to push these dynamics into instability.

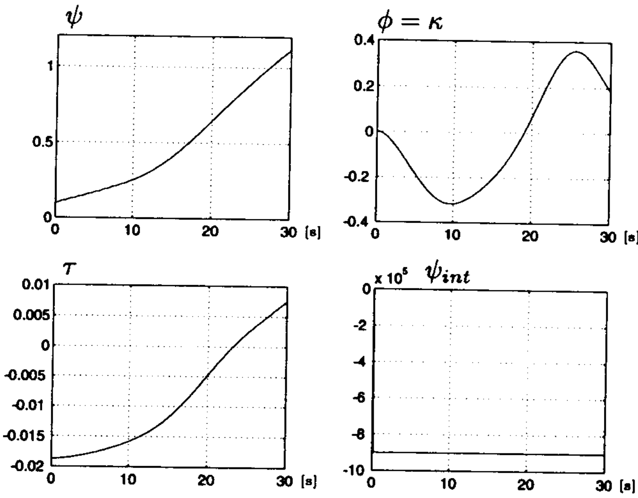

Figure 3: Simulation of the unstable internal dynamics

5 A Mechanical solution to overcome the defect

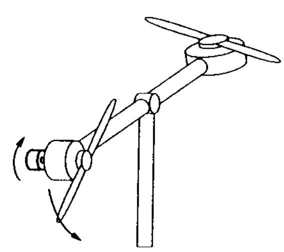

Figure 4: Mechanical compensation

As shown in the previous section, the *toycopter* has a one-dimensional defect. It is interesting to see if this defect can be overcome by changing the mechanical structure of the system retaining the fact that the aerodynamic force is controlled by varying the speed.

The key idea is to generate another term equal in magnitude and opposite in direction to the $I_r \dot{\omega}_r$ in (8). For this purpose, an inertia as shown in Figure 4 can additionally be constructed. This inertia will spin in a direction opposite to that of the propeller so that the $I_r \dot{\omega}_r$ term can be canceled. Such a transformation will alter (8) to :

$$\begin{aligned} I_\psi \ddot{\psi} &= C_m \omega_m + G_s \sin(\psi) + G_c \cos(\psi) \\ &\quad + I_m \omega_m \dot{\phi} \cos(\psi) + \frac{1}{2} I_c \sin(2\psi) \dot{\phi}^2 \end{aligned} \quad (32)$$

while the other equations remain the same. The resulting system is flat and the flat outputs are $\tau = \psi$ and $\kappa = \phi$. The system can be transformed into chains of integrators, one with 4 integrators and the other with 3.

Note that for compensation it is not necessary that the compensating inertia be equal to that of the rear propeller. The only requirement is that the rate of change of kinetic momenta should be of the same magnitude but of opposite sign. Hence, one can have a much smaller inertia turning very fast or *vice versa*.

The same idea can be used to compensate the term $I_m \dot{\omega}_m \sin(\psi)$ caused by the main propeller, so as to obtain a flat system with the flat outputs being the same as before. The only drawback is that the inertia of the main propeller is larger than that of the rear propeller. Since compensating the smallest inertia is more natural, a fixture for the rear propeller is preferable.

6 Conclusion

By analyzing various properties of the *toycopter*, it was seen that it is more difficult to control than a 2-DOF helicopter. Hence, one is forced to use more intricate control techniques, specifically due to the following reasons : (i) the nonminimum-phase character of the system, (ii) the fact that it is not possible to get the flat outputs and (iii) the internal dynamics obtained in the defect 1 case are unstable.

Possible directions for the control of the *toycopter* include, (i) control of the clock [3] (ii) hierarchical control and (iii) predictive control with a lower bound on the prediction horizon [9]. On the other hand, the authors feel that the challenge this problem poses is interesting enough to make this simple laboratory setup a benchmark for many sophisticated nonlinear control algorithms to come.

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