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A CFO-based model of contingent claims with jump risk

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Resumo

Nesta tese apresenta-se um modelo estrutural de avaliação de ativos contingentes baseado em Goldstein *et al.* (2001). Neste último, assume-se que a empresa é detentora de um projeto, cujo resultado operacional (i.e. resultado antes de impostos e juros) segue um movimento Browniano geométrico. A empresa entra em incumprimento na primeira vez que este processo toca numa barreira definida endogenamente. Os acionistas, obrigacionistas e o governo são considerados como tendo um ativo contingente na performance do referido projeto. O mesmo acontece com os custos decorrentes da falência da empresa, os quais são considerados como pertencentes a um agente fictício. A este modelo acrescentam-se agora dois aspectos. Em primeiro lugar, considera-se que a variável de estado é o fluxo de caixa operacional, o que permite a aplicação do modelo a um maior número de empresas, na medida em que este é frequentemente positivo mesmo quando o resultado operacional é negativo. Esta alteração é possivel porque os custos com capital (capex) são excluídos da variável de estado. Em alternativa, considera-se que os fornecedores de capital têm direito a receber um fluxo financeiro constante enquanto a empresa se mantiver em funcionamento. Este fluxo corresponde ao investimento necessário para garantir a taxa de crescimento projetada para o fluxo de caixa operacional. Na prática, este princípio pode ser estendido a qualquer direito sobre um custo fixo da empresa. Para além de aumentar o número de empresas a que o modelo pode ser aplicado, mostra-se nesta tese que a introdução explícita de custos fixos leva a um aumento da probabilidade do processo estocástico tocar na barreira, bem como cria um efeito de alavancagem operacional que acentua o efeito de alavancagem financeira já sobejamente conhecido na literatura. A segunda inovação desta tese consiste na introdução da possibilidade de um salto súbito de dimensão fixa na variável de estado. A introdução deste termo permite replicar melhor os *spreads* observados na prática, especialmente no caso de obrigações e *credit default swaps* de curto prazo. Para além de fornecer um formulário para a avaliação de ações, obrigações e *credit default swaps*, esta tese analisa algumas questões fundamentais da literatura de *corporate finance*, como sejam a estrutura ótima e o custo de capital, e fornece formulas quase fechadas para o preço de opções sobre ações de estilo Europeu. A literatura financeira tem quase exclusivamente tratado estes ativos de forma separada dos restantes. No entanto, esta separação é inconsistente, potencialmente levando ao surgimento de oportunidades de arbitragem que podem ser exploradas com modelos do tipo aqui apresentado. Esta separação é também ineficiente, na medida em que o preço das opções fornece informação valiosa para a calibração deste tipo de modelos.

Palavras chave: Modelos estruturais, Finanças empresariais, Avaliação de opções.
Códigos JEL: G13, G32.

Abstract

This thesis presents a structural model of contingent claims in the spirit of Goldstein *et al.* (2001). In the latter, the firm is assumed to hold a project whose earnings before interest and taxes (EBIT) follow a geometric Brownian motion with default occurring at the first time the state variable falls below an endogenously determined barrier. Shareholders, debtholders, the government and distress costs are then considered as claimants on this project. The model in this thesis adds two elements to this setup. First, the cash flow from operating activities (CFO) is considered to be the state variable. This allows the application of the model to a greater number of firms since the CFO is often positive, even when EBIT is negative. This change is possible because capital expenditures (capex) are excluded from the state variable. Instead, they are treated as a contingent claim belonging to an external claimant, the capex suppliers, who hold the right to receive a fixed stream of cash flows corresponding to the investments needed to assure the forecasted project growth rate. In practice, this principle can be extended to any claim on a fixed stream of cash flows. In addition to enlarging the number of firms for which the model can be applied, the introduction of fixed costs such as capex is shown to increase significantly the probability of the stochastic process hitting the barrier; and to create an operating leverage effect, which accentuates the financial leverage effect frequently referred in the literature. Second, the possibility of a sudden negative jump of fixed size in the state variable is added. The introduction of this term improves the capacity of the model to replicate the observed credit spreads, especially in the case of short term bonds and credit default swaps. In addition to providing pricing formulas for equity, bonds and credit default swaps, this thesis analyzes some important questions in the field of corporate finance, such as the optimal capital structure and the cost of capital, and gives almost closed-form formulas for pricing European-style equity options. Finance literature has mostly treated option pricing and the pricing of all other securities contingent on the firm's capital structure as separate research areas. This separation is inconsistent, potentially leading to arbitrage opportunities that can be exploited using models of the type presented here. This separation is also inefficient since option prices can be extremely valuable for model calibration.

Keywords: Structural models, Corporate finance, Option pricing.
JEL Codes: G13, G32.

1. Introduction

Structural models of credit risk were a major breakthrough when they were first proposed in the seventies. For the first time it was possible to have probabilities of default and losses given default derived in a single theoretical setting. In addition, since default in these models is directly related with the firm capital structure, market information on share prices could be used for model calibration. This contrasts with the methods used at that time, notably, Altman Z-scores and Logit models, where default is seen as an exogenous statistical process with no clear economic cause. They were also a big step forward in the fields of asset pricing and corporate finance. Also for the first time, it was possible to have in a single framework the price of all claims that are contingent on the firm. This provided a solid setting to analyse corporate finance theories such as the trade-off theory of capital structure.

Though theoretically appealing, the early enthusiasm around structural credit risk models decreased in late eighties. On the one hand, the empirical applications of the first generation of these models had disappointing results with the models failing to accomodate the observed bond spreads and explaining their time variation. As discussed in the literature review, there are several reasons for this. On the other hand, following Jarrow and Turnbull (1995), a new class of credit risk models emerged. These models, which were termed 'reduced-form', attracted practioners and researchers attention as they were able to use the information available in bond prices in a more tractable way. They were also able to fit better the data. In these models default is modelled as a intensity process, which is a function of some exogenous latent state variables. Similar to Logit models, there is no theoretical model of the firm capital structure and default has no clear economic rationale, which limits the use of these models.

Meanwhile, a second generation of structural credit risk models emerged. These models tried to ovecome some of the issues that were pointed as being the causes of the poor results obtained by the first models. Empirical evidence is still scarce, but results suggest that, whenever the most appropriate estimation methods are used, current models are able to fit the data significantly better. In some cases, however, this improvement came at the cost of mathematical tractability. With nowadays computers, the latter is not essential for pricing purposes as numerical techniques such as Monte Carlo simulations and finite differences allow us to price even path dependent derivatives in an efficient way. Nevertheless, closed form solutions are still useful if one wants to infer market views on latent variables (e.g. the market value of a firm assets) based on observed market prices on contingent claims such as equity and CDS spreads. This can be particularly useful for policy makers trying to understand what is going on in financial markets.

This thesis presents a model that is able to provide a closed form solution for the most important types of financial assets that are contingent on the market value of the firm, notably, equity, bonds/CDS and European-style options (quasi-closed form in this case), and simultaneously take into account two of the most relevant contributions to the original Merton (1974) model, notably, the possibility of early default and asset price jumps. This is done in a model where the state variable is the firm cash flow from operating activities. This thesis starts with a literature review on structural credit risk and corporate finance models. Chapter 2 explains the model considered in this thesis and provides the reader with some tools needed to understand the derivations in the subsequent chapters. Chapters 3 to 7 derive the pricing of the before mentioned financial assets and discusses some important corporate finance concepts such as the cost of capital and the trade-off theory of optimal capital structure. Chapter 8 illustrates the model. Chapter 9 concludes.

1.1. Literature Review

Structural models of contingent liabilities were pioneered by Merton (1974) following Black and Scholes (1973) and Merton (1973). In its seminal paper, Merton considers that a firm financed by equity and debt honours its commitments towards debtholders if the value of its assets at maturity exceed its debt. If not, the firm declares bankruptcy and all its assets are liquidated with all the proceeds accruing to creditors. Intrinsically, the equity holders hold the firm, but have the right (but not the obligation) to sell it to the debtholders at debt's nominal value. In other words, stockholders own a put option on the firm assets with strike equal to nominal debt. This option is given by the debtholders whose claim, at market price, is worth the nominal debt value discounted at the risk free rate less the value of this put option, which is interpreted as debtholders' expected loss. Under the assumption that firm assets follow a diffusion process known as geometric Brownian motion, the price of this option is straightforward to compute. Alternatively, one can consider that debtholders own the firm but shareholders have a call option on it with strike equal to the firm nominal liabilities. This equivalence follows from the Put-Call parity.

Though revolutionary and theoretically appealing, the first empirical papers on the application of Merton's structural credit risk model led to disappointing results. Jones *et al.* (1984) is the first paper that empirically assesses the validity of the model. In this paper, Merton's model is applied to a set of firms with simple capital structures during the period between 1977 and 1981. Though they found a better performance in the case of speculative bonds, these authors concluded that in general the model overstates bond prices. They also concluded that the model is heavily penalized by the assumption of constant interest rates. Subsequent studies using more recent data reached broadly the same conclusions. This is the case of Lyden and Saraniti (2001) and Eom *et al.* (2004).

The reasons behind the lack of success of Merton's model have been extensively analysed in the literature.[1] The irrealistic assumption that default could only occur at debt's maturity was one of the issues that was first addressed. In order to overcome this, Black and Cox (1976) present a model where a firm financed by a single debt issue with a fixed maturity defaults at the first time the asset process crosses a pre-specified time-varying exponential barrier. When this happens, debtholders have the right to force reorganization of the firm, receiving firm value at default time.[2] Shareholders lose everything. The existence of this barrier has several implications. First of all, the barrier sets the maximum loss the debt holder may have. However, it also leads to higher probabilities of default. The consideration of a default barrier has also important consequences for the shareholder in terms of risk taking incentives. Differently from Merton's model where equity is a monotone function of asset volatility, equity is now a concave function on volatility. In this model when shareholders take riskier projects they increase the odds of very positive outcomes, but they also have a higher chance of losing their firm. Black and Cox (1976) study the impact of this barrier on debtholders with different levels of seniority.

A fundamental question is what determines the barrier. In the first sections of their paper, Black and Cox (1976) justify the consideration of this barrier by the inclusion of safety covenants in debt contracts. These safety covenants allow debtholders to demand debt payment whenever the value of the firm goes below a certain point. This has been named stock-based insolvency. Something similar occurs in many countries with laws giving debtholders in general the right to push the firm to bankruptcy whenever assets breach some

[1] The literature on structural models of contingent liabilities is very large and this review does not intend to be exaustive. Among the extensions to the original Merton model that are out of this review are models that assume a dynamic capital structure (e.g. Fischer *et al.* (1989)), models where assets follow mean reverting processes (e.g. Collin-Dufresne and Goldstein (2001) and Sarkar and Zapatero (2003)) as well as models that take into acount liquidity risk (e.g. He and Xiong (2012)).

[2] Depending on how the barrier is specified, default can also occur at debt maturity. In those specifications where default at debt maturity is virtually impossible, the loss given default becomes constant and equal to the barrier level.

lower level. In regulated sectors, such as banks, one may also think that regulators may push shareholders to increase capital. Whenever the latter are not able to do it, the bank may be resolved. In any of these cases, there must be some trigger that is often defined based on book values since the true asset value of the firm is not observable.

A second approach arises when one sets default as the result of the firm not being able to honour its ongoing payments either through the defined payout rate or by issuing new securities. This is usually referred as flow-based insolvency. Here we may think of three cases. In a model where the asset value is perfectly observed by all agents, one possibility is that the barrier is endogenously determined by the shareholders as the result of an optimal stopping time problem. Consider that at any moment in time shareholders may inject capital in the firm avoiding default. Shareholders will be willing to do it as long as equity value after the capital injection is higher than the capital injection. Notice that in a model with strict absolute priority and no problems of information asymmetry this condition remains valid even if shareholders have no capacity to increase capital since they are better off diluted than under default.[3] This was first proposed by Black and Cox (1976) who derived the optimal barrier in a model where the firm is financed by a perpetual bond. They showed that in this case, the default barrier is independent of the current value of assets, proportional to the contractual debt service (i.e. the higher the coupon payments the higher the barrier) and a decreasing function of asset volatility. Basically, shareholders are more willing to save the firm if they see any chances of making large profits in the future. This line of research was pursued in several other papers as documented below. A second possibility arises when the assumption of information symmetry falls. In this case, the firm capacity to increase capital may depend on its shareholders capacity to inject capital. If current shareholders are not able to do it the firm becomes dependent on external prospective shareholders assessment. This shall lead to a default barrier above the one implied by the optimal stopping time problem.[4] In the limit, we may have a third case where the firm may have to service debt using only internally generated funds. Whenever the latter is not sufficient the firm defaults. This is the case of Kim *et al.* (1993).

Several years after Black and Cox (1976) seminal paper, Leland (1994a) proposes a model where a firm financed by perpetual debt continues its activity until the asset process hits a default barrier determined endogenously by shareholders willingness to capitalize the firm. In addition, taxes and distress costs are introduced in order to analyse the firm optimal capital structure. They consider that the tax benefits of debt can be seen as a security that

[3] Implicitly, we are assuming that both current and prospective shareholders observe the current value of assets. In reality none of them observe the market value of assets. The problem is more relevant in the case of prospective shareholders, though.

[4] In this case, one may either set the barrier exogenously or model the impact of information asymmetries.

pays a constant coupon equal to the tax-shaltering value of interest payments as long as the firm is solvent. Similarly, distress costs can be seen as a claim on assets whenever default occurs. In their model, firm value equals the market value of assets plus the value of tax benefits minus distress costs. Equity is then computed subtracting current debt value from the market value of the firm. Leland (1994a) is one of the most relevant papers in corporate finance literature. Even so, besides the stochastic process itself, there are three issues that deserve some comment. The first issue regards the hypothesis of asset tradability after debt being issued. This issue is recognized in footnote 11 of the paper. A second source of criticism concerns the fact that in this model an increase in taxes actually leads to an increase in firm value. This results from the fact that the asset value is treated as an exogenous variable. Finally, the assumption that shareholders pay debtholders' coupons totally from their own pockets (assets are not sold in this model) may lead the expected leverage ratio of the firm to decrease as time goes by. Subsequent articles allow for asset sales. Notice however that in this case, as noted by Goldstein *et al.* (2001), the government claim is presumed constant (a fraction of coupon payments, which are paid as long as the firm operates) implicitly leading to an oversestimation of shareholders dividend variation, which is not in line with empirical findings.

Leland pursued his line of research in Leland (1994b) and Leland and Toft (1996). These papers differ mostly on the assumptions regarding debt rollover. In Leland (1994b) the firm retires a constant fraction of the currently outstanding debt at its principal value and replaces it by new debt so that cashflow requirements for debt service are equal to a fixed coupon amount and a fixed sinking fund requirement. Similar to the perpetual coupon bond case analysed in Black and Cox (1976) and Leland (1994a), total debt has time-homogenous cash-flows in this case, which is crucial to the computation of an endogenous barrier based on shareholders willingness to capitalize the firm. In Leland and Toft (1996) the firm continuously sells a constant amount of coupon bonds with a certain maturity, which it will redeem at par. New bond principal in then issued. In spite of each debt issue having non-constant cashflows, aggregate debt has time-homogenous cash-flows. In a recent provocative article, Décamps and Villeneuve (2014) argue however that the strategic default decision problem faced by equity holders in a model with roll-over debt has never been formulated properly. Instead it is presented as a kind of natural extension of Leland (1994a) infinite maturity case. In particular, they question whether equity value can be computed by taking current debt value out of asset value and whether equity holders' problem ia a standard stopping time problem with solution given by the smooth pasting condition. In their paper they prove that equity can in fact be computed by difference. However, the smooth pasting principle is shown to be the unique optimal shareholder strategy only under some specific conditions.

The studies referred up to now assume that absolute priority holds meaning that equity holders receive zero when default occurs and that senior debtholders only lose capital when equity holders and subordinate debtholders lose everything. In a case where debtholders may not be able to push the firm into bankruptcy, shareholders may run some sort of asset substitution (i.e. take risky projects they wouldn't take if the firm was not overleveraged) or try to service debt liquidating the firm assets at fire sale prices in order to gain time to see if things go better. These agency costs may justify the violation of absolute priority rules in practice. In this case, the barrier may arise from a game between the different stakeholders with shareholders still receiving something in case of debt restructuring. This approach was followed by Anderson and Sundaresan (1996), who showed that unlike Leland (1994a), Leland (1994b) and Leland and Toft (1996), the optimal barrier, restructuing barrier in this case, is an increasing function of liquidation costs. The higher the cost of liquidating the firm, the greater is shareholders' capacity to extract value from debt-holders. Other studies assume that the absolute priority does not hold but do not model it. This is the case of Longstaff and Schwartz (1995), who assume that the firm meets all its contractual obligation as long as the market value of its assets are above a certain threhold determined exogenously. Once this barrier is broken the firm defaults on all its obligations. Each stakeholder may then receive something that is also defined exogenously based on empirical evidence. This assumption simplifies the pricing of the different debt securities as one can simply use the average loss given default observed for each type of debt security for each sector of activity.

Another source of criticism on Merton model is the assumption of constant interest rates. The first structural credit risk model to overcome this issue is the one by Shimko *et al.* (1993).[5] These authors assume that interest rates follow the Vasicek model. In their model the firm is assumed to default only at debt maturity. The models by Kim *et al.* (1993) and Longstaff and Schwartz (1995) were the first to price risky corporate debt with stochastic interest rates under a first passage time setting. The models differ on the way the barrier is set. While the barrier in Kim *et al.* (1993) is defined through the capacity of the firm to service debt based only on its internally generated funds, in the case of Longstaff and Schwartz (1995) the barrier is defined exogenously.[6] The consideration of stochastic interest rates is crucial to price any fixed income security. The impact on credit spreads is not so clear, though. In these models the drift of the asset process depends on the risk free interest rate. As result, the higher the interest rate, the higher the drift, the lower the probability of default and

[5] In trueth, Merton (1973) already foresees stochastic interest rates. In his paper, bond prices are assumed to follow a GBM.

[6] In this model the probability of default on a bond is determined by a single variable rather than the default status of other bonds. There is no need to condition on the pattern of cash payments to be made prior to the maturity of the bond. Thus, one can value coupon bonds as portfolios of discount bonds.

the lower the credit spread. In this setting, positive correlations between the interest rate and the asset process tend to generate higher spreads because the two factors move in the same direction.[7] This effect disappears when one considers that nominal liabilities and the default barrier grow at the same rate as assets, which is equivalent to say that the firm has a stationary capital structure.[8] Further effects may play their role in more complex models, though. In models where the barrier is determined endogenously, an increase in the interest rate may lead to a downwards shift in the barrier. Notice that the higher the interest rate the more valuable is shareholders option and the longer they are willing to wait for recovery. In contrast, under an incomplete market setting whenever rollover is introduced it makes sense to think that an increase in the interest rate leads to an increase in the barrier simply because shareholders wait for less time whenever borrowing costs are higher.[9]

Though the financial extensions referred were able to improve the model, empirical literature from the early twenty first century shows that these models are still unable to correctly predict spreads. Lyden and Saraniti (2001) compare Merton (1974) and Longstaff and Schwartz (1995) using a database composed of firms with only one bond outstanding. Asset values were computed as the sum of firms' equity, market value of bonds and adjusted values for other liabilities. The results were again disappointing but the authors finish their paper reffering that the classical model could still be correct in case the poor fit is due to problems in asset volatility prediction. Eom *et al.* (2004) implement the models by Merton (1974), Geske (1977), Longstaff and Schwartz (1995), Leland and Toft (1996) and Collin-Dufresne and Goldstein (2001) using a sample of 182 bonds from firms with simple capital structures during the period 1986-1997. They concluded that while Merton model leads to too low spreads, more recent models overestimate spreads on average. They also refer that the more recent models overstate the risks coming from firms with high leverage or volatility but underpredict spreads on safer bonds. They conclude that the major challenge for structural credit risk models is to increase spread predictions without overstating either volatility, leverage or coupons paid. This goes in line with the general idea that structural models underpredict spreads unless abnormal parameter values are used. This is more pronounced in investment grade bonds but true for any rating in the case of short-term bonds. Some studies have analysed whether this difference could be justified by a liquidity premium. This

[7] An increase in the risk free rate is usually associated with inflation expectations above the central bank target which often occurs when the economy is growing fast and financial assets appreciate for this reason. This fact justifies a positive correlation between asset returns and the risk free rate.

[8] This does not mean however that the firm management pursues a specific capital structure as in Collin-Dufresne and Goldstein (2001).

[9] Additionally, in the case of banks, higher interest rates are associated with the possibility of banks funding their assets at below the risk free rate through deposits. This is something that tends not to be possible when interest rates are at or near the lower bound.

is the case of Perraudin and Taylor (2003), Longstaff *et al.* (2005) and Huang and Huang (2012). Though these studies suggest that non-default issues can justify a significant part of corporate bond (investment grade) spreads vis-a-vis sovereign bonds, they were not enough to explain the empirical shortcomings of structural models. More recently, some studies have argued that the poor performance of structural models is closely related with the estimation methods used and state that independently of the model, the results obtained are significantly better whenever the parameters are estimated through maximum likelihood. This is the case of Li and Wong (2008), who calibrate Merton (1974), Longstaff and Schwartz (1995) and Collin-Dufresne and Goldstein (2001) models using the maximum likelihood estimator first proposed by Duan (1994). In contrast with Eom *et al.* (2004), these authors conclude that these structural models are very useful for pricing medium and long term bonds but they are not able to replicate the observed prices on short-term bonds. Similar conclusions are reached by Wong and Choi (2009), using the Brockman and Turtle (2003) model, and by Forte (2011) and Forte and Lovreta (2012), using a slightly modified version of Leland and Toft (1996) model. The last two studies estimate the barrier parameter iteratively, though.[10]

Despite the improvements achieved by the already referred models, the current consensus is that structural models poor performance is maily due to the assumption of a pure diffusion process. Under a diffusion process, the time of default is accessible meaning that there is an increasing sequence of stopping times that converges to the default time and thus 'foretells' the event of default. In other words, the probability of assets falling substantially goes to zero as one approximates debt maturity. In addition, the light tails of the Normal distribution circunvent the firm from defaulting unless it is already near financial distress. This is in strong contradiction with the observed data. In order to solve these issues, and following Merton (1976) seminal article, Zhou (2001) considers a first passage credit risk model where firm assets follow a jump-diffusion process with jump amplitudes from a Lognormal distribution. In this case, default can occur both in an expected way, due to the diffusion process, or unexpectedly due to jumps. The model is able to generate credit spreads in line with the ones observed. Unfortunately, the proposed model has no closed form solution. Hilberink and Rogers (2002) extend Leland (1994a) model to the case where the market value of firm assets faces only downward jumps.[11] The authors did not reach a closed form solution but debt prices can be computed numerically without too much pain. Cathcart and El-Jahel (2003) propose a first passage time structural model where default can occur in an expected manner

[10] It is interesting to note that the estimates produced by the iterative procedure proposed by Forte and Lovreta (2012) are very close to the results that come out from the application of the smooth pasting condition.

[11] As referred by the authors downward jumps are more likely than upward jumps. In addition, this assumption simplifies the problem considerably.

through the diffusion process or unexpectedly at the first jump event determined according to a stochastic hazard rate. Their paper distinguishes from most of the literature by specifying that default occurs depending on the dynamics of a signaling variable. The authors argue that this approach is more flexible as it may be applied to issuers that do not hold a clearly identifiable set of assets such as sovereigns. Moreover, their model considers that the risk free rate follows the Cox-Ingersoll-Ross (CIR) process. The latter is assumed to be uncorrelated with the asset process but affect the hazard rate of the jump event. Absolute priority is assumed to be violated. Despite providing almost closed-form solutions to bond prices this paper does not provide any guidance on equity value, which limits model calibration. More recently, several papers (e.g. Chen and Kou (2009), Huang and Huang (2012)) have proposed the use of the double exponential distribution instead of the Normal distribution to model jump sizes. Besides having heavier tails, this distribution has the advantage of having a closed-form solution.

The introduction of jumps in the asset process has not been the only way proposed in the literature to account for the fact that spreads tend to zero as debt approaches maturity. Moodys-KMV is probably the most successful commercial application of structural models. Under their model the normalized distance between asset values and the default barrier (i.e. distance to default) is first computed and then evaluated in Moodys proprietary database instead of using the Normal distribution. JPMorgan, Goldman Sachs, Deutsche Bank and the RiskMetrics Group (see Finger *et al.* (2002)) consider instead that the default boundary, though constant, is uncertain. This turns default into an unpredictable event enabling the model to produce non-zero short-term spreads even for investment grade bonds. A third alternative is to consider that the volatility term is not constant. This approach has been followed by Fouque *et al.* (2006). In this paper, it is shown that when asset volatility follows a fast mean-reverting process short term spreads increase significantly. A fourth possibility is to leave the assumption that the market value of the firm is observed. This is what occurs in Duffie and Lando (2001). In their model, investors observe at each moment in time that the firm has not defaulted and noisy accounting reports, on which their assessment on the market value of assets is based. The two values are then assumed to be joint Normal distributed.[12] Based on this assumption, they compute the distribution of the true asset value conditional on the available information. They show that for an issuer with conditionally unbiased reported assets, a kind of Jensen effect implies a lower debt price as compared with the perfect observation case. In addition, with imperfect information, credit spreads remain bounded away from zero as maturity goes to zero. Finally, in the last years the IMF has used

[12]This results from the fact that the difference between the logarithm of the true market value and the observed being assumed to be normally distributed.

in its macrofinancial risk assessments structural models with Gram-Charlier expansions (see Jobst and Gray (2013)). The latter is an approximate density function that differs slightly from the standard Normal distribution by introducing potentially non-zero skewness and excess kurtosis.

1.2. Contribution to the literature

This thesis presents a structural model of contingent claims that builds on the literature of first passage time models. The paper most similar with this thesis is the one from Goldstein *et al.* (2001). In the first part of their paper, it is assumed that the firm holds a project whose earnings before interest and taxes (EBIT) follows a geometric Brownian motion with default occurring in the first time the state variable falls below a certain level. Stakeholders on the firm receive their payoffs based on the value of this project, which in contrast with most structural models of corporate liabilities, is seen as a non-tradable asset. The implications of abandoning the tradability assumption are analysed here. This thesis further elaborates in this model by adding mainly two additional features. First, the cash flow from operating activities (CFO) is considered to be the state variable and a fixed cost parameter is introduced. The latter corresponds to the capital expenditures (capex) needed to mantain the current project growth rate. However, any type of fixed cost can be considered as long as the state variable is redefined in accordance. It is shown that this additional parameter leads to an operational leverage effect that increases firm volatility in bad times. In addition, under fixed costs the barrier level that maximizes the smooth pasting condition is not the one that maximizes firm value ex-ante. Second, the possibility of a sudden negative jump of fixed size in the project's capacity to generate earnings is added as in Realdon (2007).[13] The empirical observation that negative jumps are more likely than positive jumps justifies this choice. Examples of negative jumps include the discovery of substantial accounting misgivings that lead investors to suddenly reduce their estimates of the true value of the firm assets (e.g. Enron, Parmalat, Salad oil), natural disasters (e.g. Tepco), accidents (e.g. BP, Spanair), terrorist attacks (PanAm) or even redenomination risk (e.g. euro area sovereign debt crisis). Some negative events do not lead firms to default but hamper their financial capacity significantly leading default risk to soar. As such, the inclusion of this term contributes to the prediction

[13] Realdon (2007) derives a closed form solution for a first passage time structural model with constant interest rate where default occurs either at the first time the process hits an exogenous barrier or at the first jump of a compounded Poisson process with fixed negative amplitude. All outstanding debt is assumed to have equal priority in case of default. Equity is then computed subtracting total debt value and distress costs value from asset value as in Leland (1998).

of more realistic short term spreads without risking model tractability.[14]

In addition to providing pricing formulas for equity, debt with different levels of seniority and credit default swaps, this thesis analyzes some important questions in the field of corporate finance, such as the optimal capital structure and the cost of capital, and gives almost closed-form formulas for pricing equity options. Finance literature has mostly treated option pricing and the pricing of securities in the firm's capital structure as separate research areas. This separation though convenient is inconsistent potentially leading to arbitrage opportunities that can be exploited using the model presented here. In addition, for model calibration, the highest the amount of information the best. Stock price information is seldom enough to properly calibrate highly parameterized models. This is particularly true when one leaves out the assumption that it is possible to trade on the firm asset, which is the case in this thesis. Stock options complement the information set potentially leading to a significant improvement in model estimation.[15] Notable exceptions in the literature are the papers by Geske (1979), Toft and Prucyk (1997), Ericsson and Reneby (2003) and Realdon (2003). Geske uses Merton's model with finite maturity zero coupon debt to price options as compound options on the firm assets. Doing so, the variance of the stock returns becomes a function of the firms' leverage. In particular, when equity goes down, the debt-to-equity increases leading to a higher variance of returns, which is in line with the observed volatility skew. Toft and Prucyk (1997) also treat equity options as compound options on assets but under Leland (1994a) model. In this case, equity options are options on a down-and-out call option that expires whenever the market value of asset falls below an endogenous barrier. These authors show that the barrier level significantly affects option values and sensitivies. For example, an increase in asset volatility leads to an increase in option value due to the usual convexity in option pricing. However, in the case of an endogenous barrier it may also shift the barrier downwards further contributing to increase option value. Ericsson and Reneby (2003) consider the case where default is triggered either by the barrier or the inability to repay debt at maturity. Realdon (2003) extend Toft and Prucyk (1997) to the case where equity retains value even after assets hitting the barrier and shows that this feature can be very relevant for the pricing of out of the money put options. This thesis aims to help closing the identified gap.

[14] According to Leland the probability of investment grade firms jumping directly to default is low. See http://www.haas.berkeley.edu/groups/finance/WP/LECTURE2.pdf.

[15] General formulas for computing CDS spreads with any seniority structure are rarely provided in published papers. These can also be very relevant for model calibration.

2. The model

2.1. The asset process

Consider a firm that holds a single project that generates a certain amount of cash flow according to the following Lévy process under the physical measure $\mathbb{P}$:

$$\frac{d\delta_t}{\delta_t} = \mu_\delta dt + \sigma dW_t{}^{\mathbb{P}} - j dN_t^{\mathbb{P}}, \tag{2.1}$$

where μ_δ is the instantaneous growth rate of the project cash flows (exogenously determined), σ is the instantaneous volatility of the cash flow growth rate, $\left\{W_t^{\mathbb{P}}, t \geq 0\right\}$ is a standard Wiener process, j is the relative cash flow change when a sudden jump occurs and finally $\left\{N_t^{\mathbb{P}}, t \geq 0\right\}$ is a Poisson process with hazard rate λ. j is a constant meaning that jumps have fixed size. It is assumed that $W_t{}^{\mathbb{P}} \perp N_t^{\mathbb{P}}$ and $0 < j < 1$ meaning that the cash flow decreases after the sudden jump but remains non-negative. δ_0 is the cash flow of the firm at time 0, the initial time of the process. δ_t is interpreted as the cash flow from operations (CFO) at time t. However, one can also interpret δ_t as the EBITDA without any change in the model. The CFO excludes any capital outflows related with investments in capital assets. Capex is nevertheless needed to justify a positive growth rate in the project cash flows. For this reason capex suppliers are seen as claimants on the project.[1] The process is assumed to continue indefinitely. However, it is considered that either at time τ, the first time the process hits a lower boundary $\bar{\delta}$, or at time $\hat{\tau}$, when $N_{\hat{\tau}} = 1$, whichever occurs first, the firm ceases to exist and the project is sold to a competitor firm. This time is denoted as τ^{Solv}. As further discussed in the next section, distress costs are incurred (or not) depending on whether δ_t is below or above $\bar{\delta}$ at τ^{Solv}. Shareholders are assumed to manage this firm but they are not allowed to change the project risk profile nor the amount of liabilities and

[1]As further developed in Section 2.2, capex suppliers are seen as having a continuous fixed claim on the project. In practice, any fixed cost can be treated in this way as long as the state variable is defined in accordance. In contrast with Goldstein *et al.* (2001), this model can thus be applied even to firms with negative EBIT.

capex expenditure required to mantain the project. Agency problems are thus ignored. The solution to equation (2.1) is given by Proposition 1.

Proposition 1. *The process given by equation (2.1) has solution equal to*

$$\delta_t = \delta_0 e^{\left(\mu_\delta - 0.5\sigma^2\right)t + \sigma W_t^{\mathbb{P}} + \ln(1-j)N_t^{\mathbb{P}}}. \tag{2.2}$$

Proof. Consider $f(x) = \ln(x)$. Applying Lévy-Itô's lemma, we have

$$\begin{aligned} d\ln(\delta_t) &= \delta_t \mu_\delta \frac{1}{\delta_t} dt - \frac{\sigma^2 \delta_t^2}{2} \frac{1}{\delta_t^2} dt + \frac{1}{\delta_t} \delta_t \sigma dW_t^{\mathbb{P}} + \left[\ln\left((1-j)\delta_{t^-}\right) - \ln\left(\delta_{t^-}\right)\right] dN_t^{\mathbb{P}} \\ &= \left(\mu_\delta - 0.5\sigma^2\right) dt + \sigma dW_t^{\mathbb{P}} + \ln(1-j)\, dN_t^{\mathbb{P}}. \end{aligned} \tag{2.3}$$

Integrating,

$$\ln(\delta_t) = \ln(\delta_0) + \left(\mu_\delta - 0.5\sigma^2\right) t + \sigma W_t^{\mathbb{P}} + \ln(1-j) N_t^{\mathbb{P}}. \tag{2.4}$$

Taking the exponent in both sides of equation (2.4) one obtains equation (2.2).

□

Consider the existence of a security, A_t, capturing the market value of this project at each moment in time, whose value at time 0 and dynamics are given by Proposition 2.

Proposition 2. *The value of security A_t at the beggining of the process is given by*

$$A_0 = \frac{\delta_0}{\mu_A - g}, \tag{2.5}$$

where $g = \mu_\delta - \lambda j$ is the expected CFO growth rate including jumps. A_t dynamics are given by

$$\frac{dA_t}{A_t} = \mu_\delta dt + \sigma_A dW_t^{\mathbb{P}} - j_A dN_t^{\mathbb{P}}. \tag{2.6}$$

where σ_A and j_A correspond to σ and j, respectively.[2]

Proof. Since the δ process is assumed to live infinitely we have that

$$A_0 = E^{\mathbb{P}}\left[\left.\int_0^{+\infty} e^{-\mu_A s} \delta_s ds \right| \mathcal{F}_0\right], \tag{2.7}$$

[2]The two notations are used throughout this thesis.

where the discount rate μ_A is assumed to be constant for mathematical tractability.[3] $\mathcal{F}_t$ is the filtration generated by the δ_t process.

Now, rewrite δ_t process in martingale form

$$\frac{d\delta_t}{\delta_t} = (\mu_\delta - \lambda j)\, dt + \sigma dW_t{}^{\mathbb{P}} - jdM_t^{\mathbb{P}}, \tag{2.8}$$

where M_t is the compensated Poisson process associated with $N_t^{\mathbb{P}}$:

$$dM_t^{\mathbb{P}} = dN_t^{\mathbb{P}} - \lambda dt. \tag{2.9}$$

Solving equation (2.8) and substituting the solution in equation (2.7), one obtains

$$A_0 = \delta_0 \int_0^{+\infty} E^{\mathbb{P}} \left[e^{-\left(\mu_A - \mu_\delta + 0.5\sigma^2 + \lambda j\right)s + \sigma W_s^{\mathbb{P}} - \ln(1-j) M_s^{\mathbb{P}}} \middle| \mathcal{F}_0 \right] ds. \tag{2.10}$$

Taking the expectation and then rearranging

$$A_0 = \delta_0 \int_0^{+\infty} e^{-(\mu_A - \mu_\delta + \lambda j)s} ds. \tag{2.11}$$

Computing the integral on the right-hand side of equation (2.11) one arrives at equation (2.5).

Since A_t is a function of δ_t one can derive the dynamics of A_t by applying Lévy-Itô's lemma. Consider

$$f(x) = \frac{x}{\mu_A - g}.$$

Applying the lemma,

$$dA_t = \frac{1}{\mu_A - g} d\delta_t. \tag{2.12}$$

Substituting $d\delta_t$ according to equation (2.1),

$$dA_t = \frac{\delta_t}{\mu_A - g} \left[\mu_\delta dt + \sigma dW_t^{\mathbb{P}} - jdN_t^{\mathbb{P}} \right]. \tag{2.13}$$

Substituting δ_t by $A_t (\mu_A - g)$ one obtains equation (2.6).

[3] Accoring to Goldstein *et al.* (2001) this result can nevertheless be obtained in a model with a representative agent with a power-utility function.

□

In contrast with most corporate finance models, the growth rate of A_t is independent of the discount rate μ_A. Also, as time passes, A_t generates δ_t but its value does not decrease as a result of this. This occurs because A_t is not really a security, and thus, its value is not affected by any payout.[4] In line with Ericsson and Reneby (2002), A_t should be thought as a fictive security instead.[5] Something that is not really a security cannot be claimed to be traded as in most structural model setups. However, for illustrative purpose, this assumption will be adopted in the next paragraphs. It is then explained what is new when A_t is not traded.

Black and Scholes revolutionized the entire derivatives industry by showing that when the underlying asset follows a continuous process such as a geometric Brownian motion one could form a risk free portfolio using only the underlying asset and one derivative. Since this portfolio is instantaneoulsy risk free it should earn the risk free rate. This finding allowed them to derive a partial differential equation describing the dynamics of the option contract for the case of a geometric Brownian motion. They then solved this equation with the boundary condition associated with a European call option arriving at the famous Black-Scholes call option formula. A similar argument was followed by Merton (1973). In his paper, Merton considers forming a self-financing portfolio containing the common stock, the option and a riskless bond with the same maturity as the option contract. Since this portfolio requires zero investment one must have that under no arbitrage opportunities, the expected return on this portfolio must equal zero. The two approaches are equivalent with this section folowing the first approach.[6] When A_t dynamics are given by equation (2.6), it is evident

[4]In order to compute the project return one has to add δ_t to the asset growth rate

$$\begin{aligned}\frac{dA_t + \delta_t dt}{A_t} &= \frac{dA_t}{A_t} + \frac{\delta_t dt}{A_t}\\ &= \frac{dA_t}{A_t} + \frac{A_t\left(\mu_A - g\right)dt}{A_t}.\end{aligned}$$

Substituting $\frac{dA_t}{A_t}$ according to equation (2.6) and then cancelling μ_δ,

$$\begin{aligned}\frac{dA_t + \delta_t dt}{A_t} &= \mu_\delta dt + \sigma_A dW_t^{\mathbb{P}} - j_A dN_t^{\mathbb{P}} + (\mu_A - g)dt\\ &= \left(\mu_A + \lambda j_A\right)dt + \sigma_A dW_t^{\mathbb{P}} - j_A dN_t^{\mathbb{P}}\\ &= \mu_A dt + \sigma_A dW_t^{\mathbb{P}} - j_A dM_t^{\mathbb{P}}.\end{aligned}$$

[5]According to Ericsson and Reneby (2002) if this was not the case we would have two types of securities with conflicting claims over the same cash flows. On the one hand, we would have the holders of this asset. On the other hand there would be shareholders and debt holders that receive dividends and coupons based on the same cash flows.

[6]The contribution from each of these authors to the development of the so-called Black-Scholes-Merton

that it is impossible to form a risk free portfolio since there are two sources of randomness (i.e. the Wiener process and the Poisson process) but only one asset in addition to the risk free asset. The market is clearly incomplete. Nevertheless, one may complete the market by adding a second traded derivative contract.[7] I will now formalize this idea.

Consider a market composed by A_t, a risk free bond, B_t, and two traded derivatives (e.g two European options on the stock), which we will call χ_G and χ_S. For simplicity, assume that these derivatives have no payouts. Further, assume that the price processes of these instruments, $\Pi^G(t)$ and $\Pi^S(t)$, are given by some functions only of t and A_t with $\Pi^G(T) = G(T, A_T)$ and $\Pi^S(T) = S(T, A_T)$. Our goal is to find how these functions, denoted as G and S, must look like in order for the market $[A_t, B_t, \chi_G, \chi_S]$ to be arbitrage free.

Applying Itô's formula to $G(t, A_t)$ and $S(t, A_t)$ the price dynamics of χ_G and χ_S are obtained

$$\begin{aligned} \frac{d\Pi_t^G}{\Pi_t^G} &= \alpha_G dt + \sigma_G dW_t^{\mathbb{P}} - j_G dN_t^{\mathbb{P}} \\ \frac{d\Pi_t^S}{\Pi_t^S} &= \alpha_S dt + \sigma_S dW_t^{\mathbb{P}} - j_S dN_t^{\mathbb{P}}, \end{aligned} \tag{2.14}$$

where the drift terms are

$$\begin{aligned} \alpha_G &= \frac{G_t + \alpha_A A G_x + 0.5\sigma_A^2 A^2 G_{xx}}{G} \\ \alpha_S &= \frac{S_t + \alpha_A A S_x + 0.5\sigma_A^2 A^2 S_{xx}}{S}, \end{aligned} \tag{2.15}$$

the diffusion terms are

$$\begin{aligned} \sigma_G &= \frac{\sigma_A A G_x}{G} \\ \sigma_S &= \frac{\sigma_A A S_x}{S}, \end{aligned} \tag{2.16}$$

model is not completly clear from reading the cited papers as each paper benefited from the other. Taking Merton's words when interviewed in 2013, Black and Scholes "had the fundamental insight of undertaking a dynamic trading strategy in the underlying stock and the risk-free asset to hedge the systematic risk of an option position, and thereby create a portfolio of stock, risk-free asset, and option whose Capital Asset Pricing Model (CAPM) equilibrium expected return would equal the risk-free interest rate. In addition to naming it the Black-Scholes model, my most significant contribution to the model was to show that if you go to shorter and shorter trading intervals, their same dynamic strategy rules will eliminate all the risk, which has the implication that you have a way to synthesize the option, even if the option does not exist. By following a set of rules for trading the stock and the risk-free asset, I could create a portfolio that produced exactly the same payoff as the option."

[7]Notice, however, that in the general case where jump sizes are random we will need to add as many assets as the possible states of the jump size distribution in order to complete the market. For a continuous jump size distribution this is impossible, turning the model incomplete.

and, finally, the jump terms correspond to[8]

$$\begin{aligned} j_G &= \frac{G\left(A_{t^-}\right) - G\left(\left(1-j\right)A_{t^-}\right)}{G\left(A_{t^-}\right)} \\ j_S &= \frac{S\left(A_{t^-}\right) - S\left(\left(1-j\right)A_{t^-}\right)}{S\left(A_{t^-}\right)}. \end{aligned} \tag{2.17}$$

Following Björk (2009) notation, α_A indicates the drift of the asset under consideration minus any payout, k, which in the case of A_t is 0 and thus $\alpha_A = \mu_\delta$.

Consider a portfolio composed of A_t, χ_G and χ_S. Denoting the relative portfolio weights by w_A, w_G and w_S the following portfolio dynamics are obtained:

$$\begin{aligned} \frac{dV_t}{V_t} &= \left(w_A\left(\alpha_A + k_A\right) + w_G\left(\alpha_G + k_G\right) + w_S\left(\alpha_S + k_S\right)\right)dt + \left(w_A\sigma_A + w_G\sigma_G + w_S\sigma_S\right)dW_t^{\mathbb{P}} \\ &\quad - \left(w_A j_A + w_G j_G + w_S j_S\right)dN_t^{\mathbb{P}}, \end{aligned} \tag{2.18}$$

where $w_A + w_G + w_S = 1$ and $k_A = k_G = k_S = 0$.

One may then build a risk free portfolio by choosing portfolio weights so that

$$\begin{cases} w_A + w_G + w_S = 1 \\ w_A\sigma_A + w_G\sigma_G + w_S\sigma_S = 0 \\ w_A j_A + w_G j_G + w_S j_S = 0 \end{cases} \tag{2.19}$$

This system has the solution

$$\begin{aligned} w_A &= \frac{\sigma_G j_S - \sigma_S j_G}{\sigma_G j_S - \sigma_S j_G + \sigma_A j_G - \sigma_A j_S + \sigma_S j_A - \sigma_G j_A} \\ w_G &= \frac{\sigma_S j_A - \sigma_A j_S}{\sigma_G j_S - \sigma_S j_G + \sigma_A j_G - \sigma_A j_S + \sigma_S j_A - \sigma_G j_A} \\ w_S &= \frac{\sigma_A j_G - \sigma_G j_A}{\sigma_G j_S - \sigma_S j_G + \sigma_A j_G - \sigma_A j_S + \sigma_S j_A - \sigma_G j_A}. \end{aligned} \tag{2.20}$$

Since this portfolio is risk free, no arbitrage implies that

$$w_A\alpha_A + w_G\alpha_G + w_S\alpha_S = r, \tag{2.21}$$

where r denotes the risk free interest rate.

[8]As further discussed in Sections 3.3 and 4.2, in this model the value of the derivative contract after the jump is not simply $G\left(\left(1-j\right)A_{t^-}\right)$ because the liquidation of the firm leads to a reorganization of each claimant rights. In this section, however, A_t is treated independently of this fact.

Substituting the optimal weights in equation (2.21) and rearranging one obtains

$$\frac{\alpha_A - j_A \frac{\sigma_G(\alpha_S - r) - \sigma_S(\alpha_G - r)}{\sigma_G j_S - \sigma_S j_G} - r}{\sigma_A} = -\frac{j_G\left(\alpha_S - r\right) - j_S\left(\alpha_G - r\right)}{\sigma_G j_S - \sigma_S j_G}, \tag{2.22}$$

where the right-hand side of equation (2.22) does not depend on any parameters relative to the underlying. Rearranging in the same way for G and S one obtains similar expressions where the right-hand side does not depend on the correspondent parameters. In addition, it is possible to prove (see Appendix A.1) that under no arbitrage both the term on the right-hand side of equation (2.22) and the term multiplying the jump term on the left-hand side are equal for the two derivatives and the underlying. In Appendix A.2 it is shown that the term multiplying the jump is its hazard rate in the risk neutral measure. For this reason, it is denoted as $\overline{\lambda}$. This leads to the conclusion that under no arbitrage, and similar to the no jump case, the excess return (adjusted for the possibility of a jump) per unit of volatility risk must be the same in the underlying and in the derivatives:[9]

$$\frac{\alpha_A - j_A\overline{\lambda} - r}{\sigma_A} = \frac{\alpha_G - j_G\overline{\lambda} - r}{\sigma_G} = \frac{\alpha_S - j_S\overline{\lambda} - r}{\sigma_S}. \tag{2.23}$$

In addition, since α_A, σ_A, j_A, $\overline{\lambda}$ and r are assumed to be constant the excess return per unit of risk must be also constant across time.

Substituting α_A, α_G, σ_G and j_G on the first equality of equation (2.23), and rearranging, one obtains the following integro-differential equation (see Appendix A.1):[10]

$$G_t + \left(r + j_A\overline{\lambda}\right) AG_x + 0.5\sigma_A^2 A^2 G_{xx} - \overline{\lambda}\left[G\left(\left(1-j\right)A_{t^-}\right) - G\left(A_{t^-}\right)\right] - rG = 0. \tag{2.24}$$

The price of χ_G can then be found by solving equation (2.24) subject to

$$\Pi^G\left(T\right) = G\left(T, A_T\right). \tag{2.25}$$

As first noted by Duffie (1988), one may alternatively use Feynman-Kac theorem in order to obtain a stochastic representation formula:

$$G\left(t,x\right) = e^{-r(T-t)} E^{\mathbb{Q}}\left[G\left(T, A_T\right)\right], \tag{2.26}$$

[9] In the case where $k \neq 0$ we have to substitute α_A, α_G and α_S by μ_A, μ_G and μ_S.

[10] Notice that when $\overline{\lambda} = 0$ and the derivative contract has no explicit time dependence the term G_t vanishes and equation (2.24) becomes an ordinary differential equation (ODE). The resulting ODE is similar to equation 3 in Leland (1994a) except that here it is assumed that G has no payout, while in the case of Leland (1994a) the derivative is assumed to have a constant payout.

with the dynamics of A_t given by

$$\frac{dA_t}{A_t} = \left(r + j_A\overline{\lambda}\right) dt + \sigma_A dW_t^{\mathbb{Q}} - j_A dN_t^{\mathbb{Q}}. \tag{2.27}$$

The latter can also be written in martingale form as

$$\frac{dA_t}{A_t} = rdt + \sigma_A dW_t^{\mathbb{Q}} - j_A dM_t^{\mathbb{Q}}. \tag{2.28}$$

Equation (2.26) is the usual risk neutral valuation formula applied to derivative χ_G. It states that the market price of χ_G is simply the expected value of its payoff at time T (under some probability measure usually denoted as $\mathbb{Q}$) discounted at the risk free rate. Since this probability measure is the one used throughout this thesis it is convenient to give it a formal definition.

Definition 1. *The probability measure $\mathbb{Q}$ is a probability measure equivalent to the original probability measure $\mathbb{P}$ such that the discounted value of any asset payoffs (i.e. the asset price plus any dividends expressed in units of the numeraire money market account) is a martingale. Mathematically, for $T \geq t$ and considering a generic financial asset X one may write*

$$\frac{X_t}{\beta_t} = E^{\mathbb{Q}}\left[\left.\frac{X_T}{\beta_T}\right| \mathcal{F}_t\right], \tag{2.29}$$

where $\beta_t = e^{rt}$ is the value of the money market account at time t.

Substituting the term correspondent to the value of the money market account on equation (2.29) one obtains

$$E^{\mathbb{Q}}\left[\left. X_T \right| \mathcal{F}_t\right] = e^{r(T-t)} X_t, \tag{2.30}$$

and it is clear that measure $\mathbb{Q}$ assumes that any financial asset generates a rate of return equal to the risk free interest rate.

The derivation just done is very useful to show how equation (2.26) emerges. As an alternative, one can start with equation (2.26) and use the Girsanov theorem for jump diffusion processes in order to change the probability measure to measure $\mathbb{Q}$. This is known as the martingale approach and it is explored in Appendix A.2. The partial differential equation (PDE) approach and the martingale approach are broadly seen as equivalent, though their equivalence have only been proved under some conditions on model parameters, which the model here considered fulfills (Heath and Schweizer (2000)).[11] Once the characteristics of each derivative are taken into account, the PDE approach has the advantage of simultaneously providing the reader the so-called hedging portfolio. The latter can be very useful if one wants to use this model for capital structure arbitrage.

Up to now it has been assumed that A_t is traded. What if it is not traded? Is it still possible to price derivative contracts using no-arbitrage arguments? Can we obtain an equation similar to equation (2.26)? The answer to these questions is yes. However, as pointed in Björk (2009), arbitrage pricing is always a case of pricing a derivative in terms of the price of some underlying asset. If A_t is not traded we do not have enough underlying assets. One can, however, add a third derivative. Consider χ_F with price process $\Pi(t)^F = F(t, A_t)$ and with $\Pi^F(T) = F(A_T)$. For simplificity, let's assume that χ_F has no dividends. Further assume that A_t is observable.[12] As for other derivatives we have that

$$\begin{aligned} \alpha_F &= \frac{F_t + \alpha_A A F_x + 0.5\sigma_A^2 A^2 F_{xx}}{F} \\ \sigma_F &= \frac{\sigma_A A F_x}{F} \\ j_F &= \frac{F(A_{t-}) - F((1-j)A_{t-})}{F(A_{t-})}. \end{aligned} \tag{2.31}$$

Proceeding as previously, it is possible to find risk weights so that we are again able to build a risk free portfolio. Since this portfolio is risk free, no arbitrage implies that

$$w_F\alpha_F + w_G\alpha_G + w_S\alpha_S = r, \tag{2.32}$$

Substituting the optimal weights into equation (2.32) one obtains

$$\overline{m}_t = \frac{\alpha_F - j_F\overline{\lambda} - r}{\sigma_F} = \frac{\alpha_G - j_G\overline{\lambda} - r}{\sigma_G} = \frac{\alpha_S - j_S\overline{\lambda} - r}{\sigma_S}. \tag{2.33}$$

Equation (2.33) leads to two very important conclusions. First, though the underlying asset is not traded, the excess return adjusted for the jump per unit of volatility risk must be the same for all derivatives. This is a consistency condition in order to avoid arbitrage

[11] In trueth, based on the first fundamental theorem of asset pricing, equation (2.26) is valid as long as arbitrage opportunities are ruled out. However, according to the second fundamental theorem of asset pricing, there is no unique equivalent martingale measure if the market is not complete. Please see Harrison and Kreps (1979), Harrison and Pliska (1981) and Harrison and Pliska (1983).

[12] This thesis leaves unanswered the delicate question of whether the asset value is *de facto* observable. The latter implies that, in addition to r and δ_0, agents must also be aware of $\overline{m}$, μ_δ, σ, λ and j.

opportunities between the derivatives. Second, though under no arbitrage the excess return adjusted for the jump per unit of volatility risk must be equal for all derivatives, it does not need to be constant as in the case where A_t is assumed to be traded. In turn, $\overline{m}_t$ is a stochastic process, which is not specified within this model. The only thing we know is that $\overline{m}_t$ is some function of t and A_t, whose value affects the price of all derivative contracts on the firm. In order to determine $\overline{m}_t$ one needs an equilibrium asset pricing model where agents preferences are modelled. As argued in footnote 2, one convenient assumption is that asset prices are determined under a representative agents model with a power-utility function. In this case, $\overline{m}_t = \overline{m}$ is in fact constant, which considerably simplifies the analysis. All derivations taken in this thesis hereafter assume that $\overline{m}$ is constant.

Substituting α_G, σ_G and j_G in equation (2.33) as previously and rearranging, one arrives at

$$G_t + (\mu_\delta - \overline{m}\sigma) AG_x + 0.5\sigma_A^2 A^2 G_{xx} - rG - \overline{\lambda}\left[G\left((1-j_A)A_{t^-}\right) - G\left(A_{t^-}\right)\right] = 0. \quad (2.34)$$

Equation (2.34) is similar to equation (2.24) but with $\mu_\delta - \overline{m}\sigma_A$ instead of $r + j_A\overline{\lambda}$. Again, one can use Feynman-Kac theorem to obtain a stochastic representation formula

$$G(t,x) = e^{-r(T-t)} E^{\mathbb{Q}}\left[G(T, A_T)\right]. \quad (2.35)$$

This time the dynamics of A_t, under the martingale measure $\mathbb{Q}$, are given by

$$\frac{dA_t}{A_t} = (\mu_\delta - \overline{m}\sigma_A)\, dt + \sigma_A dW_t^{\mathbb{Q}} - j_A dN_t^{\mathbb{Q}}, \quad (2.36)$$

or equivalently, using the compensated Poisson process,

$$\frac{dA_t}{A_t} = \left(\mu_\delta - \overline{\lambda} j_A - \overline{m}\sigma_A\right) dt + \sigma_A dW_t^{\mathbb{Q}} - j_A dM_t^{\mathbb{Q}}, \quad (2.37)$$

Solving equation (2.37) as in Proposition 1, one obtains

$$A_t = A_0 e^{\left(\mu_\delta - \overline{\lambda} j_A - \overline{m}\sigma_A - 0.5\sigma_A^2\right)t + \sigma_A W_t^{\mathbb{Q}} + \ln(1-j_A) M_t^{\mathbb{Q}}}. \quad (2.38)$$

Equation (2.37) is equal to equation (3) in Goldstein *et al.* (2001) when $\lambda = 0$. While Goldstein *et al.* (2001) arrives at equation (2.37) following a martingale approach, a partial differential equation approach is followed in this thesis. The martingale approach is useful to clarify one aspect that is not completely clear so far, notably, the relation between μ_A ,

$\overline{m}$ and $\overline{\lambda}$. From the first fundamental theorem of asset pricing it is known that

$$A_0 = E^{\mathbb{P}}\left[\int_0^{+\infty} e^{-\mu_A s}\delta_s ds \middle| \mathcal{F}_0\right] = E^{\mathbb{Q}}\left[\int_0^{+\infty} e^{-rs}\delta_s ds \middle| \mathcal{F}_0\right], \tag{2.39}$$

where

$$\frac{d\delta_t}{\delta_t} = (\mu_\delta - \overline{m}\sigma)dt + \sigma dW_t^{\mathbb{Q}} - jdN_t^{\mathbb{Q}}. \tag{2.40}$$

Solving the above stochastic differential equation and substituting in equation (2.39), one obtains

$$A_0 = \frac{\delta_0}{r + \overline{m}\sigma + \overline{\lambda}j - \mu_\delta}. \tag{2.41}$$

Summing and subtracting λj in the denominator, one arrives at

$$A_0 = \frac{\delta_0}{r + \overline{m}\sigma + \left(\overline{\lambda} - \lambda\right) j - g}. \tag{2.42}$$

Thus, $\mu_A = r + \overline{m}\sigma + \left(\overline{\lambda} - \lambda\right) j$ meaning that the expected return on the project can be decomposed in three components, the risk free rate, a premium $\overline{m}$ per unit of volatility risk and a premium $\left(\overline{\lambda} - \lambda\right)$ per unit of jump risk. As will be shown in Sections 3.3 and 4.2 a similar result is obtained for equity and debt.

Before ending this section, it is important to compare the equations obtained here with those obtained in the literature when the state variable is the market value of the firm's asset. Except for the jump term, equation (2.28) is equal to equation 1 in Leland (1994a) after changing to the risk neutral measure. The latter assumes that the firm generates a return of μ_A and that shareholders pay interest from their own pockets. Since μ_A is positive this model implicitly assumes that the firm is expected to deleverage as time goes by.[13] By assuming that A_t is a traded security, however, this ends up having no effect on equity and debt valuations. Decreasing leverage and shareholders paying debtholders continuously are unreasonable assumptions, though. In order to overcome these issues, several models assume that a constant fraction k of A_t is continuously sold with shareholders putting any additional value in case it is optimal to do it. When k is below (above) μ_A the firm is implicitly assumed to deleverage (increase leverage). Though k ends up affecting debt and

[13] The same occurs in the model presented in this thesis whenever $\mu_\delta > 0$.

equity valuations through the probability of default, again the assumption that A_t is traded turns μ_A irrelevant for pricing purposes.

Though challenging, leaving the hypothesis of asset tradeability seems thus wise. In this case it can be shown that μ_A and $\overline{m}$ affect equity and debt valuations. The higher is μ_A the lower the probability of default and the higher the debt value. Computing equity by subtracting debt value from asset value as it is done in most papers may lead to counter-intuitive conclusions, though. For instance, the higher is μ_A, the lower is the probability of default of the firm, the higher is debt value for the same A_t and thus the lower is the equity value. One alternative is to price equity directly as the present value of all future dividends. Doing this there is no guarantee, however, that the asset value equals the sum of equity and debt value unless this is imposed during estimation. The latter occurs because μ_A and A_t are not set jointly in the same model. It adds that the joint estimation of μ_A and A_t is particularly difficult. A convenient and reasonable hypothesis in this case is to assume that $k = \mu_A$, which means that the firm distributes all its returns. In this case, the asset drift becomes 0 under the physical measure and debt value ends up not depending either on μ_A and k even when the asset is non-tradable. Equity and debt continue to depend on $\overline{m}$, though. It occurs that the asset process obtained in this case is exactly equal to equation (2.37) when $\mu_\delta = 0$ and $\overline{\lambda} = 0$. This is not a surprising result since the way the model is set we are intrisically assuming that all the cash flow is distributed, which is tantamount to say that $\mu_A = k$. The approach followed in this thesis brings, however, two major advantages vis-a-vis simply assuming that the latter terms cancel out. First, by setting A_t as a function of δ_t, A_t becomes a function of something that is observed. Second, A_t is now computed in a way consistent with $\overline{m}$.[14] Additionaly, this model setup allow us to better value the effect of taxes, distress costs and fixed costs. These features are discussed in the next section.

2.2. Contingent claimants and the default barrier

Traditionally, structural models of corporate liabilities assume A_t as the state variable. Taking Leland (1994a) as reference, the value of the firm then corresponds to A_t plus the tax shield arising from debt minus distress costs. As pointed by Goldstein *et al.* (2001) this approach is inconsistent in the sense that an increase in the tax rate leads to an increase in the value of the firm. In addition, the assumption that government's revenue is a constant share

[14]Someone that does not agree with equation (2.7) can still estimate A_t as a latent process. In this case, one can ignore Chapter 3 and compute equity by difference.

of the coupons paid, and thus constant, is not consistent with the empirical observation. Instead, in this model it is considered that the cash flows generated by the project have five claimants, notably, shareholders, debtholders, the government, capex suppliers and distress costs. Each of these claimants have very different payoffs depending on the project cash flows and the firm capacity to stay in operation. As long as the firm mantains its activity shareholders are considered to receive the cash flows generated by the project minus any payments to debtholders, government and capex suppliers. Payments to debtholders and capex suppliers are fixed.[15] The government receives a fixed share of shareholders payoff, which is not constant, and a fixed share of debtholders' coupons, which are constant. As established in most tax regimes, shareholders returns are taxed twice (first at the firm level, $\bar{t}^{Corp}$, and then at the investor level, $\bar{t}^{Div}$). As referred in Section 2.1 the firm is closed at time $\tau^{Solv} = Min\{\tau, \hat{\tau}\}$ where τ is the first passage time of δ_t through $\bar{\delta}$ and $\hat{\tau}$ is the time of the first jump. τ can also be defined as the first hitting time of A_t through $\bar{v}$, which is the project value associated with $\bar{\delta}$. Once the firm activity is over, two cases may occur depending on whether $A_{\tau^{Solv}} \geq \bar{v}$. If $A_{\tau^{Solv}} \geq \bar{v}$, which can only occur when the collapse of the firm is triggered by a sudden jump, distress costs are not incurred and the firm receives $\beta^{Sold} A_{\tau^{Solv}}$. β^{Sold} correspond to equity and debtholders share on the project when this is sold. One possibility is to assume that this corresponds to their current share. In this case β^{Sold} can be easily found iteratively. The remaining $\left(1 - \beta^{Sold}\right) A_{\tau^{Solv}}$ is considered to belong to external claimants on the project (see Chapter 5). The usual pecking order then applies to $\beta^{Sold} A_{\tau^{Solv}}$ meaning that debtholders receive the minimum of nominal debt and the recovered value. Shareholders only receive something if $\beta^{Sold} A_{\tau^{Solv}} \geq L$. If $A_{\tau^{Solv}} \leq \bar{v}$, the firm is considered to be economically non-viable and distress costs are incurred. This correspond to costs with lawyers and value destruction caused by fire sales and loss of intangible value. In this case, the project is again sold for $A_{\tau^{Solv}}$ but only $\beta^{Bank} A_{\tau^{Solv}}$ accrues to the firm. It is assumed that $0 \leq \beta^{Bank} \leq \beta^{Sold}$. The difference between β^{Sold} and β^{Bank} times the project value at the time the firm closes corresponds to distress costs. External claimants in this case hold $\left(1 - \beta^{Sold}\right) A_{\tau^{Solv}}$ plus these distress costs. β^{Bank} may be estimated using market prices.

As already explained, shareholders receive at each moment in time the difference between the CFO generated by the project and their duties towards other claimants. While government claim is assumed to fluctuate with the project returns, the costs with debtholders and

[15] One may claim that capex is not a fixed cost since the firm may adjust it during a crisis period. This shall affect the operating cash flows growth rate and the project value, though. As referred in footnote 1 of this chapter, as long as one defines the state variable in accordance one can consider any type of fixed costs.

capex suppliers are considered fixed meaning that whenever the cash flows generated are not enough, shareholders have to inject capital in the firm in order to avoid default. The obvious question is for how long are they willing to do it. By answering this question one can arrive at an endogenous default barrier. As first showed by Black and Cox (1976) the answer to this question is given by solving a standard optimal stopping time problem. As explained in Leland (1994a) this is the case because the above described liability structure is compatible with an environment where debt securities are time independent. The latter is essential to arrive at a closed form solution for an endogenous default barrier resulting from an optimal stopping time problem. According to Leland (1994b), this occurs only in three cases, notably, when debt is perpetual, when debt is retired at a constant rate and continuously replaced by new debt so that the cash payouts are constant and when a firm continuously sells a constant amount of new debt with the same maturity, which is then redeem at par upon maturity.[16] In this thesis, the first case is followed. Nevertheless, the computation of bond prices with arbitrary finite maturity is straightforward, based on the formulae given in Chapter 4.[17] It is assumed that each type of debt security is initially sold at par entitling its owner to a certain coupon, which is pre-determined according to the firm's risk. The latter is in contrast with Leland (1994a), which assumes the that coupon payments are chosen by the shareholders when optimizing the firm's capital structure. The coupon level is thus seen as a synonym of debt in his model. content

2.3. The distribution functions

Chapters 3 to 7 of this thesis derive the price of equity, debt, credit default swaps and European-style options on stocks following a probabilistic approach. This is done by integrating the joint density of the asset process, the time to hit the barrier and the jump time appropriately. Since the Brownian motion is assumed to be independent from the Poisson process, the joint distribution of A_t with $\tau \geq t$ is treated separately from the jump distribution. The derivation of these probabilities is presented in Sections 2.3.1 and 2.3.2.

[16]Décamps and Villeneuve (2014) show, however, that the last two cases do not correspond to standard stopping time problems and that extra assumptions are required to ensure that the smooth pasting condition leads to an unique optimal stopping time.

[17]In this case one must be aware that the usual balance sheet identity is not respected anymore as the model becomes internally inconsistent. In addition, the endogenous barrier derived in Section 3.4 is not compatible with an enviroment with rollover debt.

2.3.1. The joint distribution of A_t and $\tau \geq t$

In this section, the joint distribution of A_t and $\tau \geq t$ is derived ignoring the possibility of jumps. Though A_t follows a geometric Brownian motion, it is instructive to start with the simpler case where X_t is a stochastic variable that follows an arithmetic Brownian motion. In this case, the joint distribution of X_t with its minimum above y is given by Proposition 3.

Proposition 3. *If* $dX_t = vdt + \sigma dW_t^{\mathbb{Q}}$ *where* $v \in \mathbb{R}$, $\sigma > 0$, *and* $\left\{W_u^{\mathbb{Q}}, u \geq 0\right\}$ *is a* $\mathbb{Q}$*-measured standard Brownian motion such that* $W_0^{\mathbb{Q}} = 0$, *then for* $x \geq y$

$$\begin{aligned}\mathbb{Q}\left(X_t^{\mathbb{Q}} \leq x, \inf_{0<u\leq t} \left(X_u^{\mathbb{Q}}\right) \geq y \middle| \mathbb{F}_0\right) &= N\left(\frac{-y+vt}{\sigma\sqrt{t}}\right) - e^{\frac{2vy}{\sigma^2}} N\left(\frac{y+vt}{\sigma\sqrt{t}}\right) \\ &\quad - N\left(\frac{-x+vt}{\sigma\sqrt{t}}\right) + e^{\frac{2vy}{\sigma^2}} N\left(\frac{-x+2y+vt}{\sigma\sqrt{t}}\right)\end{aligned} \tag{2.43}$$

and

$$\mathbb{Q}\left(X_t^{\mathbb{Q}} \in dx, \inf_{0<u\leq t} \left(X_u^{\mathbb{Q}}\right) \geq y \middle| \mathcal{F}_0\right) = n\left(x; vt; \sigma\sqrt{t}\right) - e^{2\frac{vy}{\sigma^2}} n\left(x; 2y+vt; \sigma\sqrt{t}\right), \tag{2.44}$$

where $N(.)$ *and* $n(.)$ *stand for the standard normal distribution and density functions, respectively.*

Proof. See Appendix A.3. □

Ignoring the Poisson process in equation (2.36) and applying Itô's formula to $f(x) = \ln\left(\frac{x}{A_0}\right)$ one obtains

$$d\ln\left(\frac{A_t}{A_0}\right) = \left(\mu_\delta - \overline{m}\sigma_A - 0.5\sigma_A^2\right)dt + \sigma_A dW_t^{\mathbb{Q}}. \tag{2.45}$$

Taking Proposition 3 with $X_t = \ln\left(\frac{A_t}{A_0}\right)$, $y = \ln\left(\frac{\bar{v}}{A_0}\right)$ and $\bar{v} < A_0$ and further noting that $\mathbb{Q}\left(\ln\left(\frac{A_t}{A_0}\right) < x, \inf_{0<u\leq t}\left(\ln\left(\frac{A_u}{A_0}\right)\right) \geq \ln\left(\frac{\bar{v}}{A_0}\right) \middle| \mathcal{F}_0\right) = \mathbb{Q}\left(\ln\left(\frac{A_t}{A_0}\right) < x, \tau \geq t \middle| \mathcal{F}_0\right)$ thus leads to

$$\mathbb{Q}\left(\ln\left(\frac{A_t}{A_0}\right) \leq x, \tau \geq t \middle| \mathcal{F}_0\right) = N\left(\frac{-\ln(R) + v^\star t}{\sigma_A\sqrt{t}}\right) - e^{\frac{2v^\star \ln(R)}{\sigma_A^2}} N\left(\frac{\ln(R) + v^\star t}{\sigma_A\sqrt{t}}\right)$$
$$- N\left(\frac{-x + v^\star t}{\sigma_A\sqrt{t}}\right) + e^{\frac{2v^\star \ln(R)}{\sigma_A^2}} N\left(\frac{-x + 2\ln(R) + v^\star t}{\sigma_A\sqrt{t}}\right)$$
$$= N\left(-\frac{\ln(R) - v^\star t}{\sigma_A\sqrt{t}}\right) - R^{2a} N\left(\frac{\ln(R) + v^\star t}{\sigma_A\sqrt{t}}\right)$$
$$- N\left(-\frac{x - v^\star t}{\sigma_A\sqrt{t}}\right) + R^{2a} N\left(\frac{-x + 2\ln(R) + v^\star t}{\sigma_A\sqrt{t}}\right) \tag{2.46}$$

and

$$\mathbb{Q}\left(\ln\left(\frac{A_t}{A_0}\right) \in dx, \tau \geq t \middle| \mathcal{F}_0\right) = n\left(x; v^\star t; \sigma_A\sqrt{t}\right) - R^{2a} n\left(x; 2\ln(R) + v^\star t; \sigma_A\sqrt{t}\right), \tag{2.47}$$

where $v^\star = \mu_\delta - \overline{m}\sigma_A - 0.5\sigma_A^2$, $R = \frac{\bar{v}}{A_0}$ and $a = \frac{v^\star}{\sigma^2}$.

In Chapters 3 to 7 we are mostly interested in the asset distribution rather than the asset return distribution. It is well known however that whenever asset returns follow an arithmetic Brownian motion, the asset itself follows a geometric Brownian motion, whose joint distribution with $\tau \geq t$ is presented in Proposition 4.

Proposition 4. *If $\frac{dA_t}{A_t} = vdt + \sigma_A dW_t^{\mathbb{Q}}$ where $v \in \mathbb{R}$, $\sigma_A > 0$, and $\left\{W_u^{\mathbb{Q}}, u \geq 0\right\}$ is a $\mathbb{Q}$-measured standard Brownian motion such that $W_0^{\mathbb{Q}} = 0$, then*

$$\mathbb{Q}\left(A_t < x, \tau \geq t \left| \mathcal{F}_0\right.\right) = N\left(h1\left(x,t\right)\right) + R^{2a} N\left(h_2\left(x,t\right)\right) - N\left(h_1\left(\bar{v},t\right)\right) - R^{2a} N\left(h_2\left(\bar{v},t\right)\right) \tag{2.48}$$

$$\begin{aligned}\mathbb{Q}\left(A_t \in dx, \tau \geq t \left| \mathcal{F}_0\right.\right) &= \frac{d}{dx} N\left(h_1\left(x,s\right)\right) + R^{2a}\frac{d}{dx} N\left(h_2\left(x,s\right)\right) \\ &= \frac{1/x}{\sigma_A\sqrt{t}}\left[n\left(h_1\left(x,t\right)\right) - R^{2a} n\left(h_2\left(x,t\right)\right)\right],\end{aligned} \tag{2.49}$$

and

$$\mathbb{Q}\left(\tau \in du \left| \mathcal{F}_0\right.\right) = \frac{d}{du} N\left(h_1\left(\bar{v},u\right)\right) + R^{2a}\frac{d}{du} N\left(h_2\left(\bar{v},u\right)\right) \tag{2.50}$$

where $h_1(z,s) = \frac{\ln\left(\frac{z}{A}\right)-v^{\star}s}{\sigma_A\sqrt{s}}$ *and* $h_2(z,s) = \frac{\ln\left(R\frac{\bar{v}}{z}\right)+v^{\star}s}{\sigma_A\sqrt{s}}$.

Proof. The proof is based on equation (2.47) and the total probability theorem.

By integrating the density function (2.47) on all possible values of X_t above $\ln\left(\frac{x}{A_0}\right)$ one obtains the probability of X_t being above $\ln\left(\frac{x}{A_0}\right)$ and simultaneoulsy the process not hitting $\overline{v}$ up to time t. This equals

$$\begin{aligned}\mathbb{Q}\left(A_t \geq x, \tau \geq t \middle| \mathcal{F}_0\right) &= \int_{ln\left(\frac{x}{A_0}\right)}^{+\infty} n\left(x; v^{\star}t; \sigma_A\sqrt{t}\right) - R^{2a} n\left(x; 2\ln(R) + v^{\star}t; \sigma_A\sqrt{t}\right) \\ &= N\left(-\frac{\ln\left(\frac{x}{A_0}\right) - v^{\star}t}{\sigma_A\sqrt{t}}\right) - R^{2a} N\left(-\frac{\ln\left(\frac{x}{A_0}\right) - 2\ln(R) - v^{\star}t}{\sigma_A\sqrt{t}}\right) \\ &= 1 - N\left(h_1(x,t)\right) - R^{2a} N\left(h_2(x,t)\right).\end{aligned} \tag{2.51}$$

Notice that when $x = \bar{v}$, this is basically the probability of the process not hitting the barrier up to time t

$$\begin{aligned}\mathbb{Q}\left(\tau \geq t \middle| \mathcal{F}_0\right) &= \mathbb{Q}\left(A_t \geq \bar{v}, \tau \geq t \middle| \mathcal{F}_0\right) \\ &= 1 - N\left(h_1(\bar{v},t)\right) - R^{2a} N\left(h_2(\bar{v},t)\right).\end{aligned} \tag{2.52}$$

Using the total probability theorem

$$\mathbb{Q}\left(\tau \geq t \middle| \mathcal{F}_0\right) = \mathbb{Q}\left(A_s \leq x, \tau \geq t \middle| \mathcal{F}_0\right) + \mathbb{Q}\left(A_s > x, \tau \geq t \middle| \mathcal{F}_0\right). \tag{2.53}$$

And thus

$$\mathbb{Q}\left(A_s \leq x, \tau \geq t \middle| \mathcal{F}_0\right) = \mathbb{Q}\left(\tau \geq t \middle| \mathcal{F}_0\right) - \mathbb{Q}\left(A_s > x, \tau \geq t \middle| \mathcal{F}_0\right). \tag{2.54}$$

Substituting equations (2.51) and (2.52) on equation (2.54), and rearranging, equation (2.48) is obtained. Differentiating leads to

$$\mathbb{Q}\left(A_t \in dx, \tau \geq t \middle| \mathcal{F}_0\right) = \frac{d}{dx} N\left(h_1(x,t)\right) + R^{2a}\frac{d}{dx} N\left(h_2(x,t)\right) \tag{2.55}$$

$$= \frac{1/x}{\sigma_A\sqrt{t}}\left[n\left(h_1(x,t)\right) - R^{2a} n\left(h_2(x,t)\right)\right]. \tag{2.56}$$

The probability of the process hitting the barrier up to time t is given by

$$\begin{aligned} \mathbb{Q}\left(\tau < t \middle| \mathcal{F}_0\right) &= 1 - \mathbb{Q}\left(\tau \geq t \middle| \mathcal{F}_0\right) \\ &= N\left(h_1\left(\bar{v}, t\right)\right) + R^{2a} N\left(h_2\left(\bar{v}, t\right)\right). \end{aligned} \tag{2.57}$$

Taking the derivative of equation (2.57) one obtains equation (2.50).

□

2.3.2. The jump time distribution

The probability distribution that describes the time between events in a Poisson process is the exponential distribution, whose density and distribution functions are given in Proposition 5.

Proposition 5. *The density and distribution functions of an exponentially distributed random variable correspond, respectively, to*

$$\mathbb{Q}\left(\hat{\tau} \in d\hat{u} \left| \mathcal{F}_0\right.\right) = \overline{\lambda} e^{-\overline{\lambda}\hat{u}} \tag{2.58}$$

and

$$\mathbb{Q}\left(\hat{\tau} \leq \hat{u} \left| \mathcal{F}_0\right.\right) = 1 - e^{-\overline{\lambda}\hat{u}}. \tag{2.59}$$

Following from equation (2.59), the probability of a jump not occurring up to time $\hat{u}$ equals

$$\mathbb{Q}\left(\hat{\tau} \geq \hat{u} \left| \mathcal{F}_0\right.\right) = e^{-\overline{\lambda}\hat{u}}. \tag{2.60}$$

2.4. Further mathematical tools

Throughout this thesis several integrals involving the standard Normal distribution function are computed. This section presents three mathematical results that are recurrently used for this purpose.

The first integral we are interested is $\int_{z_1}^{z_2} x \frac{d}{dx} N\left(h_1\left(x, s\right)\right) dx$. This appears several times and can be computed using Proposition 6 below.

Proposition 6. *For any $s > 0$ we have, for every $z_1 > 0$ and $z_2 > 0$,*

$$\int_{z_1}^{z_2} x \frac{d}{dx} N\left(\frac{\ln(x) - \ln(A) - v^\star s}{\sigma_A \sqrt{s}}\right) dx = A e^{(v^\star + 0.5\sigma_A^2)s} \left[N\left(h_3\left(z_2, s\right)\right) - N\left(h_3\left(z_1, s\right)\right)\right], \quad (2.61)$$

where $h_3\left(z, s\right) = \frac{\ln\left(\frac{z}{A}\right) - (v^\star + \sigma_A^2)s}{\sigma_A \sqrt{s}}$.

Proof. For any $a \in \mathbb{R}$ and $b > 0$ we have, for every y>0,

$$\begin{aligned} \int_0^y x \frac{d}{dx} N\left(\frac{\ln(x) + a}{b}\right) dx &= \int_0^y x n\left(\frac{\ln(x) + a}{b}\right) \frac{1}{xb} dx \\ &= \frac{1}{b} \int_0^y n\left(\frac{\ln(x) + a}{b}\right) dx \\ &= \frac{1}{b} \int_0^y \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln(x)+a}{b}\right)^2} dx. \end{aligned} \quad (2.62)$$

Taking $w = \ln\left(x\right)$,

$$\begin{aligned} \int_0^y x \frac{d}{dx} N\left(\frac{\ln(x) + a}{b}\right) dx &= \frac{1}{b} \int_{-\infty}^{\ln(y)} \frac{1}{\sqrt{2\pi}} e^w e^{-\frac{1}{2}\left(\frac{w+a}{b}\right)^2} dw \\ &= \int_{-\infty}^{\ln(y)} \frac{1}{\sqrt{2\pi b^2}} e^{-\frac{1}{2}\left(\frac{w^2 + 2aw + a^2 - 2b^2 w}{b^2}\right)} dw \\ &= e^{-\frac{1}{2}\left(\frac{a^2 - \left(b^2 - a\right)^2}{b^2}\right)} \int_{-\infty}^{\ln(y)} \frac{1}{\sqrt{2\pi b^2}} e^{-\frac{1}{2}\left(\frac{w^2 - 2w\left(b^2 - a\right) + \left(b^2 - a\right)^2}{b^2}\right)} dw \\ &= e^{-\frac{1}{2}\left(\frac{a^2 - b^4 + 2ab^2 - a^2}{b^2}\right)} \int_{-\infty}^{\ln(y)} \frac{1}{\sqrt{2\pi b^2}} e^{-\frac{1}{2}\frac{\left(w - b^2 + a\right)^2}{b^2}} dw \\ &= \mathrm{e}^{0.5b^2 - a} N\left(\frac{\ln(y) + a - b^2}{b}\right). \end{aligned} \quad (2.63)$$

Taking $a = -\ln(A) - v^\star s$ and $b = \sigma_A\sqrt{s}$, we obtain

$$\begin{aligned}\int_{z_1}^{z_2} x\frac{d}{dx}N\left(\frac{\ln(x)-\ln(A)-v^\star s}{\sigma_A\sqrt{s}}\right)dx &= e^{0.5\sigma_A^2 s+\ln(A)+v^\star s}\left[N\left(\frac{\ln(z_2)-\ln(A)-v^\star s-\sigma_A^2 s}{\sigma_A\sqrt{s}}\right)\right.\\ &\left.-N\left(\frac{\ln(z_1)-\ln(A)-v^\star s-\sigma_A^2 s}{\sigma_A\sqrt{s}}\right)\right]\\ &= e^{\ln(A)+(v^\star+0.5\sigma_A^2)s}\left[N\left(\frac{\ln(z_2/A)-(v^\star+\sigma_A^2)s}{\sigma_A\sqrt{s}}\right)\right.\\ &\left.-N\left(\frac{\ln(z_1/A)-(v^\star+\sigma_A^2)s}{\sigma_A\sqrt{s}}\right)\right]\\ &= Ae^{(v^\star+0.5\sigma_A^2)s}\left[N\left(h_3\left(z_2,s\right)\right)-N\left(h_3\left(z_1,s\right)\right)\right].\end{aligned} \tag{2.64}$$

□

We are also often interested in $\int_{z_1}^{z_2} x\frac{d}{dx}N\left(h_2\left(x,s\right)\right)dx$, which can be computed using Proposition 7.

Proposition 7. *For any $s > 0$ we have, for every $z_1 > 0$ and $z_2 > 0$,*

$$\int_{z_1}^{z_2} x\frac{d}{dx}N\left(\frac{-\ln(x)+\ln(R\bar{v})+v^\star s}{\sigma_A\sqrt{s}}\right) = AR^2e^{(v^\star+0.5\sigma_A^2)s}\left[N\left(h_4\left(z_2,s\right)\right)-N\left(h_4\left(z_1,s\right)\right)\right], \tag{2.65}$$

where $h_4\left(z,s\right) = \frac{\ln\left(R\frac{\bar{v}}{z}\right)+\left(v^\star+\sigma_A^2\right)s}{\sigma_A\sqrt{s}}$.

Proof. Procceding as in equation (2.62) one obtains

$$\int_0^y x\frac{d}{dx}N\left(\frac{-\ln(x)+a}{b}\right) = \mathrm{e}^{0.5b^2+a}N\left(\frac{-\ln(y)+a+b^2}{b}\right). \tag{2.66}$$

Taking $a = \ln(R\bar{v}) + v^\star s$ and $b = \sigma_A\sqrt{s}$,

$$
\begin{aligned}
&\int_{z_1}^{z_2} x \frac{d}{dx} N\left(\frac{-\ln(x)+\ln(R\bar{v})+v^{\star}s}{\sigma_A\sqrt{s}}\right) dx \\
&= e^{0.5\sigma_A^2 s+\ln(R\bar{v})+v^{\star}s}\left[N\left(\frac{-\ln\left(z_2\right)+\ln\left(R\bar{v}\right)+v^{\star}s+\sigma_A^2 s}{\sigma_A\sqrt{s}}\right)\right. \\
&\left.-N\left(\frac{-\ln\left(z_1\right)+\ln\left(R\bar{v}\right)+v^{\star}s+\sigma_A^2 s}{\sigma_A\sqrt{s}}\right)\right] \\
&= e^{0.5\sigma_A^2 s-\ln(A)+\ln\left(\bar{v}^2\right)+v^{\star}s}\left[N\left(\frac{\ln\left(R\frac{\bar{v}}{z_2}\right)+\left(v^{\star}+\sigma_A^2\right)s}{\sigma_A\sqrt{s}}\right)-N\left(\frac{\ln\left(R\frac{\bar{v}}{z_1}\right)+\left(v^{\star}+\sigma_A^2\right)s}{\sigma_A\sqrt{s}}\right)\right] \\
&= \frac{\bar{v}^2}{A} e^{\left(v^{\star}+0.5\sigma_A^2\right)s}\left[N\left(h_4\left(z_2,s\right)\right)-N\left(h_4\left(z_1,s\right)\right)\right] \\
&= AR^2 e^{\left(v^{\star}+0.5\sigma_A^2\right)s}\left[N\left(h_4\left(z_2,s\right)\right)-N\left(h_4\left(z_1,s\right)\right)\right].
\end{aligned}
\tag{2.67}
$$

□

The last result of this section is used to compute integrals such as $\int_0^T e^{\varpi s}\frac{d}{ds}N\left(h_1\left(z,s\right)\right)ds$ and $\int_0^T e^{\varpi s}\frac{d}{ds}N\left(h_2\left(z,s\right)\right)ds$. This can be done using Proposition 8 below. As this type of integral appears in different forms throughout this thesis, the referred proposition defines a general function to compute it, which is called $F\left(a,b,c,y\right)$.

Proposition 8. *Let $a,b,c \in \mathbb{R}$ satisfy $b<0$ and $c^2>2a$. Then, we have for every $y>0$*

$$
F\left(a,b,c,y\right) := \int_0^y e^{ax}\frac{d}{dx}N\left(\frac{b-cx}{\sqrt{x}}\right)dx = \begin{cases} \Omega_g^+\left(a,c\right)g^+(y)+\Omega_h^+\left(a,c\right)h^+(y), b>0 \\ \Omega_g^-\left(a,c\right)g^-(y)+\Omega_h^-\left(a,c\right)h^-(y), b<0 \end{cases}, \tag{2.68}
$$

where [18]

$$\begin{aligned}
&\Omega_g^{\pm}(a,c) = \mp\tfrac{\sqrt{c^2-2a}\mp c}{2\sqrt{c^2-2a}} \\
&\Omega_h^{\pm}(a,c) = \mp\tfrac{\sqrt{c^2-2a}\pm c}{2\sqrt{c^2-2a}} \\
&g^{\pm}(y) = e^{\mp b\Psi_g^{\pm}(a,c)}N\left(\tfrac{\mp b-\sqrt{c^2-2ay}}{\sqrt{y}}\right) \\
&h^{\pm}(y) = e^{\mp b\Psi_h^{\pm}(a,c)}N\left(\tfrac{\mp b+\sqrt{c^2-2ay}}{\sqrt{y}}\right) \\
&\Psi_g^{\pm}(a,c) = \mp c - \sqrt{c^2-2a} \\
&\Psi_h^{\pm}(a,c) = \mp c + \sqrt{c^2-2a}
\end{aligned}\; . \tag{2.69}$$

Proof. For $b < 0$ see Bielecki *et al.* (2006). For $b > 0$ notice that

$$N\left(\frac{b-cx}{\sqrt{x}}\right) = 1 - N\left(\frac{-b+cx}{\sqrt{x}}\right)$$

.

Denoting

$$\begin{aligned}
b^* &= -b \\
c^* &= -c
\end{aligned}$$

and substituting, one obtains

$$\begin{aligned}
\int_0^y e^{ax}\frac{d}{dx}N\left(\frac{b-cx}{\sqrt{x}}\right)dx &= \int_0^y \mathrm{e}^{ax}\frac{d}{dx}\left[1-N\left(\frac{b^*-c^*x}{\sqrt{x}}\right)\right] \\
&= -\int_0^y \mathrm{e}^{ax}\frac{d}{dx}N\left(\frac{b^*-c^*x}{\sqrt{x}}\right).
\end{aligned} \tag{2.70}$$

Since $b^* < 0$ one can apply the result proved in Bielecki *et al.* (2006).

□

In the referred particular cases this leads to

$$\begin{aligned}
\int_0^T e^{\omega s}\frac{d}{ds}N\left(h_1(z,s)\right)ds &= F\left(\varpi, \frac{ln\left(\frac{z}{A}\right)}{\sigma_A}, \frac{v^\star}{\sigma_A}, T\right) \\
\int_0^T e^{\omega s}\frac{d}{ds}N\left(h_2(z,s)\right)ds &= F\left(\varpi, \frac{ln\left(R\frac{\overline{v}}{z}\right)}{\sigma_A}, \frac{v^\star}{\sigma_A}, T\right).
\end{aligned} \tag{2.71}$$

[18] The $\pm$ signal reflect whether we are in the case that $b > 0$ or $b < 0$. This thesis does not provide a closed-form solution for the case where $b = 0$.

2.5. Basic Securities

All securities priced in this thesis can be computed as a combination of five basic securities. These basic securities have some similarities with some type of call or put option. Nevertheless, they cannot be really seen as options for a number of reasons that differ case by case. For this reason they are called pseudo-options in most cases. This section shows how each of these basic securities can be computed.

Pseudo-asset or nothing "no liquidation" call: $AN(s)$

Consider a security whose price today is given by $AN(s) = E^{\mathbb{Q}}\left[A_s 1_{\{\tau > s, \hat{\tau} > s\}} | \mathcal{F}_0\right]$. Basically, this security pays A_s if the firm has not been closed up to time s. This security resembles a down-and-out call option with maturity s, exercise price 0 and barrier $\bar{v}$ but with two differences. First, in addition to the barrier this option only pays A_s if a negative jump has not occurred up to time s. Second, the expected payoff in this case is not discounted. The time-0 price of this security corresponds to

$$\begin{aligned} AN(s) &= \int_{\bar{v}}^{+\infty} \int_{s}^{+\infty} \int_{s}^{+\infty} x \mathbb{Q}\left(A_s \in dx, \tau \in du, \hat{\tau} \in d\hat{u} \,|\mathcal{F}_0\right) \\ &= \int_{\bar{v}}^{+\infty} \int_{s}^{+\infty} \int_{s}^{+\infty} x \mathbb{Q}\left(\hat{\tau} \in d\hat{u} \,|\mathcal{F}_0\right) \mathbb{Q}\left(A_s \in dx, \tau \in du \,|\mathcal{F}_0\right) \\ &= \int_{\bar{v}}^{+\infty} x \mathbb{Q}\left(\hat{\tau} \geq s \,|\mathcal{F}_0\right) \mathbb{Q}\left(A_s \in dx, \tau \geq s \,|\mathcal{F}_0\right). \end{aligned} \tag{2.72}$$

Using equations (2.60) and (2.49), then

$$AN(s) = \int_{\bar{v}}^{+\infty} e^{-\overline{\lambda} s} x \left[\frac{d}{dx} N\left(h_1(x,s)\right) + R^{2a} \frac{d}{dx} N\left(h_2(x,s)\right)\right] dx. \tag{2.73}$$

Using Proposition 6 and 7,

$$\begin{aligned} AN\left(s\right) = A_0 e^{\left(v^{\star}+0.5\sigma_A^2-\overline{\lambda}\right)s} \lim_{x\rightarrow+\infty} \{N\left(h_3\left(x,s\right)\right) - N\left(h_3\left(\bar{v},s\right)\right) \\ +R^{2a+2}\left[N\left(h_4\left(x,s\right)\right) - N\left(h_4\left(\bar{v},s\right)\right)\right]\right\}. \end{aligned} \tag{2.74}$$

Given that $\lim_{x\rightarrow+\infty} N\left(h_3\left(x,s\right)\right) = 1$ and $\lim_{x\rightarrow+\infty} N\left(h_4\left(x,s\right)\right) = 0$, this simplifies to

$$AN\left(s\right) = A_0 e^{\left(v^{\star}+0.5\sigma_A^2-\overline{\lambda}\right)s}\left[1 - N\left(h_3\left(\bar{v},s\right)\right) - R^{2a+2} N\left(h_4\left(\bar{v},s\right)\right)\right]. \tag{2.75}$$

Pseudo-Digital "no liquidation" call: $Dig\left(s\right)$

Define $Dig\left(s\right) = E^{\mathbb{Q}}\left[1_{\{\tau>s,\hat{\tau}>s\}}|\mathcal{F}_0\right]$. $Dig\left(s\right)$ is similar to $AN\left(s\right)$ except that the payoff now is the monetary unit instead of the underlying asset. As the expected payoff is not discounted, Dig(s) corresponds to the probability of the firm surviving up to time s.

$$\begin{aligned} Dig\left(s\right) &= \int_s^{+\infty}\int_s^{+\infty} \mathbb{Q}\left(\tau \in du, \hat{\tau} \in d\hat{u}\right|\mathcal{F}_0\left.\right) d\hat{u}du \\ &= \mathbb{Q}\left(\tau > s, \hat{\tau} > s\right|\mathcal{F}_0\left.\right) \\ &= \mathbb{Q}\left(\hat{\tau} > s\right|\mathcal{F}_0\left.\right)\mathbb{Q}\left(\tau > s\right|\mathcal{F}_0\left.\right). \end{aligned} \tag{2.76}$$

Using equations (2.60) and (2.52), one obtains

$$Dig\left(s\right) = e^{-\overline{\lambda}s}\left[1 - N\left(h_1\left(\bar{v},s\right)\right) - R^{2a} N\left(h_2\left(\bar{v},s\right)\right)\right]. \tag{2.77}$$

Digital down-and-out "no jump" put with rebate: $DigHit\left(s\right)$

Define $DigHit\left(s\right) = E^{\mathbb{Q}}\left[e^{-r\tau}1_{\{\tau<s,\hat{\tau}>\tau\}}|\mathcal{F}_0\right]$. $DigHit\left(s\right)$ can be seen as the non-deferable rebate of a put down-and-out with maturity s, exercise price and barrier equal to $\bar{v}$ and rebate equal to 1 that only pays off if a negative jump does not occur up to maturity. Notice that the probability of this option ending up in the money is zero and thus the value of this option comes exclusively from the rebate. Mathematically,

$$
\begin{aligned}
DigHit\,(s) &= \int_0^s \int_u^\infty e^{-ru}\mathbb{Q}\left(\tau \in du, \hat{\tau} \in d\hat{u} \,|\mathcal{F}_0\right) \\
&= \int_0^s \int_u^\infty e^{-ru}\mathbb{Q}\left(\hat{\tau} \in d\hat{u} \,|\mathcal{F}_0\right)\mathbb{Q}\left(\tau \in du \,|\mathcal{F}_0\right) \qquad (2.78) \\
&= \int_0^s e^{-ru}\mathbb{Q}\left(\tau \in du \,|\mathcal{F}_0\right)\mathbb{Q}\left(\hat{\tau} \geq u \,|\mathcal{F}_0\right).
\end{aligned}
$$

Using equations (2.50) and (2.60), then

$$
DigHit\,(s) = \int_0^s \mathrm{e}^{-ru} e^{-\overline{\lambda} u} \left(\frac{d}{du} N\left(h_1\left(\bar{v}, u\right)\right) + R^{2a} \frac{d}{du} N\left(h_2\left(\bar{v}, u\right)\right)\right) du. \qquad (2.79)
$$

Using Proposition 8 and denoting $\varpi = -(r + \overline{\lambda})$,

$$
DigHit\,(s) = F\left(\varpi, \frac{\ln(R)}{\sigma_A}, \frac{v^\star}{\sigma_A}, s\right) + R^{2a} F\left(\varpi, \frac{\ln(R)}{\sigma_A}, -\frac{v^\star}{\sigma_A}, s\right). \qquad (2.80)
$$

Remark 1. *Consider that* $DigHit^\star\,(s) = E^{\mathbb{Q}}\left[e^{-r\tau} 1_{\{\tau < s, \hat{\tau} > \tau\}} |\mathcal{F}_0\right]$. $DigHit^\star\,(s)$ *is given by equation (2.80) replacing* ϖ *by* $-\overline{\lambda}$.

Pseudo-range asset or nothing down-and-out "jump" call: $ANJump\left(\overline{l}, \overline{u}, s\right)$ Consider that $ANJump\left(\overline{l}, \overline{u}, s\right) = E^{\mathbb{Q}}\left[e^{-r\hat{\tau}} A_{\hat{\tau}-} 1_{\left\{\overline{l} < A_{\hat{\tau}-} < \overline{u}, \tau > \hat{\tau}, \hat{\tau} < s\right\}} |\mathcal{F}_0\right]$. This security value is very similar to an option with maturity s that pays the underlying value (just before the jump) if the jump occurs, the asset before the jump lies between $\overline{l}$ and $\overline{u}$ and, finally, if the barrier has not been hit previously. However, in this case we are discounting the payoff from the moment the jump occurs and not from time s.

$$
\begin{aligned}
ANJump\left(\bar{l},\overline{u},s\right) &= \int\limits_0^s \int\limits_{\bar{l}}^{\overline{u}} xe^{-r\hat{u}} \int\limits_{\hat{u}}^{\infty} \mathbb{Q}\left(A_{\hat{u}} \in dx, \tau \in du, \hat{\tau} \in d\hat{u} \left| \mathcal{F}_0 \right.\right) \\
&= \int\limits_0^s \int\limits_{\bar{l}}^{\overline{u}} xe^{-r\hat{u}} \int\limits_{\hat{u}}^{\infty} \mathbb{Q}\left(\hat{\tau} \in d\hat{u} \left| \mathcal{F}_0 \right.\right) \mathbb{Q}\left(A_{\hat{u}} \in dx, \tau \in du \left| \mathcal{F}_0 \right.\right) \qquad (2.81) \\
&= \int\limits_0^s \int\limits_{\bar{l}}^{\overline{u}} xe^{-r\hat{u}} \mathbb{Q}\left(\hat{\tau} \in d\hat{u} \left| \mathcal{F}_0 \right.\right) \mathbb{Q}\left(A_{\hat{u}} \in dx, \tau \geq \hat{u} \left| \mathcal{F}_0 \right.\right).
\end{aligned}
$$

Substituting the jump density from equation (2.58) and using equation (2.49),

$$
ANJump\left(\bar{l},\overline{u},s\right) = \overline{\lambda} \int\limits_0^s \int\limits_{\bar{l}}^{\overline{u}} xe^{-\left(r+\overline{\lambda}\right)\hat{u}} \left(\frac{d}{dx} N\left(h_1\left(x,\hat{u}\right)\right) + R^{2a} \frac{d}{dx} N\left(h_2\left(x,\hat{u}\right)\right) \right) d\hat{u}dx. \qquad (2.82)
$$

Using Propositions 6 and 7 and denoting $\omega = v^{\star} + 0.5\sigma_A^2 - r - \overline{\lambda}$,

$$
\begin{aligned}
ANJump\left(\bar{l},\overline{u},s\right) = \overline{\lambda} A_0 \int\limits_0^s e^{\omega\hat{u}} &\left\{ N\left(h_3\left(\overline{u},\hat{u}\right)\right) - N\left(h_3\left(\bar{l},\hat{u}\right)\right) \right. \\
&\left. + R^{2a+2} \left[N\left(h_4\left(\overline{u},\hat{u}\right)\right) - N\left(h_4\left(\bar{l},\hat{u}\right)\right) \right] \right\} d\hat{u}.
\end{aligned} \qquad (2.83)
$$

Integrating by parts

$$
\begin{aligned}
ANJump\left(\bar{l},\overline{u},s\right) = \frac{\overline{\lambda} A_0}{\omega} &\left\{ e^{\omega\hat{u}} \left[N\left(h_3\left(\overline{u},\hat{u}\right)\right) - N\left(h_3\left(\bar{l},\hat{u}\right)\right) \right] \Big|_0^s \right. \\
&- \int\limits_0^s e^{\omega\hat{u}} \frac{d}{d\hat{u}} \left[N\left(h_3\left(\overline{u},\hat{u}\right)\right) - N\left(h_3\left(\bar{l},\hat{u}\right)\right) \right] d\hat{u} \\
&+ R^{2a+2} \left[e^{\omega\hat{u}} \left[N\left(h_4\left(\overline{u},\hat{u}\right)\right) - N\left(h_4\left(\bar{l},\hat{u}\right)\right) \right] \Big|_0^s \right. \\
&\left.\left. - \int\limits_0^s e^{\omega\hat{u}} \frac{d}{d\hat{u}} \left[N\left(h_4\left(\overline{u},\hat{u}\right)\right) - N\left(h_4\left(\bar{l},\hat{u}\right)\right) \right] d\hat{u} \right] \right\}.
\end{aligned} \qquad (2.84)
$$

Using Proposition 8,

$$
\begin{aligned}
ANJump\left(\bar{l},\overline{u},s\right) = & \frac{\overline{\lambda}A_0}{\omega}\left\{e^{\omega s}\left(N\left(h_3\left(\overline{u},s\right)\right) - N\left(h_3\left(\bar{l},s\right)\right)\right)\right. \\
& - N\left(h_3\left(\overline{u},0\right)\right) + N\left(h_3\left(\bar{l},0\right)\right) \\
& - F\left(\omega, \frac{\ln\left(\frac{\overline{u}}{A}\right)}{\sigma_A}, \frac{v^\star + \sigma_A^2}{\sigma_A}, s\right) + F\left(\omega, \frac{\ln\left(\frac{\bar{l}}{A}\right)}{\sigma_A}, \frac{v^\star + \sigma_A^2}{\sigma_A}, s\right) \\
& + R^{2a+2}\left[e^{\omega s}\left(N\left(h_4\left(\overline{u},s\right)\right) - N\left(h_4\left(\bar{l},s\right)\right)\right)\right. \\
& - N\left(h_4\left(\overline{u},0\right)\right) + N\left(h_4\left(\bar{l},0\right)\right) \\
& \left.\left. - F\left(\omega, \frac{\ln\left(\frac{R\bar{v}}{\overline{u}}\right)}{\sigma_A}, -\frac{v^\star + \sigma_A^2}{\sigma_A}, s\right) + F\left(\omega, \frac{\ln\left(\frac{R\bar{v}}{\bar{l}}\right)}{\sigma_A}, -\frac{v^\star + \sigma_A^2}{\sigma_A}, s\right)\right]\right\}.
\end{aligned}
\tag{2.85}
$$

Remark 2. *Consider that* $ANJump^\star\left(\bar{l},\overline{u},s\right) = E^{\mathbb{Q}}\left[A_{\hat{\tau}}1_{\left\{\bar{l}<A_{\hat{\tau}}<\overline{u},\tau>\hat{\tau},\hat{\tau}<s\right\}}|\mathcal{F}_0\right]$. $ANJump^\star\left(\bar{l},\overline{u},s\right)$ *is given by equation (2.85) replacing* ω *by* $\omega^\star = v^\star + 0.5\sigma_A^2 - \overline{\lambda}$.

Pseudo-range digital down-and-out "jump" call: $DigJump\left(\bar{l},\overline{u},s\right)$

Finally, consider $DigJump\left(\bar{l},\overline{u},s\right) = E^{\mathbb{Q}}\left[e^{-r\hat{\tau}}1_{\left\{\bar{l}<A_{\hat{\tau}}<\overline{u},\tau>\hat{\tau},\hat{\tau}<s\right\}}|\mathcal{F}_0\right]$. This security is very similar to $ANJump\left(\bar{l},\overline{u},s\right)$ except that the payoff is the monetary unit and not the underlying asset. The value of this security can be derived as follows.

$$
\begin{aligned}
DigJump\left(\bar{l},\overline{u},s\right) &= \int_0^s \int_{\bar{l}}^{\overline{u}} e^{-r\hat{u}} \int_{\hat{u}}^{\infty} \mathbb{Q}\left(A_{\hat{u}} \in dx, \tau \in du, \hat{\tau} \in d\hat{u}\,|\mathcal{F}_0\right) \\
&= \int_0^s \int_{\bar{l}}^{\overline{u}} e^{-r\hat{u}} \int_{\hat{u}}^{\infty} \mathbb{Q}\left(\hat{\tau} \in d\hat{u}\,|\mathcal{F}_0\right)\mathbb{Q}\left(A_{\hat{u}} \in dx, \tau \in du\,|\mathcal{F}_0\right).
\end{aligned}
\tag{2.86}
$$

Substituting the jump density from equation (2.58) and using equation (2.49),

$$
DigJump\left(\bar{l},\overline{u},s\right) = \overline{\lambda}\int_0^s \int_{\bar{l}}^{\overline{u}} e^{-\left(r+\overline{\lambda}\right)\hat{u}}\left[\frac{d}{dx}N\left(h_1\left(x,\hat{u}\right)\right) + R^{2a}\frac{d}{dx}N\left(h_2\left(x,\hat{u}\right)\right)\right]d\hat{u}dx. \tag{2.87}
$$

Integrating and denoting $\varpi = -(r+\overline{\lambda})$,

$$\begin{aligned} DigJump\left(\bar{l},\overline{u},s\right) &= \overline{\lambda}\int\limits_0^s e^{\varpi\hat{u}}\left[N\left(h_1\left(x,\hat{u}\right)\right)|_{\bar{l}}^{\bar{u}} + R^{2a}\ N\left(h_2\left(x,\hat{u}\right)\right)|_{\bar{l}}^{\bar{u}}\right] d\hat{u} \\ &= \overline{\lambda}\int\limits_0^s e^{\varpi\hat{u}}\left\{N\left(h_1\left(\overline{u},\hat{u}\right)\right) - N\left(h_1\left(\bar{l},\hat{u}\right)\right)\right. \\ &\left. +R^{2a}\left[N\left(h_2\left(\overline{u},\hat{u}\right)\right) - N\left(h_2\left(\bar{l},\hat{u}\right)\right)\right]\right\} d\hat{u}. \end{aligned} \tag{2.88}$$

Integrating by parts,

$$\begin{aligned} DigJump\left(\bar{l},\overline{u},s\right) = \frac{\overline{\lambda}}{\varpi}\Bigg\{ & e^{\varpi\hat{u}}\left[N\left(h_1\left(\overline{u},\hat{u}\right)\right) - N\left(h_1\left(\bar{l},\hat{u}\right)\right)\right]\Big|_0^s \\ & - \int\limits_0^s e^{\varpi\hat{u}}\frac{d}{d\hat{u}}\left[N\left(h_1\left(\overline{u},\hat{u}\right)\right) - N\left(h_1\left(\bar{l},\hat{u}\right)\right)\right] d\hat{u} \\ & + R^{2a}\Bigg[e^{\varpi\hat{u}}\left[N\left(h_2\left(\overline{u},\hat{u}\right)\right) - N\left(h_2\left(\bar{l},\hat{u}\right)\right)\right]\Big|_0^s \\ & - \int\limits_0^s e^{\varpi\hat{u}}\frac{d}{d\hat{u}}\left[N\left(h_2\left(\overline{u},\hat{u}\right)\right) - N\left(h_2\left(\bar{l},\hat{u}\right)\right)\right] d\hat{u}\Bigg]\Bigg\}. \end{aligned} \tag{2.89}$$

Using Proposition 8,

$$\begin{aligned} DigJump\left(\bar{l},\overline{u},s\right) = \frac{\overline{\lambda}}{\varpi}\Bigg\{ & e^{\varpi s}\left[N\left(h_1\left(\overline{u},s\right)\right) - N\left(h_1\left(\bar{l},s\right)\right)\right] \\ & - N\left(h_1\left(\overline{u},0\right)\right) + N\left(h_1\left(\bar{l},0\right)\right) \\ & - F\left(\varpi,\frac{\ln\left(\frac{\overline{u}}{A}\right)}{\sigma_A},\frac{v^\star}{\sigma_A},s\right) + F\left(\varpi,\frac{\ln\left(\frac{\bar{l}}{A}\right)}{\sigma_A},\frac{v^\star}{\sigma_A},s\right) \\ & + R^{2a}\Bigg[e^{\varpi s}\left[N\left(h_2\left(\overline{u},s\right)\right) - N\left(h_2\left(\bar{l},s\right)\right)\right] \\ & - N\left(h_2\left(\overline{u},0\right)\right) + N\left(h_2\left(\bar{l},0\right)\right) \\ & - F\left(\varpi,\frac{\ln\left(\frac{R\bar{v}}{\overline{u}}\right)}{\sigma_A},-\frac{v^\star}{\sigma_A},s\right) + F\left(\varpi,\frac{\ln\left(\frac{R\bar{v}}{\bar{l}}\right)}{\sigma_A},-\frac{v^\star}{\sigma_A},s\right)\Bigg]\Bigg\}. \end{aligned} \tag{2.90}$$

Remark 3. *Consider that* $DigJump^\star\left(\bar{l},\overline{u},s\right) = E^{\mathbb{Q}}\left[1_{\left\{\bar{l}<A_{\hat{\tau}}<\overline{u},\tau>\hat{\tau},\hat{\tau}<s\right\}}|\mathcal{F}_0\right]$. $DigJump^\star\left(\bar{l},\overline{u},s\right)$ *is given by equation (2.90) replacing* ϖ *by* $-\overline{\lambda}$.

3. Equity

Chapter 2 presented the stochastic process governing the project cash flow generation and a fictive security representing the market value of the project. In the next sections, we will derive the price of equity, debt, CDS and European-style stock options as contingent claims on this project. Starting with equity, its value in this model corresponds to the sum of two components: 1) the after-tax present value of all future dividends up to the moment the firm stops its activity (i.e. $(1-\bar{t}^{Eff})Div_0$); and 2) the proceedings from selling the project at τ^{Solv} whenever these are enough to comply with all the firm liabilities after taking into account all external claimants on the project (i.e. $EqRec_0$). Mathematically,

$$E_0 = \left(1-\bar{t}^{Eff}\right) Div_0 + EqRec_0, \tag{3.1}$$

where $\left(1-\bar{t}^{Eff}\right) = \left(1-\bar{t}^{Corp}\right)\left(1-\bar{t}^{Div}\right)$.

The first two sections of this chapter cover the contribution from these two components to equity value. The third section derives the dynamics of the equity process and presents the concept of cost of equity. This chapter ends with the derivation of shareholders' optimal default barrier.

3.1. Dividends

As explained in Chapter 2, the project owned by our firm continuously generates δ_t. The project requires a continuous investment of q, though. In addition, the firm must pay debtholders coupons and government taxes. At each moment in time, depending on whether δ_t is enough to cover coupons and capex, shareholders either receive the difference or inject capital in the firm in order to avoid bankruptcy. In the latter case it is assumed that a tax provision is created. Mathematically, the before-tax present value of all future dividends received by shareholders equals

$$\begin{aligned}Div_0 &= \lim_{T\to+\infty}\int\limits_0^T e^{-rs}E^{\mathbb{Q}}\left[\left(\delta_s - cL - q\right)1_{\{\tau>s,\hat{\tau}>s\}}|\mathcal{F}_0\right]ds\\ &= \lim_{T\to+\infty}\int\limits_0^T e^{-rs}E^{\mathbb{Q}}\left[\delta_s 1_{\{\tau>s,\hat{\tau}>s\}}|\mathcal{F}_0\right]ds - (cL+q)\lim_{T\to+\infty}\int\limits_0^T e^{-rs}E^{\mathbb{Q}}\left[1_{\{\tau>s,\hat{\tau}>s\}}|\mathcal{F}_0\right]ds.\end{aligned} \tag{3.2}$$

Noticing that $\delta_s = (\mu_A - g)\,A_s$, we have that the first and second expectations correspond respectively to $(\mu_A - g)$ times $AN(s)$ and $Dig(s)$:

$$Div_0 = (\mu_A - g)\lim_{T\to+\infty}\int\limits_0^T e^{-rs}AN(s)\,ds - (cL+q)\lim_{T\to+\infty}\int\limits_0^T e^{-rs}Dig(s)\,ds, \tag{3.3}$$

where the first term corresponds to the discounted sum of all future cash flow as long as the firm exists and the remaining terms are the discounted sum of all future interest and capex costs as long as the firm exists. For this reason, these terms are called $Payout_0^T$, $Coupon_0^T$ and $Capex_0^T$

$$Div_0 = \lim_{T\to+\infty}\left[Payout_0^T - Coupon_0^T - Capex_0^T\right]. \tag{3.4}$$

Starting with $Payout_0^T$, substituting $AN(s)$ by equation (2.75) and denoting $\omega = v^{\star} + 0.5\sigma^2 - r - \overline{\lambda}$, one obtains

$$Payout_0^T = (\mu_A - g)\,A_0\int\limits_0^T e^{\omega s}\left[\left(1 - N\left(h_3\left(\bar{v},s\right)\right)\right) - R^{2a+2}N\left(h_4\left(\bar{v},s\right)\right)\right]ds. \tag{3.5}$$

Substituting A_0 by $\frac{\delta_0}{\mu_A - g}$ and integrating by parts,

$$\begin{aligned}Payout_0^T = \frac{\delta_0}{\omega}\Bigg\{ & e^{\omega s}\left[1 - N\left(h_3\left(\bar{v},s\right)\right)\right]\Big|_0^T + \int\limits_0^T e^{\omega s}\frac{d}{ds}N\left(h_3\left(\bar{v},s\right)\right)ds\\ & -R^{2a+2}\left[e^{\omega s}N\left(h_4\left(\bar{v},s\right)\right)\Big|_0^T - \int\limits_0^T e^{\omega s}\frac{d}{ds}N\left(h_4\left(\bar{v},s\right)\right)ds\right]\Bigg\}.\end{aligned} \tag{3.6}$$

Applying Proposition 8,

$$Payout_0^T = \frac{\delta_0}{\omega}\left\{e^{\omega T}\left[1 - N\left(h_3\left(\bar{v}, T\right)\right)\right] - 1 + N\left(h_3\left(\bar{v}, 0\right)\right) + F\left(\omega, \frac{\ln\left(R\right)}{\sigma}, \frac{v^{\star} + \sigma^2}{\sigma}, T\right)\right.$$
$$\left. - R^{2a+2}\left[e^{\omega T} N\left(h_4\left(\bar{v}, T\right)\right) - N\left(h_4\left(\bar{v}, 0\right)\right) - F\left(\omega, \frac{\ln\left(R\right)}{\sigma}, -\frac{v^{\star} + \sigma^2}{\sigma}, T\right)\right]\right\}. \tag{3.7}$$

Given that $N\left(h_3\left(\bar{v}, 0\right)\right) = 0$ and $N\left(h_4\left(\bar{v}, 0\right)\right) = 0$, equation (3.7) can be simplified into[1]

$$\begin{aligned} Payout_0^T = & \frac{\delta_0}{\omega}\left\{e^{\omega T}\left[1 - N\left(h_3\left(\bar{v}, T\right)\right)\right] - 1 + F\left(\omega, \frac{\ln\left(R\right)}{\sigma}, \frac{v^{\star} + \sigma^2}{\sigma}, T\right)\right. \\ & \left. - R^{2a+2}\left[e^{\omega T} N\left(h_4\left(\bar{v}, T\right)\right) - F\left(\omega, \frac{\ln\left(R\right)}{\sigma}, -\frac{v^{\star} + \sigma^2}{\sigma}, T\right)\right]\right\}. \end{aligned} \tag{3.8}$$

Considering that T goes to infinity and that $\omega < 0$, then

$$\begin{aligned} Payout_0 := & \lim_{T \to +\infty} Payout_0^T \\ = & \frac{\delta_0}{\omega} \lim_{T \to +\infty}\left[F\left(\omega, \frac{\ln\left(R\right)}{\sigma}, \frac{v^{\star} + \sigma^2}{\sigma}, T\right) + R^{2a+2} F\left(\omega, \frac{\ln\left(R\right)}{\sigma}, -\frac{v^{\star} + \sigma^2}{\sigma}, T\right) - 1\right]. \end{aligned} \tag{3.9}$$

Finally, computing the limits as explained in Appendix A.4, one obtains

$$\begin{aligned} Payout_0 = & \frac{\delta_0}{\omega}\left[\Omega_h^-\left(\omega, \frac{v^{\star} + \sigma^2}{\sigma}\right) R^{\frac{1}{\sigma}\Psi_h^-\left(\omega, \frac{v^{\star}+\sigma^2}{\sigma}\right)}\right. \\ & \left. + \Omega_h^-\left(\omega, -\frac{v^{\star} + \sigma^2}{\sigma}\right) R^{2a+2+\frac{1}{\sigma}\Psi_h^-\left(\omega, -\frac{v^{\star}+\sigma^2}{\sigma}\right)} - 1\right]. \end{aligned} \tag{3.10}$$

$Coupon_0^T$ can be computed in a similar way. Replacing $Dig\left(s\right)$ by equation (2.77) and denoting $\varpi = -(r + \overline{\lambda})$, one obtains:

$$Coupon_0^T = cL \int_0^T \mathrm{e}^{\varpi s}\left[1 - N\left(h_1\left(\bar{v}, s\right)\right) - R^{2a} N\left(h_2\left(\bar{v}, s\right)\right)\right] ds. \tag{3.11}$$

[1]Notice that

$$h_3\left(\bar{v}, 0\right) = \lim_{s \to 0} \frac{\ln(R) - (v^{\star} + \sigma^2)s}{\sigma\sqrt{s}} = \lim_{s \to 0} \frac{\ln(R)}{\sigma\sqrt{s}} - \lim_{s \to 0} \frac{(v^{\star} + \sigma^2)}{\sigma}\sqrt{s} = -\infty \Rightarrow N\left(h_3\left(\bar{v}, 0\right)\right) = 0$$
$$h_4\left(\bar{v}, 0\right) = \lim_{s \to 0} \frac{\ln(R) + (v^{\star} + \sigma^2)s}{\sigma\sqrt{s}} = \lim_{s \to 0} \frac{\ln(R)}{\sigma\sqrt{s}} + \lim_{s \to 0} \frac{(v^{\star} + \sigma^2)}{\sigma}\sqrt{s} = -\infty \Rightarrow N\left(h_4\left(\bar{v}, 0\right)\right) = 0$$

Integrating by parts and noticing that $N\left(h_1\left(\bar{v},0\right)\right)=0$ and $N\left(h_2\left(\bar{v},0\right)\right)=0$,

$$\begin{aligned}
Coupon_0^T &= \frac{cL}{\varpi}\left\{e^{\varpi s}\left[1-N\left(h_1\left(\bar{v},s\right)\right)-R^{2a}N\left(h_2\left(\bar{v},s\right)\right)\right]|_0^T\right.\\
&\quad\left.-\int_0^T \mathrm{e}^{\varpi s}\frac{d}{ds}\left[1-N\left(h_1\left(\bar{v},s\right)\right)-R^{2a}N\left(h_2\left(\bar{v},s\right)\right)\right]ds\right\}\\
&= \frac{cL}{\varpi}\left\{\mathrm{e}^{\varpi T}\left[1-N\left(h_1\left(\bar{v},T\right)\right)-R^{2a}N\left(h_2\left(\bar{v},T\right)\right)\right]-1\right.\\
&\quad\left.+\int_0^T \mathrm{e}^{\varpi s}\left[\frac{d}{ds}N\left(h_1\left(\bar{v},s\right)\right)+R^{2a}\frac{d}{ds}N\left(h_2\left(\bar{v},s\right)\right)\right]ds\right\}.
\end{aligned} \tag{3.12}$$

Using Proposition 8,

$$\begin{aligned}
Coupon_0^T &= \frac{cL}{\varpi}\left\{\mathrm{e}^{\varpi T}\left[1-N\left(h_1\left(\bar{v},T\right)\right)-R^{2a}N\left(h_2\left(\bar{v},T\right)\right)\right]-1\right.\\
&\quad\left.+F\left(\varpi,\frac{\ln(R)}{\sigma},\frac{v^\star}{\sigma},T\right)+R^{2a}F\left(\varpi,\frac{\ln(R)}{\sigma},-\frac{v^\star}{\sigma},T\right)\right\}.
\end{aligned} \tag{3.13}$$

Considering that T goes to infinity, then

$$\begin{aligned}
Coupon_0 &:= \lim_{T\to+\infty} Coupon_0^T\\
&= \frac{cL}{\varpi}\lim_{T\to+\infty}\left[F\left(\varpi,\frac{\ln(R)}{\sigma},\frac{v^\star}{\sigma},T\right)+R^{2a}F\left(\varpi,\frac{\ln(R)}{\sigma},-\frac{v^\star}{\sigma},T\right)-1\right].
\end{aligned} \tag{3.14}$$

Finally, taking the limit (please see Appendix A.4), one obtains

$$Coupon_0 = \frac{cL}{\varpi}\left[\Omega_h^-\left(\varpi,\frac{v^\star}{\sigma}\right)R^{\frac{1}{\sigma}\Psi_h^-\left(\varpi,\frac{v^\star}{\sigma}\right)}+\Omega_h^-\left(\varpi,-\frac{v^\star}{\sigma}\right)R^{2a+\frac{1}{\sigma}\Psi_h^-\left(\varpi,-\frac{v^\star}{\sigma}\right)}-1\right]. \tag{3.15}$$

Doing the same for $Capex_0$, one obtains

$$\begin{aligned}
Capex_0 &:= \lim_{T\to+\infty} Capex_0^T\\
&= \frac{q}{\varpi}\left[\Omega_h^-\left(\varpi,\frac{v^\star}{\sigma}\right)R^{\frac{1}{\sigma}\Psi_h^-\left(\varpi,\frac{v^\star}{\sigma}\right)}+\Omega_h^-\left(\varpi,-\frac{v^\star}{\sigma}\right)R^{2a+\frac{1}{\sigma}\Psi_h^-\left(\varpi,-\frac{v^\star}{\sigma}\right)}-1\right].
\end{aligned} \tag{3.16}$$

3.2. Recovered value after closing the firm

When the firm closes, the project is sold. This may occur because the project value hits the barrier or due to a jump. Thus,

$$EqRec_0 = EqRecHit_0 + EqRecJump_0, \tag{3.17}$$

where $EqRecHit_0$ and $EqRecJump_0$ are the respective contributes to equity value.

3.2.1. Recovered value after hitting the barrier

In the case that the firm hits the barrier, it is assumed that the firm incurs in distress costs and recovers only $\beta^{Bank} A_\tau$. Since debtholders have priority over shareholders they only receive something when $\bar{v} > \frac{L}{\beta^{Bank}}$.[2] Mathematically,

$$\begin{aligned} EqRecHit_0 &= \begin{cases} \lim_{T\to+\infty} E^Q \left[e^{-r\tau} \left(\beta^{Bank}\bar{v} - L\right) 1_{\{\tau<T,\hat{\tau}>\tau\}} |\mathcal{F}_0 \right], \bar{v} \geq \frac{L}{\beta^{Bank}} \\ 0, \bar{v} < \frac{L}{\beta^{Bank}} \end{cases} \\ &= \begin{cases} \left(\beta^{Bank}\bar{v} - L\right) \lim_{T\to+\infty} DigHit\left(T\right), \bar{v} \geq \frac{L}{\beta^{Bank}} \\ 0, \bar{v} < \frac{L}{\beta^{Bank}} \end{cases} . \end{aligned} \tag{3.18}$$

$DigHit\left(T\right)$ is given by equation (2.80). Taking the limit, one obtains

$$\begin{aligned} \lim_{T\to+\infty} DigHit\left(T\right) &= \lim_{T\to+\infty} \left[F\left(\varpi, \frac{\ln\left(R\right)}{\sigma}, \frac{v^\star}{\sigma}, T\right) + R^{2a} F\left(\varpi, \frac{\ln\left(R\right)}{\sigma}, -\frac{v^\star}{\sigma}, T\right)\right] \\ &= \Omega_h^- \left(\varpi, \frac{v^\star}{\sigma}\right) R^{\frac{1}{\sigma}\Psi_h^-\left(\varpi, \frac{v^\star}{\sigma}\right)} + R^{2a}\Omega_h^- \left(\varpi, -\frac{v^\star}{\sigma}\right) R^{\frac{1}{\sigma}\Psi_h^-\left(\varpi, -\frac{v^\star}{\sigma}\right)}. \end{aligned} \tag{3.19}$$

The above limits can be computed with the help of equations (A.61) and (A.62) in Appendix A.4 with ϖ replacing ω and $\frac{v^\star}{\sigma}$ instead of $\frac{v^\star+\sigma^2}{\sigma}$.

3.2.2. Recovered value after a jump

When the firm closes after a jump, depending on whether $A_{\hat{\tau}}$ is above or below $\bar{v}$, the firm recovers $\beta^{Sold} A_{\hat{\tau}}$ or incurs distress costs and recovers only $\beta^{Bank} A_{\hat{\tau}}$. This difference turns

[2] In this model, this should be seen as the special case of a firm with a very low level of debt and high fixed costs. In this case, the barrier might be higher than $\frac{L}{\beta^{Bank}}$.

the analysis substantially more complex and thus the two cases will be treated separately.

$$EqRecJump_0 = EqRecJumpBank_0 + EqRecJumpSold_0 \tag{3.20}$$

The firm is liquidated with distress costs: $EqRecJumpBank_0$

The firm is liquidated with default costs when the project value stays below the barrier after the jump (i.e. $A_{\hat{\tau}^-} \in \left[\overline{v}, \frac{\overline{v}}{1-j}\right]$). In this case, there might be two types of payoffs. These will depend on the relation between the recovered value, $\beta^{Bank}(1-j)A_{\hat{\tau}^-}$, and the amount of liabilities. Mathematically, the nominal recovered value equals

$$EqRecJumpBank_{\hat{\tau}} = \begin{cases} 0, \overline{v} < A_{\hat{\tau}^-} < \frac{L}{\beta^{Bank}(1-j)} \\ \beta^{Bank}(1-j)A_{\hat{\tau}^-} - L, \frac{L}{\beta^{Bank}(1-j)} < A_{\hat{\tau}^-} < \frac{\overline{v}}{1-j} \end{cases}. \tag{3.21}$$

Since there is no guarantee that $\overline{v} < \frac{L}{\beta^{Bank}(1-j)}$ and $\frac{L}{\beta^{Bank}(1-j)} < \frac{\overline{v}}{1-j}$ the following cases may emerge:

First case: $\overline{v} \leq \frac{L}{\beta^{Bank}(1-j)} \leq \frac{\overline{v}}{1-j}$

$$\begin{aligned} & EqRecJumpBank_0 \\ & = \lim_{T\to+\infty} E^{\mathbb{Q}}\left[e^{-r\hat{\tau}}\left[\beta^{Bank}(1-j)A_{\hat{\tau}^-} - L\right] 1_{\left\{\frac{L}{\beta^{Bank}(1-j)} < A_{\hat{\tau}^-} < \frac{\overline{v}}{1-j}, \tau > \hat{\tau}, \hat{\tau} < T\right\}} \middle| \mathcal{F}_0 \right] \\ & = \beta^{Bank}(1-j) \lim_{T\to+\infty} ANJump\left(\frac{L}{\beta^{Bank}(1-j)}, \frac{\overline{v}}{1-j}, T\right) \\ & - L \lim_{T\to+\infty} DigJump\left(\frac{L}{\beta^{Bank}(1-j)}, \frac{\overline{v}}{1-j}, T\right); \end{aligned} \tag{3.22}$$

Second case: $\frac{L}{\beta^{Bank}(1-j)} \leq \overline{v} \leq \frac{\overline{v}}{1-j}$

$$\begin{aligned} EqRecJumpBank_0 &= \lim_{T\to+\infty} E^{\mathbb{Q}}\left[e^{-r\hat{\tau}}\left[\beta^{Bank}(1-j)A_{\hat{\tau}^-} - L\right] 1_{\left\{\overline{v} < A_{\hat{\tau}^-} < \frac{\overline{v}}{1-j}, \tau > \hat{\tau}, \hat{\tau} < T\right\}} \middle| \mathcal{F}_0 \right] \\ &= \beta^{Bank}(1-j) \lim_{T\to+\infty} ANJump\left(\bar{v}, \frac{\bar{v}}{1-j}, T\right) \\ &- L \lim_{T\to+\infty} DigJump\left(\bar{v}, \frac{\bar{v}}{1-j}, T\right); \end{aligned} \tag{3.23}$$

Third case: $\overline{v} \leq \frac{\overline{v}}{1-j} \leq \frac{L}{\beta^{Bank}(1-j)}$

In this case shareholders know that they will receive nothing after a sudden jump if distress costs are incurred.

The limits above can be computed using equations (A.65) and (A.68) in Appendix A.4.

The firm is liquidated without distress costs: $EqRecJumpSold_0$

The firm is liquidated without distress costs when the project value stays above the barrier after the jump (i.e. $A_{\hat{\tau}^-} \in \left[\frac{\overline{v}}{1-j}, +\infty\right]$). Again, two types of payoffs may emerge. Shareholders either recover 0 or the difference between the firm recovered value and nominal liabilities.

$$EqRecJumpSold_{\hat{\tau}} = \begin{cases} 0, \frac{\overline{v}}{1-j} < A_{\hat{\tau}^-} < \frac{L}{\beta^{Sold}(1-j)} \\ \beta^{Sold}(1-j)A_{\hat{\tau}^-} - L, \frac{L}{\beta^{Sold}(1-j)} < A_{\hat{\tau}^-} < +\infty \end{cases} \quad (3.24)$$

Two cases may emerge depending whether $\frac{\overline{v}}{1-j}$ is lower than $\frac{L}{\beta^{Sold}(1-j)}$ or not:

First case: $\frac{\overline{v}}{1-j} \leq \frac{L}{\beta^{Sold}(1-j)} \leq +\infty$

$$\begin{aligned} &EqRecJumpSold_0 \\ &= \lim_{T\to+\infty} E^{\mathbb{Q}}\left[e^{-r\hat{\tau}}\left[\beta^{Sold}(1-j)A_{\hat{\tau}^-} - L\right]1_{\left\{\frac{L}{\beta^{Sold}(1-j)}<A_{\hat{\tau}^-}<+\infty,\tau>\hat{\tau},\hat{\tau}<T\right\}}\middle|\mathcal{F}_0\right] \\ &= \beta^{Sold}(1-j)\lim_{\overline{u},T\to+\infty} ANJump\left(\frac{L}{\beta^{Sold}(1-j)},\overline{u},T\right) \\ &- L\lim_{\overline{u},T\to+\infty} DigJump\left(\frac{L}{\beta^{Sold}(1-j)},\overline{u},T\right); \end{aligned} \quad (3.25)$$

Second case: $\frac{L}{\beta^{Sold}(1-j)} \leq \frac{\overline{v}}{1-j} \leq +\infty$

$$\begin{aligned} EqRecJumpSold_0 &= \lim_{T\to+\infty} E^{\mathbb{Q}}\left[e^{-r\hat{\tau}}\left[\beta^{Sold}(1-j)A_{\hat{\tau}^-} - L\right]1_{\left\{\frac{\overline{v}}{1-j}<A_{\hat{\tau}^-}<+\infty,\tau>\hat{\tau},\hat{\tau}<T\right\}}\middle|\mathcal{F}_0\right] \\ &= \beta^{Sold}(1-j)\lim_{\overline{u},T\to+\infty} ANJump\left(\frac{\overline{v}}{1-j},\overline{u},T\right) \\ &- L\lim_{\overline{u},T\to+\infty} DigJump\left(\frac{\overline{v}}{1-j},\overline{u},T\right); \end{aligned} \quad (3.26)$$

The limits above are given by equations (A.76) and (A.77) in Appendix A.4.

Before finishing this section, it is important to emphasize one feature of this model. While in the case where the process falls below (or hits) the barrier is justifiable to compute equity as the positive difference between the recovered value and liabilities, in the opposite case this is only motivated by mathematical tractability. Notice that the alternative would be an equity process with an indeterminate number of jumps.[3] In these cases, we are assuming that the firm is being liquidated, which is something that would not occur if the same value of A_t would have been reached through diffusion. By doing this, whenever $\beta^{Sold} A_{\hat{\tau}}$ stays above the barrier we are inappropriately eliminating shareholders' option to have a positive payoff in the future if the project perspectives improve. The size of this problem depends crucially on j. When j is sufficiently high, the probability of the project value falling below the barrier is very high eliminating the problem. However, unless $j = 1$ there is always a residual probability that A_t stays above the barrier after the jump. It is reasonable to think that the largest the value of j the lowest should be the value of λ (i.e. the largest the size of the jump the less probable it is). By setting j too high the analyst ignores the possibility of a large range of jump sizes that are virtually impossible under diffusion and whose probability may be significantly higher than the one associated with the chosen level of j. The analyst faces thus a trade-off between setting j very high and λ very low minimizing this error or, alternatively, setting a lower value of j and a higher value of λ potentially increasing this pricing error but better capturing the risks faced by the project. In this context it is important to have an idea of the potential size of the error that arises from assuming that the shareholder receives the maximum of zero or the intrinsic value of his claim. Define $\overline{EqRecJumpSold_0}$ as the correct equity recovered value when the project stays above the barrier after a sudden jump and $\overline{E}\left((1-j)\,A_{\hat{\tau}-}\right)$ as the correct equity pricing function evaluated at the project value just after the jump event. In this case we have

$$\overline{EqRecJumpSold_0} = \lim_{T\to\infty} E^{\mathbb{Q}}\left[e^{-r\hat{\tau}}\overline{E}\left((1-j)\,A_{\hat{\tau}-}\right) 1_{\left\{\bar{l}<A_{\hat{\tau}-}<+\infty,\tau>\hat{\tau},\hat{\tau}<T\right\}}\right]. \tag{3.27}$$

where $\bar{l}$ corresponds to $\frac{\bar{v}}{1-j}$ independent of the relative position vis-a-vis $\frac{L}{\beta^{Sold}(1-j)}$. Assuming that 1) the barrier value is independent of whether the shareholder receives an option over the project or the maximum of zero and the intrinsic value of shareholders' claim and 2) in case of a second jump the shareholder effectively receives the maximum of zero and the intrinsic value of his claim on the project (i.e. his option to continue running the firm is lost

[3]Equity value in this circunstance can be computed using Fast Fourier Transforms as proposed for the case of Normally distributed jumps by Carr and Madan (1999). There is no literature on the pricing of options in this case, though.

after a second jump even if the project value stays above the barrier), $\overline{E}(A_t)$ corresponds to our equity pricing function, $E(A_t)$, given by equation (3.1). Thus,

$$\overline{EqRecJumpSold_0} \approx \lim_{T\to\infty} E\left[e^{-r\hat{\tau}} E\left((1-j)\,A_{\hat{\tau}^-}\right) 1_{\left\{\bar{l}<A_{\hat{\tau}^-}<+\infty, \tau>\hat{\tau}, \hat{\tau}<T\right\}}\right]. \tag{3.28}$$

Procceding as in equations (2.81) and (2.82),

$$\begin{aligned}\overline{EqRecJumpSold_0} \approx \bar{\lambda} \lim_{T,\bar{u}\to\infty} \int_0^T \int_{\bar{l}}^{\bar{u}} E\left((1-j)\,x\right) e^{-\left(r+\bar{\lambda}\right)\hat{u}} \bigg(\frac{d}{dx} N\left(h_1\left(x,\hat{u}\right)\right) \\ +R^{2a} \frac{d}{dx} N\left(h_2\left(x,\hat{u}\right)\right)\bigg) \, d\hat{u}dx.\end{aligned} \tag{3.29}$$

Unfortunetly, it is not possible to apply Propositions 6 and 7 because this time the derivative of the Normal function is multiplied by a function of x instead of x. To the best of my knowledge the only way to compute the above integral is thus numerically. The difference between $EqRecJumpSold_0$ and $\overline{EqRecJumpSold_0}$ gives the analyst a good idea of the pricing error that arises from the assumption that the firm closes after the jump even if the project value stays above the barrier after the jump.

3.3. The equity process and the cost of equity

Sections 3.1 and 3.2 showed how to price equity as a contingent claim on the firm's cash flows. Equally interesting is to look at the stochastic process governing equity value. In the presented model, it has been shown that equity is a function only of A_t, with its dynamics under the physical and risk neutral measure given by equations (2.6) and (2.37), respectively. Taking the latter, applying Ito's lemma to function $E(A_t)$ and dividing by E_{t^-} one obtains

$$\frac{dE_t}{E_{t^-}} = \left(\left(\mu_\delta - \overline{m}\sigma_A\right) \frac{\partial E}{\partial A} \frac{A_t}{E_{t^-}} + 0.5 \frac{\partial^2 E}{\partial A^2} \frac{A_t^2}{E_{t^-}} \sigma_A^2\right) dt + \sigma_{E_t} dW_t^{\mathbb{Q}} - j_{E_t} dN_t^{\mathbb{Q}}, \tag{3.30}$$

where

$$\begin{aligned}\sigma_{E_t} &= \frac{\partial E}{\partial A} \frac{A_t}{E_{t^-}} \sigma_A \\ j_{E_t} &= \frac{E\left(A_{t^-}\right) - max\left\{\left[\beta^{Bank} 1_{\left\{(1-j_A)A_{t^-}\leq\bar{v}\right\}} + \beta^{Sold} 1_{\left\{(1-j_A)A_{t^-}>\bar{v}\right\}}\right]\left(1-j_A\right)A_{t^-} - L, 0\right\}}{E\left(A_{t^-}\right)}.\end{aligned} \tag{3.31}$$

In the above application of Ito's lemma, notice that $E\left((1-j)\,A_{t^-}\right)$ was replaced by $max\{\beta^{Sold}\,(1-j)\,A_t - L, 0\}$. This occurs because the firm is assumed to be closed and thus equity value after the jump is not equivalent to replacing A_t by $(1-j)\,A_t$ in the equity function.[4] Substituting N_t by $M_t+\lambda dt$ one arrives at the following martingale representation:

$$\frac{dE_t}{E_{t^-}} = \left((\mu_\delta - \overline{m}\sigma_A) \frac{\partial E}{\partial A} \frac{A_t}{E_{t^-}} + 0.5 \frac{\partial^2 E}{\partial A^2} \frac{A_t^2}{E_{t^-}} \sigma_A^2 - \overline{\lambda} j_{E_t} \right) dt + \sigma_{E_t} dW_t^{\mathbb{Q}} - j_{E_t} dM_t^{\mathbb{Q}}. \quad (3.32)$$

Equation (3.32) requires the computation of the first and second derivatives of the equity function. One can avoid the computation of the second derivative by using measure $\mathbb{Q}$ definition, though. Following Definition 1, any financial asset generates a rate of return equal to the risk free interest rate under measure $\mathbb{Q}$. In the case of equity, its return corresponds to

$$\frac{dE_t + (\delta_t - cL - q)\,dt}{E_t} = \frac{dE_t}{E_t} + \frac{(\delta_t - cL - q)\,dt}{E_t} \quad (3.33)$$

$$= \frac{dE_t}{E_t} + k_{E_t} dt, \quad (3.34)$$

Hence, under measure $\mathbb{Q}$, $\frac{dE_t}{E_t}$ must have a drift equal to $r - k_{E_t}$, implying that[5]

$$(\mu_\delta - \overline{m}\sigma_A) \frac{\partial E}{\partial A} \frac{A_t}{E_{t^-}} + 0.5 \frac{\partial^2 E}{\partial A^2} \frac{A_t^2}{E_{t^-}} \sigma_A^2 - \overline{\lambda} j_{E_t} = r - k_{E_t}. \quad (3.35)$$

Equation (3.32) can thus be rewritten as

$$\frac{dE_t}{E_{t^-}} = (r - k_{E_t})\,dt + \sigma_{E_t} dW_t^{\mathbb{Q}} - j_{E_t} dM_t^{\mathbb{Q}}, \quad (3.36)$$

where only the first derivative of the equity function is required to compute σ_{E_t}. This corresponds to

$$\frac{\partial E}{\partial A} = \left(1 - \overline{t}^{Eff}\right) \frac{\partial Div}{\partial A} + \frac{\partial EqRec}{\partial A}. \quad (3.37)$$

Starting with the derivative of the dividend function and using equation (3.4),

$$\frac{\partial Div}{\partial A} = \frac{\partial Payout_0}{\partial A} - \frac{\partial Coupon_0}{\partial A} - \frac{\partial Capex_0}{\partial A}. \quad (3.38)$$

[4]The difference between $E\left((1-j)\,A_{t^-}\right)$ and $max\{\beta^{Sold}\,(1-j)\,A_t - L, 0\}$ corresponds to the pricing 'error' incurred by assuming that the firm is closed after the jump even if the value of the project stays above $\overline{v}$.

[5]The validity of the above equation was confirmed numerically.

The derivatives above follow from equations (3.10), (3.15) and (3.16). The derivative of the Payout function is given by

$$
\begin{aligned}
\frac{\partial Payout_0}{\partial A} &= \frac{\mu_A - g}{\omega}\left[\Omega_h^-\left(\omega, \frac{v^\star + \sigma^2}{\sigma}\right) R^{\frac{1}{\sigma}\Psi_h^-\left(\omega, \frac{v^\star+\sigma^2}{\sigma}\right)}\right. \\
&\left. +\Omega_h^-\left(\omega, -\frac{v^\star + \sigma^2}{\sigma}\right) R^{2a+2+\frac{1}{\sigma}\Psi_h^-\left(\omega, -\frac{v^\star+\sigma^2}{\sigma}\right)} - 1\right] \\
&+ \frac{(\mu_A - g)\, A}{\omega}\left[\Omega_h^-\left(\omega, \frac{v^\star + \sigma^2}{\sigma}\right) \frac{\partial}{\partial A} R^{\frac{1}{\sigma}\Psi_h^-\left(\omega, \frac{v^\star+\sigma^2}{\sigma}\right)}\right. \\
&\left. +\Omega_h^-\left(\omega, -\frac{v^\star + \sigma^2}{\sigma}\right) \frac{\partial}{\partial A} R^{2a+2+\frac{1}{\sigma}\Psi_h^-\left(\omega, -\frac{v^\star+\sigma^2}{\sigma}\right)}\right],
\end{aligned}
\tag{3.39}
$$

where

$$
\begin{aligned}
\frac{\partial}{\partial A} R^{\frac{1}{\sigma}\Psi_h^-\left(\omega, \frac{v^\star+\sigma^2}{\sigma}\right)} &= \frac{-\frac{1}{\sigma}\Psi_h^-\left(\omega, \frac{v^\star+\sigma^2}{\sigma}\right)}{\bar{v}} \left(\frac{A}{\bar{v}}\right)^{-1-\frac{1}{\sigma}\Psi_h^-\left(\omega, \frac{v^\star+\sigma^2}{\sigma}\right)} \\
&= \frac{-\frac{1}{\sigma}\Psi_h^-\left(\omega, \frac{v^\star+\sigma^2}{\sigma}\right)}{\bar{v}} R^{1+\frac{1}{\sigma}\Psi_h^-\left(\omega, \frac{v^\star+\sigma^2}{\sigma}\right)}
\end{aligned}
\tag{3.40}
$$

and

$$
\begin{aligned}
\frac{\partial}{\partial A} R^{2a+2+\frac{1}{\sigma}\Psi_h^-\left(\omega, -\frac{v^\star+\sigma^2}{\sigma}\right)} &= \frac{-2a - 2 - \frac{1}{\sigma}\Psi_h^-\left(\omega, -\frac{v^\star+\sigma^2}{\sigma}\right)}{\bar{v}} \left(\frac{A}{\bar{v}}\right)^{-2a-3-\frac{1}{\sigma}\Psi_h^-\left(\omega, -\frac{v^\star+\sigma^2}{\sigma}\right)} \\
&= \frac{-2a - 2 - \frac{1}{\sigma}\Psi_h^-\left(\omega, -\frac{v^\star+\sigma^2}{\sigma}\right)}{\bar{v}} R^{2a+3+\frac{1}{\sigma}\Psi_h^-\left(\omega, -\frac{v^\star+\sigma^2}{\sigma}\right)}.
\end{aligned}
\tag{3.41}
$$

The derivative of the coupon function is

$$
\frac{\partial Coupon_0}{\partial A} = \frac{cL}{\varpi}\left[\Omega_h^-\left(\varpi, \frac{v^\star}{\sigma}\right) \frac{\partial}{\partial A} R^{\frac{1}{\sigma}\Psi_h^-\left(\varpi, \frac{v^\star}{\sigma}\right)} + \Omega_h^-\left(\varpi, -\frac{v^\star}{\sigma}\right) \frac{\partial}{\partial A} R^{2a+\frac{1}{\sigma}\Psi_h^-\left(\varpi, \frac{v^\star}{\sigma}\right)}\right],
\tag{3.42}
$$

where

$$
\frac{\partial}{\partial A} R^{\frac{1}{\sigma}\Psi_h^-\left(\varpi, \frac{v^\star}{\sigma}\right)} = \frac{-\frac{1}{\sigma}\Psi_h^-\left(\varpi, \frac{v^\star}{\sigma}\right)}{\bar{v}} R^{1+\frac{1}{\sigma}\Psi_h^-\left(\varpi, \frac{v^\star}{\sigma}\right)}
\tag{3.43}
$$

and

$$\frac{\partial}{\partial A}R^{2a+\frac{1}{\sigma}\Psi_h^-\left(\varpi,-\frac{v^\star}{\sigma}\right)}=\frac{-2a-\frac{1}{\sigma}\Psi_h^-\left(\varpi,-\frac{v^\star}{\sigma}\right)}{\bar{v}}R^{2a+1+\frac{1}{\sigma}\Psi_h^-\left(\varpi,-\frac{v^\star}{\sigma}\right)}. \tag{3.44}$$

Doing the same for the capex,

$$\frac{\partial Capex_0}{\partial A}=\frac{q}{\varpi}\left[\Omega_h^-\left(\varpi,\frac{v^\star}{\sigma}\right)\frac{\partial}{\partial A}R^{\frac{1}{\sigma}\Psi_h^-\left(\varpi,\frac{v^\star}{\sigma}\right)}+\Omega_h^-\left(\varpi,-\frac{v^\star}{\sigma}\right)\frac{\partial}{\partial A}R^{2a+\frac{1}{\sigma}\Psi_h^-\left(\varpi,\frac{v^\star}{\sigma}\right)}\right]. \tag{3.45}$$

For the recovered value after the firm closing, we have that

$$\frac{\partial EqRec_0}{\partial A}=\frac{\partial EqRecHit_0}{\partial A}+\frac{\partial EqRecJump_0}{\partial A}. \tag{3.46}$$

The derivative of the equity recovery hit function (3.18) corresponds to

$$\begin{aligned}\frac{\partial EqRecHit_0}{\partial A}&=\begin{cases}\left(\beta^{Bank}\bar{v}-L\right)\frac{\partial}{\partial A}\lim_{T\to+\infty}DigHit\left(T\right),\bar{v}\geq\frac{L}{\beta^{Bank}}\\ 0,\bar{v}<\frac{L}{\beta^{Bank}}\end{cases}\\ &=\begin{cases}\left(\beta^{Bank}\bar{v}-L\right)\left[\Omega_h^-\left(\varpi,\frac{v^\star}{\sigma}\right)\frac{\partial}{\partial A}R^{\frac{1}{\sigma}\Psi_h^-\left(\varpi,\frac{v^\star}{\sigma}\right)}\right.\\ \left.+\Omega_h^-\left(\omega,-\frac{v^\star}{\sigma}\right)\frac{\partial}{\partial A}R^{2a+\frac{1}{\sigma}\Psi_h^-\left(\omega,-\frac{v^\star}{\sigma}\right)}\right],\bar{v}\geq\frac{L}{\beta^{Bank}}\\ 0,\bar{v}<\frac{L}{\beta^{Bank}}\end{cases},\end{aligned} \tag{3.47}$$

where the above derivatives are given by equations (3.43) and (3.44).

For the derivative of the equity recovery jump function, equation (3.20), we have

$$\frac{\partial EqRecJump_0}{\partial A}=\frac{\partial EqRecJumpBank_0}{\partial A}+\frac{\partial EqRecJumpSold_0}{\partial A}. \tag{3.48}$$

For the first term above, it is necessary to derive either equations (3.22), (3.23) or zero (case 3). For the second term above, it is required to derive either equation (3.25) or (3.26). In all cases, one needs to compute the derivative of the limits of the $ANJump\left(\bar{l},\overline{u},T\right)$ and $DigJump\left(\bar{l},\overline{u},T\right)$ functions. These are given by equations (A.78) and (A.81), for the case where only T goes to $+\infty$, and by equations (A.82) and (A.84) for the cases where both $\overline{u}$ and T go to $+\infty$. Both are presented in Appendix A.5.

Notice that when $\lambda=0$, equation (3.36) is very similar to the one obtained in the Black-Scholes model. However, in contrast with the GBM process, it does not exhibit constant volatility because $\frac{\partial E}{\partial A}\frac{A_t}{E_{t^-}}$ is not constant. Since there is still only one source of uncertainty (i.e. the Brownian motion) the process is usually referred to be locally stochastic.

Taking the derivative of σ_{E_t} in order to A_t one observes a negative relation (i.e. equity volatility increases as the asset and stock prices decrease.). This is known in the literature as the leverage effect. Notice, however, that in constrast with models whose state variable is A_t (e.g. Toft and Prucyk (1997)) here σ_{E_t} is locally stochastic even if the firm has no liabilities. This occurs because the leverage effect results not only from financial leverage (i.e. debt) but also from operating leverage due to fixed costs such as capex expenditures.

Corporate managers frequently have to take decisions on whether to take or not a project. These decisions are usually called capital budgeting. In doing so they must compare the expected return on the project and their cost of capital (i.e. the rate of return stakeholders expect from them). Whenever managers take projects with rates of return below these expectations, they are intrinsically destroying value. But how much is that rate? Firms have two main classes of financial stakeholders, notably, shareholders and debtholders. Each one requires a different rate of return depending on risk. These are called the cost of equity and the cost of debt, respectively. Mathematically, these correspond to the drift of each of these assets under measure $\mathbb{P}$ plus the respective payouts. Taking equation (3.36) and changing the probability measure through Girsanov theorem one obtains

$$\frac{dE_t}{E_{t^-}} = \left(r - k_{E_t} + \overline{m}\sigma_{E_t} + \left(\overline{\lambda} - \lambda\right) j_{E_t}\right) dt + \sigma_{E_t} dW_t^{\mathbb{P}} - j_{E_t} dM_t^{\mathbb{P}}. \tag{3.49}$$

and thus the cost of equity equals $r + \overline{m}\sigma_{E_t} + \left(\overline{\lambda} - \lambda\right) j_{E_t}$.

Notice that, similarly to equation (2.42) regarding the asset rate of return, the cost of equity is simply equal to the risk free rate plus a premium per unit of volatility risk and a premium per unit of jump risk.

3.4. The endogenous barrier

At each moment in time, whenever the cash flow generated by the project is not enough to pay all duties, shareholders must inject capital in the firm. In a model with no information issues and where shareholders face no liquidity constraints, it is plausible to think that equity holders choose the default time τ strategically by solving the following stopping time problem:

$$\sup_{\tau \in \tau_{[0,+\infty]}} E_0(\tau), \tag{3.50}$$

where $E_0(\tau)$ is given by equation (3.1) as a function of τ. According to Décamps and Villeneuve (2014) it can be shown that this time-homogenous property implies that the optimal stopping time solution to this problem is a barrier strategy $\tau^\star = \inf\left\{s : \delta_t \leq \overline{\delta}\right\}$ where $\overline{\delta}$ is a positive constant, which can be determined by the classical smooth-pasting condition. More recently, Kyprianou and Surya (2007) prove that in the case of Lévy processes this depends on whether the process has unbounded variation leading to the following theorem.

Proposition 9. *Consider that $A_t = A_0 e^{X_t}$, where X_t is a spectrally negative Lévy process. If X_t has unbounded variation so that 0 is regular for the lower half-line $(-\infty, 0)$, then the bankruptcy-triggering asset level $\bar{v}$ satisfies the condition of smooth pasting; that is to say that $\overline{v}$ is chosen to satisfy*

$$\left.\frac{\partial E}{\partial A}\right|_{A=\overline{v}} = 0.$$

Proof. See Kyprianou and Surya (2007). □

As the process here considered satisfies the conditions described in Proposition 9, the optimal barrier is derived in the usual way. First take the derivative of equation (3.1), then substitute A_t by $\bar{v}$ and finally equate to 0. Based on equation (3.46), the first derivarive of equation (3.1) is simply the derivative of the payout function minus the derivatives of the coupon and capex functions plus the derivative of the equity recovery function.

The first is given by substituting equations (3.40) and (3.41) into equation (3.39). Then, replacing A by $\bar{v}$ one obtains

$$\begin{aligned}\left.\frac{\partial Payout_0}{\partial A}\right|_{A=\overline{v}} &= \frac{\mu_A - g}{\omega}\left[\Omega_h^-\left(\omega, \frac{v^\star + \sigma^2}{\sigma}\right) + \Omega_h^-\left(\omega, -\frac{v^\star + \sigma^2}{\sigma}\right) - 1\right] \\ &+ \frac{(\mu_A - g)\,\overline{v}}{\omega}\left[\Omega_h^-\left(\omega, \frac{v^\star + \sigma^2}{\sigma}\right)\frac{-\frac{1}{\sigma}\Psi_h^-\left(\omega, \frac{v^\star+\sigma^2}{\sigma}\right)}{\overline{v}}\right. \\ &\left. + \Omega_h^-\left(\omega, -\frac{v^\star + \sigma^2}{\sigma}\right)\frac{-2a - 2 - \frac{1}{\sigma}\Psi_h^-\left(\omega, -\frac{v^\star+\sigma^2}{\sigma}\right)}{\overline{v}}\right].\end{aligned} \tag{3.51}$$

The latter simplifies to

$$\begin{aligned}\left.\frac{\partial Payout_0}{\partial A}\right|_{A=\overline{v}} = \frac{\mu_A - g}{\omega}&\left\{\Omega_h^-\left(\omega, \frac{v^\star + \sigma^2}{\sigma}\right)\left[1 - \frac{1}{\sigma}\Psi_h^-\left(\omega, \frac{v^\star + \sigma^2}{\sigma}\right)\right]\right. \\ &\left. + \Omega_h^-\left(\omega, -\frac{v^\star + \sigma^2}{\sigma}\right)\left[-2a - 1 - \frac{1}{\sigma}\Psi_h^-\left(\omega, -\frac{v^\star + \sigma^2}{\sigma}\right)\right] - 1\right\}.\end{aligned} \tag{3.52}$$

Replacing equations (3.43) and (3.44) into equations (3.42) and (3.45) and then replacing A_t by $\bar{v}$, one obtains the correspondent terms for the coupon and capex functions:

$$\left.\frac{\partial Coupon_0}{\partial A}\right|_{A=\bar{v}} = \frac{cL}{\varpi\bar{v}}\left\{-\frac{1}{\sigma}\Omega_h^-\left(\varpi,\frac{v^\star}{\sigma}\right)\Psi_h^-\left(\varpi,\frac{v^\star}{\sigma}\right)\right. \\ \left.+\Omega_h^-\left(\varpi,-\frac{v^\star}{\sigma}\right)\left[-2a-\frac{1}{\sigma}\Psi_h^-\left(\varpi,-\frac{v^\star}{\sigma}\right)\right]\right\} \tag{3.53}$$

$$\left.\frac{\partial Capex_0}{\partial A}\right|_{A=\bar{v}} = \frac{q}{\varpi\bar{v}}\left\{-\frac{1}{\sigma}\Omega_h^-\left(\varpi,\frac{v^\star}{\sigma}\right)\Psi_h^-\left(\varpi,\frac{v^\star}{\sigma}\right)\right. \\ \left.+\Omega_h^-\left(\varpi,-\frac{v^\star}{\sigma}\right)\left[-2a-\frac{1}{\sigma}\Psi_h^-\left(\varpi,-\frac{v^\star}{\sigma}\right)\right]\right\}. \tag{3.54}$$

The derivative of the equity recovery function corresponds to the sum of the derivative of the equity recovery hit function and the equity recovery jump function. Starting with the hit function, substituting equations (3.43) and (3.44) into equation (3.47) and replacing A by $\bar{v}$ one obtains

$$\frac{\partial EqRecHit_0}{\partial A} = \begin{cases} \left(\beta^{Bank}-\frac{L}{\bar{v}}\right)\left[-\frac{1}{\sigma}\Psi_h^-\left(\varpi,\frac{v^\star}{\sigma}\right)\Omega_h^-\left(\varpi,\frac{v^\star}{\sigma}\right)\right. \\ \left.+\left(-2a-\frac{1}{\sigma}\Psi_h^-\left(\varpi,-\frac{v^\star}{\sigma}\right)\right)\Omega_h^-\left(\omega,-\frac{v^\star}{\sigma}\right)\right], \bar{v}\geq\frac{L}{\beta^{Bank}} \\ \\ 0, \bar{v}<\frac{L}{\beta^{Bank}} \end{cases}. \tag{3.55}$$

The derivative of the equity jump recovery term is given by summing either the derivative of equations (3.22), (3.23) or zero (case 3) with the derivative of either equation (3.25) or (3.26). As already referred, this requires the computation of the derivative of the limits of the $ANJump\left(\bar{l},\overline{u},T\right)$ and $DigJump\left(\bar{l},\overline{u},T\right)$ functions with two cases emerging depending on whether only T goes to $+\infty$ or both $\overline{u}$ and T go to $+\infty$.

Only T goes to $+\infty$

The derivative of $ANJump\left(\bar{l},\overline{u},T\right)$ evaluated at $A=\bar{v}$ when only T goes to $+\infty$ is computed taking equation (A.78) and replacing A by $\bar{v}$:[6]

[6]Notice that $\bar{l}$ and $\overline{u}$ are always bigger than $\bar{v}$ for all cases we are interested.

$$
\begin{aligned}
&\frac{\partial}{\partial A}\lim_{T\to+\infty} ANJump\left(\bar{l},\overline{u},T\right)\\
&=\frac{\overline{\lambda}}{\omega}\left\{-1-\lim_{T\to+\infty}F\left(\omega,\frac{\ln\left(\frac{\overline{u}}{\bar{v}}\right)}{\sigma},\frac{v^{\star}+\sigma^2}{\sigma},T\right)+\lim_{T\to+\infty}F\left(\omega,\frac{\ln\left(\frac{\bar{l}}{\bar{v}}\right)}{\sigma},\frac{v^{\star}+\sigma^2}{\sigma},T\right)\right.\\
&\left.+\left[-\lim_{T\to+\infty}F\left(\omega,\frac{\ln\left(\frac{\bar{v}}{\overline{u}}\right)}{\sigma},-\frac{v^{\star}+\sigma^2}{\sigma},T\right)+\lim_{T\to+\infty}F\left(\omega,\frac{\ln\left(\frac{\bar{v}}{\bar{l}}\right)}{\sigma},-\frac{v^{\star}+\sigma^2}{\sigma},T\right)\right]\right\}\\
&+\frac{\overline{\lambda}\bar{v}}{\omega}\left\{-\frac{\partial}{\partial A}\lim_{T\to+\infty}F\left(\omega,\frac{\ln\left(\frac{\overline{u}}{\bar{v}}\right)}{\sigma},\frac{v^{\star}+\sigma^2}{\sigma},T\right)+\frac{\partial}{\partial A}\lim_{T\to+\infty}F\left(\omega,\frac{\ln\left(\frac{\bar{l}}{\bar{v}}\right)}{\sigma},\frac{v^{\star}+\sigma^2}{\sigma},T\right)\right.\\
&+\frac{-2a-2}{\bar{v}}\left[-\lim_{T\to+\infty}F\left(\omega,\frac{\ln\left(\frac{\bar{v}}{\overline{u}}\right)}{\sigma},-\frac{v^{\star}+\sigma^2}{\sigma},T\right)+\lim_{T\to+\infty}F\left(\omega,\frac{\ln\left(\frac{\bar{v}}{\bar{l}}\right)}{\sigma},-\frac{v^{\star}+\sigma^2}{\sigma},T\right)\right]\\
&\left.+\left[-\frac{\partial}{\partial A}\lim_{T\to+\infty}F\left(\omega,\frac{\ln\left(\frac{\bar{v}}{\overline{u}}\right)}{\sigma},-\frac{v^{\star}+\sigma^2}{\sigma},T\right)+\frac{\partial}{\partial A}\lim_{T\to+\infty}F\left(\omega,\frac{\ln\left(\frac{\bar{v}}{\bar{l}}\right)}{\sigma},-\frac{v^{\star}+\sigma^2}{\sigma},T\right)\right]\right\}.
\end{aligned}
\tag{3.56}
$$

where the first two limits are given by equation (A.66), replacing A by $\bar{v}$, and the second two limits are given by equation (A.67) replacing R by 1. The derivatives are, respectively, given by equations (A.79) and (A.80), again replacing A by $\bar{v}$ and R by 1.

Doing the same for the derivative of $DigJump\left(\bar{l},\overline{u},T\right)$ when ony T goes to $+\infty$,[7]

$$
\begin{aligned}
&\frac{\partial}{\partial A}\lim_{T\to+\infty} DigJump\left(\bar{l},\overline{u},T\right)\\
&=\frac{\overline{\lambda}}{\varpi}\left\{-\frac{\partial}{\partial A}\lim_{T\to+\infty}F\left(\varpi,\frac{\ln\left(\frac{\overline{u}}{\bar{v}}\right)}{\sigma},\frac{v^{\star}}{\sigma},T\right)+\frac{\partial}{\partial\bar{v}}\lim_{T\to+\infty}F\left(\varpi,\frac{\ln\left(\frac{\bar{l}}{\bar{v}}\right)}{\sigma},\frac{v^{\star}}{\sigma},T\right)\right.\\
&+\frac{-2a}{\bar{v}}\left[-\lim_{T\to+\infty}F\left(\varpi,\frac{\ln\left(\frac{\bar{v}}{\overline{u}}\right)}{\sigma},-\frac{v^{\star}}{\sigma},T\right)+\lim_{T\to+\infty}F\left(\varpi,\frac{\ln\left(\frac{\bar{v}}{\bar{l}}\right)}{\sigma},-\frac{v^{\star}}{\sigma},T\right)\right]\\
&\left.+\left[-\frac{\partial}{\partial A}\lim_{T\to+\infty}F\left(\varpi,\frac{\ln\left(\frac{\bar{v}}{\overline{u}}\right)}{\sigma},-\frac{v^{\star}}{\sigma},T\right)+\frac{\partial}{\partial A}\lim_{T\to+\infty}F\left(\varpi,\frac{\ln\left(\frac{\bar{v}}{\bar{l}}\right)}{\sigma},-\frac{v^{\star}}{\sigma},T\right)\right]\right\},
\end{aligned}
\tag{3.57}
$$

[7]Notice again that $\bar{l}$ and $\overline{u}$ are always bigger than $\bar{v}$ for all cases we are interested.

where the first two limits can be computed using equation (A.66) replacing ω by ϖ, $\frac{v^{\star}+\sigma^2}{\sigma}$ by $\frac{v^{\star}}{\sigma}$ and A by $\bar{v}$ and the second two limits are given by equation A.67 doing the same replacements (this implies replacing R by 1). The derivatives are obtained doing the same substitutions in equations (A.79) and (A.80).

Both $\overline{u}$ and T go to $+\infty$

For the derivative of $ANJump\left(\bar{l},\overline{u},T\right)$ when both $\overline{u}$ and T go to $+\infty$, substituting equations (3.41) and (A.83) into equation (A.82) and replacing A by $\bar{v}$, one obtains[8]

$$\begin{aligned}
&\frac{\partial}{\partial A}\lim_{\bar{u},T\to+\infty} ANJump\left(\frac{L}{\beta^{Sold}\left(1-j\right)},\bar{u},T\right)\Bigg|_{A=\overline{v}} \\
&=\frac{\overline{\lambda}}{\omega}\left\{\Omega_h^+\left(\omega,\frac{v^{\star}+\sigma^2}{\sigma}\right)\left(\frac{\bar{l}}{\bar{v}}\right)^{-\frac{1}{\sigma}\Psi_h^+\left(\omega,\frac{v^{\star}+\sigma^2}{\sigma}\right)}+\Omega_h^-\left(\omega,-\frac{v^{\star}+\sigma^2}{\sigma}\right)\left(\frac{\bar{v}}{\bar{l}}\right)^{\frac{1}{\sigma}\Psi_h^-\left(\omega,-\frac{v^{\star}+\sigma^2}{\sigma}\right)}\right\} \\
&+\frac{\overline{\lambda}\overline{v}}{\omega}\left\{\Omega_h^+\left(\omega,\frac{v^{\star}+\sigma^2}{\sigma}\right)\frac{\frac{1}{\sigma}\Psi_h^+\left(\omega,\frac{v^{\star}+\sigma^2}{\sigma}\right)}{\bar{l}}\left(\frac{\bar{l}}{\bar{v}}\right)^{1-\frac{1}{\sigma}\Psi_h^+\left(\omega,\frac{v^{\star}+\sigma^2}{\sigma}\right)}\right. \\
&+\Omega_h^-\left(\omega,\frac{v^{\star}+\sigma^2}{\sigma}\right)\frac{-\frac{1}{\sigma}\Psi_h^-\left(\omega,\frac{v^{\star}+\sigma^2}{\sigma}\right)}{\bar{l}}\left(\frac{\bar{l}}{\bar{v}}\right)^{1\mp\frac{1}{\sigma}\Psi_h^-\left(\omega,\frac{v^{\star}+\sigma^2}{\sigma}\right)} \\
&\left.+\left(\frac{\bar{v}}{\bar{l}}\right)^{\frac{1}{\sigma}\Psi_h^-\left(\omega,-\frac{v^{\star}+\sigma^2}{\sigma}\right)}\Omega_h^-\left(\omega,-\frac{v^{\star}+\sigma^2}{\sigma}\right)\frac{-2a-2-\frac{1}{\sigma}\Psi_h^-\left(\omega,-\frac{v^{\star}+\sigma^2}{\sigma}\right)}{\bar{v}}\right\}.
\end{aligned} \tag{3.58}$$

Cancelling $\overline{v}$ on the last three expressions and factoring out, one obtains

$$\begin{aligned}
&\frac{\partial}{\partial A}\lim_{\bar{u},T\to+\infty} ANJump\left(\bar{l},\overline{u},T\right)\Bigg|_{A=\overline{v}} \\
&=\frac{\bar{\lambda}}{\omega}\left\{\Omega_h^+\left(\omega,\frac{v^{\star}+\sigma^2}{\sigma}\right)\left(\frac{\bar{l}}{\bar{v}}\right)^{-\frac{1}{\sigma}\Psi_h^+\left(\omega,\frac{v^{\star}+\sigma^2}{\sigma}\right)}\left[1+\frac{1}{\sigma}\Psi_h^+\left(\omega,\frac{v^{\star}+\sigma^2}{\sigma}\right)\right]\right. \\
&\left.+\Omega_h^-\left(\omega,-\frac{v^{\star}+\sigma^2}{\sigma}\right)\left(\frac{\bar{v}}{\bar{l}}\right)^{\frac{1}{\sigma}\Psi_h^-\left(\omega,-\frac{v^{\star}+\sigma^2}{\sigma}\right)}\left[1-2a-2-\frac{1}{\sigma}\Psi_h^-\left(\omega,-\frac{v^{\star}+\sigma^2}{\sigma}\right)\right]\right\}.
\end{aligned} \tag{3.59}$$

Noticing that $\Omega_h^+\left(\omega,\frac{v^{\star}+\sigma^2}{\sigma}\right)=-\Omega_h^-\left(\omega,-\frac{v^{\star}+\sigma^2}{\sigma}\right)$ and $\Psi_h^+\left(\omega,\frac{v^{\star}+\sigma^2}{\sigma}\right)=\Psi_h^-\left(\omega,-\frac{v^{\star}+\sigma^2}{\sigma}\right)$

[8]Notice that $\bar{l}$ is always bigger than $\bar{v}$ for all cases we are interested.

one can factor out further:

$$
\begin{aligned}
&\frac{\partial}{\partial A}\lim_{\bar{u},T\to+\infty} ANJump\left(\bar{l},\bar{u},T\right)\bigg|_{A=\bar{v}} \\
&= \frac{\bar{\lambda}}{\omega}\Omega_h^+\left(\omega,\frac{v^\star+\sigma^2}{\sigma}\right)\left(\frac{\bar{l}}{\bar{v}}\right)^{-\frac{1}{\sigma}\Psi_h^+\left(\omega,\frac{v^\star+\sigma^2}{\sigma}\right)}\left[\frac{2}{\sigma}\Psi_h^+\left(\omega,\frac{v^\star+\sigma^2}{\sigma}\right)+2a+2\right].
\end{aligned}
\tag{3.60}
$$

Finally, for the derivative of $DigJump\left(\bar{l},\overline{u},T\right)$ when both $\overline{u}$ and T go to $+\infty$, substituting equations (A.85) and (3.44) into equation (A.84) and replacing A by $\bar{v}$, one obtains

$$
\begin{aligned}
&\frac{\partial}{\partial A}\lim_{\bar{u},T\to+\infty} DigJump\left(\bar{l},\overline{u},T\right)\bigg|_{A=\overline{v}} \\
&= \frac{\bar{\lambda}}{\varpi}\left\{\Omega_h^+\left(\varpi,\frac{v^\star}{\sigma}\right)\frac{\frac{1}{\sigma}\Psi_h^+\left(\varpi,\frac{v^\star}{\sigma}\right)}{\bar{l}}\left(\frac{\bar{l}}{\overline{v}}\right)^{1-\frac{1}{\sigma}\Psi_h^+\left(\varpi,\frac{v^\star}{\sigma}\right)}\right. \\
&\left.+\left(\frac{\overline{v}}{\bar{l}}\right)^{\frac{1}{\sigma}\Psi_h^-\left(\varpi,-\frac{v^\star}{\sigma}\right)}\Omega_h^-\left(\varpi,-\frac{v^\star}{\sigma}\right)\frac{-2a-\frac{1}{\sigma}\Psi_h^-\left(\varpi,-\frac{v^\star}{\sigma}\right)}{\overline{v}}\right\}.
\end{aligned}
\tag{3.61}
$$

Multiplying and dividing the first term by $\overline{v}$ and simplifying the expression yields

$$
\begin{aligned}
&\tfrac{\partial}{\partial A}\lim_{\bar{u},T\to+\infty} DigJump\left(\bar{l},\overline{u},T\right)\bigg|_{A=\bar{v}} \\
&= \tfrac{\bar{\lambda}}{\varpi\bar{v}}\left\{\Omega_h^+\left(\varpi,\tfrac{v}{\sigma}\right)\tfrac{1}{\sigma}\Psi_h^+\left(\varpi,\tfrac{v^\star}{\sigma}\right)\left(\tfrac{\bar{l}}{\bar{v}}\right)^{-\frac{1}{\sigma}\Psi_h^+\left(\varpi,\frac{v^\star}{\sigma}\right)}\right. \\
&\left.+\left(\tfrac{\bar{l}}{\bar{v}}\right)^{-\frac{1}{\sigma}\Psi_h^-\left(\varpi,-\frac{v^\star}{\sigma}\right)}\Omega_h^-\left(\varpi,-\tfrac{v^\star}{\sigma}\right)\left[-2a-\tfrac{1}{\sigma}\Psi_h^-\left(\varpi,-\tfrac{v^\star}{\sigma}\right)\right]\right\}.
\end{aligned}
\tag{3.62}
$$

Finally, noting that $\Omega_h^+\left(\varpi,\frac{v^\star}{\sigma}\right) = -\Omega_h^-\left(\varpi,-\frac{v^\star}{\sigma}\right)$ and $\Psi_h^+\left(\varpi,\frac{v^\star}{\sigma}\right) = \Psi_h^-\left(\varpi,-\frac{v^\star}{\sigma}\right)$, then

$$
\begin{aligned}
&\frac{\partial}{\partial A}\lim_{\bar{u},T\to+\infty} DigJump\left(\bar{l},\overline{u},T\right)\bigg|_{A=\bar{v}} \\
&= \frac{\bar{\lambda}}{\varpi\bar{v}}\Omega_h^+\left(\varpi,\frac{v^\star}{\sigma}\right)\left(\frac{\bar{l}}{\bar{v}}\right)^{-\frac{1}{\sigma}\Psi_h^+\left(\varpi,\frac{v^\star}{\sigma}\right)}\left[\frac{2}{\sigma}\Psi_h^+\left(\varpi,\frac{v^\star}{\sigma}\right)+2a\right].
\end{aligned}
\tag{3.63}
$$

Taking the derivative of the equity function, equating to 0 and then solving it numerically, one obtains shareholders' optimal default barrier.

4. Debt

The value of debt corresponds to the sum of the market value of all the firm debt issues, which in this model are assumed to be perpetual. In order to simplify model presentation, most articles in contingent claim analysis assume a liability structure where all debt claims have the same level of seniority. However, for model calibration it is very useful to take seniority into account. For most firms, considering a liability structure with one or two types of debt issues is a good approximation of reality. For larger firms, however, we may have several layers of priority. For instance, in the case of banks, we may think of covered bonds[1], deposits (below and above the levels defined by deposit guarantee schemes), unsecured senior bond holders, subordinate debtholders or even preferred equity. Though absolute priority is not always respected in practice, the pecking order should be taken into account while using market information to calibrate model parameters. For this computation one only needs to know the amount of liabilities that are senior to the one being considered. We will denote it as X. Debt value can then be computed as the sum of two components: 1) the present value of all future after-tax coupons up to the moment the firm stops its activity ($(1-\bar{t}^{Debt})Coupon_0^{\star}$); and 2) the recovered value whenever the firm closes ($DbRec_0^{\star}$). Mathematically,

$$D_0^{\star} = \left(1 - \bar{t}^{Debt}\right) Coupon_0^{\star} + DbRec_0^{\star}. \tag{4.1}$$

The present value of all future coupons up to the moment the firm stops its activity is given by equation (3.14) with the specific issue coupon rate replacing c. * is used as superscript to differentiate from the case where the entire debt of the firm is considered. The value of the remaining component is given in the first section of this chapter. Similar to Chapter 3, the debt process and the cost of debt is then analysed in the second section of this chapter, which ends up with the derivation of the probability of default in this model.

[1] In the case where the firm has only one covered bond issue we can think of this covered bond as the most senior claim.

4.1. Recovered value after closing the firm

Similar to $EqRec_0$, $DbRec_0$ corresponds to the sum of the contributions from the cases where the project value hits the barrier and when the jump occurs.

$$DbRec_0^{\star} = DbRecHit_0^{\star} + DbRecJump_0^{\star}. \tag{4.2}$$

4.1.1. Recovered value after hitting the barrier

We may have three mutually exclusive cases depending on the relationship between the recovered value at the barrier, the amount of liabilities that are senior to the considered debt classes and the size of the debt issue. The first case arises when the recovered value at the barrier is smaller than the amount of senior liabilities. In this case, all the recovered value accrues to senior debtholders, implying that subordinate debtholders receive nothing. The second case arises when the recovered value at the barrier is sufficient to pay all debtholders that are senior to the one being priced but is not enough to cover all nominal liabilities. In this case, the debtholder under consideration receives the difference. Finally, whenever the recovered value is enough to cover X and $L^{\star}$, the contribution to the value of our debt issue is simply the discounted nominal value. Mathematically, these three cases correspond to

$$\begin{aligned} DbRecHit_0^{\star} &= \lim_{T\to+\infty} \begin{cases} 0, \beta^{Bank}\bar{v} \leq X \\ \left(\beta^{Bank}\bar{v} - X\right) E^{\mathbb{Q}}\left[e^{-r\tau}1_{\{\tau<T,\hat{\tau}>\tau\}} \left|\mathcal{F}_0\right.\right], X < \beta^{Bank}\bar{v} \leq X + L^{\star} \\ L^{\star} E^{\mathbb{Q}}\left[e^{-r\tau}1_{\{\tau<T,\hat{\tau}>\tau\}} \left|\mathcal{F}_0\right.\right], \beta^{Bank}\bar{v} > X + L^{\star} \end{cases} \\ &= \lim_{T\to+\infty} \begin{cases} 0, \beta^{Bank}\bar{v} \leq X \\ \left(\beta^{Bank}\bar{v} - X\right) DigHit\left(T\right), X < \beta^{Bank}\bar{v} \leq X + L^{\star} \\ L^{\star} DigHit\left(T\right), \beta^{Bank}\bar{v} > X + L^{\star} \end{cases}, \end{aligned} \tag{4.3}$$

where $\lim\limits_{T\to+\infty} DigHit\left(T\right)$ is given by equation (3.19). The limits of $F\left(.\right)$ are computed using equations (A.61) and (A.62) in Appendix A.4 with ϖ instead of ω and $\frac{v}{\sigma}$ replacing $\frac{v+\sigma^2}{\sigma}$.

4.1.2. Recovered value after jump

Depending on the relation between the asset value after the jump and the barrier, we may have two cases:

$$DbRecJump_0^{\star} = DbRecJumpBank_0^{\star} + DbRecJumpSold_0^{\star}. \tag{4.4}$$

The firm is liquidated with distress costs: $DbRecJumpBank_0^\star$

When the project value stays below the barrier after the jump, meaning that $A_{\hat{\tau}^-} \in \left[\overline{v}, \frac{\overline{v}}{1-j}\right]$ there might be three types of payoffs. These will depend on the relation between the recovered value, $\beta^{Bank}(1-j)A_{\hat{\tau}^-}$, the amount of senior liabilities and the size of the debt issue considered. Mathematically, the nominal recovered value equals

$$DbRecJumpBank_{\hat{\tau}}^\star = \begin{cases} 0, \overline{v} < A_{\hat{\tau}^-} < \frac{X}{\beta^{Bank}(1-j)} \\ \beta^{Bank}(1-j)A_{\hat{\tau}^-} - X, \frac{X}{\beta^{Bank}(1-j)} < A_{\hat{\tau}^-} < \frac{X+L^\star}{\beta^{Bank}(1-j)} \\ L^\star, \frac{X+L^\star}{\beta^{Bank}(1-j)} < A_{\hat{\tau}^-} < \frac{\overline{v}}{1-j} \end{cases}. \quad (4.5)$$

However, there is no guarantee that $\frac{X}{\beta^{Bank}(1-j)} > \overline{v}$ and that $\frac{\overline{v}}{1-j} \geq \frac{X+L^\star}{\beta^{Bank}(1-j)}$. In addition, it may occur that $\frac{\overline{v}}{1-j} < \frac{X}{\beta^{Bank}(1-j)}$ and $\overline{v} > \frac{X+L}{\beta^{Bank}(1-j)}$. As a result, six cases may emerge depending on the parameter values. I will start with the most general case where all payoffs are possible. I will then move to the two cases where two types of payoffs are possible and finish with the three extreme cases where only one type of payoff may occur.

First case: $\overline{v} \leq \frac{X}{\beta^{Bank}(1-j)} \leq \frac{X+L^\star}{\beta^{Bank}(1-j)} \leq \frac{\overline{v}}{1-j}$

In this case, all payoffs are possible depending on the asset value when the event occurs. Therefore,

$$\begin{aligned} & DbRecJumpBank_0^\star \\ &= \lim_{T\to+\infty} E^{\mathbb{Q}}\left[e^{-r\hat{\tau}}\left[\beta^{Bank}(1-j)A_{\hat{\tau}^-} - X\right] 1_{\left\{\frac{X}{\beta^{Bank}(1-j)} < A_{\hat{\tau}^-} < \frac{X+L^\star}{\beta^{Bank}(1-j)}, \tau > \hat{\tau}, \hat{\tau} < T\right\}} \middle| \mathcal{F}_0 \right] \\ &+ L^\star \lim_{T\to+\infty} E^{\mathbb{Q}}\left[e^{-r\hat{\tau}} 1_{\left\{\frac{X+L^\star}{\beta^{Bank}(1-j)} < A_{\hat{\tau}^-} < \frac{\overline{v}}{1-j}, \tau > \hat{\tau}, \hat{\tau} < T\right\}} \middle| \mathcal{F}_0 \right] \\ &= \beta^{Bank}(1-j) \lim_{T\to+\infty} ANJump\left(\frac{X}{\beta^{Bank}(1-j)}, \frac{X+L^\star}{\beta^{Bank}(1-j)}, T\right) \\ &- X \lim_{T\to+\infty} DigJump\left(\frac{X}{\beta^{Bank}(1-j)}, \frac{X+L^\star}{\beta^{Bank}(1-j)}, T\right) \\ &+ L^\star \lim_{T\to+\infty} DigJump\left(\frac{X+L^\star}{\beta^{Bank}(1-j)}, \frac{\overline{v}}{1-j}, T\right). \end{aligned} \quad (4.6)$$

Second case: $\overline{v} \leq \frac{X}{\beta^{Bank}(1-j)} \leq \frac{\overline{v}}{1-j} \leq \frac{X+L^\star}{\beta^{Bank}(1-j)}$

Two types of payoff are possible. The debtholder either receives the discounted recovered value (after senior debtholders share) or nothing. Hence,

$$\begin{aligned}
&DbRecJumpBank_0^{\star} \\
&= \lim_{T\to+\infty} E^{\mathbb{Q}}\left[e^{-r\hat{\tau}} \left[\beta^{Bank}(1-j) A_{\hat{\tau}^-} - X\right] 1_{\left\{\frac{X}{\beta^{Bank}(1-j)} < A_{\hat{\tau}^-} < \frac{\bar{v}}{1-j}, \tau>\hat{\tau}, \hat{\tau}<T\right\}} \middle| \mathcal{F}_0 \right] \\
&= \beta^{Bank}(1-j) \lim_{T\to+\infty} ANJump\left(\frac{X}{\beta^{Bank}(1-j)}, \frac{\bar{v}}{1-j}, T\right) \\
&- X \lim_{T\to+\infty} DigJump\left(\frac{X}{\beta^{Bank}(1-j)}, \frac{\bar{v}}{1-j}, T\right).
\end{aligned} \tag{4.7}$$

Third case: $\frac{X}{\beta^{Bank}(1-j)} \leq \bar{v} \leq \frac{X+L^{\star}}{\beta^{Bank}(1-j)} \leq \frac{\bar{v}}{1-j}$

Again, two types of payoff are possible. This time the debtholder either receives everything or the discounted recovered value (after senior debtholders share):

$$\begin{aligned}
&DbRecJumpBank_0^{\star} \\
&= \lim_{T\to+\infty} E^{\mathbb{Q}}\left[e^{-r\hat{\tau}} \left[\beta^{Bank}(1-j) A_{\hat{\tau}^-} - X\right] 1_{\left\{\bar{v} < A_{\hat{\tau}^-} < \frac{X+L^{\star}}{\beta^{Bank}(1-j)}, \tau>\hat{\tau}, \hat{\tau}<T\right\}} \middle| \mathcal{F}_0 \right] \\
&+ L^{\star} \lim_{T\to+\infty} E^{\mathbb{Q}}\left[e^{-r\hat{\tau}} 1_{\left\{\frac{X+L^{\star}}{\beta^{Bank}(1-j)} < A_{\hat{\tau}^-} < \frac{\bar{v}}{1-j}, \tau>\hat{\tau}, \hat{\tau}<T\right\}} \middle| \mathcal{F}_0 \right] \\
&= \beta^{Bank}(1-j) \lim_{T\to+\infty} ANJump\left(\bar{v}, \frac{X+L^{\star}}{\beta^{Bank}(1-j)}, T\right) \\
&- X \lim_{T\to+\infty} DigJump\left(\bar{v}, \frac{X+L^{\star}}{\beta^{Bank}(1-j)}, T\right) \\
&+ L^{\star} \lim_{T\to+\infty} DigJump\left(\frac{X+L^{\star}}{\beta^{Bank}(1-j)}, \frac{\bar{v}}{1-j}, T\right).
\end{aligned} \tag{4.8}$$

Fourth case: $\bar{v} \leq \frac{\bar{v}}{1-j} \leq \frac{X}{\beta^{Bank}(1-j)} \leq \frac{X+L^{\star}}{\beta^{Bank}(1-j)}$

In this case, the debtholder knows beforehand that it will always receive nothing after a sudden jump if distress costs are incurred. Thus,

$$DbRecJumpBank_0^{\star} = 0.$$

Fifth case: $\frac{X}{\beta^{Bank}(1-j)} \leq \bar{v} \leq \frac{\bar{v}}{1-j} \leq \frac{X+L^{\star}}{\beta^{Bank}(1-j)}$

In this case, the debtholder knows that it will always receive the discounted recovered value:

$$DbRecJumpBank_0^\star = \lim_{T\to+\infty} E^{\mathbb{Q}}\left[e^{-r\hat{\tau}}\left[\beta^{Bank}(1-j)A_{\hat{\tau}-} - X\right]1_{\left\{\bar{v}<A_{\hat{\tau}-}<\frac{\bar{v}}{1-j},\tau>\hat{\tau},\hat{\tau}<T\right\}}\middle|\mathcal{F}_0\right]$$
$$= \beta^{Bank}(1-j)\lim_{T\to+\infty} ANJump\left(\bar{v},\frac{\overline{v}}{1-j},T\right)$$
$$- X\lim_{T\to+\infty} DigJump\left(\bar{v},\frac{\overline{v}}{1-j},T\right). \tag{4.9}$$

Sixth case: $\frac{X}{\beta^{Bank}(1-j)} \leq \frac{X+L^\star}{\beta^{Bank}(1-j)} \leq \overline{v} \leq \frac{\overline{v}}{1-j}$

In the last case, the debtholders know that they will always recover everything, meaning that their investment has no credit risk:

$$DbRecJumpBank_0^\star = L^\star \lim_{T\to+\infty} E^{\mathbb{Q}}\left[e^{-r\hat{\tau}}1_{\left\{\bar{v}<A_{\hat{\tau}-}<\frac{\bar{v}}{1-j},\tau>\hat{\tau},\hat{\tau}<T\right\}}\middle|\mathcal{F}_0\right]$$
$$= L^\star \lim_{T\to+\infty} DigJump\left(\bar{v},\frac{\overline{v}}{1-j},T\right). \tag{4.10}$$

All limits presented can be computed using either equation (A.65) or (A.68) in Appendix A.4.

The firm is liquidated without distress costs $DbRecJumpSold_0^\star$

Again, there might be three types of payoffs depending on the relation between the asset value before the jump event, the amount of senior liabilities and the size of the debt issue considered. The intervals considered for each type of payoff are nevertheless different since distress costs are not incurred in this case. Thus,

$$DbRecJumpSold_{\hat{\tau}}^\star = \begin{cases} 0, \frac{\bar{v}}{1-j} < A_{\hat{\tau}-} < \frac{X}{\beta^{Sold}(1-j)} \\ (1-j)A_{\hat{\tau}-}, \frac{X}{\beta^{Sold}(1-j)} < A_{\hat{\tau}-} < \frac{X+L^\star}{\beta^{Sold}(1-j)} \\ L^\star, \frac{X+L^\star}{\beta^{Sold}(1-j)} < A_{\hat{\tau}-} < +\infty \end{cases}. \tag{4.11}$$

Once more, there is no guarantee that $\frac{X}{\beta^{Sold}(1-j)} > \frac{\bar{v}}{1-j}$. In addition it can occur that $\frac{\overline{v}}{1-j} > \frac{X+L^\star}{\beta^{Sold}(1-j)}$. Notwithstanding, we know that $\frac{X}{\beta^{Sold}(1-j)} \leq \frac{X+L^\star}{\beta^{Sold}(1-j)} \leq \frac{L}{\beta^{Sold}(1-j)}$ and thus this time we have only three cases.

First case: $\frac{\overline{v}}{1-j} \leq \frac{X}{\beta^{Sold}(1-j)} \leq \frac{X+L^\star}{\beta^{Sold}(1-j)} \leq +\infty$

Depending on the project value when the jump occurs, all payoffs are possible. Therefore,

$$
\begin{aligned}
&DbRecJumpSold_0^\star \\
&= \lim_{T\to+\infty} E^{\mathbb{Q}}\left[e^{-r\hat{\tau}} \left[\left(\beta^{Sold}(1-j)\right) A_{\hat{\tau}^-} - X \right] 1_{\left\{ \frac{X}{\beta^{Sold}(1-j)} < A_{\hat{\tau}^-} < \frac{X+L^\star}{\beta^{Sold}(1-j)}, \tau > \hat{\tau}, \hat{\tau} < T \right\}} \middle| \mathcal{F}_0 \right] \\
&+ L^\star \lim_{T\to+\infty} E^{\mathbb{Q}}\left[e^{-r\hat{\tau}} 1_{\left\{ \frac{X+L^\star}{\beta^{Sold}(1-j)} < A_{\hat{\tau}^-} < +\infty, \tau > \hat{\tau}, \hat{\tau} < T \right\}} \middle| \mathcal{F}_0 \right] \\
&= \beta^{Sold}(1-j) \lim_{T\to+\infty} ANJump\left(\frac{X}{\beta^{Sold}(1-j)}, \frac{X+L^\star}{\beta^{Sold}(1-j)}, T \right) \\
&- X \lim_{T\to+\infty} DigJump\left(\frac{X}{\beta^{Sold}(1-j)}, \frac{X+L^\star}{\beta^{Sold}(1-j)}, T \right) \\
&+ L^\star \lim_{\overline{u},T\to+\infty} DigJump\left(\frac{X+L^\star}{\beta^{Sold}(1-j)}, \overline{u}, T \right).
\end{aligned}
\tag{4.12}
$$

Second case: $\frac{X}{\beta^{Sold}(1-j)} \leq \frac{\bar{v}}{1-j} \leq \frac{X+L^\star}{\beta^{Sold}(1-j)} \leq +\infty$

In this case, the debtholder knows beforehand that his payoff will be either the nominal debt value or the discounted asset value after deducting the senior debtholders payoff:

$$
\begin{aligned}
&DbRecJumpSold_0^\star \\
&= \lim_{T\to+\infty} E^{\mathbb{Q}}\left[e^{-r\hat{\tau}} \left[\left(\beta^{Sold}(1-j)\right) A_{\hat{\tau}^-} - X \right] 1_{\left\{ \frac{\bar{v}}{1-j} < A_{\hat{\tau}^-} < \frac{X+L^\star}{\beta^{Sold}(1-j)}, \tau > \hat{\tau}, \hat{\tau} < T \right\}} \middle| \mathcal{F}_0 \right] \\
&+ L^\star \lim_{T\to+\infty} E^{\mathbb{Q}}\left[e^{-r\hat{\tau}} 1_{\left\{ \frac{X+L^\star}{\beta^{Sold}(1-j)} < A_{\hat{\tau}^-} < +\infty, \tau > \hat{\tau}, \hat{\tau} < T \right\}} \middle| \mathcal{F}_0 \right] \\
&= \beta^{Sold}(1-j) \lim_{T\to+\infty} ANJump\left(\frac{\bar{v}}{1-j}, \frac{X+L^\star}{\beta^{Sold}(1-j)}, T \right) \\
&- X \lim_{T\to+\infty} DigJump\left(\frac{\bar{v}}{1-j}, \frac{X+L^\star}{\beta^{Sold}(1-j)}, T \right) \\
&+ L^\star \lim_{\overline{u},T\to+\infty} DigJump\left(\frac{X+L^\star}{\beta^{Sold}(1-j)}, \overline{u}, T \right).
\end{aligned}
\tag{4.13}
$$

Third case: $\frac{X}{\beta^{Sold}(1-j)} \leq \frac{X+L^\star}{\beta^{Sold}(1-j)} \leq \frac{\bar{v}}{1-j} \leq +\infty$

In the third case the debtholder knows that his payoff will be always equal to the nominal debt value. Hence,

$$
\begin{aligned}
DbRecJumpSold_0^{\star} &= L^{\star} \lim_{T\to+\infty} E^{\mathbb{Q}}\left[e^{-r\hat{\tau}} 1_{\left\{\frac{\bar{v}}{1-j}<A_{\hat{\tau}-}<+\infty,\tau>\hat{\tau},\hat{\tau}<T\right\}} \middle| \mathcal{F}_0 \right] \\
&= L^{\star} \lim_{\overline{u},T\to+\infty} DigJump\left(\frac{\bar{v}}{1-j},\overline{u},T\right).
\end{aligned}
\tag{4.14}
$$

Again, all limits presented can be computed using equation (A.77), (A.65) and (A.68) in Appendix A.4.

Similar to equity, the simplifying assumption that the firm is closed after a sudden negative jump, even if the project value is above the barrier, affects debt pricing. Again, it is not possible to have a precise number for the impact of this assumption. However, under the hypothesis that 1) the barrier is independent from this fact and 2) the firm is liquidated after a second jump, no matter the project value, one can have an idea of the impact of this assumption. Define $\overline{DbRecJumpSold_0}$ as the correct debt recovered value when the project stays above the barrier after a sudden jump and $\overline{D}(A)$ as the correct debt pricing function for each value of A_t. Approximating $\overline{D}(A)$ with $D(A)$, which is given by equation (4.1), then

$$
\overline{DbRecJumpSold_0} \approx \lim_{T\to\infty} E\left[e^{-r\hat{\tau}} D\left((1-j)\,A_{\hat{\tau}-}\right) 1_{\left\{\bar{l}<A_{\hat{\tau}-}<+\infty,\tau>\hat{\tau},\hat{\tau}<T\right\}}\right]. \tag{4.15}
$$

Procceding as in equations (2.81) and (2.82),

$$
\begin{aligned}
\overline{DbRecJumpSold_0} \approx \bar{\lambda} \lim_{T,\bar{u}\to\infty} \int_0^T \int_{\bar{l}}^{\bar{u}} & D\left((1-j)\,x\right) e^{-\left(r+\bar{\lambda}\right)\hat{u}} \left(\frac{d}{dx} N\left(h_1\left(x,\hat{u}\right)\right)\right. \\
&\left. + R^{2a} \frac{d}{dx} N\left(h_2\left(x,\hat{u}\right)\right)\right) d\hat{u}dx.
\end{aligned}
\tag{4.16}
$$

Similar to equity, to the best of my knowledge the only way to solve this integral is numerically. The difference between $DbRecJumpSold_0$ and $\overline{DbRecJumpSold_0}$ gives the analyst a good idea of the pricing error that arises from the assumption that the firm closes after the jump even if the project value stays above the barrier after the jump.

4.2. The debt process and the cost of debt

Similarly to Section 3.3, the dynamics of debt can be derived by applying Itô's lemma to the debt function. In this case, one obtains

$$\frac{dD_t}{D_{t^-}} = \left((\mu_\delta - \overline{m}\sigma_A) \frac{\partial D}{\partial A} \frac{A_t}{D_{t^-}} + 0.5 \frac{\partial^2 D}{\partial A^2} \frac{A_t^2}{D_{t^-}} \sigma_A^2 - \overline{\lambda} j_{D_t} \right) dt + \sigma_{D_t} dW_t^{\mathbb{Q}} - j_{D_t} dM_t^{\mathbb{Q}}, \quad (4.17)$$

where

$$\sigma_{D_t} = \frac{\partial D}{\partial A} \frac{A_t}{D_{t^-}} \sigma_A$$

$$j_{D_t} = \frac{D(A_{t^-}) - min\{ \left[\beta^{Bank} 1_{\{(1-j_A)A_{t^-} \leq \bar{v}\}} + \beta^{Sold} 1_{\{(1-j_A)A_{t^-} > \bar{v}\}} \right] (1 - j_A) A_{t^-} - X, L^\star \}}{D(A_{t^-})}. \quad (4.18)$$

Alternatively, one can rely on measure $\mathbb{Q}$ definition. In this case, we have that

$$\frac{dD_t}{D_{t^-}} = (r - k_{D_t}) dt + \sigma_{D_t} dW_t^{\mathbb{Q}} - j_{D_t} dM_t^{\mathbb{Q}}, \quad (4.19)$$

where

$$k_{D_t} = \frac{cLdt}{D_t}. \quad (4.20)$$

Using equation (4.19) one avoids the computation of the second derivative of the debt function. For the first derivative, and using equation (4.1), we have that

$$\frac{\partial D}{\partial A} = \left(1 - \overline{t}^{Debt}\right) \frac{\partial Coupon_0}{\partial A} + \frac{\partial DbRec_0}{\partial A}, \quad (4.21)$$

where the first term in equation (4.21) is given by equation (3.42) and the second term corresponds to

$$\frac{\partial DbRec}{\partial A} = \frac{\partial DbRecHit_0}{\partial A} + \frac{\partial DbRecJump_0}{\partial A}. \quad (4.22)$$

The derivative of the debt recovery hit function is

$$\frac{\partial DbRecHit_0}{\partial A} = \begin{cases} 0, \beta^{Bank}\overline{v} \leq X \\ \left(\beta^{Bank}\overline{v} - X\right) \frac{\partial}{\partial A} \lim_{T \to +\infty} DigHit\left(T\right), X < \beta^{Bank}\overline{v} \leq X + L^{\star} \\ L^{\star} \frac{\partial}{\partial A} \lim_{T \to +\infty} DigHit\left(T\right), \beta^{Bank}\overline{v} > X + L^{\star} \end{cases}, \quad (4.23)$$

where

$$\frac{\partial}{\partial A} \lim_{T \to +\infty} DigHit\left(T\right) = \Omega_h^{-}\left(\varpi, \frac{v^{\star}}{\sigma}\right) \frac{\partial}{\partial A} R^{\frac{1}{\sigma}\Psi_h^{-}\left(\varpi, \frac{v^{\star}}{\sigma}\right)} + \Omega_h^{-}\left(\omega, -\frac{v^{\star}}{\sigma}\right) \frac{\partial}{\partial A} R^{2a + \frac{1}{\sigma}\Psi_h^{-}\left(\omega, -\frac{v^{\star}}{\sigma}\right)}, \quad (4.24)$$

with the above derivatives given by equations (3.43) and (3.44).

For the derivative of the debt recovery jump function, using equation (4.4), we have that

$$\frac{\partial DbRecJump_0}{\partial A} = \frac{\partial DbRecJumpBank_0}{\partial A} + \frac{\partial DbRecJumpSold_0}{\partial A}. \quad (4.25)$$

Similarly to the derivatives of the equity recovery jump function, the above terms require the computation of the derivative of the limits of the $ANJump\left(\overline{l}, \overline{u}, T\right)$ and $DigJump\left(\overline{l}, \overline{u}, T\right)$ functions, which are given by equations (A.78) and (A.81) for the case where only T goes to $+\infty$ and equations (A.82) and (A.84) for the cases where both $\overline{u}$ and T go to $+\infty$. Both are given in Appendix A.5.

Changing the probability measure to $\mathbb{P}$ and summing k_{D_t} to the drift in equation (4.19) one obtains the cost of debt, which corresponds to $r + \overline{m}\sigma_{D_t} + \left(\overline{\lambda} - \lambda\right) j_{D_t}$.

4.3. The probability of default

The first two sections of this chapter presented formulas for computing the market price of debt issues. This was done by summing the present value of all future coupons with the present value of the recovered values. Both under measure $\mathbb{Q}$. As further explained in Chapter 6, the difference between the nominal promissed value and the recovered value is interpreted as the expected loss. A common standpoint in credit risk literature is to decompose the expected loss between the probability of default and the loss given default. This decomposition can be very useful to understand the firm's credit risk profile. This section shows how the probability of default can be computed in this model under measure

$\mathbb{Q}$, so that this decomposition can be made. For someone interested in the true probability of the firm defaulting on its obligations, one should use measure $\mathbb{P}$ instead. This can be easily obtained replacing $v^{\star} = \mu_{\delta} - \overline{m}\sigma - 0.5\sigma^2$ by $v^{\star} = \mu_{\delta} - 0.5\sigma^2$ and $\overline{\lambda}$ by λ in Propositions 4 and 5.[2] It is interesting to note that in both cases the probability of default depends on risk pricing parameters through the project value.

Before all else, it is important to define default. In this model, it is considered that the firm defaults if the debtholders receive less than L when the firm closes its business. Notice that default does not need to imply distress costs. In the case where the asset jumps to below L but stays above $\bar{v}$ the firm defaults but there is no default costs. As in previous sections we will decompose the probability of default in two components:

$$PD(T) = PD^{Hit}(T) + PD^{Jump}(T), \tag{4.26}$$

where $PD^{Hit}(T)$ and PD^{Jump} are, respectively, the probability of the firm defaulting after the process hitting the barrier and after a sudden jump up to time T.

4.3.1. Probability of default after hitting the barrier

Regarding $PD^{Hit}(T)$, it is clear that this is 0 whenever $\bar{v} > \frac{L}{\beta^{Bank}}$. In this case, diffusion is not able to lead the firm to default. Thus, we have that

$$PD^{Hit}(T) = \begin{cases} 0, \bar{v} \geq \frac{L}{\beta^{Bank}} \\ \mathbb{Q}\left(\tau < T, \hat{\tau} > \tau \,|\mathcal{F}_0\right), \bar{v} < \frac{L}{\beta^{Bank}} \end{cases} \tag{4.27}$$

$$= \begin{cases} 0, \bar{v} \geq \frac{L}{\beta^{Bank}} \\ DigHit^{\star}\left(T\right), \bar{v} < \frac{L}{\beta^{Bank}} \end{cases}, \tag{4.28}$$

where $DigHit^{\star}\left(T\right)$ is given by equation (2.80).

4.3.2. Probability of default after a jump

Regarding $PD^{Jump}(T)$, one should consider two cases: 1) default with distress costs; and default without distress costs. Therefore,

$$PD^{Jump}(T) = PD^{JumpBank}(T) + PD^{JumpSold}(T). \tag{4.29}$$

[2]The terms $DigHit^{\star}\left(T\right)$ and $DigJump^{\star}\left(\bar{l}, \overline{u}, T\right)$ in equations (4.27)-(4.32) must then be computed accordingly.

These correspond to the cases where shareholders do not receive anything after the jump. The formulae for the probability of default is thus straightforward if one follows the opposite logic to Section 3.2.2.

Probability of default with distress costs

Similarly to Section 3.2.2, we may have three cases here.

First case: $\overline{v} \leq \frac{L}{\beta^{Bank}(1-j)} \leq \frac{\overline{v}}{1-j}$

$$\begin{aligned} PD^{JumpBank}(T) &= \mathbb{Q}\left(\bar{v} < A_{\hat{\tau}^-} < \frac{L}{\beta^{Bank}(1-j)}, \tau > \hat{\tau}, \hat{\tau} < T\right) \\ &= DigJump^{\star}\left(\bar{v}, \frac{L}{\beta^{Bank}(1-j)}, T\right). \end{aligned} \tag{4.30}$$

Second case: $\frac{L}{\beta^{Bank}(1-j)} \leq \overline{v} \leq \frac{\overline{v}}{1-j}$

In this case, the probability of default is always 0.

Third case: $\overline{v} \leq \frac{\overline{v}}{1-j} \leq \frac{L}{\beta^{Bank}(1-j)}$

In this case the firm always defaults after incurring distress costs and thus

$$\begin{aligned} PD^{JumpBank}(T) &= \mathbb{Q}\left(\bar{v} < A_{\hat{\tau}^-} < \frac{\bar{v}}{1-j}, \tau > \hat{\tau}, \hat{\tau} < T\right) \\ &= DigJump^{\star}\left(\bar{v}, \frac{\bar{v}}{1-j}, T\right). \end{aligned} \tag{4.31}$$

Probability of default without distress costs

Following again Section 3.2.2, we may have two cases here.

First case: $\frac{\overline{v}}{1-j} \leq \frac{L}{\beta^{Sold}(1-j)} \leq +\infty$

$$\begin{aligned} PD^{JumpSold}(T) &= \mathbb{Q}\left(\frac{\bar{v}}{1-j} < A_{\hat{\tau}^-} < \frac{L}{\beta^{Sold}(1-j)}, \tau > \hat{\tau}, \hat{\tau} < T\right) \\ &= DigJump^{\star}\left(\frac{\bar{v}}{1-j}, \frac{L}{\beta^{Sold}(1-j)}, T\right). \end{aligned} \tag{4.32}$$

Second case: $\frac{L}{\beta^{Sold}(1-j)} \leq \frac{\overline{v}}{1-j} \leq +\infty$

In this case, the firm never defaults without distress costs.

5. External claimants, firm value and the optimal capital structure

Chapters 3 and 4 presented formulas for computing shareholders and debtholders' claims on the project. These are not the only agents whose payoff depends on the latter. The government payoff increases with the value of the project. Also, capex suppliers are better off when the firm is running than after bankruptcy. In contrast, distress costs only occur when the firm goes bankrupt. The first section of this chapter presents formulas for computing the value of these external claims. The second and third sections of these chapter discuss two very important corporate finance concepts, notably, the optimal capital structure and the cost of capital.

5.1. External claimants

Our project has three external claimants, notably, the government, capex suppliers and distress costs. Government and capex suppliers receive a continuous stream of cash flows as long as the firm exhists. In the case of the government, this is a variable stream (taxes). In the case of capex suppliers, it is a fixed stream. In contrast, distress costs only occur when it is optimal for the shareholders to close the firm. Notice that whenever the firm closes, the project is sold for the project value at that time with the firm receiving $\beta^{Sold} A_{\tau^{Solv}}$. Depending on whether distress costs are incurred or not (i.e. whether the project value is lower or equal to $\bar{v}$), the firm loses $\left(\beta^{Sold} - \beta^{Bank}\right) A_{\tau^{Solv}}$ due to distress costs. As referred in Section 2.2, these correspond to costs with lawyers and value destruction caused by fire sales and loss of intangible value. Since the project continues, in either case there is a residual claim that does not belong to the firm; it belongs to external claimants.[1] This corresponds to

[1]The hypothesis that the project continues forever might look strange at first sight. However, one may look to any project as being composed by tangible and intangible assets. While the intangible assets may be destroyed when the firm closes, tangible assets such as land, buildings or machinery continue in the

$\left(1-\beta^{Sold}\right) A_{\tau^{Solv}}$. As already referred, these external claimants are the government, capex suppliers and distress costs. The latter are still a claimant because the project buyer can close its activity, too. By splitting $\left(1-\beta^{Sold}\right) A_{\tau^{Solv}}$ to each of these claimants we obtain the fundamental accounting identity. In other words, the value of the project equals the value attributed to all its claimants. The way this value is split is not of interest for us, though.[2]

The government, capex suppliers and distress costs claims correspond, respectively, to

$$\begin{aligned} GovClaim_0 = & \lim_{T\to+\infty} \tau^{Eff} \int_0^T e^{-rs} E^{\mathbb{Q}}\left[\left(\delta_s - cL - q\right) 1_{\{\tau>s,\hat{\tau}>s\}} | \mathcal{F}_0\right] ds \\ & + \lim_{T\to+\infty} \tau^{Debt} \int_0^T e^{-rs} E^{\mathbb{Q}}\left[cL1_{\{\tau>s,\hat{\tau}>s\}} | \mathcal{F}_0\right] ds + \overline{Gov} Z_0 \\ & = \tau^{Eff}\left(Payout_0 - Coupon_0 - Capex_0\right) + \tau^{Debt} Coupon_0 + \overline{Gov} Z_0, \end{aligned} \tag{5.1}$$

$$CapClaim_0 = Capex_0 + \overline{q} Z_0, \tag{5.2}$$

and

$$\begin{aligned} DCClaim_0 = & \left(\beta^{Sold} - \beta^{Bank}\right) \lim_{T\to+\infty} \left[E^{\mathbb{Q}}\left[e^{-r\tau} \bar{v} 1_{\{\tau<T,\hat{\tau}>\tau\}} | \mathcal{F}_0\right]\right. \\ & \left. + E^{\mathbb{Q}}\left[e^{-r\hat{\tau}}\left(1-j\right) A_{\hat{\tau}^-} 1_{\left\{\bar{v}<A_{\hat{\tau}^-}<\frac{\bar{v}}{1-j},\tau>\hat{\tau},\hat{\tau}<T\right\}} | \mathcal{F}_0\right]\right] + \overline{DC} Z_0 \\ & = \left(\beta^{Sold} - \beta^{Bank}\right) \lim_{T\to+\infty} \left[\bar{v} DigHit\left(T\right) + \left(1-j\right) ANJump\left(\bar{v}, \frac{\bar{v}}{1-j}, T\right)\right] \\ & + \overline{DC} Z_0, \end{aligned} \tag{5.3}$$

where Z_0 is external claimants share on the discounted expected value of the project when

market indefinitely.

[2] One possible way to split this residual claim is to consider that the government, capex and distress costs share in the firm that buys the project equals the one in the firm under consideration. In this case one can simply run the model once with some initial values and then substitute by the ones obtained.

the firm closes and $\overline{Gov}$, $\overline{q}$ and $\overline{DC}$ are the correspondent shares of Z_0. Z_0 corresponds to

$$\begin{aligned}
Z_0 = &\left(1-\beta^{Sold}\right) \lim_{T\to+\infty} \left\{E^{\mathbb{Q}}\left[e^{-r\tau}\bar{v}1_{\{\tau<T,\hat{\tau}>\tau\}}\left|\mathcal{F}_0\right.\right]\right. \\
&+E^{\mathbb{Q}}\left[e^{-r\hat{\tau}}\left(1-j\right)A_{\hat{\tau}-}1_{\left\{\bar{v}<A_{\hat{\tau}-}<\frac{\bar{v}}{1-j},\tau>\hat{\tau},\hat{\tau}<T\right\}}\left|\mathcal{F}_0\right.\right] \\
&\left.+E^{\mathbb{Q}}\left[e^{-r\hat{\tau}}\left(1-j\right)A_{\hat{\tau}-}1_{\left\{\frac{\bar{v}}{1-j}<A_{\hat{\tau}-}<+\infty,\tau>\hat{\tau},\hat{\tau}<T\right\}}\middle|\mathcal{F}_0\right]\right\} \\
=&\left(1-\beta^{Sold}\right)\left[\bar{v}\lim_{T\to+\infty} DigHit\left(T\right)+\left(1-j\right)\lim_{T\to+\infty} ANJump\left(\bar{v},\frac{\bar{v}}{1-j},T\right)\right. \\
&\left.+\left(1-j\right)\lim_{\overline{u},T\to+\infty} ANJump\left(\frac{\bar{v}}{1-j},\overline{u},T\right)\right],
\end{aligned} \tag{5.4}$$

where $\lim_{T\to+\infty} DigHit\left(T\right)$ is given by equation (3.19), $\lim_{T\to+\infty} ANJump\left(\bar{v},\frac{\bar{v}}{1-j},T\right)$ is given by equation (A.65) and, finally, $\lim_{\overline{u},T\to+\infty} ANJump\left(\frac{\bar{v}}{1-j},\overline{u},T\right)$ is given by equation (A.76).

5.2. Firm value and the optimal capital structure

The question of whether there is an optimal capital structure and its determination has been at the core of corporate finance literature since the early days. What is known as the modern theory of capital structure began with Modigliani and Miller (1958) paper. The latter sets the conditions under which the capital structure is irrelevant for firm valuation. Currently, three theories dominate bookshelves, notably, the trade-off theory, the pecking order theory and the market-timing theory. The trade-off theory postulates that debt is able to increase the value of the firm because it is tax deductible. However, the higher the debt level the higher the probability of bankruptcy and distress costs. The optimal debt level is the one where the marginal tax benefit equals the marginal cost arising from bankruptcy costs. In contrast with the trade-off theory, the pecking order theory does not set an optimal capital structure level. Instead, it asserts that the firm maximizes its value by choosing to finance new projects with the cheapest source of funding with this depending critically on information asymmetry issues. Internal funding is considered to be the cheapest form of financing, followed by debt and then equity. Finally, market-timing theory indicates that firms may create value by issuing equity when this is seen as overvalued and issue debt when equity is undervalued. Again, there is no such thing as an optimal leverage ratio in this theory.

Among the referred theories, the trade-off theory is thus the only one that suggests an optimal leverage ratio. Following this theory, we are interested in choosing what is the level of L that maximizes firm value, which corresponds to the sum of shareholders and debtholders claims on the project. Since the project value is invariant to the choice of L, the level of L that maximizes firm value is also the one that minimizes external claimants value. In addition, since debt is issued at time 0, this is also the value of L that maximizes shareholders claim. Notice that at time 0 shareholders hold the firm and any proccedings from issuing debt are distributed as an extraordinary dividend. So, the higher value of debt, the higher the dividend received. In principle, the optimal value of L can be found by differentiating equity and debt functions in order to L, summing and equaling to zero. However, this turns out to be particularly difficult to do because, differently from Leland (1994a), we have no closed form solution for $\bar{v}$, which depends on L. Chapter 8 provides numerical solutions, though.

5.3. The firm process and the cost of capital

The firm value corresponds to the sum of shareholders and debtholders' positions. Sections 3.3 and 4.2 derived the dynamics of these claims by applying Ito's lemma to the equity and debt functions, respectively. Doing the same for the value of the firm, one obtains

$$\frac{dV_t}{V_{t^-}} = (r - k_{V_t})\,dt + \sigma_{V_t} dW_t^{\mathbb{Q}} - j_{V_t} dM_t^{\mathbb{Q}}, \tag{5.5}$$

where

$$k_{V_t} = \frac{\delta_t - q}{V_t}, \tag{5.6}$$

$$\begin{aligned}\sigma_{V_t} &= \frac{\partial V}{\partial A}\frac{A_t}{V_{t^-}}\sigma_A \\ &= \left(\frac{\partial E}{\partial A} + \frac{\partial D}{\partial A}\right)\frac{A_t}{V_{t^-}}\sigma_A,\end{aligned} \tag{5.7}$$

and

$$j_{V_t} = \frac{V\left(A_{t^-}\right) - \left(\beta^{Bank} 1_{\left\{(1-j_A)A_{t^-} \leq \bar{v}\right\}} + \beta^{Sold} 1_{\left\{(1-j_A)A_{t^-} > \bar{v}\right\}}\right)\left(1 - j_A\right)A_{t^-}}{D\left(A_{t^-}\right)}. \tag{5.8}$$

It is clear from the above equations that neither the drift, the volatility or the jump terms are constants. Instead, they are functions of the underlying stochastic process. Changing the probability measure to $\mathbb{P}$ and summing k_{V_t} to the drift in equation (5.5) one obtains the cost of capital, which corresponds to $r + \overline{m}\sigma_{V_t} + \left(\overline{\lambda} - \lambda\right) j_{V_t}$.

6. CDS

A credit default swap (CDS) is a contract by which the seller of the CDS agrees to compensate the buyer in case of a credit event. In return, and as long as the underlying entity does not default, the buyer of the CDS makes a series of payments to the seller, the CDS spread. These streams of cash flows are usually called the protection leg and the coupon leg, respectively. All CDS contracts have a notional value. When a credit event occurs, the CDS contract must be settled, which may happen either physically or in cash. When the CDS contract is settled physically, the protection buyer delivers a bond to the seller in exchange for the par value of that bond. If the protection buyer simultaneously holds the underlying debt obligation and CDS contracts with the same notional value he is basically eliminating the credit risk from his portfolio. In this case, CDS works as an insurance mechanism. In practice, however, there are many more CDS contracts then bond holdings. In order to solve the problem, it is usually organized an auction. This auction has two phases. In the first phase, those willing to settle physically place orders for the company's debt. The range of prices received is then used to calculate the, so called, inside market midpoint (IMM). The IMM is used to set how much protection sellers have to give to protection buyers on the second phase. For those contracts settling in cash, the protection buyer must receive from the protection seller one minus the IMM on the par value.

The CDS spread (cds) for a contract with maturity t^{cds}, nominal value L^{cds} and underlying debt security $L^{\star}$ corresponds to the coupon value that turns the coupon leg equal to the protection leg. In this model, this coupon is assumed to be paid continuously. Thus, there is no accrued interest. Mathematically, the coupon leg equals

$$CouponLeg\left(t^{cds}, L^{cds}\right) = cds \times L^{cds} \int_{0}^{t^{cds}} E^{\mathbb{Q}}\left[\mathrm{e}^{-rs} 1_{\{\tau > s, \hat{\tau} > s\}} \,|\mathcal{F}_0\right] ds. \tag{6.1}$$

Following the same steps as in the case of $Coupon_0^T$, equation (6.1) can be rewritten as

$$\begin{aligned} CouponLeg\left(t^{cds}, L^{cds}\right) = \frac{cds \times L^{cds}}{\varpi} \Big\{ \mathrm{e}^{\varpi t^{cds}} \left[1 - N\left(h_1\left(\bar{v}, t^{cds}\right)\right) - R^{2a} N\left(h_2\left(\bar{v}, t^{cds}\right)\right)\right] - 1 \\ + F\left(\varpi, \frac{\ln(R)}{\sigma}, \frac{v^\star}{\sigma}, t^{cds}\right) + R^{2a} F\left(\varpi, \frac{\ln(R)}{\sigma}, -\frac{v^\star}{\sigma}, t^{cds}\right)\Big\}, \end{aligned} \tag{6.2}$$

which corresponds to equation (3.13) substituting L by L^{cds}, c by cds and T by t^{cds}.

The protection leg value corresponds to the expected loss, which is the difference between the promissed value L^{cds} and the recovered value (in case of default) discounted from the default time. The recovered value depends on the seniority of the debt issue that underlies the CDS contract. As previously, the expected loss can be computed as the sum of the contribution from the case where the asset process hits the barrier and the case where the sudden jump occurs. Thus, the expected loss between now and t^{cds} corresponds to

$$EL_0^{t^{cds}} = ELHit_0^{t^{cds}} + ELJump_0^{t^{cds}}. \tag{6.3}$$

The expected loss up to time t^{cds} when the process hits the barrier equals

$$\begin{aligned} ELHit_0^{t^{cds}} &= L^{cds} E^{\mathbb{Q}}\left[e^{-r\tau} 1_{\{\tau < t^{cds}, \hat{\tau} > \tau\}} \, |\mathcal{F}_0\right] - \frac{L^{cds}}{L^\star} RecHit_0^{\star t^{cds}} \\ &= L^{cds} DigHit\left(t^{cds}\right) - \frac{L^{cds}}{L^\star} RecHit_0^{\star t^{cds}}, \end{aligned} \tag{6.4}$$

where $RecHit_0^{t^{cds}}$ can be computed using equation (4.3), replacing T by t^{cds} while not taking the limit.

Regarding the expected loss up to time t^{cds} when the sudden jump occurs, we have to consider two cases:

$$ELJump_0^{t^{cds}} = ELJumpBank_0^{t^{cds}} + ELJumpSold_0^{t^{cds}}, \tag{6.5}$$

where

$$\begin{aligned} ELJumpBank_0^{t^{cds}} &= L^{cds} E^{\mathbb{Q}}\left[e^{-r\hat{\tau}} 1_{\left\{\bar{v} < A_{\hat{\tau}} < \frac{\bar{v}}{1-j}, \tau > \hat{\tau}, \hat{\tau} < t^{cds}\right\}} \, |\mathcal{F}_0\right] - \frac{L^{cds}}{L^\star} RecJumpBank_0^{t^{cds}} \\ &= L^{cds} DigJump\left(\bar{v}, \frac{\bar{v}}{1-j}, t^{cds}\right) - \frac{L^{cds}}{L^\star} RecJumpBank_0^{t^{cds}}, \end{aligned} \tag{6.6}$$

and

$$\begin{aligned} ELJumpSold_0^{t^{cds}} &= L^{cds} E^{\mathbb{Q}} \left[e^{-r\hat{\tau}} 1_{\left\{ \frac{\bar{v}}{1-j} < A_{\hat{\tau}} < +\infty, \tau > \hat{\tau}, \hat{\tau} < t^{cds} \right\}} | \mathcal{F}_0 \right] - \frac{L^{cds}}{L^\star} RecJumpSold_0^{t^{cds}} \\ &= L^{cds} \lim_{\overline{u} \to +\infty} DigJump \left(\frac{\bar{v}}{1-j}, \overline{u}, t^{cds} \right) - \frac{L^{cds}}{L^\star} RecJumpSold_0^{t^{cds}}. \end{aligned} \tag{6.7}$$

$RecJumpBank_0^{t^{cds}}$ can be computed using equations (4.6) to (4.10) given in Section 4.1.2. $DigJump\left(\bar{v}, \frac{\bar{v}}{1-j}, t^{cds}\right)$ is given by equation (2.90). $RecJumpSold_0^{t^{cds}}$ can be computed using equations (4.12) to (4.14) in Section 4.1.2, while replacing T by t^{cds} and not taking the limit.

7. European Call and Put options

7.1. Call options

Typically the time 0 value of a European call option on a stock corresponds to the discounted expected value under measure $\mathbb{Q}$ of the maximum of zero and the stock price minus the strike price at maturity. In this model, however, the equity process may stop before the option maturity either because the asset process hits the barrier or due to a sudden jump.[1] In these cases, equity value corresponds to the recovered value as presented in Chapter 3. Mathematically,

$$\begin{aligned} C_0\left(E_S, K, S\right) = e^{-rS} \Big\{ & E^{\mathbb{Q}}\left[\left(E_S - K\right) 1_{\left\{E_S > K, \tau^{Solv} > S\right\}} \left| \mathcal{F}_0 \right.\right] \\ & + E^{\mathbb{Q}}\left[\left(EqRec_{\tau^{Solv}} - K\right) 1_{\left\{K < EqRec_{\tau^{Solv}}, \tau^{Solv} < S\right\}} \left| \mathcal{F}_0 \right.\right] \Big\}, \end{aligned} \tag{7.1}$$

where E_S can be computed using the E function derived in Chapter 3 replacing A by A_S. The next two subsections calculate the two terms above.

7.1.1. The firm closes after option maturity

Consider the following function that gives the intrinsic value of a call option at its maturity:

$$IV^C(A_S) = E_S(A_S) - K. \tag{7.2}$$

The call option is in the money for $IV^C(A_S) > 0$. Substituting on the first term in equation (7.1), then

$$\begin{aligned} E^{\mathbb{Q}}\left[\left(E_S - K\right) 1_{\left\{E_S > K, \tau^{Solv} > S\right\}} \left| \mathcal{F}_0 \right.\right] = & E^{\mathbb{Q}}\left[E_S 1_{\left\{IV^C(A_S) > 0, \tau > S, \hat{\tau} > S\right\}} \left| \mathcal{F}_0 \right.\right] \\ & - K\mathbb{Q}\left(IV^C\left(A_S\right) > 0, \tau > S, \hat{\tau} > S \right| \mathcal{F}_0\left.\right). \end{aligned} \tag{7.3}$$

[1] It is assumed in these cases that the equity value does not change between τ^{Solv} and S.

Define $\overline{A^C}$ as the value of A_S that solves $IV^C(A_S) = 0$. We are basically interested in the values of $A_S > max\left(\bar{v}, \overline{A^C}\right)$.[2] The first term in equation (7.3) can thus be rewritten as

$$E^{\mathbb{Q}}\left[E_S 1_{\{IV^C>0,\tau>S,\hat{\tau}>S\}}\left|\mathcal{F}_0\right.\right] = \int_{max\left(\bar{v},\overline{A^C}\right)}^{+\infty} \int_S^{+\infty}\int_S^{+\infty} E_S\left(A_S\right)\mathbb{Q}\left(A_S \in dx, \tau \in du, \hat{\tau} \in d\hat{u}\left|\mathcal{F}_0\right.\right)$$
$$= \int_{max\left(\bar{v},\overline{A^C}\right)}^{+\infty} E_S\left(A_S\right)\mathbb{Q}\left(\hat{\tau} \geq S\left|\mathcal{F}_0\right.\right)\mathbb{Q}\left(A_S \in dx, \tau \geq S\left|\mathcal{F}_0\right.\right), \tag{7.4}$$

where $\overline{A^C}$ can be determined numerically.[3]

Using equations (2.60) and (2.49) we end up with an expression that can be solved easily numerically:

$$E^{\mathbb{Q}}\left[E_S 1_{\{IV^C>0,\tau>S,\hat{\tau}>S\}}\left|\mathcal{F}_0\right.\right]$$
$$= e^{-\bar{\lambda}S}\int_{max\left(\bar{v},\overline{A^C}\right)}^{+\infty} \frac{1/x}{\sigma\sqrt{S}} E_S\left(x\right)\left[n\left(h_1\left(x,S\right)\right) - R^{2a} n\left(h_2\left(x,S\right)\right)\right]dx. \tag{7.5}$$

Doing the same for the second term,

$$K\mathbb{Q}\left(IV^C\left(A_S\right) > 0, \tau > S, \hat{\tau} > S\middle|\mathcal{F}_0\right) = K\int_{max\left(\bar{v},\overline{A^C}\right)}^{+\infty}\int_S^{+\infty}\int_S^{+\infty}\mathbb{Q}\left(A_S \in dx, \tau \in du, \hat{\tau} \in d\hat{u}\left|\mathcal{F}_0\right.\right)$$
$$= K\int_{max\left(\bar{v},\overline{A^C}\right)}^{+\infty}\mathbb{Q}\left(\hat{\tau} \geq S\left|\mathcal{F}_0\right.\right)\mathbb{Q}\left(A_S \in dx, \tau \geq S\left|\mathcal{F}_0\right.\right). \tag{7.6}$$

[2] Notice that we need to restrict A_S to be higher than $\bar{v}$ because, under certain parameter values, shareholders may still recover something when the process hits the barrier. It is thus possible that $\overline{A^C} < \bar{v}$.

[3] Since $IV^{C^{-1}}$ is a monotone function of the intrinsic value we have that $IV^{C^{-1}}(+\infty) = +\infty$.

Using equations (2.60) and (2.49) and then taking the integral,

$$\begin{aligned}
&K\mathbb{Q}\left(IV^C\left(A_S\right)>0,\tau>S,\hat{\tau}>S\middle|\mathcal{F}_0\right)\\
&=e^{-\overline{\lambda}S}K\int\limits_{max\left(\bar{v},\overline{A^C}\right)}^{+\infty}\frac{d}{dx}N\left(h_1\left(x,S\right)\right)+R^{2a}\frac{d}{dx}N\left(h_2\left(x,S\right)\right)dx\\
&=e^{-\overline{\lambda}S}K\left\{\lim_{x\to+\infty}N\left(h_1\left(x,S\right)\right)-N\left(h_1\left(max\left(\bar{v},\overline{A^C}\right),S\right)\right)+\right.\\
&\left.+R^{2a}\left[\lim_{x\to+\infty}N\left(h_2\left(x,S\right)\right)-N\left(h_2\left(max\left(\bar{v},\overline{A^C}\right),S\right)\right)\right]\right\}.
\end{aligned} \tag{7.7}$$

Finally, taking notice that $\lim_{x\to+\infty}N\left(h_1\left(x,S\right)\right)=1$ and $\lim_{x\to+\infty}N\left(h_2\left(x,S\right)\right)=0$,

$$\begin{aligned}
K\mathbb{Q}\left(IV^C\left(A_S\right)>0,\tau>S,\hat{\tau}>S\middle|\mathcal{F}_0\right)=e^{-\overline{\lambda}S}K\left[1-N\left(h_1\left(max\left(\bar{v},\overline{A^C}\right),S\right)\right)\right.\\
\left.-R^{2a}N\left(h_2\left(max\left(\bar{v},\overline{A^C}\right),S\right)\right)\right].
\end{aligned} \tag{7.8}$$

7.1.2. The firm closes before option maturity

The firm may close before option maturity either because the asset process hits the barrier or due to a sudden jump. As a result,

$$\begin{aligned}
&E^{\mathbb{Q}}\left[\left(EqRec_{\tau^{Solv}}-K\right)1_{\left\{K<EqRec_{\tau^{Solv}},\tau^{Solv}<S\right\}}|\mathcal{F}_0\right]\\
&=E^{\mathbb{Q}}\left[\left(EqRecHit_{\tau}-K\right)1_{\left\{K<EqRecHit_{\tau},\tau<S,\hat{\tau}>\tau\right\}}|\mathcal{F}_0\right]\\
&+E^{\mathbb{Q}}\left[\left(EqRecJump_{\hat{\tau}}-K\right)1_{\left\{K<EqRecJump_{\hat{\tau}},\tau>\hat{\tau},\hat{\tau}<S\right\}}|\mathcal{F}_0\right].
\end{aligned} \tag{7.9}$$

The asset process hits the barrier before option maturity

In the case where the asset process hits the barrier,

$$\begin{aligned}
&E^{\mathbb{Q}}\left[\left(EqRecHit_{\tau}-K\right)1_{\left\{K<EqRecHit_{\tau},\tau<S,\hat{\tau}>\tau\right\}}\middle|\mathcal{F}_0\right]\\
&=\begin{cases}E^{\mathbb{Q}}\left[\left[\left(\beta^{Bank}\bar{v}-L\right)-K\right]1_{\{\tau<S,\hat{\tau}>\tau\}}\middle|\mathcal{F}_0\right],\bar{v}\geq\frac{L+K}{\beta^{Bank}}\\0,\bar{v}<\frac{L+K}{\beta^{Bank}}\end{cases}\\
&=\begin{cases}\left(\beta^{Bank}\bar{v}-L-K\right)\mathbb{Q}\left(\tau<S,\hat{\tau}>\tau\middle|\mathcal{F}_0\right),\bar{v}\geq\frac{L+K}{\beta^{Bank}}\\0,\bar{v}<\frac{L+K}{\beta^{Bank}}\end{cases}\\
&=\begin{cases}\left(\beta^{Bank}\bar{v}-L-K\right)DigHit^{\star}(S),\bar{v}\geq\frac{L+K}{\beta^{Bank}}\\0,\bar{v}<\frac{L+K}{\beta^{Bank}}.\end{cases}.
\end{aligned} \tag{7.10}$$

A sudden jump in asset value occurs before the option maturity

When the firm closes due to a sudden jump, we must distinguish two cases depending on whether distress costs are incurred or not:

$$\begin{aligned}
&E^{\mathbb{Q}}\left[(EqRecJump_{\hat{\tau}}-K)\,1_{\{K<EqRecJump_{\hat{\tau}},\tau>\hat{\tau},\hat{\tau}<S\}}\middle|\,\mathcal{F}_0\right]\\
&=E^{\mathbb{Q}}\left[(EqRecJumpBank_{\hat{\tau}}-K)\,1_{\{K<EqRecJumpBank_{\hat{\tau}},\tau>\hat{\tau},\hat{\tau}<S\}}\middle|\,\mathcal{F}_0\right]\\
&+E^{\mathbb{Q}}\left[(EqRecJumpSold_{\hat{\tau}}-K)\,1_{\{K<EqRecJumpSold_{\hat{\tau}},\tau>\hat{\tau},\hat{\tau}<S\}}\middle|\,\mathcal{F}_0\right].
\end{aligned} \tag{7.11}$$

When distress costs are incurred we may have three cases depending on the relative position of the barrier.

First case: $\bar{v}\leq\frac{L+K}{\beta^{Bank}(1-j)}\leq\frac{\bar{v}}{1-j}$

$$\begin{aligned}
&E^{\mathbb{Q}}\left[(EqRecJumpBank_{\hat{\tau}}-K)\,1_{\{K<EqRecJumpBank_{\hat{\tau}},\tau>\hat{\tau},\hat{\tau}<S\}}\middle|\,\mathcal{F}_0\right]\\
&=E^{\mathbb{Q}}\left[\left[\left[\beta^{Bank}(1-j)\,A_{\hat{\tau}^-}-L\right]-K\right]1_{\left\{\frac{L+K}{\beta^{Bank}(1-j)}<A_{\hat{\tau}^-}<\frac{\bar{v}}{1-j},\tau>\hat{\tau},\hat{\tau}<S\right\}}\middle|\,\mathcal{F}_0\right]\\
&=\beta^{Bank}(1-j)\,E^{\mathbb{Q}}\left[A_{\hat{\tau}^-}1_{\left\{\frac{L+K}{\beta^{Bank}(1-j)}<A_{\hat{\tau}^-}<\frac{\bar{v}}{1-j},\tau>\hat{\tau},\hat{\tau}<S\right\}}\middle|\,\mathcal{F}_0\right]\\
&-(L+K)\,\mathbb{Q}\left(\frac{L+K}{\beta^{Bank}(1-j)}<A_{\hat{\tau}^-}<\frac{\bar{v}}{1-j},\tau>\hat{\tau},\hat{\tau}<S\middle|\,\mathcal{F}_0\right)\\
&=\beta^{Bank}(1-j)\,ANJump^{*}\left(\frac{L+K}{\beta^{Bank}(1-j)},\frac{\bar{v}}{1-j},S\right)-(L+K)\,DigJump^{*}\left(\frac{L+K}{\beta^{Bank}(1-j)},\frac{\bar{v}}{1-j},S\right).
\end{aligned} \tag{7.12}$$

Second case: $\frac{L+K}{\beta^{Bank}(1-j)}\leq\bar{v}\leq\frac{\bar{v}}{1-j}$

$$\begin{aligned}
&E^{\mathbb{Q}}\left[(EqRecJumpBank_{\hat{\tau}}-K)\,1_{\{K<EqRecJumpBank_{\hat{\tau}},\tau>\hat{\tau},\hat{\tau}<S\}}\middle|\,\mathcal{F}_0\right]\\
&=E^{\mathbb{Q}}\left[\left[\left[\beta^{Bank}(1-j)\,A_{\hat{\tau}^-}-L\right]-K\right]1_{\left\{\bar{v}<A_{\hat{\tau}^-}<\frac{\bar{v}}{1-j},\tau>\hat{\tau},\hat{\tau}<S\right\}}\middle|\,\mathcal{F}_0\right]\\
&=\beta^{Bank}(1-j)\,E^{\mathbb{Q}}\left[A_{\hat{\tau}^-}1_{\left\{\bar{v}<A_{\hat{\tau}^-}<\frac{\bar{v}}{1-j},\tau>\hat{\tau},\hat{\tau}<S\right\}}\middle|\,\mathcal{F}_0\right]\\
&-(L+K)\,\mathbb{Q}\left(\bar{v}<A_{\hat{\tau}^-}<\frac{\bar{v}}{1-j},\tau>\hat{\tau},\hat{\tau}<S\middle|\,\mathcal{F}_0\right)\\
&=\beta^{Bank}(1-j)\,ANJump^{*}\left(\bar{v},\frac{\bar{v}}{1-j},S\right)-(L+K)\,DigJump^{*}\left(\bar{v},\frac{\bar{v}}{1-j},S\right).
\end{aligned} \tag{7.13}$$

Third case: $\bar{v}\leq\frac{\bar{v}}{1-j}\leq\frac{L+K}{\beta^{Bank}(1-j)}$

$$E^{\mathbb{Q}}\left[(EqRecJumpBank_{\hat{\tau}}-K)\,1_{\{K<EqRecJumpBank_{\hat{\tau}},\tau>\hat{\tau},\hat{\tau}<S\}}\middle|\,\mathcal{F}_0\right]=0. \tag{7.14}$$

When distress costs are not incurred we may have two cases.
First case: $\frac{\bar{v}}{1-j} \leq \frac{L+K}{\beta^{Sold}(1-j)} \leq +\infty$

$$\begin{aligned}
&E^{\mathbb{Q}}\left[(EqRecJumpSold_{\hat{\tau}} - K)\, 1_{\{K<EqRecJumpSold_{\hat{\tau}},\tau>\hat{\tau},\hat{\tau}<S\}} \middle| \mathcal{F}_0 \right] \\
&= E^{\mathbb{Q}}\left[\left(\left[\beta^{Sold}(1-j)\, A_{\hat{\tau}} - L \right] - K \right) 1_{\{K<\beta^{Sold}(1-j)A_{\hat{\tau}}-L,\tau>\hat{\tau},\hat{\tau}<S\}} \,|\mathcal{F}_0 \right] \\
&= \beta^{Sold}(1-j)\, E^{\mathbb{Q}}\left[A_{\hat{\tau}} 1_{\left\{ A_{\hat{\tau}} > \frac{L+K}{\beta^{Sold}(1-j)},\tau>\hat{\tau},\hat{\tau}<S \right\}} \,|\mathcal{F}_0 \right] \\
&- (L+K)\, \mathbb{Q}\left(A_{\hat{\tau}} > \frac{L+K}{\beta^{Sold}(1-j)}, \tau > \hat{\tau}, \hat{\tau} < S \middle| \mathcal{F}_0 \right) \\
&= \beta^{Sold}(1-j) \lim_{\overline{u}\to+\infty} ANJump^{\star}\left(\frac{L+K}{\beta^{Sold}(1-j)}, \overline{u}, S \right) \\
&- (L+K) \lim_{\overline{u}\to+\infty} DigJump^{\star}\left(\frac{L+K}{\beta^{Sold}(1-j)}, \overline{u}, S \right).
\end{aligned} \tag{7.15}$$

Second case: $\frac{L+K}{\beta^{Sold}(1-j)} \leq \frac{\bar{v}}{1-j} \leq +\infty$

$$\begin{aligned}
&E^{\mathbb{Q}}\left[(EqRecJumpSold_{\hat{\tau}} - K)\, 1_{\{K<EqRecJumpSold_{\hat{\tau}},\tau>\hat{\tau},\hat{\tau}<S\}} \middle| \mathcal{F}_0 \right] \\
&= E^{\mathbb{Q}}\left[\left(\left[\beta^{Sold}(1-j)\, A_{\hat{\tau}} - L \right] - K \right) 1_{\left\{ A_{\hat{\tau}} > \frac{\bar{v}}{1-j},\tau>\hat{\tau},\hat{\tau}<S \right\}} \,|\mathcal{F}_0 \right] \\
&= \beta^{Sold}(1-j)\, E^{\mathbb{Q}}\left[A_{\hat{\tau}} 1_{\left\{ A_{\hat{\tau}} > \frac{\bar{v}}{1-j},\tau>\hat{\tau},\hat{\tau}<S \right\}} \,|\mathcal{F}_0 \right] \\
&- e^{-rS}(L+K)\, \mathbb{Q}\left(A_{\hat{\tau}} > \frac{\bar{v}}{1-j}, \tau > \hat{\tau}, \hat{\tau} < S \middle| \mathcal{F}_0 \right) \\
&= \beta^{Sold}(1-j) \lim_{\overline{u}\to+\infty} ANJump^{\star}\left(\frac{\bar{v}}{1-j}, \overline{u}, S \right) \\
&- (L+K) \lim_{\overline{u}\to+\infty} DigJump^{\star}\left(\frac{\bar{v}}{1-j}, \overline{u}, S \right).
\end{aligned} \tag{7.16}$$

7.2. Put options

The procedure for determining the price of put options is similar to the one for call options. Again, we have to consider both the hypothesis that the asset process stops after and before the option maturity:

$$\begin{aligned}
P_0(E_S, K, S) = e^{-rS} \Big\{ & E^{\mathbb{Q}}\left[(K - E_S)\, 1_{\{E_S<K,\tau^{Solv}>S\}} \,|\mathcal{F}_0 \right] \\
& + E^{\mathbb{Q}}\left[(K - EqRec_{\tau^{Solv}})\, 1_{\{K>EqRec_{\tau^{Solv}},\tau^{Solv}>S\}} \,|\mathcal{F}_0 \right] \Big\}.
\end{aligned} \tag{7.17}$$

The next two sections cover how these two terms can be computed.[4]

7.2.1. The firm closes after option maturity

Consider the following function that gives the intrinsic value of a put option at its maturity:

$$IV^P(A_S) = K - E_S(A_S). \tag{7.18}$$

The put option is in the money when $IV^P(A_S) > 0$. Substituting on the first term in equation (7.17), we obtain:

$$\begin{aligned} E^{\mathbb{Q}}\left[(K - E_S)\, 1_{\{E_S<K,\tau^{Solv}>S\}}\, |\mathcal{F}_0\right] &= K\mathbb{Q}\left(IV^P\left(A_S\right) > 0, \tau > S, \hat{\tau} > S \middle| \mathcal{F}_0\right) \\ &\quad - E^{\mathbb{Q}}\left[E_S 1_{\{IV^P(A_S)<0,\tau>S,\hat{\tau}>S\}}\, |\mathcal{F}_0\right]. \end{aligned} \tag{7.19}$$

The strategy to compute these terms is very similar to the one for call options. This time, however, we have to take into account that the intrinsic value of the put option can never be higher than K, which occurs when E_S is zero. The latter can only occur when A_t approaches the barrier or never occur depending on the barrier value. Define $\overline{A^P}$ as the value of A_S that solves $IV^P(A_S) = 0$. Taking this into account, the first term in equation (7.21) equals

$$\begin{aligned} &K\mathbb{Q}\left(IV^P\left(A_S\right) > 0, \tau > S, \hat{\tau} > S \middle| \mathcal{F}_0\right) \\ &= K \int_{\bar{v}}^{max\left(\bar{v},\overline{A^P}\right)} \int_S^{+\infty} \int_S^{+\infty} \mathbb{Q}\left(\hat{\tau} \in d\hat{u}\, |\mathcal{F}_0\right) \mathbb{Q}\left(A_S \in dx, \tau \in du\, |\mathcal{F}_0\right) \\ &= K \int_{\bar{v}}^{max\left(\bar{v},\overline{A^P}\right)} \mathbb{Q}\left(\hat{\tau} > S\, |\mathcal{F}_0\right) \mathbb{Q}\left(A_S \in dx, \tau > S\, |\mathcal{F}_0\right). \end{aligned} \tag{7.20}$$

Again, using equations (2.60) and (2.49), then

$$\begin{aligned} &K\mathbb{Q}\left(IV^P\left(A_S\right) > 0, \tau > S, \hat{\tau} > S \middle| \mathcal{F}_0\right) \\ &= e^{-\overline{\lambda} S} K \int_{\bar{v}}^{max\left(\bar{v},\overline{A^P}\right)} \frac{d}{dx} N\left(h_1\left(x, S\right)\right) + R^{2a} \frac{d}{dx} N\left(h_2\left(x, S\right)\right) dx. \end{aligned} \tag{7.21}$$

[4] Similar to call options, it is assumed that the recovered value stays constant between τ^{Solv} and S whenever $\tau^{Solv} < S$.

This leads to

$$
\begin{aligned}
&K\mathbb{Q}\left(IV^{P}\left(A_{S}\right)>0,\tau>S,\hat{\tau}>S\middle|\mathcal{F}_{0}\right)\\
&=e^{-\overline{\lambda}S}K\left\{N\left(h_{1}\left(max\left(\bar{v},\overline{A^{P}}\right),S\right)\right)-N\left(h_{1}\left(\bar{v},S\right)\right)+\right.\\
&\left.+R^{2a}\left[N\left(h_{2}\left(max\left(\bar{v},\overline{A^{P}}\right),S\right)\right)-N\left(h_{2}\left(\bar{v},S\right)\right)\right]\right\}.
\end{aligned}
\tag{7.22}
$$

The computation of the second term is very similar to equation (7.5):

$$
E^{\mathbb{Q}}\left[E_{S}1_{\left\{E_{S}<K,\tau>S,\hat{\tau}>S\right\}}|\mathcal{F}_{0}\right]=e^{-\overline{\lambda}S}\int_{\bar{v}}^{max\left(\bar{v},\overline{A^{P}}\right)}\frac{1/x}{\sigma\sqrt{S}}E_{S}\left(x\right)\left[n\left(h_{1}\left(x,S\right)\right)-R^{2a}n\left(h_{2}\left(x,S\right)\right)\right]dx.
\tag{7.23}
$$

7.2.2. The firm closes before the option maturity

Again, the firm may close before option maturity either because the asset process hits the barrier or due to a sudden jump.

$$
\begin{aligned}
&E^{\mathbb{Q}}\left[\left(K-EqRec_{\tau^{Solv}}\right)1_{\left\{K>EqRec_{\tau^{Solv}},\tau^{Solv}<S\right\}}|\mathcal{F}_{0}\right]\\
&=E^{\mathbb{Q}}\left[\left(K-EqRecHit_{\tau}\right)1_{\left\{K>EqRecHit_{\tau},\tau<S,\hat{\tau}>\tau\right\}}|\mathcal{F}_{0}\right]\\
&+E^{\mathbb{Q}}\left[\left(K-EqRecJump_{\hat{\tau}}\right)1_{\left\{K>EqRecJump_{\hat{\tau}},\tau>\hat{\tau},\hat{\tau}<S\right\}}|\mathcal{F}_{0}\right].
\end{aligned}
\tag{7.24}
$$

The asset hits the barrier before the option maturity

In the case where the asset process hits the barrier

$$
\begin{aligned}
&E^{\mathbb{Q}}\left[\left(K-EqRecHit_{\tau}\right)1_{\left\{K>EqRecHit_{\tau},\tau<S,\hat{\tau}>\tau\right\}}\middle|\mathcal{F}_{0}\right]\\
&=\begin{cases}0,\bar{v}>\frac{L+K}{\beta^{Bank}}\\ E^{\mathbb{Q}}\left[\left[K-\left(\beta^{Bank}\bar{v}-L\right)\right]1_{\left\{\tau<S,\hat{\tau}>\tau\right\}}\middle|\mathcal{F}_{0}\right],\frac{L}{\beta^{Bank}}\leq\bar{v}\leq\frac{L+K}{\beta^{Bank}}\\ E^{\mathbb{Q}}\left[K1_{\left\{\tau<S,\hat{\tau}>\tau\right\}}\middle|\mathcal{F}_{0}\right],\bar{v}<\frac{L}{\beta^{Bank}}\end{cases}\\
&=\begin{cases}0,\bar{v}>\frac{L+K}{\beta^{Bank}}\\ \left(L+K-\beta^{Bank}\bar{v}\right)\mathbb{Q}\left(\tau<S,\hat{\tau}>\tau\middle|\mathcal{F}_{0}\right),\frac{L}{\beta^{Bank}}\leq\bar{v}\leq\frac{L+K}{\beta^{Bank}}\\ K\mathbb{Q}\left(\tau<S,\hat{\tau}>\tau\middle|\mathcal{F}_{0}\right),\bar{v}<\frac{L}{\beta^{Bank}}\end{cases}\\
&=\begin{cases}0,\bar{v}>\frac{L+K}{\beta^{Bank}}\\ \left(L+K-\beta^{Bank}\bar{v}\right)DigHit^{\star}(S),\frac{L}{\beta^{Bank}}\leq\bar{v}\leq\frac{L+K}{\beta^{Bank}}\\ KDigHit^{\star}(S),\bar{v}<\frac{L}{\beta^{Bank}}\end{cases}.
\end{aligned}
\tag{7.25}
$$

A sudden jump in asset value occurs before the option maturity

When the firm closes due to a sudden jump, we must distinguish two cases depending on whether distress costs occur or not:

$$\begin{aligned}
&E^{\mathbb{Q}}\left[\left(K - EqRecJump_{\hat{\tau}}\right) 1_{\{K > EqRecJump_{\hat{\tau}}, \tau > \hat{\tau}, \hat{\tau} < S\}} \middle| \mathcal{F}_0\right] \\
&= E^{\mathbb{Q}}\left[\left(K - EqRecJumpBank_{\hat{\tau}}\right) 1_{\{K > EqRecJumpBank_{\hat{\tau}}, \tau > \hat{\tau}, \hat{\tau} < S\}} \middle| \mathcal{F}_0\right] \\
&+ E^{\mathbb{Q}}\left[\left(K - EqRecJumpSold_{\hat{\tau}}\right) 1_{\{K > EqRecJumpSold_{\hat{\tau}}, \tau > \hat{\tau}, \hat{\tau} < S\}} \middle| \mathcal{F}_0\right].
\end{aligned} \tag{7.26}$$

When distress costs are incurred we may have three cases depending on the relative position of the barrier.

First case: $\overline{v} \leq \frac{L+K}{\beta^{Bank}(1-j)} \leq \frac{\overline{v}}{1-j}$

$$\begin{aligned}
&E^{\mathbb{Q}}\left[\left(K - EqRecJumpBank_{\hat{\tau}}\right) 1_{\{K > EqRecJumpBank_{\hat{\tau}}, \tau > \hat{\tau}, \hat{\tau} < S\}} \middle| \mathcal{F}_0\right] \\
&= E^{\mathbb{Q}}\left[\left[K - \left[\beta^{Bank}(1-j) A_{\hat{\tau}^-} - L\right]\right] 1_{\left\{\bar{v} < A_{\hat{\tau}^-} < \frac{L+K}{\beta^{Bank}(1-j)}, \tau > \hat{\tau}, \hat{\tau} < S\right\}} \middle| \mathcal{F}_0\right] \\
&= (L+K)\, \mathbb{Q}\left(\bar{v} < A_{\hat{\tau}^-} < \frac{L+K}{\beta^{Bank}(1-j)}, \tau > \hat{\tau}, \hat{\tau} < S \middle| \mathcal{F}_0\right) \\
&- \beta^{Bank}(1-j)\, E^{\mathbb{Q}}\left[A_{\hat{\tau}^-} 1_{\left\{\bar{v} < A_{\hat{\tau}^-} < \frac{L+K}{\beta^{Bank}(1-j)}, \tau > \hat{\tau}, \hat{\tau} < S\right\}} \middle| \mathcal{F}_0\right] \\
&= (L+K)\, DigJump^*\left(\bar{v}, \frac{L+K}{\beta^{Bank}(1-j)}, S\right) - \beta^{Bank}(1-j)\, ANJump^*\left(\bar{v}, \frac{L+K}{\beta^{Bank}(1-j)}, S\right).
\end{aligned} \tag{7.27}$$

Second case: $\frac{L+K}{\beta^{Bank}(1-j)} \leq \overline{v} \leq \frac{\overline{v}}{1-j}$

$$E^{\mathbb{Q}}\left[\left(K - EqRecJumpBank_{\hat{\tau}}\right) 1_{\{K > EqRecJumpBank_{\hat{\tau}}, \tau > \hat{\tau}, \hat{\tau} < S\}} \middle| \mathcal{F}_0\right] = 0. \tag{7.28}$$

Third case: $\bar{v} \leq \frac{\bar{v}}{1-j} \leq \frac{L+K}{\beta^{Bank}(1-j)}$

$$\begin{aligned}
&E^{\mathbb{Q}}\left[(K - EqRecJumpBank_{\hat{\tau}})\, 1_{\{K > EqRecJumpBank_{\hat{\tau}}, \tau > \hat{\tau}, \hat{\tau} < S\}} \middle| \mathcal{F}_0 \right] \\
&= E^{\mathbb{Q}}\left[\left[K - \left[\beta^{Bank}(1-j)\, A_{\hat{\tau}-} - L\right]\right] 1_{\left\{\bar{v} < A_{\hat{\tau}-} < \frac{\bar{v}}{1-j}, \tau > \hat{\tau}, \hat{\tau} < S\right\}} \middle| \mathcal{F}_0 \right] \\
&= (L+K)\, \mathbb{Q}\left(\bar{v} < A_{\hat{\tau}-} < \frac{\bar{v}}{1-j}, \tau > \hat{\tau}, \hat{\tau} < S \middle| \mathcal{F}_0 \right) \\
&-\beta^{Bank}(1-j)\, E^{\mathbb{Q}}\left[A_{\hat{\tau}-} 1_{\left\{\bar{v} < A_{\hat{\tau}-} < \frac{\bar{v}}{1-j}, \tau > \hat{\tau}, \hat{\tau} < S\right\}} \middle| \mathcal{F}_0 \right] \\
&= (L+K)\, DigJump^{*}\left(\bar{v}, \frac{\bar{v}}{1-j}, S\right) - \beta^{Bank}(1-j)\, ANJump^{*}\left(\bar{v}, \frac{\bar{v}}{1-j}, S\right).
\end{aligned} \tag{7.29}$$

When distress costs are not incurred we may have two cases.

First case: $\frac{\bar{v}}{1-j} \leq \frac{L+K}{\beta^{Sold}(1-j)} \leq +\infty$

$$\begin{aligned}
&E^{\mathbb{Q}}\left[(K - EqRecJumpSold_{\hat{\tau}})\, 1_{\{K > EqRecJumpSold_{\hat{\tau}}, \tau > \hat{\tau}, \hat{\tau} < S\}} \middle| \mathcal{F}_0 \right] \\
&= E^{\mathbb{Q}}\left[\left(K - \left[\beta^{Sold}(1-j)\, A_{\hat{\tau}} - L\right]\right) 1_{\left\{\frac{\bar{v}}{1-j} < A_{\hat{\tau}-} < \frac{L+K}{\beta^{Sold}(1-j)}, \tau > \hat{\tau}, \hat{\tau} < S\right\}} |\mathcal{F}_0 \right] \\
&= (L+K)\, \mathbb{Q}\left(\frac{\bar{v}}{1-j} < A_{\hat{\tau}} < \frac{L+K}{\beta^{Sold}(1-j)}, \tau > \hat{\tau}, \hat{\tau} < S \right) \\
&- (1-j)\, E^{\mathbb{Q}}\left[A_{\hat{\tau}} 1_{\left\{\frac{\bar{v}}{1-j} < A_{\hat{\tau}-} < \frac{L+K}{\beta^{Sold}(1-j)}, \tau > \hat{\tau}, \hat{\tau} < S\right\}} |\mathcal{F}_0 \right] \\
&= (L+K)\, DigJump^{\star}\left(\frac{\bar{v}}{1-j}, \frac{L+K}{\beta^{Sold}(1-j)}, S \right) \\
&- \beta^{Sold}(1-j)\, ANJump^{\star}\left(\frac{\bar{v}}{1-j}, \frac{L+K}{\beta^{Sold}(1-j)}, S \right).
\end{aligned} \tag{7.30}$$

Second case: $\frac{L}{\beta^{Sold}(1-j)} \leq \frac{L+K}{\beta^{Sold}(1-j)} \leq \frac{\bar{v}}{1-j} \leq +\infty$

$$E^{\mathbb{Q}}\left[(K - EqRecJumpSold_{\hat{\tau}})\, 1_{\{K > EqRecJumpSold_{\hat{\tau}}, \tau > \hat{\tau}, \hat{\tau} < S\}} \middle| \mathcal{F}_0 \right] = 0. \tag{7.31}$$

8. Numerical analysis

The previous chapters presented the valuation model and have shown how to price contingent claims. In this chapter the model is illustrated through comparative statics. This chapter is divided into five sections. The first three sections address corporate finance issues, notably, the value of each claimant holding, the optimal capital structure and the cost of capital. The fourth section turns to credit risk metrics such as the probablity of default, the recovery rate and the term structure of credit spreads. The effect of seniority on CDS spreads is also studied. The last section compares the option prices produced by this model with those obtained in the Black-Scholes model.

8.1. Project valuation and stakeholders holdings

As the base case, consider the following parameter values: $\delta_0 = 400$, $\mu_\delta = 0.04$, $\sigma = 0.15$, $\lambda = 0.01$, $j = 0.6$, $L = 1200$, $q = 100$, $r = 0.01$, $\beta^{Sold} = 0.4$, $\beta^{Bank} = 0.3$, $\overline{m} = 0.4$, $\overline{\lambda} - \lambda = 0.012$, $\overline{q} = 0.50$, $\overline{Gov} = 0.45$ and $\overline{DC} = 0.05$.[1] In addition, consider that $\overline{t}^{Div} = 0.28$, $\overline{t}^{Debt} = 0.28$ and $\overline{t}^{Corp} = 0.21$. So, $\overline{t}^{Eff} = 0.43$. Table 8.1 shows the value of each claim in the base case and for several other parameter values. Table 8.2 provides key information to understand the results obtained, notably, ρ, which corresponds to the optimal barrier normalized by L, the distance to the barrier, the recovery rate at the barrier, the optimal coupon rate and earnings before taxes (EBT) when the barrier is hit.[2]

Starting with the base case, the project under analysis is split $40\% - 60\%$ between firm investors and external claimants. This is in line with the assumption of $\beta^{Sold} = 0.4$.[3] The risk pricing parameters, $\overline{m}$ and $\overline{\lambda} - \lambda$, have a very significant impact both on the project

[1] Several of the numbers presented as our base case are rounded numbers obtained from financial documents of a real firm in Portugal. β^{Sold} was chosen to be 0.4 after running the model once with a different value. Distress costs in the literature are often referred as 20% of the asset value. For this reason β^{Bank} was considered to equal 0.3. The analyst can however take better estimates either doing more research on the firm, looking to sector data or calibrating the model to market prices, if possible.

[2] EBT was computed as the operating cash flow minus the interest expense minus the fixed capex expenditure.

[3] This split is stable for most of the parameter values considered. The exceptions are $q = 50$ and $q = 150$.

Table 8.1.: Contingent claims on the project

	Asset	Equity		Debt		Capex		Gov		DC	
	Eur	Eur	%	Eur	%	Eur	%	Eur	%	Eur	%
Base case	9 259	2 516	27.2	1 200	13.0	2 760	29.8	2 618	28.3	166	1.8
$\overline{m} = 0.3$	14 184	4 775	33.7	1 200	8.5	3 746	26.4	4 291	30.2	172	1.2
$\overline{m} = 0.5$	6 873	1 500	21.8	1 200	17.5	2 137	31.1	1 853	27.0	183	2.7
$\overline{\lambda} - \lambda = 0$	11 111	3 192	28.7	1 200	10.8	3 387	30.5	3 138	28.2	194	1.7
$\overline{\lambda} - \lambda = 0.024$	7 937	2 021	25.5	1 200	15.1	2 326	29.3	2 241	28.2	148	1.9
$\sigma = 0.1$	17 241	6 129	35.5	1 200	7.0	4 464	25.9	5 272	30.6	176	1.0
$\sigma = 0.2$	6 329	1 360	21.5	1 200	19.0	1 853	29.3	1 737	27.4	179	2.8
$\lambda = 0$	13 333	3 977	29.8	1 200	9.0	4 166	31.2	3 757	28.2	233	1.7
$\lambda = 0.02$	7 092	1 702	24.0	1 200	16.9	2 054	29.0	1 997	28.2	139	2.0
$j = 0.4$	10 309	2 914	28.3	1 200	11.6	3 101	30.1	2 913	28.3	181	1.8
$j = 0.8$	8 403	2 215	26.4	1 200	14.3	2 449	29.1	2 362	28.1	177	2.1
$r = 0$	12 048	3 471	28.8	1 200	10.0	3 734	31.0	3 415	28.3	228	1.9
$r = 0.02$	7 519	1 904	25.3	1 200	16.0	2 170	28.9	2 110	28.1	135	1.8
$\delta_0 = 300$	6 944	1 373	19.8	1 200	17.3	2 391	34.4	1 779	25.6	201	2.9
$\delta_0 = 500$	11 574	3 698	31.9	1 200	10.4	3 026	26.1	3 498	30.2	153	1.3
$\mu_\delta = 0.03$	7 519	1 767	23.5	1 200	16.0	2 320	30.9	2 055	27.3	177	2.4
$\mu_\delta = 0.05$	12 048	3 778	31.4	1 200	10.0	3 349	27.8	3 557	29.5	164	1.4
$q = 50$	9 259	3 157	34.1	1 200	13.0	1 759	19.0	3 043	32.9	99	1.1
$q = 150$	9 259	1 984	21.4	1 200	13.0	3 532	38.1	2 293	24.8	250	2.7
$L = 1000$	9 259	2 680	28.9	1 000	10.8	2 794	30.2	2 634	28.4	151	1.6
$L = 1400$	9 259	2 352	25.4	1 400	15.1	2 723	29.4	2 601	28.1	183	2.0
$\beta^{Bank} = 0.25$	9 259	2 475	26.7	1 200	13.0	2 748	29.7	2 613	28.2	223	2.4
$\beta^{Bank} = 0.35$	9 259	2 553	27.6	1 200	13.0	2 771	29.9	2 623	28.3	113	1.2
$\bar{t}^{Div} = 0.23$	9 259	2 682	29.0	1 200	13.0	2 760	29.8	2 452	26.5	166	1.8
$\bar{t}^{Div} = 0.33$	9 259	2 349	25.4	1 200	13.0	2 760	29.8	2 785	30.1	166	1.8
$\bar{t}^{Debt} = 0.23$	9 259	2 543	27.5	1 200	13.0	2 768	29.9	2 586	27.9	162	1.8
$\bar{t}^{Debt} = 0.33$	9 259	2 484	26.8	1 200	13.0	2 750	29.7	2 655	28.7	170	1.8
$\bar{t}^{Corp} = 0.16$	9 259	2 667	28.8	1 200	13.0	2 760	29.8	2 467	26.6	166	1.8
$\bar{t}^{Corp} = 0.26$	9 259	2 364	25.5	1 200	13.0	2 760	29.8	2 770	29.9	166	1.8

The first row corresponds to the base case where: $\delta_0 = 400$, $\mu_\delta = 0.04$, $\sigma = 0.15$, $\lambda = 0.01$, $j = 0.6$, $L = 1200$, $q = 100$, $r = 0.01$, $\beta^{Sold} = 0.4$, $\beta^{Bank} = 0.3$, $\overline{m} = 0.4$, $\overline{\lambda} - \lambda = 0.012$, $\overline{q} = 0.50$, $\overline{Gov} = 0.45$ and $\overline{DC} = 0.05$, $\bar{t}^{Div} = 0.28$, $\bar{t}^{Debt} = 0.28$ and $\bar{t}^{Corp} = 0.21$. All remaining rows correspond to the base case with only the mentioned parameters changed.

Table 8.2.: Endogenous barrier and the optimal coupon rate

	ρ	$A/\bar{v}$	Recovery rate at $\bar{v}$ (%)	Coupon rate (%)	EBT at $\bar{v}$
Base case	1.89	4.08	56.7	3.0	-38
$\overline{m} = 0.3$	2.22	5.33	66.5	1.9	-48
$\overline{m} = 0.5$	1.75	3.28	52.4	4.6	-33
$\overline{\lambda} - \lambda = 0$	2.09	4.43	62.8	2.5	-40
$\overline{\lambda} - \lambda = 0.024$	1.75	3.79	52.4	3.6	-38
$\sigma = 0.1$	2.87	5.01	86.0	1.5	-38
$\sigma = 0.2$	1.58	3.34	47.4	5.7	-49
$\lambda = 0$	2.35	4.74	70.4	2.1	-21
$\lambda = 0.02$	1.67	3.54	50.1	4.2	-52
$j = 0.4$	1.99	4.31	59.8	2.4	-36
$j = 0.8$	1.92	3.65	57.6	4.4	-43
$r = 0$	2.04	4.92	61.2	1.3	-35
$r = 0.02$	1.80	3.47	54.1	4.7	-42
$\delta_0 = 300$	2.01	2.88	60.2	3.7	-40
$\delta_0 = 500$	1.83	5.27	54.9	2.7	-37
$\mu_\delta = 0.03$	1.78	3.52	53.4	4.0	-34
$\mu_\delta = 0.05$	2.08	4.84	62.3	2.2	-44
$q = 50$	1.20	6.45	35.9	3.0	-24
$q = 150$	2.55	3.03	76.5	2.8	-51
$L = 1000$	2.11	4.39	63.3	2.7	-35
$L = 1400$	1.75	3.78	52.5	3.3	-41
$\beta^{Bank} = 0.25$	1.94	3.98	48.4	3.3	-39
$\beta^{Bank} = 0.35$	1.85	4.18	64.7	2.7	-37
$\overline{t}^{Div} = 0.23$	1.89	4.08	56.7	3.0	-38
$\overline{t}^{Div} = 0.33$	1.89	4.08	56.7	3.0	-38
$\overline{t}^{Debt} = 0.23$	1.86	4.15	55.7	2.8	-37
$\overline{t}^{Debt} = 0.33$	1.93	4.01	57.8	3.2	-39
$\overline{t}^{Corp} = 0.16$	1.89	4.08	56.7	3.0	-38
$\overline{t}^{Corp} = 0.26$	1.89	4.08	56.7	3.0	-38

The first row corresponds to the base case where: $\delta_0 = 400$, $\mu_\delta = 0.04$, $\sigma = 0.15$, $\lambda = 0.01$, $j = 0.6$, $L = 1200$, $q = 100$, $r = 0.01$, $\beta^{Sold} = 0.4$, $\beta^{Bank} = 0.3$, $\overline{m} = 0.4$, $\overline{\lambda} - \lambda = 0.012$, $\overline{q} = 0.50$, $\overline{Gov} = 0.45$ and $\overline{DC} = 0.05$, $\overline{t}^{Div} = 0.28$, $\overline{t}^{Debt} = 0.28$ and $\overline{t}^{Corp} = 0.21$. All remaining rows correspond to the base case with only the mentioned parameters changed.

value and on the way it is split. A lower price of risk increases project value and equity holders get a higher absolute and percentual claim both on the project and on the firm. The same occurs with the government. Debt value stays the same for all parameter values since it is sold at par. However, its stake on the project and on the firm decreases with a lower level of risk aversion. Capex suppliers and distress costs claims increase in absolute value but their relative stake decreases. The distance to the barrier increases substantially even though the barrier itself increases, leading to a higher recovery rate at the barrier. As a result of a lower probability of default and lower LGD, the coupon rate decreases. The opposite occurs when the price of risk increases. However, all effects tend to be significantly smaller except in the coupon rate, where the opposite occurs (i.e the effect is stronger when the price of risk increases).

Similarly to the price of risk parameters, volatility has a very significant negative effect on the project value. In addition, a lower volatility level leads to a higher equity and government stake in the project. This is in sharp contrast with the original Merton model where an increase in volatility leads to a higher equity value due to its standard call features. Again, the distance to the barrier increases despite the barrier increasing leading to a higher recovery rate at the barrier. This is in contrast with Leland (1994a) where a decrease in volatility does not affect the asset value but shifts the barrier upwards, thus decreasing the distance to the barrier. The coupon rate decreases when volatility decreases. The opposite occurs when volatility increases, with a very significant asymetric effect being observed.

The effect of the jump hazard rate is similar to the volatility, though less significant in the case considered here. It is noteworthy that the distance to the barrier decreases whenever the jump hazard rate increases, independently of the jump risk being priced or not (the latter is not shown in Table 8.2). It is also surprising that an increase in the jump hazard rate actually leads to a lower distress costs claim. This occurs because the negative effect on the project value dominates the positive effect arising from a higher default probability and a lower recovery rate at the barrier. This does not occur when project volatility increases. In this case, a slight increase in distress costs is observed. Multiplying the current project value by $1 - j$ and comparing with the barrier value one can better understand the reason for this. For a jump size of 0.6, it is highly probable that the jump does not push the project value below the barrier. In this case, distress costs are not incurred.

Decreasing the jump size increases the project value but has little impact on the way this value is split. The barrier level is unchanged, though the distance to the barrier increases as a consequence of a higher project value. The coupon rate decreases, though the effect is much stronger in the opposite direction. It is interesting to note that changes in j have slighly more effect on the capex suppliers than on equity holders. This occurs because equity holders leave the project immediately after the jump, often with zero value due to limited liability. In contrast, capex suppliers still hold a percentual claim on the project value after the jump since the project continues.

The interest rate has an effect similar to the price of risk. The lower the interest rate the higher the project value. It is remarkable that $1p.p.$ change in the risk free rate produces a variation of $20\% - 30\%$ in project value. This emphasizes the importance of models with stochastic interest rates when the project value is endogenous. The use of the current short term interest rate, though theoretically appealing, may lead to significant overvaluation (or undervaluation) of the project value whenever this is far from its long-term mean. The lower the interest rate the lower the coupon rate and the higher the equity claim. This goes in contrast with models that assume traded assets where a decrease in the risk free rate decrease the asset drift leading to a higher probability of default and a lower equity value. Similarly to equity, a decrease in the interest rate leads to an increase in government's claim. However, in contrast with changes in the price of risk, this time governments' relative share decreases because a reduction in the interest rate leads to a lower tax revenue on coupons. As for risk pricing and risk parameters, the effect of increases and decreases in the interest rate is not symmetric. This occurs besides the effect on the coupon rate being of equal size, which is in contrast with the latter. The reason for this is that equity is intrinsically a leveraged way of being exposed to the risk of the project. So, in addition to the coupon rate effect, we have that the higher the interest rate the higher the advantage of not having to hold the entire project. This mitigates the negative impact on equity value arising from positive changes in the coupon rate, leading to a non-symmetric effect.

Negative changes in δ_0 decrease the project value significantly, affecting especially shareholders and the government. In the case of shareholders, their claim falls from 2516 to 1373 as consequence of a 1) decrease in project value, 2) an increase in the coupon rate and 3) a subsequent decrease in the distance to the barrier. So, a 25% decrease in initial operating cash flow leads to a reduction of 45% in equity value. This huge reduction is closely related with the perpetual debt assumption and the fact that the process here considered is both Markovian and non-mean reverting.[4] This turns the equity value very sensitive to the current operating cash flow level.[5] In addition, it contributes to profitability levels close to the barrier that are likely above those observed in practice.[6] The opposite occurs when δ_0

[4]When the coupon rate and the barrier are kept unchanged equity value falls 42%.

[5]Whenever the current operating cash flow is affected by one-off effects it may be wise to adjust it. For a model with mean reverting EBITDA see Sarkar and Zapatero (2003). The latter does not allow for a trend in earnings.

[6]There are other factors that may justify this. Examples include stiky costs, the need to rollover debt at a likely higher rate and debtholders potential preference for renewing loans instead of pushing the firm to default. Regarding the first, notice that a simultaneous increase in δ_0 and q of 100 leads EBT at the

increases. However, in contrast with changes in the price of risk and the risk free rate, an increase in δ_0 produced an impact of similar absolute size for the parameters considered.

The CFO growth rate, μ_δ, has an impact symmetric to the risk free rate on the project value, meaning that a decrease in the μ_δ decreases the project value and decreases equity holders stake. The effect on equity valuation is much stronger in this case, though. This occurs besides the interest rate having a much stronger effect on the coupon rate than μ_δ. Notice that while interest rate reductions mitigate the benefit of leverage, increases in μ_δ amplify this benefit.

Fixed costs such as those with capex suppliers have no effect on the project value. They have clear distributive effects, though. Equity holders and the government increase their stake with a lower level of capex. The effect is slightly asymmetric. Capex expenditures have a very strong impact on the barrier level. The lower the capex, the lower the barrier because shareholders are willing to inject capital in the firm for a longer period of time hoping that the project goes well. In consequence, the recovery rate when default effectively materializes is the lowest observed under the tested parameter values. Besides this significant decrease in the recovery rate, the coupon rate is unchanged, signaling that the probability of default and the recovery effect cancel out. The opposite occurs when capex costs increase. This time, however, the coupon rate decreases despite the distance to the barrier decreasing significantly.

Similarly to capex expenditures, firm liabilities have no impact on the project but change each claimant holdings. A decrease in L increases equity value as interest expense declines. Government and capex suppliers claims increase slightly as the firm is expected to stay more years in operation. In the case of the government, notice that an increase in L leads to less taxes on coupons but more taxes on dividends. These cancel out since $\bar{t}^{Div} = \bar{t}^{Debt}$.

Regarding distress costs, Table 8.1 considers the case where these correspond to 15% (i.e. $0.40 - 0.25$) and 5% (i.e. $0.40 - 0.35$) of the project value. Since the firm value is around 40% of the project value, these correspond to 37.5% and 12.5% of the firm value, respectively. In none of the cases distress costs have a sizable impact on the contingent claims value except for the distress costs fictive holder. This low impact should be the result of a low probability of default on a reasonably long horizon of time. In spite of this, the coupon rate increased $0.3p.p.$ and the barrier value increased.

Taxation has no impact on the project value but affects the way this is distributed in an obvious way. Except for $\bar{t}^{Debt}$, other taxes have no impact on the distance to the barrier, coupon rate and recovery at the barrier. This would not be the case in a model where the firm may lose its tax shields. Taxes on debt coupons have a slight effect on the coupon rate

barrier to fall from -38 to -63. This suggests that the model can be improved by an in depth analysis of the expenditure profile.

and on the barrier. The lower the tax level, the lower the coupon rate and the lower the barrier.

The current model assumes that the firm is closed after a jump even if the project value is higher than the barrier. As argued in Section 3.2, this assumption simplifies tremendously all computations. However, we are deliberately misestimating the recovered value in case of a jump. In addition, whenever this misestimation affects significantly debt value, the produced estimates on the coupon rate are also erroneous, affecting shareholders decision over the endogenous barrier. It is thus important to have an idea of the impact that arises from this assumption. Figures 8.1 and 8.2 show the difference between the recovered value assumed in the model and the numeric estimates produced by equations (3.29) and (4.16) divided by equity and debt values. This is done for each level of j between 0.2 and 1 in the base case, $\delta_0 = 200$ and $\delta_0 = 600$. The barrier and the coupon rate are not reestimated after computing the numeric estimates. Starting with the base case, surprisingly, a positive difference is obtained for most of the considered values of j both in the case of equity and debt. Differences are significantly higher in the case of debt. This result can be understood with the help of Figures 8.3 and 8.4, which compare the intrinsic value of shareholders and debtholders claims on the project (with and without distress costs) with the value given by the equity and debt pricing functions derived in this thesis. This is done for several values of δ under the base case (i.e. the barrier and the coupon rate used correspond to the values obtained when $\delta_0 = 400$). These figures show that, despite equity and debt pricing functions producing always values above the intrinsic value of these claimants holdings when distressed costs are taken into account, this is not always the case when distressed costs are not deducted. This occurs because shareholders have no possibility to close the firm without distress costs. So, their optimal stopping time is based on $\beta^{Bank} A_{\tau^{Solv}}$ and not $\beta^{Sold} A_{\tau^{Solv}}$. For a sufficiently high level of distress costs and $A_{\hat{\tau}^-} \geq \frac{\bar{v}}{1-j}$, by considering that shareholders receive $Max\left\{\beta^{Sold} A_{\tau^{Solv}} - L, 0\right\}$ instead of $Max\left\{\beta^{Bank} A_{\tau^{Solv}} - L, 0\right\}$ after a jump, we are more than compensating the loss of their option to continue running the firm. Regarding debtholders, after a jump, they typically hold a perpetual debt security whose risk is not totally reflected in the coupon rate. Recovering $Min\left\{\beta^{Bank} A_{\tau^{Solv}}, L\right\}$ is thus typically better than holding the debt security. Receiving $Min\left\{\beta^{Sold} A_{\tau^{Solv}}, L\right\}$ is even better. When $\delta = 130$, the latter is worth 56% more than a risky debt security with a coupon rate of 3.01% (as set when $\delta_0 = 400$). This difference decreases for values of δ near the barrier and near 400. The large difference observed is closely related with the assumption that debt is perpetual and the relatively low interest rate level assumed.

The significant differences observed in debt values for a large class of δ values suggest

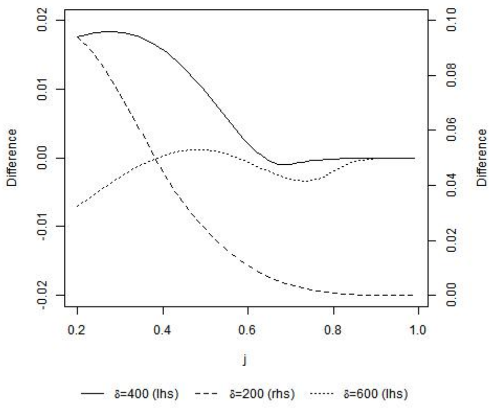


Figure 8.1.: Difference in equity recovered value resulting from assuming the firm is closed after a sudden jump even if the project value stays above the barrier.

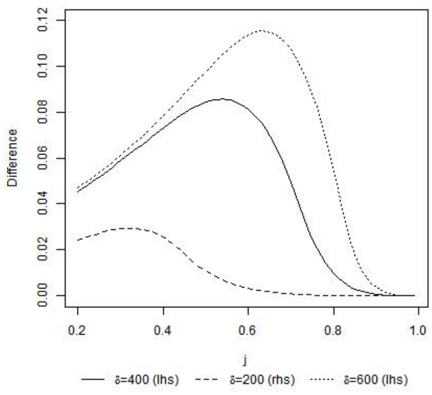


Figure 8.2.: Difference in debt recovered value resulting from assuming the firm is closed after a sudden jump even if the project value stays above the barrier.

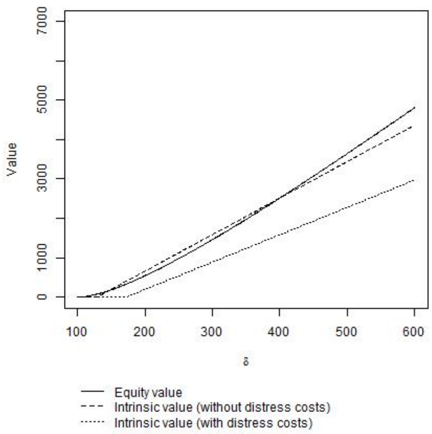


Figure 8.3.: Comparison between equity value and intrinsic value of shareholders claim with and without distress costs.

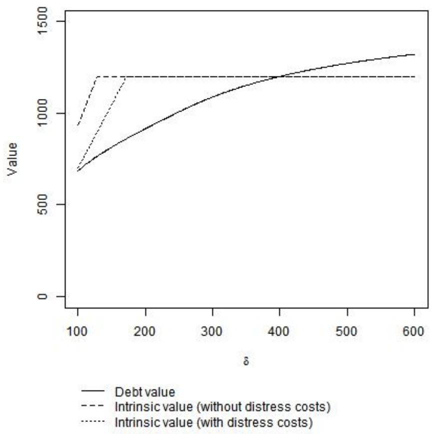


Figure 8.4.: Comparison between debt value and intrinsic value of debtholders claim with and without distress costs.

that it is important to check the impact of the assumption that the firm is always closed after a sudden jump on the coupon rate and the barrier. Unfortunetly, a closed form solution for the equity value has not been obtained for the case where the shareholder may continue running the firm after a sudden jump and thus it is impossible to solve for the smooth pasting condition. Figure 8.3 shows, however, that for reasonable parameter values, shareholders are almost indifferent between recovering the intrinsic value of their claim (without subtracting distress costs) and an option to continue running the firm. This suggests that it is reasonable to compute the optimal barrier as previoulsy but adjusting the coupon rate to reflect the new expected loss value. Figure 8.5 shows the difference between the equity value assumed in the model and the numeric estimates resulting from reestimating the coupon rate, the barrier and the recovered value after a jump for each value of j. Figure 8.6 shows the impact on the barrier and the coupon rate in the base case. The impact on the barrier and the coupon rate when $\delta_0 = 200$ and $\delta_0 = 600$ is presented in Figures 8.7 and 8.8. The difference in equity price is very small for most values of j (always smaller than 2.5% in absolute terms). It is interesting to note that the relation between the difference in equity valuation and j is clearly non-monotonic. In the base case, this difference is very small for low and very high values of j and increases slighly for medium values of j. This was already expected based on Figure 8.4. Besides the small differences observed in equity valuations, Figure 8.6 points to non-negligible differences in terms of the endogenous barrier and the coupon rate when $j < 0.8$ in the base case. This implies an underestimation of the probability of default whenever j is set below 0.8.[7] This problem only occurs for $j < 0.5$ when $\delta_0 = 200$ and it is slightly intensified when $\delta_0 = 600$.

[7] For the base case the new ρ is 1.96. Using this value, the 10-year probability of hitting the barrier under measure $\mathbb{P}$ increases from 0.04% to 0.05%.

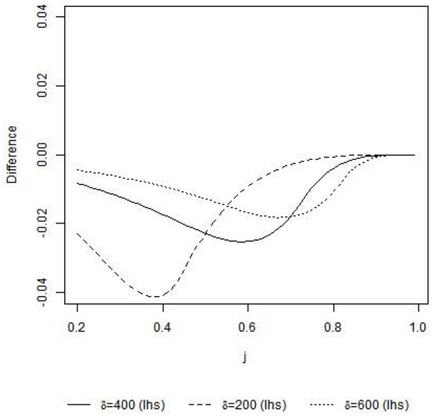


Figure 8.5.: Difference in equity value resulting from assuming that the firm is closed after a sudden jump even if the project value stays above the barrier. Base case, $\delta_0 = 200$ and $\delta_0 = 600$.

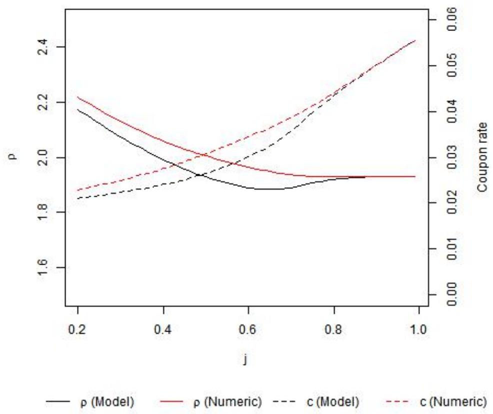


Figure 8.6.: Impact on ρ and on the coupon rate, under the base case, of the assumption that the firm is closed after a sudden jump even if the project value stays above the barrier.

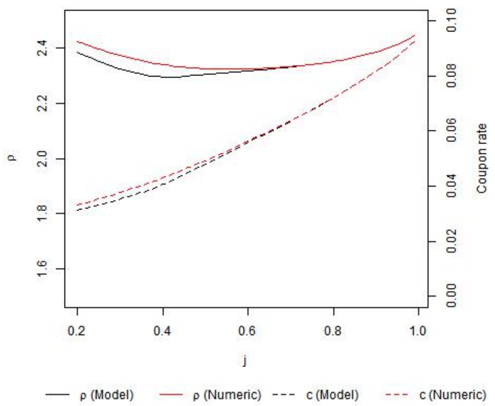


Figure 8.7.: Impact on ρ and on the coupon rate, for $\delta_0 = 200$, of the assumption that the firm is closed after a sudden jump even if the project value stays above the barrier.

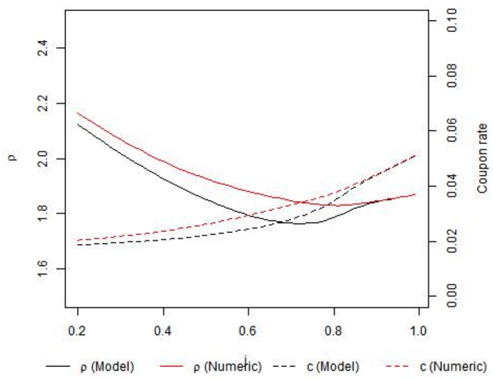


Figure 8.8.: Impact on ρ and on the coupon rate, for $\delta_0 = 600$, of the assumption that the firm is closed after a sudden jump even if the project value stays above the barrier.

8.2. The optimal capital structure

The previous section computed each claimant value for several parameter values. As it is clear from Chapter 2, financial debt in this model does not affect the project value. However, in line with the trade-off theory of capital structure, it affects the way this is distributed. Table 8.3 presents firm value with no debt and at optimal debt level, the difference between the two (debt benefit), the optimal debt-to-equity level, interest expense over operating cash flow at the optimal debt level, the optimal debt level and the maximum amount of debt the firm can issue. Depending on parameter values, debt is able to increase firm value between 5% (when $\bar{t}^{Div} = 0.23$ and $\bar{t}^{Debt} = 0.33$) and 13% (when $\lambda = 0$). These values are significantly below those reported by Leland (1994a), though for a significantly lower corporate tax rate and risk free rate.[8] It is interesting to note that lower prices of diffusion risk decrease slightly debt capacity to create value while a lower price of jump risk do the opposite. The same occurs with risk parameters (i.e. lower volatility reduces debt benefit but lower jump risk increases). Also interesting, reductions in j lead to almost no change in debt benefit vis-a-vis the base case but higher values of j produced one of the highest percentual increases in firm value. These effects should be related with the assumption that the firm is closed even if the project value stays above the barrier (see Figure 8.4). The lower is δ_0 and μ_δ the higher is the benefit of debt, though the optimal L is substantially lower due to the lower project value effect. The same occurs with $\bar{t}^{Debt}$ but this time the optimal L is higher since the project value is not changed. The opposite occurs with q, β^{Bank}, $\bar{t}^{Div}$ and $\bar{t}^{Corp}$.

Table 8.3 presents two leverage measures, notably, the debt-to-equity ratio and interest expense over operating cash flow. The latter can be seen as a proxy for the inverse of the EBITDA interest coverage ratio. The first adjusts for changes in prices while the second is based only on nominal figures. Debt-to-equity values at optimal debt values ranged from 3.9 (in the case of $q = 150$) to 1.3 (when $q = 50$ and $\beta^{Bank} = 0.25$). Diffusion risk aversion showed a positive relation with optimal debt-to-equity while jump risk aversion had almost no impact. Higher risk parameters produced slightly higher debt-to-equity levels, though L is lower in all cases. The same occurs with r, q, β^{Bank}, $\bar{t}^{Div}$ and $\bar{t}^{Corp}$. In contrast, lower δ_0, μ_δ and $\bar{t}^{Debt}$ produced higher debt-to-equity values. Interest expense share on operating cash flow ranged from 27.6% when $\beta^{Bank} = 0.25$ to 52% when $\beta^{Bank} = 0.35$. Most parameters had an impact on this metric similar to the one observed in the debt-to-equity ratio. The only exceptions are risk and risk pricing parameters. An increase in diffusion and jump risk aversion produced a reduction in interest expenses over operating cash flow. The same occurs

[8] Leland (1994a) refers that under reasonable parameter values debt can increase firm value by 25% to 40%. Taking $r = 6\%$, $\beta^{Bank} = 0.2$, $\bar{t}^{Corp} = 35\%$ and $q = 0$ in this model (i.e. parameters in line with those assumed in the referred paper) debt is able to increase firm value by 18%.

after an increases in σ and λ.

The last column of Table 8.3 presents figures on the maximum amount of financial debt the firm can issue at time 0. Notice that as L increases the default boundary also increases, eventually reaching the current project value. Since it is not possible to have $A_0 < \bar{v}$, this sets the maximum amount of debt the firm can issue. On average, the optimal level of debt tends to be around 80% of the firm maximum debt capacity. This figure varies from a minumum of 71% ($q = 50$ and $\beta^{Bank} = 0.25$) to a maximum of 88% ($q = 150$). As a rule of thumb, the higher the debt benefit the lower the slack between the optimal and maximum debt levels. Goldstein *et al.* (2001) shows that whenever it is possible to issue further debt in the future, optimal leverage tend to be smaller. The same occurs when the firm has to roll-over its debt (see Leland and Toft (1996)). In this model, however, the fact that the firm only issues perpetual debt once is probably pushing up the optimal debt level. The small slack between the two is a signal that the optimal debt level suggested by the model is probably too high.

Figure 8.9 shows firm value, capex, goverment and distress costs claims for several debt values using base case parameters. Figure 8.10 provides additional information on the default boundary and the coupon rate for each debt value. Except for very low levels of debt, firm value increases gradually up to the optimal level. As predicted by the trade-off theory, this increase occurs amid a decrease in the government's claim and an increase in distress costs claim. It is interesting to note that, in addition to the government's claim, capex claim also decreases. The higher the debt value the higher the default barrier and the probability the firm being closed. Since, in this case, capex suppliers only receive a share of the project value at the time of liquidation, they are typically worst off in case of liquidation. In contrast with the government's claim, which is almost monotonic on the debt level, the capex claim rises initially before starting a gradual decrease. In addition, for very high levels of debt (and thus project value very near the barrier), the capex claim rises again, signaling that capex suppliers are better off with a fixed share over the recovered value on the project than with a risky claim on a fixed stream. Notice that in the latter case, there is a non-negative probability of a negative jump, in which case capex suppliers would receive even less. The initial increase in the capex claim is related with the decrease in the endogenous barrier. For low levels of debt, interest expenditure is low and the decision to either continue running the firm or closing it is mostly determined by fixed costs like capex expenditure. In spite of the distress costs involved, for sufficiently high levels of fixed costs, the shareholder may be better off closing the firm at an early stage and still recover something than waiting to see whether the project improves. When this occurs, equity value near the barrier may be significantly above zero. When the firm starts issuing debt, shareholders optimal barrier falls because debtholders have priority whenever the firm closes. This turns the hypothesis to

Table 8.3.: Optimal and maximum leverage

	V (no debt)	V(L^{Opt})	Debt benefit (%)	D/E	cL/δ_0 (%)	L^{Opt}	L^{Max}
Base case	3 594	3 896	8.4	2.1	37.4	2 640	3 320
$\overline{m} = 0.3$	5 888	6 285	6.7	1.6	40.8	3 890	5 220
$\overline{m} = 0.5$	2 541	2 792	9.9	2.5	35.7	1 990	2 420
$\overline{\lambda} - \lambda = 0$	4 244	4 692	10.6	2.1	38.7	3 190	3 990
$\overline{\lambda} - \lambda = 0.024$	3 108	3 332	7.2	2.1	36.8	2 260	2 850
$\sigma = 0.1$	7 230	7 662	6.0	2.0	45.0	5 090	6 490
$\sigma = 0.2$	2 383	2 623	10.1	2.2	36.8	1 800	2 240
$\lambda = 0$	4 993	5 653	13.2	2.1	39.9	3 830	4 780
$\lambda = 0.02$	2 791	2 973	6.5	2.2	36.9	2 030	2 560
$j = 0.4$	4 022	4 329	7.6	2.2	35.8	2 960	3 650
$j = 0.8$	3 237	3 600	11.2	2.3	41.8	2 500	3 060
$r = 0$	4 611	4 967	7.7	2.0	33.9	3 310	4 240
$r = 0.02$	2 950	3 214	9.0	2.2	39.6	2 200	2 740
$\delta_0 = 300$	2 439	2 682	10.0	3.2	42.1	2 040	2 380
$\delta_0 = 500$	4 785	5 153	7.7	1.6	33.9	3 200	4 280
$\mu_\delta = 0.03$	2 820	3 086	9.4	2.4	36.2	2 170	2 660
$\mu_\delta = 0.05$	4 882	5 238	7.3	1.8	39.3	3 360	4 400
$q = 50$	4 220	4 502	6.7	1.3	32.1	2 570	3 600
$q = 150$	3 101	3 439	10.9	3.9	44.0	2 740	3 100
$\beta^{Bank} = 0.25$	3 548	3 770	6.3	1.3	27.6	2 150	3 040
$\beta^{Bank} = 0.35$	3 648	4 060	11.3	3.3	52.0	3 110	3 630
$\bar{t}^{Div} = 0.23$	3 789	3 992	5.3	1.4	29.9	2 360	3 320
$\bar{t}^{Div} = 0.33$	3 400	3 816	12.3	2.9	43.5	2 830	3 320
$\bar{t}^{Debt} = 0.23$	3 594	4 008	11.5	2.5	40.6	2 860	3 430
$\bar{t}^{Debt} = 0.33$	3 594	3 790	5.4	1.6	31.8	2 330	3 220
$\bar{t}^{Corp} = 0.16$	3 772	3 983	5.6	1.5	30.4	2 380	3 320
$\bar{t}^{Corp} = 0.26$	3 417	3 823	11.9	2.8	42.8	2 810	3 320

The first row corresponds to the base case where: $\delta_0 = 400$, $\mu_\delta = 0.04$, $\sigma = 0.15$, $\lambda = 0.01$, $j = 0.6$, $L = 1200$, $q = 100$, $r = 0.01$, $\beta^{Sold} = 0.4$, $\beta^{Bank} = 0.3$, $\overline{m} = 0.4$, $\overline{\lambda} - \lambda = 0.012$, $\overline{q} = 0.50$, $\overline{Gov} = 0.45$ and $\overline{DC} = 0.05$, $\bar{t}^{Div} = 0.28$, $\bar{t}^{Debt} = 0.28$ and $\bar{t}^{Corp} = 0.21$. All remaining rows correspond to the base case with only the mentioned parameters changed.

wait and see more tempting vis-a-vis closing the firm. However, shareholders shall receive almost nothing in case they decide to close the firm. This should justify the lower slope in firm value for low levels of debt. Nothing of this occurs when $q = 0$; in this case the barrier starts at 0 and increases as the coupon rate increases.

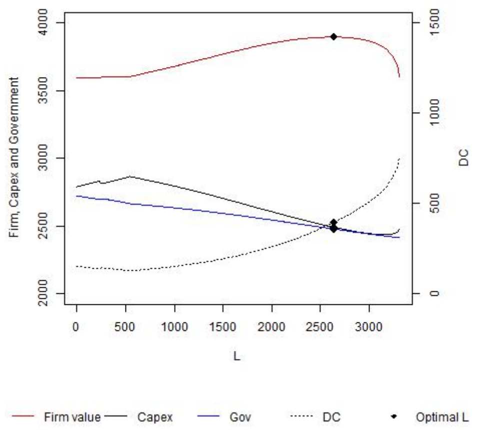


Figure 8.9.: Contingent claims for several values of L.

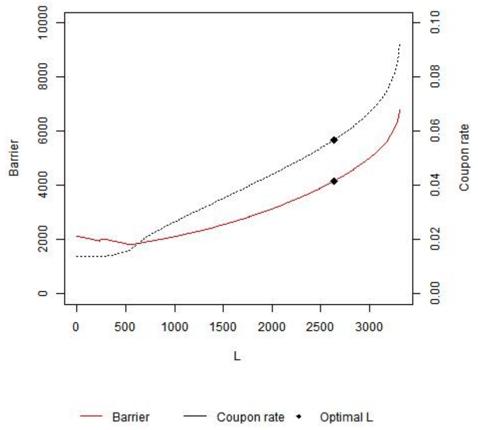


Figure 8.10.: Default barrier and coupon rate for several values of L.

8.3. The cost of capital

Table 8.4 presents the cost of equity, debt and capital for different parameter values at time 0. The volatility, jump size and payout terms are also supplied. The cost of equity ranges between 7.7% when $\sigma = 0.1$ and 18.0% when $\sigma = 0.2$. Changes in project volatility produced the highest movements in terms of cost of equity followed closely by changes in the market price of diffusion risk. Significantly smaller differences were observed for all other parameters. Regarding the cost of debt, values between 1.3% and 4.7% were observed. As expected, the cost of debt is lower than the coupon rate due to the probability of default. Volatility and its market price were also the parameters with more impact (between 1*p.p.* and 2*p.p.*). However, this time there were other parameters that produced variations of almost the same size. This is the case of the risk free rate and the jump hazard rate. Increases in the latter, independently of being priced or not, lead to significant increases in debt volatility with effect on the cost of debt. It is interesting to note that most parameters had a more significant impact, at least in relative terms, in the cost of debt than in the cost of equity. The most notable cases are j, β^{Bank} and L, as a result of sizable changes in debt volatility (in the first two cases absolute changes in debt volatility were higher than equity volatility). In the case of an increase in j, j_D moves from 0 (no imminent risk of losses resulting from a jump) to a potential loss of 58%. In the opposite direction, changes in q produced substantially higher variations in the cost of equity than in the cost of debt.

Table 8.4 presented the impact of changing the initial operating cash flow value, δ_0, on the required return, volatility and jump size of equity, debt and the firm. Nevertheless, δ_t is a stochastic process and it is thus interesting to have a look on the impact of variations of δ_t in the cost of capital. In this case, the coupon rate and the barrier are kept constant. The results are presented in Figures 8.11, 8.12 and 8.13. As expected, the cost of equity and the cost of debt are very high for low levels of δ. In these cases, small increases in δ lead to very significant reductions in investors required return. For very large values of δ (in this case, above 600) the cost of equity and the cost of debt become very stable. The cost of capital (i.e. the rate of return firms should use to discount their projects) is relatively stable until very close to the barrier, ranging between 8.5% and 10%. For $\delta < 130$ we observe a very steep rise in the cost of capital, though. Notice that the cost of capital is a weighted average of the cost of equity and the cost of debt. In spite of these two figures increasing as δ decreases, debtholders own an increasing share of the firm avoiding a steeper increase in the cost of capital until very close to the barrier. There is a point, however, where the increase in the cost of equity is so high that the cost of capital soars.

Figures 8.12 and 8.13 help us understand better the cost of capital behaviour by looking at its determinants. Starting with volatility, reading the graph from the right to the left, we observe a gradual increase as δ decreases. This was already expected and resembles what is known in the literature as a leverage effect (i.e. volatility increases as the stock price decreases). It is interesting to note that even for $\delta = 1000$, σ_V is above σ despite the probability of incurring in distress costs being null in this case. This occurs due to the fixed costs with capex. Only for very high levels of δ, these fixed costs are diluted and the volatility of the firm converges to the volatility of the project. This suggests that in addition to a leverage effect, this model is able to capture a kind of operating leverage effect. As δ decreases the probability of incurring distress costs increases and the already referred operating leverage effect becomes more relevant leading to an increase in σ_E, σ_D and σ_V. This occurs up to $\delta = 212$. At this point, σ_V starts decreasing suggesting that at this level the fact that debtholders hold an increasing stake of the firm dominates. For values of δ next to the barrier firm volatility explodes, though.[9]

[9]When $\beta^{Bank} = \beta^{Sold}$ and simultaneously $q = 0$ the volatility of the firm returns is close to 0.15 for very

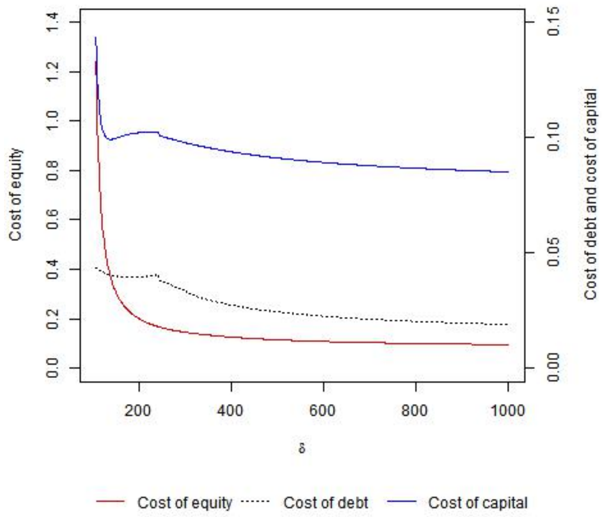


Figure 8.11.: Cost of equity, debt and capital for different values of δ.

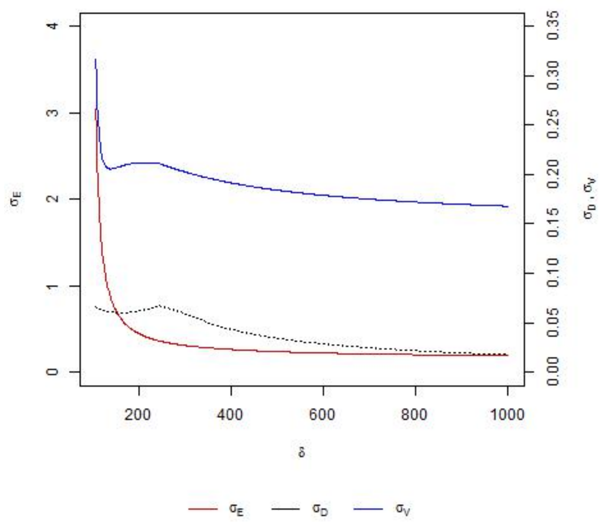


Figure 8.12.: Volatility of equity, debt and firm returns for different values of δ.

Table 8.4.: Drift, volatility and jump terms of the equity, debt and firm processes

	Required return			Volatility			Jump			Payout		
	r_{E_0}	r_{D_0}	r_{V_0}	σ_{E_0}	σ_{D_0}	σ_{V_0}	j_{E_0}	j_{D_0}	j_{V_0}	k_{E_0}	k_{D_0}	k_{V_0}
Base case	12.6	2.7	9.4	0.26	0.04	0.19	0.89	0.00	0.60	6.0	2.2	6.4
$\overline{m} = 0.3$	8.7	1.6	7.3	0.23	0.02	0.18	0.78	0.00	0.62	3.3	1.4	4.0
$\overline{m} = 0.5$	17.0	4.3	11.4	0.30	0.06	0.19	1.00	0.08	0.59	9.3	3.3	8.8
$\overline{\lambda} - \lambda = 0$	11.2	2.3	8.7	0.25	0.03	0.19	0.82	0.00	0.60	4.8	1.8	5.4
$\overline{\lambda} - \lambda = 0.024$	14.1	3.2	10.1	0.27	0.06	0.19	0.97	0.00	0.61	7.2	2.6	7.4
$\sigma = 0.1$	7.7	1.3	6.6	0.15	0.01	0.12	0.75	0.00	0.62	2.6	1.1	3.2
$\sigma = 0.2$	18.0	4.7	11.8	0.40	0.09	0.25	1.00	0.16	0.60	9.7	4.1	9.3
$\lambda = 0$	10.9	1.9	8.8	0.25	0.02	0.20	0.77	0.00	0.59	3.9	1.5	4.6
$\lambda = 0.02$	14.4	3.9	10.1	0.28	0.07	0.19	1.00	0.05	0.61	8.3	3.1	8.2
$j = 0.4$	11.8	2.1	9.0	0.25	0.03	0.19	0.56	0.00	0.40	5.3	1.7	5.8
$j = 0.8$	13.0	3.6	9.7	0.27	0.05	0.19	1.00	0.58	0.85	6.4	3.1	6.9
$r = 0$	10.8	1.5	8.4	0.25	0.04	0.19	0.79	0.00	0.59	4.7	0.9	5.1
$r = 0.02$	14.4	4.0	10.4	0.28	0.05	0.19	1.00	0.00	0.61	7.3	3.4	7.6
$\delta_0 = 300$	15.1	3.5	9.7	0.32	0.06	0.20	1.00	0.07	0.57	6.4	2.7	6.1
$\delta_0 = 500$	11.3	2.4	9.1	0.23	0.03	0.18	0.82	0.00	0.62	5.7	1.9	6.5
$\mu_\delta = 0.03$	13.6	3.3	9.4	0.28	0.06	0.19	1.00	0.00	0.59	8.1	2.9	8.0
$\mu_\delta = 0.05$	11.5	2.1	9.2	0.24	0.03	0.19	0.81	0.00	0.61	4.1	1.6	4.8
$q = 50$	10.8	2.7	8.6	0.22	0.04	0.17	0.91	0.00	0.66	5.7	2.2	6.3
$q = 150$	14.6	2.5	10.0	0.31	0.04	0.21	0.86	0.00	0.53	6.2	2.0	6.2
$L = 1000$	12.1	2.4	9.4	0.25	0.03	0.19	0.82	0.00	0.60	5.8	1.9	6.4
$L = 1400$	13.0	3.1	9.3	0.27	0.05	0.19	0.97	0.00	0.61	6.1	2.4	6.3
$\beta^{Bank} = 0.25$	12.7	3.0	9.5	0.27	0.05	0.20	0.89	0.00	0.60	6.0	2.4	6.4
$\beta^{Bank} = 0.35$	12.4	2.5	9.3	0.26	0.04	0.19	0.89	0.00	0.61	5.9	2.0	6.3
$\bar{t}^{Div} = 0.23$	12.5	2.7	9.5	0.26	0.04	0.19	0.90	0.00	0.62	6.0	2.2	6.1
$\bar{t}^{Div} = 0.33$	12.6	2.7	9.3	0.26	0.04	0.19	0.88	0.00	0.58	5.9	2.2	6.7
$\bar{t}^{Debt} = 0.23$	12.5	2.7	9.4	0.26	0.04	0.19	0.89	0.00	0.60	6.0	2.2	6.3
$\bar{t}^{Debt} = 0.33$	12.7	2.7	9.4	0.26	0.04	0.19	0.89	0.00	0.60	6.0	2.2	6.4
$\bar{t}^{Corp} = 0.16$	12.5	2.7	9.5	0.26	0.04	0.19	0.89	0.00	0.62	6.0	2.2	6.5
$\bar{t}^{Corp} = 0.26$	12.6	2.7	9.3	0.26	0.04	0.19	0.88	0.00	0.58	5.9	2.2	6.2

The first row corresponds to the base case where: $\delta_0 = 400$, $\mu_\delta = 0.04$, $\sigma = 0.15$, $\lambda = 0.01$, $j = 0.6$, $L = 1200$, $q = 100$, $r = 0.01$, $\beta^{Sold} = 0.4$, $\beta^{Bank} = 0.3$, $\overline{m} = 0.4$, $\overline{\lambda} - \lambda = 0.012$, $\overline{q} = 0.50$, $\overline{Gov} = 0.45$ and $\overline{DC} = 0.05$, $\bar{t}^{Div} = 0.28$, $\bar{t}^{Debt} = 0.28$ and $\bar{t}^{Corp} = 0.21$. All remaining rows correspond to the base case with only the mentioned parameters changed.

high levels of δ and decreases slightly to around 0.14 as it approaches the barrier. Volatility then explodes next to the barrier.

Moving to Figure 8.13, again from the right to the left, j_E increases as δ decreases. For values of δ such that $A_t < \frac{L}{\beta^{Sold}(1-j)}$ (i.e. $\delta < 324$), j_E equals 1. In contrast, j_D decreases up to this point. This occurs because when $\delta = 1000$ debt is clearly above par (the coupon rate was defined with $\delta = 400$). A jump represents thus a big loss for debtholders. As δ approaches 400, j_D goes to 0 meaning that in case of a jump debtholders recover the full amount of their investment. For values of δ between 400 and 324, j_D continues decreasing, reaching negative numbers, which means that debtholders are better off after the jump. This occurs because debtholders recover the full amount invested despite debt being below par value. j_D starts then increasing as δ decreases because debtholders recover gradually less after a jump. When A_t falls below $\frac{\bar{v}}{(1-j)}$ (i.e. $\delta < 245$), j_D increases drastically because at this level a jump leads to default costs. j_D continues then increasing gradually up to 0.6. Regarding j_V, this starts at broadly the same level as j_E. However, similarly to j_D, j_V decreases gradually, though at a lower pace. For $\delta < 400$ we have that $j_V < 0.6$, meaning that external claimants are proportionaly more afected by the jump than the firm. This changes when $\delta < 130$ because distress costs become a reality after a jump. j_V then continues decreasing reaching 0.6 near the barrier.

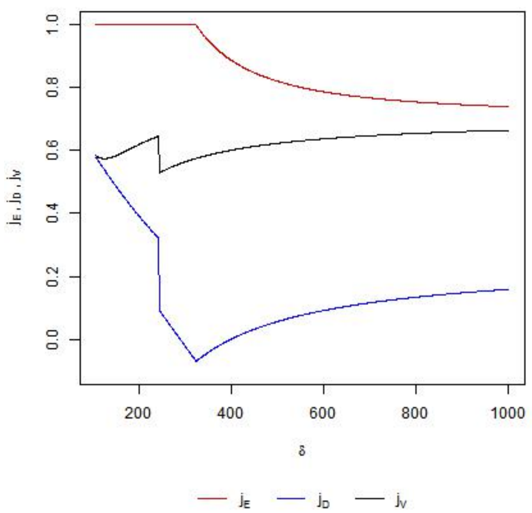


Figure 8.13.: Jump size of equity, debt and firm returns for different values of δ.

8.4. Credit risk

This thesis presented three credit risk metrics, notably, the probability of default, the recovery rate and the CDS spread. These are computed in this section. Starting with the probability of default, this is presented under measure $\mathbb{P}$ in Table 8.5. Measure $\mathbb{Q}$ is later analysed when the lights turn to credit default swaps. As previously argued, $\mathbb{P}$ is the correct probability measure for someone interested in the true probability of a debtor defaulting. The latter is decomposed between the probability of defaulting after hitting the barrier and after a jump. Except for the case where $\sigma = 0.2$, the probability of the process hitting the barrier is always very near zero in the first 5 years. Even in the 10-year maturity, only $\sigma = 0.2$ and $\delta_0 = 300$ produce cumulative default probabilities above 0.5%.[10] This was already expected since we are assuming a constant debt level and a positive CFO growth rate. The fact that shareholders are the ones who chose the default point may also help explain the results obtained, though. Figure 8.14 presents the coupon rate, the probability of default and the probability of default after hitting the barrier for several values of ρ. The optimal barrier is indicated with a small circle. The figure shows that shareholders' optimal decision is to set the barrier at a low level despite paying a significantly higher coupon rate in exchange (almost the maximum level in the figure). Why is the barrier set so low? Figure 8.15 helps answering this question. The latter shows equity value as a function of ρ before and after debt issuance. In contrast with Leland (1994b), choosing $\rho = 1.89$ does not maximize equity value ex-ante but is the optimal default point based on the smooth pasting condition.[11] Suppose shareholders convince debtholders that they will leave the firm whenever δ reaches 108 (i.e at the level that maximizes the equity function ex-ante, which is $\rho = 2.08$). By doing this the coupon rate falls slightly from 3.01 to 2.96. It occurs that when δ approaches the agreed value it is optimal for the shareholder not to respect his commitment and wait a little bit more. Figure 8.15 shows that the ρ value that maximizes equity value when δ is close to 108 is extremely close to the initial solution to the smooth pasting condition (i.e. $\rho = 1.89$). As δ decreases they will become equal. The same occurs with any level of ρ different from 1.89. Debtholders know this. Thus, unless they have a mechanism to force shareholders to

[10] Moody's attributes a rating of $Ba2$ to the firm that served as inspiration for the base case. Their report acknowledges the firm strong profitability margin and strong cash flow generation but it is also referred that the firm faces risks coming from low product diversification and parent company reliance on a constant dividend flow. Standard and Poor's emphasizes the same issues. On a stand-alone basis the firm credit profile is seen as BB+ but the rating is capped at BB based on the overall group credit profile. No reference has been found regarding the effect of country based rating caps, which suggests that these are not binding. Notice that the type of risks faced by this firm are probably better captured by a jump term than a diffusion term.

[11] This is justified by the introduction of fixed costs. Despite the value of ρ that maximizes equity value being higher than the one that comes out from the smooth pasting condition, this does not imply that fixed costs lead to a decrease in the barrier. Table 8.2 showed exactly the opposite.

abandon the firm whenever the pre-agreed level has been reached, they will charge a coupon rate in line with the smooth pasting condition. In the real world, these mechanisms are often introduced through debt covenants in the contracts. In this model, the latter are able to reduce the firm interest burden but they are not optimal to shareholders. However, this may not be the case in a model where agency costs are considered.

The low probability of default after hitting the barrier contrasts with the probability of default after a jump. Even for the 6-month maturity, a non-annualized default probability near 0.5% is observed in several cases. λ and j are the parameters with stronger impact. Notice that in the base case with $\lambda = 0.01$ and $j = 0.6$ the firm is able to resist a negative jump at time 0. When λ increases the project value decreases substantially turning the firm vulnerable to a jump. When $j = 0.8$ the project decreases a little but the high jump size turns default almost certain after a jump. In addition to these, several other parameters lead to sizable variations in the probability of default. In particular, it is interesting to note the strong effect of risk aversion even under measure $\mathbb{P}$. This occurs because the higher the risk aversion the lower the project value and the lower the distance to L. As a result, when $\overline{m} = 0.5$ the 10-year cumulative probability of the jump leading to default increases more than $3p.p.$. Summing the terms relative to the probability of hitting the barrier and the probability of the jump leading to default, one obtains the total probability of default, which is clearly dominated by the latter. This may not be true, however, throughout the firm lifetime as illustrated in Figure 8.16, which shows the probability of default after hitting the barrier and after a jump in the base case for several values of δ. For values of δ below 180 the probability of hitting the barrier in 10 years is higher than the probability of defaulting after a jump.

Credit default swaps and recovery rates are presented in Table 8.6 together with the probability of default under measure $\mathbb{Q}$.[12][13] For all parameters considered, an upward shape term structure is observed as a result of lower recovered values and a higher probability of hitting the barrier for longer maturities. For most parameter values, the spreads start at 0

[12] As expected, the probabilities of default are substantially higher in this case. Even so, the probability of the process hitting the barrier is very close to zero up to 5-years. The only exception is again $\sigma = 0.2$. The 10-year figures are substantially higher, reaching values above 5% in the case of $\overline{m} = 0.5$, $\delta_0 = 300$, $q = 150$ and above 20% in the case that $\sigma = 0.2$

[13] The reader might fell tempted to compare the 10-year CDS spread with the coupon rate presented in Table 8.2. Notice that the coupon rate presented is the value that the firm spends with coupon payments. This includes the risk free rate and the associated spread received by the debtholder but also taxes that must be then deducted to compute debtholders claim on the project. This contrasts with CDS spreads, which are assumed to be free of taxation. So, in order to compare the values obtained in Table 8.2 with those obtained in Table 8.6 one needs to add the interest rate and then divide by $1 - \bar{t}^{Debt}$.

Table 8.5.: Probability of default (PD) - Measure $\mathbb{P}$

	PD $\mathbb{P}$-measure Hit (%)					PD $\mathbb{P}$-measure Jump (%)				
Time	0.5	1	2	5	10	0.5	1	2	5	10
Base case	0.00	0.00	0.00	0.00	0.04	0.00	0.02	0.10	0.48	1.17
$\overline{m} = 0.3$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.10
$\overline{m} = 0.5$	0.00	0.00	0.00	0.01	0.21	0.43	0.78	1.37	2.78	4.54
$\overline{\lambda} - \lambda = 0$	0.00	0.00	0.00	0.00	0.02	0.00	0.00	0.01	0.12	0.43
$\overline{\lambda} - \lambda = 0.024$	0.00	0.00	0.00	0.00	0.07	0.09	0.22	0.51	1.34	2.52
$\sigma = 0.1$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$\sigma = 0.2$	0.00	0.00	0.00	0.36	2.78	0.47	0.88	1.59	3.35	5.68
$\lambda = 0$	0.00	0.00	0.00	0.00	0.01	0.00	0.00	0.00	0.00	0.00
$\lambda = 0.02$	0.00	0.00	0.00	0.00	0.11	0.77	1.37	2.40	4.86	7.88
$j = 0.4$	0.00	0.00	0.00	0.00	0.02	0.00	0.00	0.00	0.01	0.06
$j = 0.8$	0.00	0.00	0.00	0.00	0.10	0.50	1.00	1.98	4.74	8.52
$r = 0$	0.00	0.00	0.00	0.00	0.01	0.00	0.00	0.00	0.06	0.27
$r = 0.02$	0.00	0.00	0.00	0.00	0.14	0.22	0.43	0.83	1.87	3.25
$\delta_0 = 300$	0.00	0.00	0.00	0.04	0.53	0.42	0.75	1.32	2.68	4.39
$\delta_0 = 500$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.09	0.34
$\mu_\delta = 0.03$	0.00	0.00	0.00	0.01	0.24	0.23	0.45	0.88	2.05	3.74
$\mu_\delta = 0.05$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.04	0.19
$q = 50$	0.00	0.00	0.00	0.00	0.00	0.00	0.02	0.10	0.48	1.17
$q = 150$	0.00	0.00	0.00	0.02	0.38	0.00	0.03	0.13	0.55	1.28
$L = 1000$	0.00	0.00	0.00	0.00	0.02	0.00	0.00	0.01	0.12	0.43
$L = 1400$	0.00	0.00	0.00	0.00	0.07	0.09	0.22	0.51	1.34	2.52
$\beta^{Bank} = 0.25$	0.00	0.00	0.00	0.00	0.05	0.00	0.02	0.10	0.48	1.17
$\beta^{Bank} = 0.35$	0.00	0.00	0.00	0.00	0.03	0.00	0.02	0.10	0.48	1.17
$\bar{t}^{Div} = 0.23$	0.00	0.00	0.00	0.00	0.04	0.00	0.02	0.10	0.48	1.17
$\bar{t}^{Div} = 0.33$	0.00	0.00	0.00	0.00	0.04	0.00	0.02	0.10	0.48	1.17
$\bar{t}^{Debt} = 0.23$	0.00	0.00	0.00	0.00	0.03	0.00	0.02	0.10	0.48	1.17
$\bar{t}^{Debt} = 0.33$	0.00	0.00	0.00	0.00	0.05	0.00	0.02	0.10	0.48	1.17
$\bar{t}^{Corp} = 0.16$	0.00	0.00	0.00	0.00	0.04	0.00	0.02	0.10	0.48	1.17
$\bar{t}^{Corp} = 0.26$	0.00	0.00	0.00	0.00	0.04	0.00	0.02	0.10	0.48	1.17

The first row corresponds to the base case where: $\delta_0 = 400$, $\mu_\delta = 0.04$, $\sigma = 0.15$, $\lambda = 0.01$, $j = 0.6$, $L = 1200$, $q = 100$, $r = 0.01$, $\beta^{Sold} = 0.4$, $\beta^{Bank} = 0.3$, $\overline{m} = 0.4$, $\overline{\lambda} - \lambda = 0.012$, $\overline{q} = 0.50$, $\overline{Gov} = 0.45$ and $\overline{DC} = 0.05$, $\bar{t}^{Div} = 0.28$, $\bar{t}^{Debt} = 0.28$ and $\bar{t}^{Corp} = 0.21$. All remaining rows correspond to the base case with only the mentioned parameters changed.

and increase gradually. The only exception is $j = 0.8$. In this case the 6-month spread is already 128 b.p. rising only to 145 b.p. in the 10-year maturity. The model did not generate an inverted term structure in any case. This is not completely surprising since an inverted term structure signals that default is highly probable in a short period of time. But when this occurs the firm is not able to issue debt. This does not mean thus that the model is not able to generate all types of term structures during the CDS lifetime. Figure 8.17 shows the term structure of CDS spreads in the base case for different values of δ that may be observed after time 0. For values of δ close to the barrier (i.e. $\overline{\delta} = 98$) the CDS term structure becomes inverted.

Table 8.7 procceeds the analysis by looking to the term structure of a CDS contract that grants protection over the firms' subordinated debt, which is assumed to be 700. So, senior debt amounts to 500. As expected, CDS spreads increase significantly as a consequence of a lower recovered value. This pattern is more pronounced for medium-long maturities. In the case of short term maturities the difference is smaller as the recovery rate is very similar in most cases. The most relevant exception is $j = 0.8$, which produces a recovery rate of only 2% that is in contrast with 42% in the non-subordinated case for the 6-month maturity. This is the only case under analysis that produces a slightly inverted (humped) term structure of spreads (though it rises again in the later years). Increasing capex to 150 has a strong impact on the recovery rate. This already occurred in Table 8.6, though at a lesser extent. The fact that the barrier value, in this case, is the second highest considered suggests that even though the probability of hitting the barrier is low, the probability of the jump leading the firm to distress costs is higher, resulting in a lower recovered value. Notice, however, that the CDS spread is near 0 due to the low probability of default. $\sigma = 0.1$ and $\sigma = 0.2$ also present significant differences vis-a-vis the no-subordination case at the 6-month horizon time, though for different reasons. While for $\sigma = 0.2$ this should be caused by the higher probability of reaching lower asset values, in the case of $\sigma = 0.1$ this should be motivated by the same reason that justifies the strong capex effect. In this case, despite the barrier being the highest in this study, the lower volatility leads to a recovery value higher as compared to the capex case.

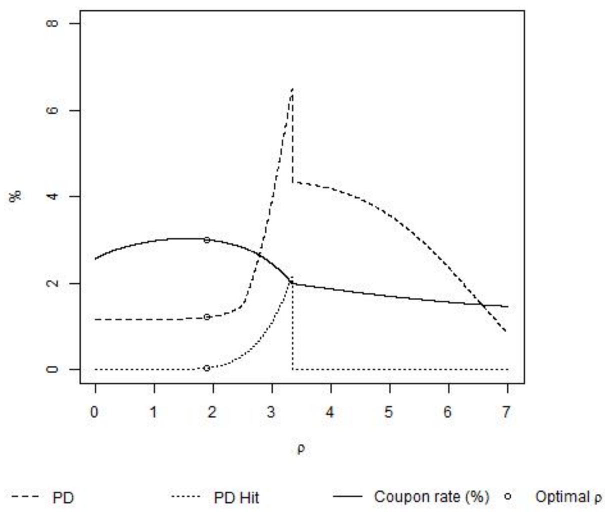
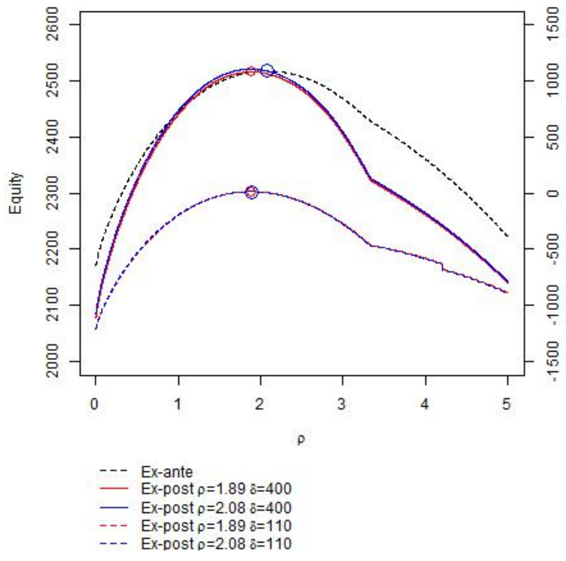


Figure 8.14.: Coupon rate and the cumulative 10-year probability of default under measure $\mathbb{P}$ for different values of ρ.



The red and the blue circles indicate, respectively, the solution to the smooth pasting condition and the value of ρ that maximizes the equity function.

Figure 8.15.: Equity value for different values of ρ and δ before and after debt issuance.

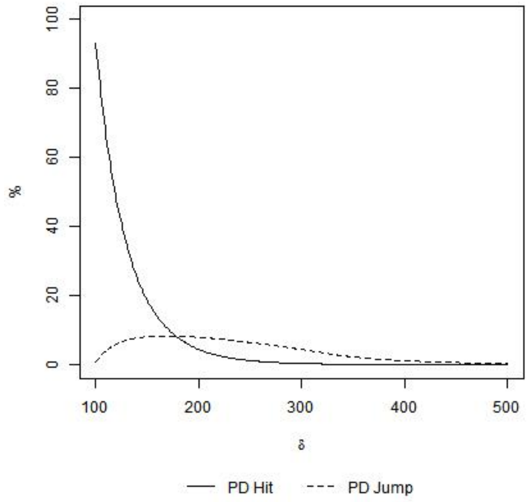


Figure 8.16.: Probability of default after hitting the barrier and after a jump for different values of δ in the base case (10-year cumulative).

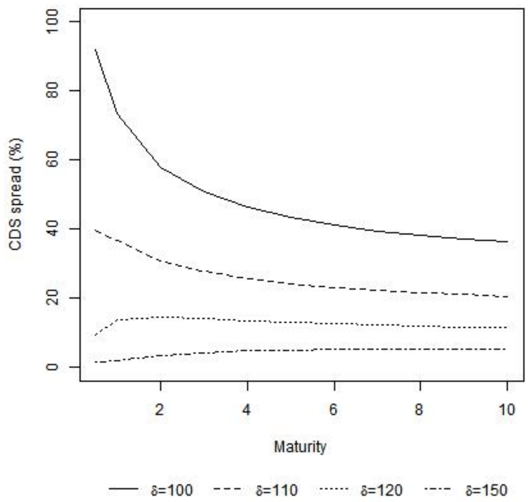


Figure 8.17.: CDS term structure for different values of δ when the barrier and the coupon rate are set in the base case.

Table 8.6.: CDS spreads, probabilities of default (PD) and recovery rates

	PD - Measure $\mathbb{Q}$ (%)					Recovery rate (%)					CDS spread (%)				
Time	0.5	1	2	5	10	0.5	1	2	5	10	0.5	1	2	5	10
Base case	0.01	0.09	0.48	2.64	8.88	96.6	94.5	90.7	81.8	71.4	0.00	0.01	0.02	0.10	0.30
$\overline{m} = 0.3$	0.00	0.00	0.00	0.13	1.20	98.4	96.7	92.6	83.1	74.7	0.00	0.00	0.00	0.00	0.04
$\overline{m} = 0.5$	1.00	1.90	3.61	8.91	24.26	89.6	87.4	83.3	73.7	63.1	0.21	0.24	0.31	0.51	1.06
$\overline{\lambda} - \lambda = 0$	0.00	0.00	0.03	0.48	2.99	97.7	95.7	91.8	82.0	70.8	0.00	0.00	0.00	0.02	0.10
$\overline{\lambda} - \lambda = 0.024$	0.38	1.04	2.58	7.78	18.40	95.1	92.8	89.1	80.8	71.3	0.04	0.08	0.15	0.33	0.65
$\sigma = 0.1$	0.00	0.00	0.00	0.00	0.08	85.3	84.8	84.0	82.0	79.8	0.00	0.00	0.00	0.00	0.00
$\sigma = 0.2$	1.06	2.02	3.86	12.21	37.14	82.7	80.3	75.7	62.4	55.3	0.37	0.41	0.48	1.00	2.07
$\lambda = 0$	0.00	0.00	0.00	0.00	0.75	97.6	93.4	71.4	71.7	72.8	0.00	0.00	0.00	0.00	0.02
$\lambda = 0.02$	1.80	3.36	6.30	14.55	28.91	91.9	89.8	86.4	78.9	70.4	0.30	0.35	0.45	0.70	1.11
$j = 0.4$	0.00	0.00	0.00	0.11	2.21	99.2	97.3	94.9	85.9	71.8	0.00	0.00	0.00	0.00	0.07
$j = 0.8$	1.09	2.18	4.30	10.43	22.06	42.0	41.9	42.0	42.2	44.0	1.28	1.28	1.29	1.31	1.45
$r = 0$	0.00	0.00	0.02	0.62	3.65	98.0	96.2	92.7	83.1	71.4	0.00	0.00	0.00	0.02	0.12
$r = 0.02$	0.57	1.18	2.45	6.49	16.33	94.4	91.9	87.9	79.5	70.6	0.06	0.10	0.15	0.29	0.59
$\delta_0 = 300$	0.97	1.82	3.42	8.55	22.74	89.7	86.3	81.4	72.3	65.5	0.20	0.25	0.33	0.51	0.93
$\delta_0 = 500$	0.00	0.00	0.04	0.79	3.97	97.8	96.1	93.3	85.9	75.9	0.00	0.00	0.00	0.02	0.11
$\mu_\delta = 0.03$	0.58	1.22	2.55	6.91	18.68	94.3	91.7	87.4	77.8	66.5	0.07	0.10	0.17	0.33	0.74
$\mu_\delta = 0.05$	0.00	0.00	0.02	0.49	2.81	98.0	96.3	92.8	83.9	74.7	0.00	0.00	0.00	0.02	0.08
$q = 50$	0.01	0.09	0.48	2.63	7.53	96.6	94.5	91.4	85.2	78.1	0.00	0.00	0.02	0.08	0.19
$q = 150$	0.01	0.12	0.57	3.22	13.97	73.8	72.2	70.0	67.1	69.7	0.01	0.03	0.09	0.23	0.50
$L = 1000$	0.00	0.00	0.07	1.01	5.15	97.7	95.6	91.2	81.1	71.5	0.00	0.00	0.00	0.04	0.17
$L = 1400$	0.25	0.68	1.69	5.22	13.52	95.1	92.7	89.0	80.7	70.2	0.02	0.05	0.10	0.22	0.47
$\beta^{Bank} = 0.25$	0.01	0.09	0.48	2.65	9.08	96.6	94.3	89.9	79.1	66.6	0.00	0.01	0.03	0.12	0.36
$\beta^{Bank} = 0.35$	0.01	0.09	0.48	2.64	8.71	96.6	94.5	91.2	83.8	75.3	0.00	0.01	0.02	0.09	0.25
$\bar{t}^{Div} = 0.23$	0.01	0.09	0.48	2.64	8.88	96.6	94.5	90.7	81.8	71.4	0.00	0.01	0.02	0.10	0.30
$\bar{t}^{Div} = 0.33$	0.01	0.09	0.48	2.64	8.87	96.6	94.5	90.7	81.8	71.4	0.00	0.01	0.02	0.10	0.30
$\bar{t}^{Debt} = 0.23$	0.01	0.09	0.48	2.64	8.75	96.6	94.5	90.9	82.2	71.7	0.00	0.01	0.02	0.10	0.29
$\bar{t}^{Debt} = 0.33$	0.01	0.09	0.48	2.65	9.03	96.6	94.4	90.5	81.3	70.9	0.00	0.01	0.02	0.11	0.31
$\bar{t}^{Corp} = 0.16$	0.01	0.09	0.48	2.64	8.88	96.6	94.5	90.7	81.8	71.4	0.00	0.01	0.02	0.10	0.30
$\bar{t}^{Corp} = 0.26$	0.01	0.09	0.48	2.64	8.87	96.6	94.5	90.7	81.8	71.4	0.00	0.01	0.02	0.10	0.30

The first row corresponds to the base case where: $\delta_0 = 400$, $\mu_\delta = 0.04$, $\sigma = 0.15$, $\lambda = 0.01$, $j = 0.6$, $L = 1200$, $q = 100$, $r = 0.01$, $\beta^{Sold} = 0.4$, $\beta^{Bank} = 0.3$, $\overline{m} = 0.4$, $\overline{\lambda} - \lambda = 0.012$, $\overline{q} = 0.50$, $\overline{Gov} = 0.45$ and $\overline{DC} = 0.05$, $\bar{t}^{Div} = 0.28$, $\bar{t}^{Debt} = 0.28$ and $\bar{t}^{Corp} = 0.21$. All remaining rows correspond to the base case with only the mentioned parameters changed.

Table 8.7.: CDS spreads and recovery rates - Subordinated debt

	Recovery rate (%)					CDS spread (%)				
Time	0.5	1	2	5	10	0.5	1	2	5	10
Base case	94.2	90.5	84.1	68.9	51.7	0.00	0.01	0.04	0.18	0.50
$\overline{m} = 0.3$	97.3	94.4	87.1	70.7	56.7	0.00	0.00	0.00	0.01	0.06
$\overline{m} = 0.5$	82.2	78.4	71.4	55.6	38.5	0.36	0.42	0.53	0.86	1.77
$\overline{\lambda} - \lambda = 0$	96.0	92.5	85.2	67.9	49.8	0.00	0.00	0.00	0.03	0.17
$\overline{\lambda} - \lambda = 0.024$	91.6	87.6	81.3	67.4	52.2	0.06	0.13	0.25	0.56	1.09
$\sigma = 0.1$	75.1	74.3	72.9	69.5	65.7	0.00	0.00	0.00	0.00	0.00
$\sigma = 0.2$	70.4	66.2	58.8	37.4	25.6	0.63	0.70	0.82	1.66	3.45
$\lambda = 0$	96.0	88.6	51.0	51.5	53.5	0.00	0.00	0.00	0.00	0.04
$\lambda = 0.02$	86.1	82.6	76.8	64.2	50.8	0.51	0.60	0.77	1.19	1.85
$j = 0.4$	98.2	95.3	91.3	75.6	51.6	0.00	0.00	0.00	0.01	0.13
$j = 0.8$	2.4	3.4	5.0	8.3	13.6	2.15	2.14	2.11	2.07	2.24
$r = 0$	96.5	93.5	87.5	71.0	51.5	0.00	0.00	0.00	0.04	0.20
$r = 0.02$	90.4	86.1	79.3	65.2	50.7	0.11	0.17	0.26	0.50	0.99
$\delta_0 = 300$	82.3	76.5	68.1	53.0	42.0	0.35	0.43	0.56	0.87	1.57
$\delta_0 = 500$	96.3	93.3	88.5	75.8	59.3	0.00	0.00	0.00	0.04	0.19
$\mu_\delta = 0.03$	90.2	85.7	78.5	62.4	44.0	0.12	0.18	0.28	0.56	1.23
$\mu_\delta = 0.05$	96.5	93.6	87.6	72.4	56.9	0.00	0.00	0.00	0.03	0.14
$q = 50$	94.2	90.6	85.3	74.8	63.5	0.00	0.01	0.04	0.14	0.32
$q = 150$	55.2	52.5	48.7	43.8	48.5	0.01	0.06	0.15	0.39	0.86
$L = 1000$	95.3	91.2	82.3	62.6	45.1	0.00	0.00	0.01	0.08	0.33
$L = 1400$	92.4	88.7	82.9	70.0	54.0	0.04	0.08	0.15	0.34	0.73
$\beta^{Bank} = 0.25$	94.2	90.3	82.7	64.8	44.9	0.00	0.01	0.04	0.20	0.59
$\beta^{Bank} = 0.35$	94.2	90.6	84.8	72.3	58.0	0.00	0.01	0.04	0.16	0.43
$\bar{t}^{Div} = 0.23$	94.2	90.5	84.1	68.9	51.7	0.00	0.01	0.04	0.18	0.50
$\bar{t}^{Div} = 0.33$	94.2	90.5	84.1	68.9	51.7	0.00	0.01	0.04	0.18	0.50
$\bar{t}^{Debt} = 0.23$	94.2	90.5	84.3	69.7	52.4	0.00	0.01	0.04	0.17	0.49
$\bar{t}^{Debt} = 0.33$	94.2	90.4	83.7	68.1	51.0	0.00	0.01	0.04	0.18	0.52
$\bar{t}^{Corp} = 0.16$	94.2	90.5	84.1	68.9	51.7	0.00	0.01	0.04	0.18	0.50
$\bar{t}^{Corp} = 0.26$	94.2	90.5	84.1	68.9	51.7	0.00	0.01	0.04	0.18	0.50

The first row corresponds to the base case where: $\delta_0 = 400$, $\mu_\delta = 0.04$, $\sigma = 0.15$, $\lambda = 0.01$, $j = 0.6$, $L = 1200$, $q = 100$, $r = 0.01$, $\beta^{Sold} = 0.4$, $\beta^{Bank} = 0.3$, $\overline{m} = 0.4$, $\overline{\lambda} - \lambda = 0.012$, $\overline{q} = 0.50$, $\overline{Gov} = 0.45$ and $\overline{DC} = 0.05$, $\bar{t}^{Div} = 0.28$, $\bar{t}^{Debt} = 0.28$ and $\bar{t}^{Corp} = 0.21$. All remaining rows correspond to the base case with only the mentioned parameters changed.

8.5. Option prices

The final section of this chapter illustrates the option pricing model presented in Chapter 7 by comparing it with the standard Black-Scholes option pricing model with dividends. In order to compare the two models one needs to determine the dividend yield to use in the Black-Scholes model. As suggested by Toft and Prucyk (1997), this can be done by solving the equation $F_0 = e^{(r-d)} E_0$ where F_0 is the forward price of equity, which corresponds to $F_0 = e^{rT} C_0 \, (K = 0)$.[14] Figure 8.18 and 8.19 plot call and put option values as functions of the equity value for this model and for the Black-Scholes model. In both cases, the model here presented leads to slightly higher option values. These differences are higher in the case of put options. In the case of the call option, the positive difference is caused by the higher drift in the risk neutral measure due to the jump. This can be seen in equation (3.36) replacing $dM_t^{\mathbb{Q}}$ by $dN_t^{\mathbb{Q}} - \overline{\lambda}$. So, under the risk neutral measure equity continuously generates a return of $\overline{\lambda} j_E$ to compensate for a potential loss due to the jump. For higher equity values this is partially offset by a slight reduction in volatility as illustrated in Figure 8.12 and by the possibility of the jump itself. In the case of the put option, the possibility of a jump increases significantly the option value, especially in the case of far out of the money options. In addition, as equity falls equity volatility rises leading to an increase in option value. So, the leverage effect has a negative impact on call option values and a positive impact on put options. The latter tend to be stronger given the non-linear relation between σ_E and δ_t. These effects are partially offset by the already referred higher drift, which reduces the probability of the put option ending up in the money.

Figures 8.20 and 8.21 compare implied volatilies obtained from 1-year call options for the base case and the usual parameter changes except for changes in β^{Bank}, $\bar{t}^{Div}$, $\bar{t}^{Debt}$ and $\bar{t}^{Corp}$ which had a very low impact on option prices. A clear volatility skew is observed in the base case with implied volatilities ranging from 0.37 (for a strike price of 60% of equity value) and 0.26 (for a strike price of 140% of equity value). This skew was observed for all parameters tested, though with different levels of intensity. In the case of call implied volatilities, in none of the cases the model presented a volatility smirk. For put implied volatilities, a volatility smirk is observed for $j = 0.8$.[15]

[14] The dividend yield computed in this way is different from k_E with the latter only taking into account the current value of the state variable δ_t. Also notice that, in contrast with the Black-Scholes model, the dividend yield is not a constant fraction of equity value. The latter can be either negative or positive, exhibiting a significant level of volatility when the project value is near the barrier.

[15] Notice that, in this model, put implied volatilities do not match call implied volatilities because the put-call parity does not hold with uncertain dividends. The difference is almost 0 for the base case, $\overline{m} = 0.3$, $\overline{\lambda} - \lambda = 0$, $\sigma = 0.1$, $\lambda = 0$, $j = 0.4$, $r = 0$, $\delta_0 = 500$, $\mu_\delta = 0.05$ $q = 50$ and $L = 1000$. For all other

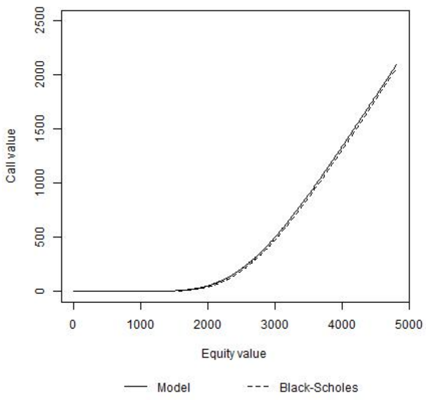


Figure 8.18.: Call option value as a function of equity value.

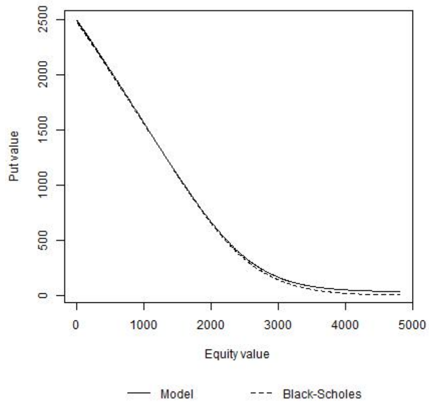


Figure 8.19.: Put option value as a function of equity value.

All parameters except δ and μ_δ presented a positive relation with implied volatility. For the cases under analysis, σ and λ produced the largest variations while L produced the smallest. The impact on implied volatility is asymmetric in the cases of σ, λ, j, δ, μ_δ and q. Except for j, this asymetric impact is only clear for far out of the money options. In the cases of λ and j, the impact is stronger for far out of the money options. The opposite occurs in the cases of σ, δ, μ_δ and q. It is interesting to note that for $\lambda = 0$ the jump effect disappears and we are left only with the leverage effect, which is small for the values considered.

parameter values tested, put implied volatilities were always higher with differences ranging from $0.1p.p.$ (in the case of $q = 150$) and $4.3p.p.$ (in the case of $j = 0.8$) for far in the money put options.

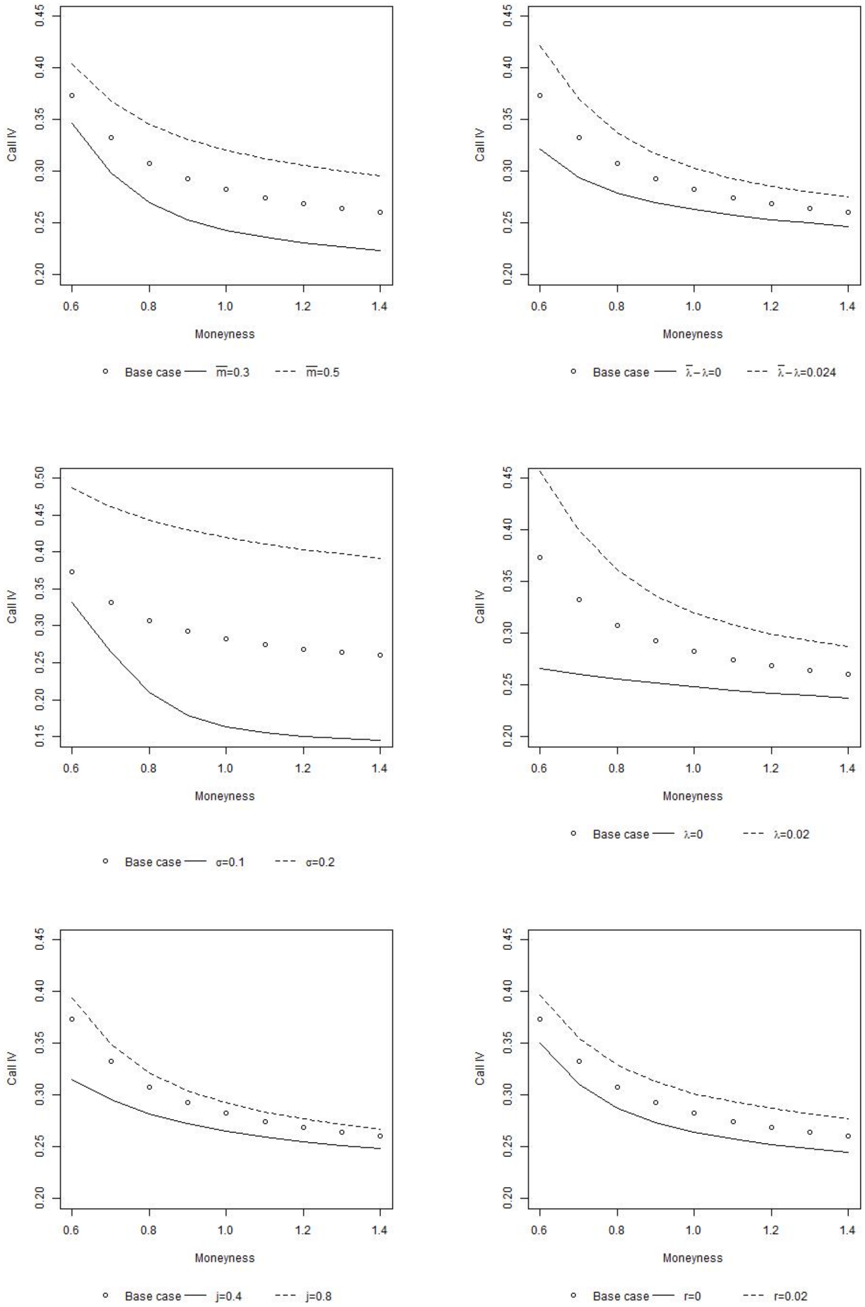


Figure 8.20.: Impact of changes in $\overline{m}$, $\overline{\lambda} - \lambda$, σ, λ, j and r on call implicit volatilities, for different levels of moneyness.

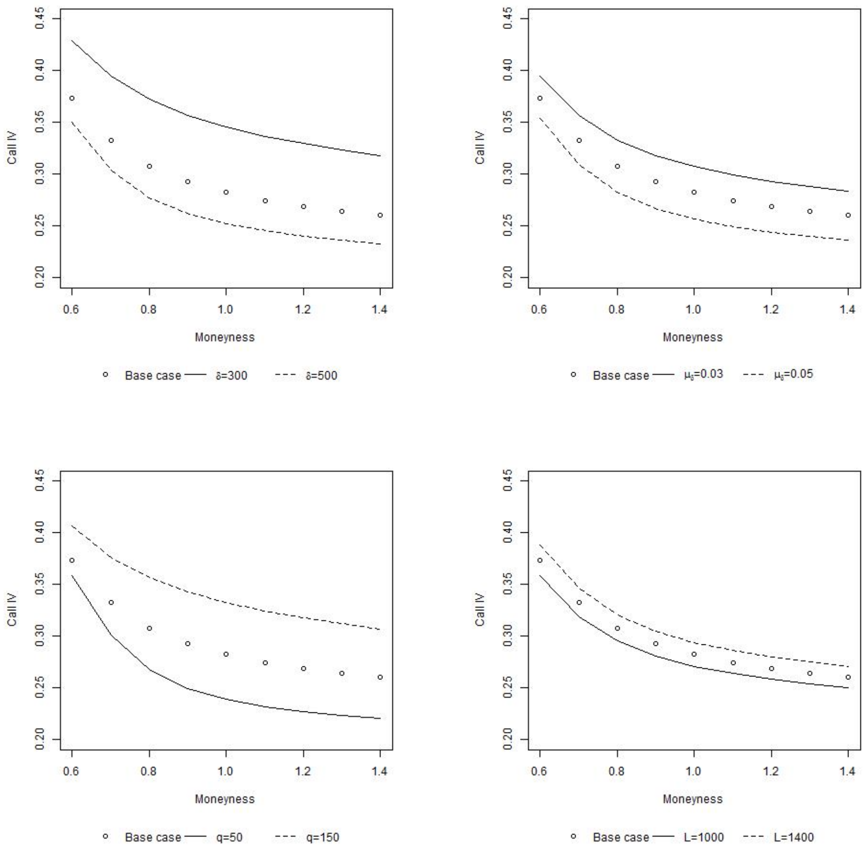


Figure 8.21.: Impact of changes in δ, μ_δ, q and L on call implicit volatilities, for different levels of moneyness.

9. Conclusion

This thesis presented a comprehensive model of contingent claims where the state variable is the operating cash flow generated by the firm. The proposed model adds two elements to the static version of Goldstein *et al.* (2001) model. First, a fixed cost parameter is introduced allowing the application of the model even for firms with negative EBIT. Second, negative jumps of fixed size are introduced leading to more realistic short term spreads. Pricing formulas for equity, debt (with any seniority level), CDS and European-style options were then derived under the assumption that the firm is closed whenever the project value hits a constant lower boundary or at the time of the first jump, whatever occurs first. To the best of my knowledge, there are very few papers where all these claims are priced in a single setting. None of them considers jumps. In addition, in none of these papers the asset value is endogenous. Though this has not been explored in this thesis, pricing all contingent claims under the same model opens new possibilities for capital structure arbitrage, a popular strategy among hedge funds. The availability of closed-form formulas for all the referred instruments (*quasi*-closed form in the case of options) and the fact that the state variable is not a latent process (as in most contingent claims models) suggest that this model may be easier to calibrate than several other models in the literature. This has not been attempted, though.

This thesis also addressed several corporate finance and credit risk issues, such as the determination of the optimal capital structure, the cost of capital, the probability of default and the loss-given-default. The numerical analysis suggests that, similarly to several other models, the one proposed in this thesis leads to leverage ratios above those observed in practice. This result may be related with the assumption of constant perpetual debt. The literature has pointed that leverage ratios tend to be significantly closer to those observed in reality whenever the possibility of issuing further debt in the future is introduced. The same occurs when debt has to be rollover. The fact that debt does not need to be rolled over also helps justifying the low probability of hitting the barrier observed for the case studied. Finally, the capacity of the model to capture the impact of both financial and operating leverage on the cost of capital was emphasized.

To conclude, in addition to relaxing the constant perpetual debt assumption, there are at least three issues that require further research. First, the model here presented assumes that the risk free rate is constant. This is an over-simplifying assumption in a model where the project value of the firm is treated as an endogenous variable. In addition, this precludes the application of the model to the valuation of financial firms. In particular, in the case of banks, higher interest rates are associated with the possibility of banks funding their assets at below the risk free rate through deposits. This is something that tends not to be possible when interest rates are at or near the lower bound, significantly affecting their equity value and shareholders willingess to inject capital. Second, the model assumes that the project cash flows follow a geometric Brownian motion. In practice, operating cash flows in several sectors exhibit some degree of mean reversion as result of business cycles. To the best of my knowledge, there is no published paper considering simultaneously the possibility of mean-reversion and a trend in cash flows. Finally, throughout this thesis agency problems and information issues are ignored. In a world where the process characteristics are not observed and can be changed by managers in secret, these may take an important role.

A. Appendix

A.1. The integro-differential equation

This appendix complements the derivation of equations (2.23) and (2.24) of Chapter 2.

Regarding equation (2.23), substituting the optimal weights given in equation (2.20) in equation (2.21) one obtains

$$\frac{(\sigma_G j_S - \sigma_S j_G)\,\alpha_A + (\sigma_S j_A - \sigma_A j_S)\,\alpha_G + (\sigma_A j_G - \sigma_G j_A)\,\alpha_S}{\sigma_G j_S - \sigma_S j_G + \sigma_A j_G - \sigma_A j_S + \sigma_S j_A - \sigma_G j_A} = r. \tag{A.1}$$

Passing the denominator from the left-hand side to the right-hand side, and then collecting the terms,

$$(\sigma_G j_S - \sigma_S j_G)\,(\alpha_A - r) = -\,(\sigma_S j_A - \sigma_A j_S)\,(\alpha_G - r) - (\sigma_A j_G - \sigma_G j_A)\,(\alpha_S - r)\,. \tag{A.2}$$

Factoring out σ_A and j_A,

$$\alpha_A - r = j_A \frac{\sigma_G\,(\alpha_S - r) - \sigma_S\,(\alpha_G - r)}{\sigma_G j_S - \sigma_S j_G} - \sigma_A \frac{j_G\,(\alpha_S - r) - j_S\,(\alpha_G - r)}{\sigma_G j_S - \sigma_S j_G}. \tag{A.3}$$

Rearranging,

$$\frac{\alpha_A - j_A \frac{\sigma_G(\alpha_S - r) - \sigma_S(\alpha_G - r)}{\sigma_G j_S - \sigma_S j_G} - r}{\sigma_A} = -\frac{j_G\,(\alpha_S - r) - j_S\,(\alpha_G - r)}{\sigma_G j_S - \sigma_S j_G}. \tag{A.4}$$

Doing the same for the derivative contracts, one obtains similar equations:

$$\frac{\alpha_G - j_G \frac{\sigma_S(\alpha_A - r) - \sigma_A(\alpha_S - r)}{\sigma_S j_A - \sigma_A j_S} - r}{\sigma_G} = -\frac{j_S\,(\alpha_A - r) - j_A\,(\alpha_S - r)}{\sigma_S j_A - \sigma_A j_S} \tag{A.5}$$

and

$$\frac{\alpha_S - j_S \frac{(\sigma_G(\alpha_A - r) - \sigma_A(\alpha_G - r))}{\sigma_G j_A - \sigma_A j_G} - r}{\sigma_S} = -\frac{j_G\left(\alpha_A - r\right) - j_A\left(\alpha_G - r\right)}{\sigma_G j_A - \sigma_A j_G}. \tag{A.6}$$

It is possible to show that the terms multiplying the jump terms are equal for the three equations. Taking equations (A.4) and (A.5) as example.

$$\frac{\sigma_G\left(\alpha_S - r\right) - \sigma_S\left(\alpha_G - r\right)}{\sigma_G j_S - \sigma_S j_G} = \frac{\sigma_S\left(\alpha_A - r\right) - \sigma_A\left(\alpha_S - r\right)}{\sigma_S j_A - \sigma_A j_S}. \tag{A.7}$$

Moving the denominators to the opposite side,

$$\left[\sigma_G\left(\alpha_S - r\right) - \sigma_S\left(\alpha_G - r\right)\right]\left(\sigma_S j_A - \sigma_A j_S\right) = \left[\sigma_S\left(\alpha_A - r\right) - \sigma_A\left(\alpha_S - r\right)\right]\left(\sigma_G j_S - \sigma_S j_G\right). \tag{A.8}$$

Applying the distributive rule,

$$\begin{aligned} &\sigma_G\sigma_S j_A\left(\alpha_S - r\right) - \sigma_S^2 j_A\left(\alpha_G - r\right) - \sigma_G\sigma_A j_S\left(\alpha_S - r\right) + \sigma_S\sigma_A j_S\left(\alpha_G - r\right) \\ &= \sigma_S\sigma_G j_S\left(\alpha_A - r\right) - \sigma_A\sigma_G j_S\left(\alpha_S - r\right) - \sigma_S^2 j_G\left(\alpha_A - r\right) + \sigma_A\sigma_S j_G\left(\alpha_S - r\right), \end{aligned} \tag{A.9}$$

and collecting alike terms,

$$\left(\alpha_S - r\right)\left[\sigma_G\sigma_S j_A - \sigma_A\sigma_S j_G\right] = \left(\alpha_A - r\right)\left[\sigma_S\sigma_G j_S - \sigma_S^2 j_G\right] + \left(\alpha_G - r\right)\left[\sigma_S^2 j_A - \sigma_S\sigma_A j_S\right]. \tag{A.10}$$

Canceling out σ_S and rearranging the order,

$$\left(\alpha_A - r\right)\left(\sigma_G j_S - \sigma_S j_G\right) = \left(\alpha_S - r\right)\left(\sigma_G j_A - \sigma_A j_G\right) - \left(\alpha_G - r\right)\left(\sigma_S j_A - \sigma_A j_S\right). \tag{A.11}$$

Dividing by $\sigma_G j_S - \sigma_S j_G + \sigma_A j_G - \sigma_A j_S + \sigma_S j_A - \sigma_G j_A$, one obtains

$$\left(\alpha_A - r\right) w_A = \left(\alpha_S - r\right) w_S - \left(\alpha_G - r\right) w_G. \tag{A.12}$$

Rearranging and noting that $w_A + w_G + w_S = 1$, one arrives at equation (2.21), which is our no arbitrage condition. So, the terms multiplying j_A and j_G are equal as long as there are no arbitrage opportunities.

In addition, it is also possible to prove that the right-hand side of equations (A.4), (A.5) and (A.6) is also equal. Taking again the first two as example, we want to prove that

$$-\frac{j_G(\alpha_S - r) - j_S(\alpha_G - r)}{\sigma_G j_S - \sigma_S j_G} = -\frac{j_S(\alpha_A - r) - j_A(\alpha_S - r)}{\sigma_S j_A - \sigma_A j_S}. \tag{A.13}$$

The above equation is very similar to equation (A.7). Moving the denominators to the opposite side, applying the distributive rule and collecting the terms one obtains:

$$(\alpha_S - r)\left[\sigma_G j_S j_A - \sigma_A j_S j_G\right] = (\alpha_A - r)\left[j_S^2 \sigma_G - j_S \sigma_S j_G\right] + (\alpha_G - r)\left[j_S \sigma_S j_{AA} - j_S^2 \sigma_A\right]. \tag{A.14}$$

Canceling out j_S and rearranging, one arrives at equation (A.11), which it is known to be true as long as there are no arbitrage opportunities. This ends the proof of equation (2.23).

Regarding equation (2.24), and replacing α_G, σ_G and j_G as given by equations (2.15), (2.16) and (2.17) on the first equation in (2.23), then

$$\begin{aligned}\frac{\alpha_A - j_A\overline{\lambda} - r}{\sigma_A} &= \frac{\frac{G_t + \alpha_A A G_x + 0.5\sigma_A^2 A^2 G_{xx}}{G} - \overline{\lambda}\left[\frac{G\left((1-j)A_{t^-}\right) - G\left(A_{t^-}\right)}{G\left(A_{t^-}\right)}\right] - r}{\frac{\sigma_A A G_x}{G}} \\ &= \frac{G_t + \alpha_A A G_x + 0.5\sigma_A^2 A^2 G_{xx} - \overline{\lambda}\left[G\left((1-j)A_{t^-}\right) - G\left(A_{t^-}\right)\right] - rG}{\sigma_A A G_x}.\end{aligned} \tag{A.15}$$

Multiplying and dividing the left-hand side by AG_x,

$$\begin{aligned}\alpha_A A G_x - j_A \overline{\lambda} A G_x - rAG_x = G_t + \alpha_A A G_x + 0.5\sigma_A^2 A^2 G_{xx} - \overline{\lambda}\left[G\left((1-j)A_{t^-}\right) - G\left(A_{t^-}\right)\right] \\ - rG.\end{aligned} \tag{A.16}$$

Eliminating $\alpha_A A G_x$ and factoring out one obtains equation (2.24).

In the case where A_t is not traded, one has

$$\overline{m} = \frac{G_t + \alpha_A A G_x + 0.5\sigma_A^2 A^2 G_{xx} - \overline{\lambda}\left[G\left((1-j)A_{t^-}\right) - G\left(A_{t^-}\right)\right] - rG}{\sigma_A A G_x}. \tag{A.17}$$

Rearranging

$$\overline{m}\sigma_A A G_x = G_t + \alpha_A A G_x + 0.5\sigma_A^2 A^2 G_{xx} - \overline{\lambda}\left[G\left((1-j)\,A_{t^-}\right) - G\left(A_{t^-}\right)\right] - rG. \qquad \text{(A.18)}$$

Substituting α_A by μ_δ and factoring out, one obtains equation (2.34).

A.2. The martingale approach

As stated in Proposition 1, under measure $\mathbb{Q}$ the discounted value of any asset payoff is a martingale. Thus, following the martingale approach, one just has to change the probability measure so that that the discounted value of A_t becomes a martingale. This can be done using the Girsanov theorem for jump-diffusion processes presented below.

Proposition 10. *Girsanov theorem for jump- difusion processes. Consider the filtered probability space* $(\Omega, \mathbb{F}, \mathbb{P}, \boldsymbol{F})$ *and assume that* $\{N_t, t \geq 0\}$ *is an optional counting process with predictable intensity* λ_t*. Assume furthermore* $\{W_t, t \geq 0\}$ *is a standard* $(\boldsymbol{F}, \mathbb{P})$*-Wiener process. Let* h_t *be a predictable process with* $h_t \geq -1$ $\mathbb{P}$*-a.s. and let* g_t *be an optional process.*

Define the process L on [0,T] by

$$\begin{cases} dL_t = L_t g_t dW_t + L_{t^-} h_t \left(dN_t - \lambda_t dt\right) \\ L_0 = 1 \end{cases}$$

and assume that $E^{\mathbb{P}}\left[L_T\right] = 1$*.*

Define the measure $\mathbb{Q}$ *on* $\mathbb{F}_t$ *by* $d\mathbb{Q} = L_t d\mathbb{P}$*, meaning that* L_T *is the Radon-Nikodym derivative. Then, the following holds:*

$$dW_t^{\mathbb{P}} = g_t dt + dW_t^{\mathbb{Q}} \qquad \text{(A.19)}$$

$$\lambda_t^{\mathbb{Q}} = (1 + h_t)\,\lambda_t^{\mathbb{P}} \qquad \text{(A.20)}$$

where $\lambda_t^{\mathbb{Q}}$ *and* $\lambda_t^{\mathbb{P}}$ *are, respectively, the intensity of the* N_t *process under measure* $\mathbb{Q}$ *and* $\mathbb{P}$*.*

Proof. See Björk (2009). □

Based on Girsanov theorem one just has to find g and h. Proposition 11 states what is needed in order for the discounted value of A_t to become a martingale.

Proposition 11. *The discounted asset process,* $\widetilde{A}_t$*, is a martingale if and only if* g *and* h

are set so that

$$\mu_\delta + \sigma_A g - (1+h)\lambda j_A = r. \tag{A.21}$$

Proof. Based on the Girsanov theorem (Proposition 10), $dW_t^{\mathbb{P}}$ and $dM_t^{\mathbb{P}}$ can be stated under measure $\mathbb{Q}$ as

$$dW_t^{\mathbb{Q}} = dW_t^{\mathbb{P}} - g_t dt \tag{A.22}$$

$$dM_t^{\mathbb{Q}} = dN_t^{\mathbb{P}} - (1+h_t)\,\lambda_t^{\mathbb{P}} dt, \tag{A.23}$$

where $dW_t^{\mathbb{Q}}$ and $dM_t^{\mathbb{Q}}$ are now martingales under measure $\mathbb{Q}$.

Rearranging and substituting on equation (2.6), one obtains the dynamics of A_t under measure $\mathbb{Q}$[1]

$$\frac{dA_t}{A_t} = (\mu_\delta + \sigma_A g - (1+h)\lambda j_A)\,dt + \sigma_A dW_t^{\mathbb{Q}} - j_A dM_t^{\mathbb{Q}}. \tag{A.24}$$

Consider now that $f(x) = e^{-rt}x$. Applying Levy-Ito's lemma to equation (A.24) we have that

$$d\widetilde{A}_t = -re^{-rt}A_t dt + e^{-rt} dA_t. \tag{A.25}$$

Substituting dA_t by equation (A.24), one obtains

$$\begin{aligned} d\widetilde{A_t} &= -re^{-rt}A_t dt + e^{-rt}\left[(\mu_\delta + \sigma_A g - j_A(1+h)\lambda)\,A_t dt + \sigma_A A_t dW_t^{\mathbb{Q}} - j_A A_t dN_t^{\mathbb{Q}}\right] \\ &= (\mu_\delta + \sigma_A g - j_A(1+h)\lambda - r)\,\widetilde{A}_t dt + \sigma_A \widetilde{A_t} dW_t^{\mathbb{Q}} - j_A \widetilde{A_t} dN_t^{\mathbb{Q}}, \end{aligned} \tag{A.26}$$

and thus

$$\frac{d\widetilde{A_t}}{\widetilde{A_t}} = (\mu_\delta + \sigma_A g - j_A(1+h)\lambda - r)\,dt + \sigma_A dW_t^{\mathbb{Q}} - j_A dM_t^{\mathbb{Q}}. \tag{A.27}$$

Given that both dW_t and dM_t are martingales under measure $\mathbb{Q}$, we have that $\widetilde{A_t}$ is a martingale if and only if $\mu_\delta + \sigma_A g - (1+h)\lambda j_A - r = 0$.

□

Equation (A.21) has two unknowns implying an infinite number of martingale measures.

[1] σ and j were substituted by σ_A and j_A, respectively.

However, one may find a similar expression for any traded derivative. In particular, in the case of χ_G, no arbitrage implies that

$$\alpha_G + \sigma_G g - (1+h)\lambda j_G = r. \tag{A.28}$$

Solving the system composed by equations (A.21) and (A.28) one obtains

$$\begin{cases} g = \frac{-j_A(\alpha_G - r) + j_G(\mu_\delta - r)}{-\sigma_A j_G + j_A \sigma_G} \\ (1+h)\,\lambda = \frac{r(\sigma_A - \sigma_G) + (\mu_\delta \sigma_G - \alpha_G \sigma_A)}{-\sigma_A j_G + j_A \sigma_G} \end{cases} . \tag{A.29}$$

where one should notice that $(1+h)\lambda$ is equal to the term multiplying j_A, j_G and j_S in equations (A.4), (A.5) and (A.6) of Appendix A.1. This goes in line with the previous interpretation that this term was the hazard rate of the process under the risk neutral measure. The fact that one can find unique values for g and h implies that $\mathbb{Q}$ is the unique martingale measure. In addition, following the second funtamental theorem of asset pricing, this indicates that the market is complete. Susbtituting g and h on equation (A.24), and denoting $\overline{\lambda} = (1+h)\,\lambda$, the drift of the A_t process under measure $\mathbb{Q}$ becomes

$$\mu_\delta + \sigma_A \frac{-j_A\left(\alpha_G - r\right) + j_G\left(\mu_\delta - r\right)}{-\sigma_A j_G + j_A \sigma_G} - j_A \overline{\lambda}. \tag{A.30}$$

Multiplying and dividing the first two terms by $-\sigma_A j_G + j_A \sigma_G$,

$$\frac{\mu_\delta\left(-\sigma_A j_G + j_A \sigma_G\right) - \sigma_A j_A\left(\alpha_G - r\right) + \sigma_A j_G\left(\mu_\delta - r\right)}{-\sigma_A j_G + j_A \sigma_G} - j_A \overline{\lambda}. \tag{A.31}$$

Applying the distributive rule and factoring out,

$$\frac{-\sigma_A j_G r + j_A \sigma_G \mu_\delta - \sigma_A j_A\left(\alpha_G - r\right)}{j_A \sigma_G - \sigma_A j_G} - j_A \overline{\lambda}. \tag{A.32}$$

Substituting $\alpha_G = \sigma_G \frac{\mu_\delta - j_A \overline{\lambda} - r}{\sigma_A} + j_G \overline{\lambda} + r$ as stated in equation (2.23),

$$\frac{-\sigma_A j_G r + j_A \sigma_G \mu_\delta - j_A \sigma_G\left(\mu_\delta - j_A \overline{\lambda} - r\right) - j_A \sigma_A j_G \overline{\lambda}}{j_A \sigma_G - \sigma_A j_G} - j_A \overline{\lambda}. \tag{A.33}$$

Factoring out again and then simplifying one obtains

$$\begin{aligned}\frac{j_A\sigma_G\left[r+j_A\overline{\lambda}\right]-\sigma_A j_G\left[r+j_A\overline{\lambda}\right]}{j_A\sigma_G-\sigma_A j_G}-j_A\overline{\lambda}&=\frac{\left[r+j_A\overline{\lambda}\right]\left[j_A\sigma_G-\sigma_A j_G\right]}{j_A\sigma_G-\sigma_A j_G}-j_A\overline{\lambda}\\&=r.\end{aligned}\tag{A.34}$$

Thus, the asset process has a drift equal to r under measure $\mathbb{Q}$ as stated in equation (2.28).

One can also use the martingale approach in the case that A_t is not traded. Again, one has to to find g and h at each moment in time such that the discounted asset price under measure $\mathbb{Q}$ is a martingale. g and h are given by the system in equation (A.29) replacing the A-terms by the correspondent F-terms:

$$\begin{cases} g=\frac{-j_F(\alpha_G-r)+j_G(\alpha_F-r)}{-\sigma_F j_G+j_F\sigma_G}\\ (1+h)\,\lambda=\frac{r(\sigma_F-\sigma_G)+(\alpha_F\sigma_G-\alpha_G\sigma_F)}{-\sigma_F j_G+j_F\sigma_G}\end{cases}\tag{A.35}$$

As in the case where the underlying is traded, g and h are still uniquely determined. The fact that this measure change is unique implies a certain relation between derivative prices but does not imply a unique price for the derivatives. Replacing the obtained solutions in equation (A.24) one obtains

$$\frac{dA_t}{A_t}=\left(\mu_\delta+g^*\sigma_A-(1+h^*)\lambda j_A\right)dt+\sigma_A dW_t{}^{\mathbb{Q}}-j_A dM_t^{\mathbb{Q}},\tag{A.36}$$

where g^* and h^* correspond to the solutions to the system in equation (A.35). Equation (A.36) is equal to equation (2.37) with $g^*\sigma_A$ instead of $-\overline{m}\sigma_A$ and $(1+h^*)\lambda$ instead of $\overline{\lambda}$.

A.3. The joint distribution of X_t and $\tau \geq t$

This appendix proves Proposition 3 on the joint distribution of X_t and $\tau \geq t$. The proof follows from the Girsanov theorem and the joint distribution of a standard Brownian motion and the first passage time. I will start by showing how to derive the latter using the reflection principle and the total probability theorem, and only then move to the case of an arithmetic Brownian motion.

From the reflection principle, it is known that for every sample path that hits level y before time t but finishes below level x at time t, there is another equally probable path that hits y before time t and then travels upwards at least $y-x$ units to finish above level $2y-x$. Mathematically, this is equivalent to say

$$\mathbb{Q}\left(W_t^{\mathbb{Q}} < x, \sup_{0<u\leq t} \left(W_u^{\mathbb{Q}}\right) > y \middle| \mathbb{F}_0 \right) = \mathbb{Q}\left(W_t^{\mathbb{Q}} > 2y - x, \sup_{0<u\leq t} \left(W_u^{\mathbb{Q}}\right) > y \middle| \mathbb{F}_0 \right). \quad \text{(A.37)}$$

Since y>x,

$$\mathbb{Q}\left(\sigma W_t^{\mathbb{Q}} > 2y - x, \sup_{0<u\leq t} \left(\sigma W_u^{\mathbb{Q}}\right) > y \middle| \mathbb{F}_0 \right) = \mathbb{Q}\left(\sigma W_t^{\mathbb{Q}} > 2y - x \middle| \mathbb{F}_0 \right). \quad \text{(A.38)}$$

Dividing the right-hand side of equation (A.38) by the standard deviation of $\sigma W_t^{\mathbb{Q}}$,

$$\begin{aligned} \mathbb{Q}\left(\frac{\sigma W_t^{\mathbb{Q}}}{\sigma\sqrt{t}} > \frac{2y - x}{\sigma\sqrt{t}} \middle| \mathbb{F}_0 \right) &= 1 - \mathbb{Q}\left(\frac{\sigma W_t^{\mathbb{Q}}}{\sigma\sqrt{t}} < \frac{2y - x}{\sigma\sqrt{t}} \middle| \mathbb{F}_0 \right) \\ &= 1 - N\left(\frac{2y - x}{\sigma\sqrt{t}} \right) = N\left(\frac{x - 2y}{\sigma\sqrt{t}} \right). \end{aligned} \quad \text{(A.39)}$$

Using this result in equation (A.37),

$$\mathbb{Q}\left(\sigma W_t^{\mathbb{Q}} < x, \sup_{0<u\leq t} \left(\sigma W_u^{\mathbb{Q}}\right) > y \middle| \mathbb{F}_0 \right) = N\left(\frac{x - 2y}{\sigma\sqrt{t}} \right). \quad \text{(A.40)}$$

It is also known from the total probability theorem that

$$\mathbb{Q}\left(\sigma W_t^{\mathbb{Q}} < x \middle| \mathbb{F}_0 \right) = \mathbb{Q}\left(\sigma W_t^{\mathbb{Q}} < x, \sup_{0<u\leq t} \left(\sigma W_u^{\mathbb{Q}}\right) > y \middle| \mathbb{F}_0 \right) \quad \text{(A.41)}$$

$$+ \mathbb{Q}\left(\sigma W_t^{\mathbb{Q}} < x, \sup_{0<u\leq t} \left(\sigma W_u^{\mathbb{Q}}\right) \leq y \middle| \mathbb{F}_0 \right), \quad \text{(A.42)}$$

where

$$\mathbb{Q}\left(\sigma W_t^{\mathbb{Q}} < x \middle| \mathbb{F}_0 \right) = N\left(\frac{x}{\sigma\sqrt{t}} \right). \quad \text{(A.43)}$$

Replacing equations (A.40) and (A.43) on equation (A.41), one obtains

$$\mathbb{Q}\left(X_t^{\mathbb{Q}} \leq x, \underset{0<u\leq t}{sup} \left(X_u^{\mathbb{Q}}\right) \leq y \middle| \mathbb{F}_0 \right) = N\left(\frac{x}{\sigma\sqrt{t}} \right) - N\left(\frac{x - 2y}{\sigma\sqrt{t}} \right) \quad \text{(A.44)}$$

for $y \geq 0$ and $x \leq y$.

Applying the Leibniz rule to equation (A.44),

$$\begin{aligned}
\mathbb{Q}\left(\sigma W_t^{\mathbb{Q}} \in dx, \sup_{0<u\leq t}\left(\sigma W_u^{\mathbb{Q}}\right) \leq y \middle| \mathbb{F}_0\right) &= \frac{d}{dx}\mathbb{Q}\left(\sigma W_t^{\mathbb{Q}} < x, \sup_{0<u\leq t}\left(\sigma W_u^{\mathbb{Q}}\right) \leq y \middle| \mathbb{F}_0\right) \\
&= \frac{d}{dx}\left[\int_{-\infty}^{\frac{x}{\sigma\sqrt{t}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du - \int_{-\infty}^{\frac{x-2y}{\sigma\sqrt{t}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du\right] \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}\left(\frac{x}{\sigma\sqrt{t}}\right)^2} \frac{1}{\sigma\sqrt{t}} - \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-2y}{\sigma\sqrt{t}}\right)^2} \frac{1}{\sigma\sqrt{t}} \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{x^2}{\sigma^2(t)}} - \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-2y)^2}{\sigma^2(t)}} \\
&= n\left(x; 0; \sigma\sqrt{t}\right) - n\left(x; 2y; \sigma\sqrt{t}\right).
\end{aligned} \tag{A.45}$$

Now consider that

$$dX_t = vdt + \sigma dW_t^{\mathbb{Q}}, \tag{A.46}$$

This is equivalent to say that $dX_t = \sigma dW_t^{\mathbb{Q}^\star}$ where $W_u^{\mathbb{Q}^\star}$ is a $\mathbb{Q}^*$-measured standard Brownian motion and

$$dW_t^{\mathbb{Q}^\star} = dW_t^{\mathbb{Q}} + \frac{v}{\sigma}dt. \tag{A.47}$$

Changing the probability measure leads to

$$\begin{aligned}
\mathbb{Q}\left(X_t \leq x, \sup_{0<u\leq t}(X_u) \leq y \middle| \mathbb{F}_0\right) &= E^{\mathbb{Q}}\left[1_{\left\{X_t \leq x, \sup_{0<u\leq t}(X_u) \leq y\right\}} \middle| \mathbb{F}_0\right] \\
&= E^{\mathbb{Q}^\star}\left[\frac{dQ}{d\mathbb{Q}^\star} 1_{\left\{\sigma W_t^{\mathbb{Q}^\star} \leq x, \sup_{0<u\leq t}\left(\sigma W_t^{\mathbb{Q}^\star}\right) \leq y\right\}} \middle| \mathbb{F}_0\right],
\end{aligned} \tag{A.48}$$

where $\frac{dQ}{d\mathbb{Q}^\star}$ can be computed using the Girsanov theorem:

$$\begin{aligned}\frac{dQ}{d\mathbb{Q}^\star}\bigg|\mathbb{F}_t &= e^{-\int\limits_0^T(-\frac{v}{\sigma})dW_s^{\mathbb{Q}^\star}-\frac{1}{2}\int\limits_0^T(-\frac{v}{\sigma})^2ds} \\ &= e^{\frac{v}{\sigma}W_t^{\mathbb{Q}^\star}-\frac{1}{2}(\frac{v}{\sigma})^2t} \\ &= e^{\frac{v}{\sigma^2}\sigma W_t^{\mathbb{Q}^\star}-\frac{1}{2}\frac{v^2}{\sigma^2}t}.\end{aligned} \tag{A.49}$$

Thus,

$$\mathbb{Q}\left(X_t \leq x, \sup_{0<u\leq t}(X_u)\leq y\bigg|\mathbb{F}_0\right) = E^{\mathbb{Q}^\star}\left[e^{\frac{v}{\sigma^2}\sigma W_t^{\mathbb{Q}^\star}-\frac{1}{2}\frac{v^2}{\sigma^2}t}\,1_{\left\{\sigma W_t^{\mathbb{Q}^\star}\leq x,\ \sup\limits_{0<u\leq t}\left(\sigma W_t^{\mathbb{Q}^\star}\right)\leq y\right\}}\bigg|\mathbb{F}_0\right]. \tag{A.50}$$

Let $z = \sigma W_t^{\mathbb{Q}^\star}$. Integrating for values of z between $-\infty$ and x,

$$\mathbb{Q}\left(X_t \leq x, \sup_{0<u\leq t}(X_u)\leq y\bigg|\mathbb{F}_0\right) = \int\limits_{-\infty}^{x} e^{\frac{v}{\sigma^2}z-\frac{1}{2}\frac{v^2}{\sigma^2}t}\left[n\left(z;0;\sigma\sqrt{t}\right)-n\left(z;2y;\sigma\sqrt{t}\right)\right]dz. \tag{A.51}$$

Applying Leibniz rule and rearranging,

$$\begin{aligned}\mathbb{Q}\left(X_t^{\mathbb{Q}} \in dx, \sup_{0<u\leq t}\left(X_u^{\mathbb{Q}}\right)\leq y\bigg|\mathbb{F}_0\right) &= e^{\frac{v}{\sigma^2}x-\frac{1}{2}\frac{v^2}{\sigma^2}t}\left[n\left(x;0;\sigma\sqrt{t}\right)-n\left(x;2y;\sigma\sqrt{t}\right)\right] \\ &= n\left(x;vt;\sigma\sqrt{t}\right)-e^{\frac{2vy}{\sigma^2}}n\left(x;2y+vt;\sigma\sqrt{t}\right).\end{aligned} \tag{A.52}$$

Alternatively, integrating equation (A.51), one obtains the joint distribution function of $X_t^{\mathbb{Q}}$ and $\sup\limits_{0<u\leq t}\left(X_u^{\mathbb{Q}}\right)\leq y$:

$$\mathbb{Q}\left(X_t^{\mathbb{Q}} < x, \sup_{0<u\leq t}\left(X_u^{\mathbb{Q}}\right)\leq y\bigg|\mathbb{F}_0\right) = N\left(\frac{x-vt}{\sigma\sqrt{t}}\right)-e^{\frac{2\mu y}{\sigma^2}}N\left(\frac{x-2y-vt}{\sigma\sqrt{t}}\right). \tag{A.53}$$

For the joint distribution function of $X_t^{\mathbb{Q}}$ and $\inf\limits_{0<u\leq t}\left(X_u^{\mathbb{Q}}\right)\geq y$, notice that

$$\sup_{0<u\leq t}\left(-X_u^{\mathbb{Q}}\right) = -\inf_{0<u\leq t}\left(X_u^{\mathbb{Q}}\right). \tag{A.54}$$

Hence,

$$\begin{aligned}\mathbb{Q}\left(X_t^{\mathbb{Q}} \leq x, \inf_{0<u\leq t}\left(X_u^{\mathbb{Q}}\right) \geq y \middle| \mathbb{F}_0\right) &= \mathbb{Q}\left(-X_t^{\mathbb{Q}} \geq -x, -\inf_{0<u\leq t}\left(X_u^{\mathbb{Q}}\right) \leq -y \middle| \mathbb{F}_0\right) \\ &= \mathbb{Q}\left(-X_t^{\mathbb{Q}} \geq -x, \sup_{0<u\leq t}\left(-X_u^{\mathbb{Q}}\right) \leq -y \middle| \mathbb{F}_0\right).\end{aligned} \tag{A.55}$$

Consider

$$\begin{aligned}Z_t &= -X_t \\ \overline{x} &= -x \\ \overline{y} &= -y\end{aligned}$$

Applying Itô's lemma to $Z_t = -X_t$, then

$$dZ_t = -vdt + (-\sigma)\, dW_t^{\mathbb{Q}}. \tag{A.56}$$

Given equation (A.52), one obtains

$$\begin{aligned}&\mathbb{Q}\left(X_t^{\mathbb{Q}} \leq x, \inf_{0<u\leq t}\left(X_u^{\mathbb{Q}}\right) \geq y \middle| \mathbb{F}_0\right) \\ &= \mathbb{Q}\left(Z_t^{\mathbb{Q}} \geq \bar{x}, \sup_{0<u\leq t}\left(Z_u^{\mathbb{Q}}\right) \leq \bar{y} \middle| \mathbb{F}_0\right) \\ &= \int_{\bar{x}}^{+\infty}\left[n\left(z; -vt; \sqrt{(-\sigma)^2 t}\right) - e^{\frac{2(-v)\overline{y}}{(-\sigma)^2}} n\left(z; 2\overline{y} - vt; \sqrt{(-\sigma)^2 t}\right)\right] 1_{\{z\leq\overline{y}\}} dz \\ &= N\left(\frac{\overline{y}+vt}{\sigma\sqrt{t}}\right) - e^{-\frac{2v\overline{y}}{\sigma^2}} N\left(\frac{\overline{y}-2\overline{y}+vt}{\sigma\sqrt{t}}\right) - N\left(\frac{\overline{x}+vt}{\sigma\sqrt{t}}\right) + e^{-\frac{2v\overline{y}}{\sigma^2}} N\left(\frac{\overline{x}-2\overline{y}+vt}{\sigma\sqrt{t}}\right).\end{aligned} \tag{A.57}$$

Substituting $\overline{x}$ and $\overline{y}$, the above simplifies to equation (2.43). Differentiating one obtains

$$\begin{aligned}\mathbb{Q}\left[X_t^{\mathbb{Q}} \in dx, \inf_{0<u\leq t}\left(X_u^{\mathbb{Q}}\right) \geq y\right] &= \frac{\partial}{\partial x}\mathbb{Q}\left[X_t^{\mathbb{Q}} \leq x, \inf_{0<u\leq t}\left(X_u^{\mathbb{Q}}\right) \geq y\right] \\ &= -\frac{\partial}{\partial x}N\left(\frac{-x+vt}{\sigma\sqrt{t}}\right) + e^{\frac{2vy}{\sigma^2}}\frac{\partial}{\partial x}N\left(\frac{-x+2y+vt}{\sigma\sqrt{t}}\right) \\ &= \frac{\partial}{\partial x}N\left(\frac{x-vt}{\sigma\sqrt{t}}\right) - e^{\frac{2vy}{\sigma^2}}\frac{\partial}{\partial x}N\left(\frac{x-2y-vt}{\sigma\sqrt{t}}\right) \\ &= n\left(x; vt; \sigma\sqrt{t}\right) - e^{\frac{2vy}{\sigma^2}} n\left(x; 2y+vt; \sigma\sqrt{t}\right).\end{aligned} \tag{A.58}$$

A.4. The limits of the F(.) function

This section shows how to compute the limits of the $F(.)$ function.

Before proceeding notice that

$$\lim_{y\to+\infty} F(a,b,c,y) = \begin{cases} \Omega_g^+(a,c)\lim_{y\to+\infty} g^+(y) + \Omega_h^+(a,c)\lim_{y\to+\infty} h^+(y), b>0 \\ \Omega_g^-(a,c)\lim_{y\to+\infty} g^-(y) + \Omega_h^-(a,c)\lim_{y\to+\infty} h^-(y), b<0 \end{cases}. \tag{A.59}$$

Since c^2-2a is always positive we have that

$$\begin{aligned}
&\lim_{y\to+\infty} g^+(y) = e^{-b\Psi_g^+(a,c)}\lim_{y\to+\infty} N\left(\tfrac{-b-y\sqrt{c^2-2a}}{\sqrt{y}}\right) = 0 \\
&\lim_{y\to+\infty} h^+(y) = e^{-b\Psi_h^+(a,c)}\lim_{y\to+\infty} N\left(\tfrac{-b+y\sqrt{c^2-2a}}{\sqrt{y}}\right) = e^{-b\left(\Psi_h^+(a,c)\right)} \\
&\lim_{y\to+\infty} g^-(y) = e^{b\Psi_g^-(a,c)}\lim_{y\to+\infty} N\left(\tfrac{b-y\sqrt{c^2-2a}}{\sqrt{y}}\right) = 0 \\
&\lim_{y\to+\infty} h^-(y) = e^{b\Psi_h^-(a,c)}\lim_{y\to+\infty} N\left(\tfrac{b+y\sqrt{c^2-2a}}{\sqrt{y}}\right) = e^{b\left(\Psi_h^-(a,c)\right)}
\end{aligned}$$

and thus

$$\lim_{y\to+\infty} F(a,b,c,y) = \begin{cases} \Omega_h^+(a,c)\, e^{-b\Psi_h^+(a,c)}, b>0 \\ \Omega_h^-(a,c)\, e^{b\Psi_h^-(a,c)}, b<0 \end{cases}. \tag{A.60}$$

So, for any fixed finite value of b, $\lim_{T\to+\infty} F(.)$ can be computed using equation (A.60).

In the case of $\lim_{T\to+\infty} F\left(\omega, \frac{\ln(R)}{\sigma}, \frac{v^\star+\sigma^2}{\sigma}, T\right)$ and $\lim_{T\to+\infty} F\left(\omega, \frac{\ln(R)}{\sigma}, -\frac{v^\star+\sigma^2}{\sigma}, T\right)$ in equation (3.9), since $\frac{\ln(R)}{\sigma} < 0$ we have

$$\begin{aligned}
\lim_{T\to+\infty} F\left(\omega, \frac{\ln(R)}{\sigma}, \frac{v^\star+\sigma^2}{\sigma}, T\right) &= \Omega_h^-\left(\omega, \frac{v^\star+\sigma^2}{\sigma}\right) e^{\frac{\ln(R)}{\sigma}\Psi_h^-\left(\omega,\frac{v^\star+\sigma^2}{\sigma}\right)} \\
&= \Omega_h^-\left(\omega, \frac{v^\star+\sigma^2}{\sigma}\right) R^{\frac{1}{\sigma}\Psi_h^-\left(\omega,\frac{v^\star+\sigma^2}{\sigma}\right)}
\end{aligned} \tag{A.61}$$

and

$$\lim_{T\to+\infty} F\left(\omega, \frac{\ln(R)}{\sigma}, -\frac{v^\star+\sigma^2}{\sigma}, T\right) = \Omega_h^-\left(\omega, -\frac{v^\star+\sigma^2}{\sigma}\right) R^{\frac{1}{\sigma}\Psi_h^-\left(\omega,-\frac{v^\star+\sigma^2}{\sigma}\right)}. \tag{A.62}$$

The same rationale applies to $\lim_{T\to+\infty} F\left(\varpi, \frac{\ln(R)}{\sigma}, \frac{v^\star}{\sigma}, T\right)$ and $\lim_{T\to+\infty} F\left(\varpi, \frac{\ln(R)}{\sigma}, -\frac{v^\star}{\sigma}, T\right)$ in equation (3.14), with ϖ appearing instead of ω and $\frac{v^\star}{\sigma}$ replacing $\frac{v^\star+\sigma^2}{\sigma}$.

Regarding the limits of the $ANJump\left(\bar{l},\overline{u},T\right)$ and $DigJump\left(\bar{l},\overline{u},T\right)$ functions, one may have two cases depending on whether we are interested in the limit when both $\overline{u}$ and T go to $+\infty$ or only T.

Only T goes to $+\infty$

Using equation (2.85),

$$\begin{aligned}
&\lim_{T\to+\infty} ANJump\left(\bar{l},\overline{u},T\right)\\
&=\lim_{T\to+\infty}\frac{\overline{\lambda}A}{\omega}\left\{e^{\omega T}\left[N\left(h_3\left(\overline{u},T\right)\right)-N\left(h_3\left(\bar{l},T\right)\right)\right]-N\left(h_3\left(\overline{u},0\right)\right)+N\left(h_3\left(\bar{l},0\right)\right)\right.\\
&-F\left(\omega,\frac{\ln\left(\frac{\overline{u}}{A}\right)}{\sigma},\frac{v^{\star}+\sigma^2}{\sigma},T\right)+F\left(\omega,\frac{\ln\left(\frac{\bar{l}}{A}\right)}{\sigma},\frac{v^{\star}+\sigma^2}{\sigma},T\right)\\
&+R^{2a+2}\left\{e^{\omega T}\left[N\left(h_4\left(\overline{u},T\right)\right)-N\left(h_4\left(\bar{l},T\right)\right)\right]-N\left(h_4\left(\overline{u},0\right)\right)+N\left(h_4\left(\bar{l},0\right)\right)\right.\\
&\left.\left.-F\left(\omega,\frac{\ln\left(\frac{R\bar{v}}{\overline{u}}\right)}{\sigma},-\frac{v^{\star}+\sigma^2}{\sigma},T\right)F\left(\omega,\frac{\ln\left(\frac{R\bar{v}}{\bar{l}}\right)}{\sigma},-\frac{v^{\star}+\sigma^2}{\sigma},T\right)\right\}\right\}.
\end{aligned}\tag{A.63}$$

Since $\omega\leq 0$ the exponential terms disappear in the limit. In addition,

$$\begin{array}{ll}
\forall x>A\Rightarrow N\left(h_3\left(x,0\right)\right)=1 & \forall x>R\overline{v}\Rightarrow N\left(h_4\left(x,0\right)\right)=0\\
\forall x<A\Rightarrow N\left(h_3\left(x,0\right)\right)=0 & \forall x<R\overline{v}\Rightarrow N\left(h_4\left(x,0\right)\right)=1\\
\forall x=A\Rightarrow N\left(h_3\left(x,0\right)\right)=0.5 & \forall x=R\overline{v}\Rightarrow N\left(h_4\left(x,0\right)\right)=0.5
\end{array}.\tag{A.64}$$

Noticing that for all intervals of interest $\bar{l}$ and $\overline{u}$ are higher than $R\bar{v}$, we have that $N\left(h_4\left(\bar{l},0\right)\right)$ and $N\left(h_4\left(\overline{u},0\right)\right)$ also disappear. As result,

$$\lim_{T\to+\infty} ANJump\left(\bar{l},\overline{u},T\right)$$
$$= \frac{\overline{\lambda}A}{\omega}\left\{-\left(1_{\{\overline{u}>A\}}+0.5\times 1_{\{\overline{u}=A\}}\right)+\left(1_{\left\{\bar{l}>A\right\}}+0.5\times 1_{\left\{\bar{l}=A\right\}}\right)\right.$$
$$-\lim_{T\to+\infty} F\left(\omega,\frac{\ln\left(\frac{\overline{u}}{A}\right)}{\sigma},\frac{v^{\star}+\sigma^2}{\sigma},T\right)+\lim_{T\to+\infty} F\left(\omega,\frac{\ln\left(\frac{\bar{l}}{A}\right)}{\sigma},\frac{v^{\star}+\sigma^2}{\sigma},T\right)$$
$$\left.+R^{2a+2}\left[-\lim_{T\to+\infty} F\left(\omega,\frac{\ln\left(\frac{R\bar{v}}{\overline{u}}\right)}{\sigma},-\frac{v^{\star}+\sigma^2}{\sigma},T\right)+\lim_{T\to+\infty} F\left(\omega,\frac{\ln\left(\frac{R\bar{v}}{\bar{l}}\right)}{\sigma},-\frac{v^{\star}+\sigma^2}{\sigma},T\right)\right]\right\}. \tag{A.65}$$

Following equation (A.60), the first two limits can be computed using

$$\lim_{T\to+\infty} F\left(\omega,\frac{\ln\left(\frac{\bar{u}}{A}\right)}{\sigma},\frac{v^{\star}+\sigma^2}{\sigma},T\right)=\begin{cases}\Omega_h^{+}\left(\omega,\frac{v^{\star}+\sigma^2}{\sigma}\right)\left(\frac{\bar{u}}{A}\right)^{-\frac{1}{\sigma}\Psi_h^{+}\left(\omega,\frac{v^{\star}+\sigma^2}{\sigma}\right)},\bar{u}>A\\ \Omega_h^{-}\left(\omega,\frac{v^{\star}+\sigma^2}{\sigma}\right)\left(\frac{\bar{u}}{A}\right)^{\frac{1}{\sigma}\Psi_h^{-}\left(\omega,\frac{v^{\star}+\sigma^2}{\sigma}\right)},\bar{u}<A\end{cases}, \tag{A.66}$$

and the remaining two limits can be computed using

$$\lim_{T\to+\infty} F\left(\omega,\frac{\ln\left(\frac{R\bar{v}}{\bar{u}}\right)}{\sigma},-\frac{v^{\star}+\sigma^2}{\sigma},T\right)=\begin{cases}\Omega_h^{+}\left(\omega,-\frac{v^{\star}+\sigma^2}{\sigma}\right)\left(\frac{R\bar{v}}{\bar{u}}\right)^{-\frac{1}{\sigma}\Psi_h^{+}\left(\omega,-\frac{v^{\star}+\sigma^2}{\sigma}\right)},\bar{u}<R\bar{v}\\ \Omega_h^{-}\left(\omega,-\frac{v^{\star}+\sigma^2}{\sigma}\right)\left(\frac{R\bar{v}}{\bar{u}}\right)^{\frac{1}{\sigma}\Psi_h^{-}\left(\omega,-\frac{v^{\star}+\sigma^2}{\sigma}\right)},\bar{u}>R\bar{v}\end{cases}. \tag{A.67}$$

For $\lim_{T\to+\infty} DigJump\left(\bar{l},\overline{u},T\right)$, and following the same steps, we arrive at

$$\lim_{T\to+\infty} DigJump\left(\bar{l},\overline{u},T\right)$$
$$= \frac{\overline{\lambda}}{\varpi}\left\{-\left(1_{\{\overline{u}>A\}}+0.5\times 1_{\{\overline{u}=A\}}\right)+\left(1_{\left\{\bar{l}>A\right\}}+0.5\times 1_{\left\{\bar{l}=A\right\}}\right)\right.$$
$$-\lim_{T\to+\infty} F\left(\varpi,\frac{\ln\left(\frac{\overline{u}}{A}\right)}{\sigma},\frac{v^{\star}}{\sigma},T\right)+\lim_{T\to+\infty} F\left(\varpi,\frac{\ln\left(\frac{\bar{l}}{A}\right)}{\sigma},\frac{v^{\star}}{\sigma},T\right) \tag{A.68}$$
$$\left.+R^{2a}\left[-\lim_{T\to+\infty} F\left(\varpi,\frac{\ln\left(\frac{R\bar{v}}{\overline{u}}\right)}{\sigma},-\frac{v^{\star}}{\sigma},T\right)+\lim_{T\to+\infty} F\left(\varpi,\frac{\ln\left(\frac{R\bar{v}}{\bar{l}}\right)}{\sigma},-\frac{v^{\star}}{\sigma},T\right)\right]\right\},$$

where the limits can be computed using equations (A.66) and (A.67) replacing ω by ϖ and $\frac{v^\star+\sigma^2}{\sigma}$ by $\frac{v}{\sigma}$.

Both $\overline{u}$ and T go to $+\infty$

Using equation (2.85), we have that

$$\begin{aligned}
&\lim_{\overline{u},T\to+\infty} ANJump\left(\bar{l},\overline{u},T\right) \\
&= \lim_{\overline{u},T\to+\infty} \frac{\overline{\lambda}A}{\omega}\left\{e^{\omega T}\left[N\left(h_3\left(\overline{u},T\right)\right)-N\left(h_3\left(\bar{l},T\right)\right)\right]-N\left(h_3\left(\overline{u},0\right)\right)+N\left(h_3\left(\bar{l},0\right)\right)\right. \\
&-F\left(\omega,\frac{\ln\left(\frac{\overline{u}}{A}\right)}{\sigma},\frac{v^\star+\sigma^2}{\sigma},T\right)+F\left(\omega,\frac{\ln\left(\bar{l}/A\right)}{\sigma},\frac{v^\star+\sigma^2}{\sigma},T\right) \\
&+R^{2a+2}\left[e^{\omega T}\left(N\left(h_4\left(\overline{u},T\right)\right)-N\left(h_4\left(\bar{l},T\right)\right)\right)-N\left(h_4\left(\overline{u},0\right)\right)+N\left(h_4\left(\bar{l},0\right)\right)\right. \\
&\left.\left.-F\left(\omega,\frac{\ln\left(\frac{R\bar{v}}{\overline{u}}\right)}{\sigma},-\frac{v^\star+\sigma^2}{\sigma},T\right)+F\left(\omega,\frac{\ln\left(\frac{R\bar{v}}{\bar{l}}\right)}{\sigma},-\frac{v^\star+\sigma^2}{\sigma},T\right)\right]\right\}.
\end{aligned} \tag{A.69}$$

Since $\omega\leq 0$ and noticing that $\lim_{\overline{u}\to+\infty} N\left(h_3\left(\overline{u},0\right)\right)=1$, $N\left(h_3\left(\bar{l},0\right)\right)=1_{\left\{\bar{l}>A\right\}}+0.5\times 1_{\left\{\bar{l}=A\right\}}$ and that $\lim_{\overline{u}\to+\infty} N\left(h_4\left(\overline{u},0\right)\right)=N\left(h_4\left(\bar{l},0\right)\right)=0$, this simplifies to[2]

$$\begin{aligned}
&\lim_{\overline{u},T\to+\infty} ANJump\left(\bar{l},\overline{u},T\right) \\
&=\frac{\overline{\lambda}A}{\omega}\left\{-1+1_{\left\{\bar{l}>A\right\}}+0.5\times 1_{\left\{\bar{l}=A\right\}}\right. \\
&-\lim_{\overline{u},T\to+\infty} F\left(\omega,\frac{\ln\left(\frac{\overline{u}}{A}\right)}{\sigma},\frac{v^\star+\sigma^2}{\sigma},T\right)+\lim_{T\to+\infty} F\left(\omega,\frac{\ln\left(\frac{\bar{l}}{A}\right)}{\sigma},\frac{v^\star+\sigma^2}{\sigma},T\right) \\
&\left.+R^{2a+2}\left[-\lim_{\overline{u},T\to+\infty} F\left(\omega,\frac{\ln\left(\frac{R\bar{v}}{\overline{u}}\right)}{\sigma},-\frac{v^\star+\sigma^2}{\sigma},T\right)+\lim_{T\to+\infty} F\left(\omega,\frac{\ln\left(\frac{R\bar{v}}{\bar{l}}\right)}{\sigma},-\frac{v^\star+\sigma^2}{\sigma},T\right)\right]\right\}.
\end{aligned} \tag{A.70}$$

The computation of the second and fourth limit is similar to equations (A.61) and (A.62). Notice, however, that while it is known beforehand that $R\bar{v}<\bar{l}$ for all values of $\bar{l}$ we are interested in, implying that $\frac{\ln\left(\frac{R\bar{v}}{\bar{l}}\right)}{\sigma}<0$, we do not know whether $\frac{\ln\left(\frac{\bar{l}}{A}\right)}{\sigma}$ is positive or negative.

[2] All values of $\bar{l}$ and $\overline{u}$ that we are interested are bigger than $\bar{v}$ and thus $N\left(h_4\left(.,0\right)\right)=0$.

Thus,

$$\lim_{T\to+\infty} F\left(\omega, \frac{\ln\left(\frac{\bar{l}}{A}\right)}{\sigma}, \frac{v^\star+\sigma^2}{\sigma}, T\right) = \begin{cases} \Omega_h^+\left(\omega, \frac{v^\star+\sigma^2}{\sigma}\right)\left(\frac{\bar{l}}{A}\right)^{-\frac{1}{\sigma}\Psi_h^+\left(\omega,\frac{v^\star+\sigma^2}{\sigma}\right)}, \bar{l} > A \\ \Omega_h^-\left(\omega, \frac{v^\star+\sigma^2}{\sigma}\right)\left(\frac{\bar{l}}{A}\right)^{\frac{1}{\sigma}\Psi_h^-\left(\omega,\frac{v^\star+\sigma^2}{\sigma}\right)}, \bar{l} < A \end{cases} \tag{A.71}$$

and

$$\lim_{T\to+\infty} F\left(\omega, \frac{\ln\left(\frac{R\bar{v}}{\bar{l}}\right)}{\sigma}, -\frac{v^\star+\sigma^2}{\sigma}, T\right) = \Omega_h^-\left(\omega, -\frac{v^\star+\sigma^2}{\sigma}\right)\left(\frac{R\bar{v}}{\bar{l}}\right)^{\frac{1}{\sigma}\Psi_h^-\left(\omega,-\frac{v^\star+\sigma^2}{\sigma}\right)}. \tag{A.72}$$

In the case of the first and third limits, one must take into account that b is not fixed. However, we know that it is always positive in the first limit and always negative in the third limit. So, we must be always either in the first branch or in the second branch of equation (A.59).

Since $c^2 - 2a > 0$, it is clear that for $b > 0$ (first limit) we have

$$\lim_{b,y\to+\infty} g^+(y) = \lim_{b,y\to+\infty} e^{-b\Psi_g^+(a,c)} N\left(\frac{-b - y\sqrt{c^2-2a}}{\sqrt{y}}\right) = 0,$$

and thus,

$$\lim_{b,y\to+\infty} F(a,b,c,y) = \Omega_h^+(a,c) \lim_{b,y\to+\infty} h^+(y).$$

Dividing the numerator and the denominator in $h^+(y)$ by y, one obtains

$$\begin{aligned} \lim_{b,y\to+\infty} h^+(y) &= \lim_{b,y\to+\infty} e^{-b\Psi_h^+(a,c)} N\left(\frac{-b + y\sqrt{c^2-2a}}{\sqrt{y}}\right) \\ &= \lim_{b,y\to+\infty} e^{-b\Psi_h^+(a,c)} N\left(\frac{-\frac{b}{y} + \sqrt{c^2-2a}}{\frac{1}{\sqrt{y}}}\right). \end{aligned} \tag{A.73}$$

Further noticing that when $b = \ln\left(\frac{\overline{u}}{A}\right)$, $-\ln\left(\frac{\overline{u}}{A}\right)$ goes to $-\infty$ faster than y goes to $+\infty$, then

$$\lim_{b,y\to+\infty} h^+(y) = 0.$$

The same rationale can be applied for $b < 0$ (third limit). Again, since $c^2 - 2a > 0$ we have that

$$\lim_{b,y\to+\infty} g^{-}(y) = \lim_{b,y\to+\infty} e^{b\Psi_g^{-}(a,c)} N\left(\frac{b - y\sqrt{c^2 - 2a}}{\sqrt{y}}\right) = 0,$$

and thus,

$$\lim_{b,y\to-\infty} F(a,b,c,y) = \Omega_h^{-}(a,c) \lim_{b,y\to+\infty} h^{-}(y).$$

Dividing the numerator and the denominator in $h^{-}(y)$ by y, one obtains

$$\begin{aligned}\lim_{b,y\to-\infty} F(a,b,c,y) &= \Omega_h^{-}(a,c) \lim_{b,y\to+\infty} h^{-}(y)\\ &= \Omega_h^{-}(a,c) \lim_{b,y\to+\infty} e^{b\Psi_h^{-}(a,c)} N\left(\frac{b + y\sqrt{c^2 - 2a}}{\sqrt{y}}\right)\\ &= \Omega_h^{-}(a,c) \lim_{b,y\to+\infty} e^{b\Psi_h^{-}(a,c)} N\left(\frac{\frac{b}{y} + \sqrt{c^2 - 2a}}{\frac{1}{\sqrt{y}}}\right).\end{aligned}$$

Since $b = \ln\left(\frac{R\bar{v}}{\overline{u}}\right)$ goes to $-\infty$ faster than y goes to $+\infty$, we have that

$$\lim_{b,y\to+\infty} h^{-}(y) = 0.$$

As a result,

$$\lim_{\overline{u},T\to+\infty} F\left(\omega, \frac{\ln\left(\frac{\overline{u}}{A}\right)}{\sigma}, \frac{v^{\star} + \sigma^2}{\sigma}, T\right) = 0 \tag{A.74}$$

and

$$\lim_{\overline{u},T\to+\infty} F\left(\omega, \frac{\ln\left(\frac{R\bar{v}}{\overline{u}}\right)}{\sigma}, -\frac{v^{\star} + \sigma^2}{\sigma}, T\right) = 0. \tag{A.75}$$

Substituting equations (A.71)-(A.75) into equation (A.70) one obtains

$$\begin{aligned}
\lim_{\overline{u},T\to+\infty} ANJump\left(\bar{l},\overline{u},T\right) = \frac{\overline{\lambda}A}{\omega}\Bigg[&-1+1_{\{\bar{l}>A\}}+0.5\times 1_{\{\bar{l}=A\}}\\
&+1_{\{\bar{l}>A\}}\Omega_h^{+}\left(\omega,\frac{v^\star+\sigma^2}{\sigma}\right)\left(\frac{\bar{l}}{A}\right)^{-\frac{1}{\sigma}\Psi_h^{+}\left(\omega,\frac{v^\star+\sigma^2}{\sigma}\right)}\\
&+1_{\{\bar{l}<A\}}\Omega_h^{-}\left(\omega,\frac{v^\star+\sigma^2}{\sigma}\right)\left(\frac{\bar{l}}{A}\right)^{\frac{1}{\sigma}\Psi_h^{-}\left(\omega,\frac{v^\star+\sigma^2}{\sigma}\right)}\\
&+R^{2a+2}\Omega_h^{-}\left(\omega,-\frac{v^\star+\sigma^2}{\sigma}\right)\left(\frac{R\bar{v}}{\bar{l}}\right)^{\frac{1}{\sigma}\Psi_h^{-}\left(\omega,-\frac{v^\star+\sigma^2}{\sigma}\right)}\Bigg].
\end{aligned} \tag{A.76}$$

Using equation (2.90) and taking the same procedure, with ϖ and $\frac{v^\star}{\sigma}$ replacing ω and $\frac{v^\star+\sigma^2}{\sigma}$, respectively, one obtains

$$\begin{aligned}
\lim_{\overline{u},T\to+\infty} DigJump\left(\bar{l},\overline{u},T\right) = \frac{\lambda}{\varpi}\Bigg[&-1+1_{\{\bar{l}>A\}}+0.5\times 1_{\{\bar{l}=A\}}\\
&+1_{\{\bar{l}>A\}}\Omega_h^{+}\left(\varpi,\frac{v^\star}{\sigma}\right)\left(\frac{\bar{l}}{A}\right)^{-\frac{1}{\sigma}\Psi_h^{+}\left(\varpi,\frac{v^\star}{\sigma}\right)}\\
&+1_{\{\bar{l}<A\}}\Omega_h^{-}\left(\varpi,\frac{v^\star}{\sigma}\right)\left(\frac{\bar{l}}{A}\right)^{\frac{1}{\sigma}\Psi_h^{-}\left(\varpi,\frac{v^\star}{\sigma}\right)}\\
&+R^{2a}\Omega_h^{-}\left(\varpi,-\frac{v^\star}{\sigma}\right)\left(\frac{R\bar{v}}{\bar{l}}\right)^{\frac{1}{\sigma}\Psi_h^{-}\left(\varpi,-\frac{v^\star}{\sigma}\right)}\Bigg].
\end{aligned} \tag{A.77}$$

A.5. The first derivative of the F(.) function

In this section, the first derivative of the limit of the F(.) function when T and u goes to $+\infty$ is derived. Similarly to Appendix A.4, we might have two cases depending on whether both $\overline{u}$ and T go to $+\infty$ or only T.

Only T goes to $+\infty$

Using equation (A.65),

$$
\begin{aligned}
&\frac{\partial}{\partial A}\lim_{T\to+\infty} ANJump\left(\bar{l},\bar{u},T\right)\\
&=\frac{\overline{\lambda}}{\omega}\Bigg\{-\left(1_{\{\overline{u}>A\}}+0.5\times 1_{\{\overline{u}=A\}}\right)+\left(1_{\{\bar{l}>A\}}+0.5\times 1_{\{\bar{l}=A\}}\right)\\
&-\lim_{T\to+\infty}F\left(\omega,\frac{\ln\left(\frac{\overline{u}}{A}\right)}{\sigma},\frac{v^\star+\sigma^2}{\sigma},T\right)+\lim_{T\to+\infty}F\left(\omega,\frac{\ln\left(\frac{\bar{l}}{A}\right)}{\sigma},\frac{v^\star+\sigma^2}{\sigma},T\right)\\
&+R^{2a+2}\left[-\lim_{T\to+\infty}F\left(\omega,\frac{\ln\left(\frac{R\bar{v}}{\overline{u}}\right)}{\sigma},-\frac{v^\star+\sigma^2}{\sigma},T\right)+\lim_{T\to+\infty}F\left(\omega,\frac{\ln\left(\frac{R\bar{v}}{\bar{l}}\right)}{\sigma},-\frac{v^\star+\sigma^2}{\sigma},T\right)\right]\Bigg\}\\
&+\frac{\overline{\lambda}A}{\omega}\Bigg\{-\frac{\partial}{\partial A}\lim_{T\to+\infty}F\left(\omega,\frac{\ln\left(\frac{\overline{u}}{A}\right)}{\sigma},\frac{v^\star+\sigma^2}{\sigma},T\right)+\frac{\partial}{\partial A}\lim_{T\to+\infty}F\left(\omega,\frac{\ln\left(\frac{\bar{l}}{A}\right)}{\sigma},\frac{v^\star+\sigma^2}{\sigma},T\right)\\
&+\frac{-2a-2}{\bar{v}}R^{2a+3}\left[-\lim_{T\to+\infty}F\left(\omega,\frac{\ln\left(\frac{R\bar{v}}{\overline{u}}\right)}{\sigma},-\frac{v^\star+\sigma^2}{\sigma},T\right)\right.\\
&\left.+\lim_{T\to+\infty}F\left(\omega,\frac{\ln\left(\frac{R\bar{v}}{\bar{l}}\right)}{\sigma},-\frac{v^\star+\sigma^2}{\sigma},T\right)\right]+R^{2a+2}\left[-\frac{\partial}{\partial A}\lim_{T\to+\infty}F\left(\omega,\frac{\ln\left(\frac{R\bar{v}}{\overline{u}}\right)}{\sigma},-\frac{v^\star+\sigma^2}{\sigma},T\right)\right.\\
&\left.+\frac{\partial}{\partial A}\lim_{T\to+\infty}F\left(\omega,\frac{\ln\left(\frac{R\bar{v}}{\bar{l}}\right)}{\sigma},-\frac{v^\star+\sigma^2}{\sigma},T\right)\right]\Bigg\}.
\end{aligned}
\tag{A.78}
$$

The limits above are given by equations (A.66) and (A.67), while the derivatives are given by

$$
\begin{aligned}
&\frac{\partial}{\partial A}\lim_{T\to+\infty}F\left(\omega,\frac{\ln\left(\frac{\bar{u}}{A}\right)}{\sigma},\frac{v^\star+\sigma^2}{\sigma},T\right)\\
&=\begin{cases}\Omega_h^+\left(\omega,\frac{v^\star+\sigma^2}{\sigma}\right)\frac{\partial}{\partial A}\left(\frac{\bar{u}}{A}\right)^{-\frac{1}{\sigma}\Psi_h^+\left(\omega,\frac{v^\star+\sigma^2}{\sigma}\right)},\bar{u}>A\\ \Omega_h^-\left(\omega,\frac{v^\star+\sigma^2}{\sigma}\right)\frac{\partial}{\partial A}\left(\frac{\bar{u}}{A}\right)^{\frac{1}{\sigma}\Psi_h^-\left(\omega,\frac{v^\star+\sigma^2}{\sigma}\right)},\bar{u}<A\end{cases}\\
&=\begin{cases}\Omega_h^+\left(\omega,\frac{v^\star+\sigma^2}{\sigma}\right)\frac{\frac{1}{\sigma}\Psi_h^+\left(\omega,\frac{v^\star+\sigma^2}{\sigma}\right)}{\overline{u}}\left(\frac{\overline{u}}{A}\right)^{1-\frac{1}{\sigma}\Psi_h^+\left(\omega,\frac{v^\star+\sigma^2}{\sigma}\right)},\bar{u}>A\\ \Omega_h^-\left(\omega,\frac{v^\star+\sigma^2}{\sigma}\right)\frac{-\frac{1}{\sigma}\Psi_h^-\left(\omega,\frac{v^\star+\sigma^2}{\sigma}\right)}{\overline{u}}\left(\frac{\overline{u}}{A}\right)^{1+\frac{1}{\sigma}\Psi_h^-\left(\omega,\frac{v^\star+\sigma^2}{\sigma}\right)},\bar{u}<A\end{cases}
\end{aligned}
\tag{A.79}
$$

and

$$\frac{\partial}{\partial A}\lim_{T\to+\infty}F\left(\omega,\frac{\ln\left(\frac{R\bar{v}}{\bar{u}}\right)}{\sigma},-\frac{v^{\star}+\sigma^2}{\sigma},T\right)$$
$$=\begin{cases}\Omega_h^+\left(\omega,-\frac{v^{\star}+\sigma^2}{\sigma}\right)\frac{\partial}{\partial A}\left(\frac{R\bar{v}}{\bar{u}}\right)^{-\frac{1}{\sigma}\Psi_h^+\left(\omega,-\frac{v^{\star}+\sigma^2}{\sigma}\right)},\bar{u}<R\bar{v}\\ \Omega_h^-\left(\omega,-\frac{v^{\star}+\sigma^2}{\sigma}\right)\frac{\partial}{\partial A}\left(\frac{R\bar{v}}{\bar{u}}\right)^{\frac{1}{\sigma}\Psi_h^-\left(\omega,-\frac{v^{\star}+\sigma^2}{\sigma}\right)},\bar{u}>R\bar{v}\end{cases}$$
$$=\begin{cases}\left(\frac{\bar{v}}{\bar{u}}\right)^{-\frac{1}{\sigma}\Psi_h^+\left(\omega,-\frac{v^{\star}+\sigma^2}{\sigma}\right)}\Omega_h^+\left(\omega,-\frac{v^{\star}+\sigma^2}{\sigma}\right)\frac{\partial}{\partial A}R^{-\frac{1}{\sigma}\Psi_h^+\left(\omega,-\frac{v^{\star}+\sigma^2}{\sigma}\right)},\bar{u}<R\bar{v}\\ \left(\frac{\bar{v}}{\bar{u}}\right)^{\frac{1}{\sigma}\Psi_h^-\left(\omega,-\frac{v^{\star}+\sigma^2}{\sigma}\right)}\Omega_h^-\left(\omega,-\frac{v^{\star}+\sigma^2}{\sigma}\right)\frac{\partial}{\partial A}R^{\frac{1}{\sigma}\Psi_h^-\left(\omega,-\frac{v^{\star}+\sigma^2}{\sigma}\right)},\bar{u}>R\bar{v}\end{cases}$$
$$=\begin{cases}\left(\frac{\bar{v}}{\bar{u}}\right)^{-\frac{1}{\sigma}\Psi_h^+\left(\omega,-\frac{v^{\star}+\sigma^2}{\sigma}\right)}\Omega_h^+\left(\omega,-\frac{v^{\star}+\sigma^2}{\sigma}\right)\frac{\frac{1}{\sigma}\Psi_h^+\left(\omega,-\frac{v^{\star}+\sigma^2}{\sigma}\right)}{\bar{v}}R^{1-\frac{1}{\sigma}\Psi_h^+\left(\omega,-\frac{v^{\star}+\sigma^2}{\sigma}\right)},\bar{u}<R\bar{v}\\ \left(\frac{\bar{v}}{\bar{u}}\right)^{\frac{1}{\sigma}\Psi_h^-\left(\omega,-\frac{v^{\star}+\sigma^2}{\sigma}\right)}\Omega_h^-\left(\omega,-\frac{v^{\star}+\sigma^2}{\sigma}\right)\frac{-\frac{1}{\sigma}\Psi_h^-\left(\omega,-\frac{v^{\star}+\sigma^2}{\sigma}\right)}{\bar{v}}R^{1+\frac{1}{\sigma}\Psi_h^-\left(\omega,-\frac{v^{\star}+\sigma^2}{\sigma}\right)},\bar{u}>R\bar{v}\end{cases}. \tag{A.80}$$

The derivative $\frac{\partial}{\partial A}\lim_{T\to+\infty}DigJump\left(\bar{l},\bar{u},T\right)$ is obtained from equation (A.68):

$$\frac{\partial}{\partial A}\lim_{T\to+\infty}DigJump\left(\bar{l},\bar{u},T\right)$$
$$=\frac{\overline{\lambda}}{\varpi}\left\{-\frac{\partial}{\partial A}\lim_{T\to+\infty}F\left(\varpi,\frac{\ln\left(\frac{\bar{u}}{A}\right)}{\sigma},\frac{v^{\star}}{\sigma},T\right)+\frac{\partial}{\partial A}\lim_{T\to+\infty}F\left(\varpi,\frac{\ln\left(\frac{\bar{l}}{A}\right)}{\sigma},\frac{v^{\star}}{\sigma},T\right)\right.$$
$$+\frac{-2a}{\bar{v}}R^{2a+1}\left[-\lim_{T\to+\infty}F\left(\varpi,\frac{\ln\left(\frac{R\bar{v}}{\bar{u}}\right)}{\sigma},-\frac{v^{\star}}{\sigma},T\right)+\lim_{T\to+\infty}F\left(\varpi,\frac{\ln\left(\frac{R\bar{v}}{\bar{l}}\right)}{\sigma},-\frac{v^{\star}}{\sigma},T\right)\right]$$
$$\left.+R^{2a}\left[-\frac{\partial}{\partial A}\lim_{T\to+\infty}F\left(\varpi,\frac{\ln\left(\frac{R\bar{v}}{\bar{u}}\right)}{\sigma},-\frac{v^{\star}}{\sigma},T\right)+\frac{\partial}{\partial A}\lim_{T\to+\infty}F\left(\varpi,\frac{\ln\left(\frac{R\bar{v}}{\bar{l}}\right)}{\sigma},-\frac{v^{\star}}{\sigma},T\right)\right]\right\}. \tag{A.81}$$

The limits above can be computed using equations (A.66) and (A.67), replacing ω by ϖ and $\frac{v^{\star}+\sigma^2}{\sigma}$ by $\frac{v^{\star}}{\sigma}$. The derivatives are given by equations (A.79) and (A.80) also replacing ω by ϖ and $\frac{v^{\star}+\sigma^2}{\sigma}$ by $\frac{v^{\star}}{\sigma}$.

Both $\overline{u}$ and T go to $+\infty$

Following equation (A.76),

$$
\begin{aligned}
&\frac{\partial}{\partial A}\lim_{\overline{u},T\to+\infty} ANJump\left(\bar{l},\overline{u},T\right)\\
&=\frac{\overline{\lambda}}{\omega}\Bigg[-1+1_{\left\{\bar{l}>A\right\}}+0.5\times 1_{\left\{\bar{l}=A\right\}}+1_{\left\{\bar{l}>A\right\}}\Omega_h^{+}\left(\omega,\frac{v^\star+\sigma^2}{\sigma}\right)\left(\frac{\bar{l}}{A}\right)^{-\frac{1}{\sigma}\Psi_h^{+}\left(\omega,\frac{v^\star+\sigma^2}{\sigma}\right)}\\
&+1_{\left\{\bar{l}<A\right\}}\Omega_h^{-}\left(\omega,\frac{v^\star+\sigma^2}{\sigma}\right)\left(\frac{\bar{l}}{A}\right)^{\frac{1}{\sigma}\Psi_h^{-}\left(\omega,\frac{v^\star+\sigma^2}{\sigma}\right)}\\
&+R^{2a+2}\Omega_h^{-}\left(\omega,-\frac{v^\star+\sigma^2}{\sigma}\right)\left(\frac{R\overline{v}}{\bar{l}}\right)^{\frac{1}{\sigma}\Psi_h^{-}\left(\omega,-\frac{v^\star+\sigma^2}{\sigma}\right)}\Bigg]\\
&+\frac{\overline{\lambda}A}{\omega}\Bigg[1_{\left\{\bar{l}>A\right\}}\Omega_h^{+}\left(\omega,\frac{v^\star+\sigma^2}{\sigma}\right)\frac{\partial}{\partial A}\left(\frac{\bar{l}}{A}\right)^{-\frac{1}{\sigma}\Psi_h^{+}\left(\omega,\frac{v^\star+\sigma^2}{\sigma}\right)}\\
&+1_{\left\{\bar{l}<A\right\}}\Omega_h^{-}\left(\omega,\frac{v^\star+\sigma^2}{\sigma}\right)\frac{\partial}{\partial A}\left(\frac{\bar{l}}{A}\right)^{\frac{1}{\sigma}\Psi_h^{-}\left(\omega,\frac{v^\star+\sigma^2}{\sigma}\right)}\\
&+\left(\frac{\overline{v}}{\bar{l}}\right)^{\frac{1}{\sigma}\Psi_h^{-}\left(\omega,-\frac{v^\star+\sigma^2}{\sigma}\right)}\Omega_h^{-}\left(\omega,-\frac{v^\star+\sigma^2}{\sigma}\right)\frac{\partial}{\partial A}R^{2a+2+\frac{1}{\sigma}\Psi_h^{-}\left(\omega,-\frac{v^\star+\sigma^2}{\sigma}\right)}\Bigg],
\end{aligned}
\tag{A.82}
$$

where the above derivatives are given by

$$
\frac{\partial}{\partial A}\left(\frac{\bar{l}}{A}\right)^{\mp\frac{1}{\sigma}\Psi_h^{\pm}\left(\omega,\frac{v^\star+\sigma^2}{\sigma}\right)}=\frac{\pm\frac{1}{\sigma}\Psi_h^{\pm}\left(\omega,\frac{v^\star+\sigma^2}{\sigma}\right)}{\bar{l}}\left(\frac{\bar{l}}{A}\right)^{1\mp\frac{1}{\sigma}\Psi_h^{\pm}\left(\omega,\frac{v^\star+\sigma^2}{\sigma}\right)}
\tag{A.83}
$$

and by equation (3.41).

Doing the same for the derivative of equation (A.77), one obtains

$$\frac{\partial}{\partial A}\lim_{\overline{u},T\to+\infty} DigJump\left(\bar{l},\overline{u},T\right) = \frac{\overline{\lambda}}{\varpi}\left\{1_{\{\bar{l}>A\}}\Omega_h^+\left(\varpi,\frac{v^\star}{\sigma}\right)\frac{\partial}{\partial A}\left(\frac{\bar{l}}{A}\right)^{-\frac{1}{\sigma}\Psi_h^+\left(\varpi,\frac{v^\star}{\sigma}\right)}\right.$$
$$+1_{\{\bar{l}<A\}}\Omega_h^-\left(\varpi,\frac{v^\star}{\sigma}\right)\frac{\partial}{\partial A}\left(\frac{\bar{l}}{A}\right)^{\frac{1}{\sigma}\Psi_h^-\left(\varpi,\frac{v^\star}{\sigma}\right)}$$
$$\left.+\left(\frac{\overline{v}}{\bar{l}}\right)^{\frac{1}{\sigma}\Psi_h^-\left(\varpi,-\frac{v^\star}{\sigma}\right)}\Omega_h^-\left(\varpi,-\frac{v^\star}{\sigma}\right)\frac{\partial}{\partial A}R^{2a+\frac{1}{\sigma}\Psi_h^-\left(\varpi,-\frac{v^\star}{\sigma}\right)}\right\}, \tag{A.84}$$

where

$$\frac{\partial}{\partial A}\left(\frac{\bar{l}}{A}\right)^{\mp\frac{1}{\sigma}\Psi_h^\pm\left(\varpi,\frac{v^\star}{\sigma}\right)} = \frac{\pm\frac{1}{\sigma}\Psi_h^\pm\left(\varpi,\frac{v^\star}{\sigma}\right)}{\bar{l}}\left(\frac{\bar{l}}{A}\right)^{1\mp\frac{1}{\sigma}\Psi_h^\pm\left(\varpi,\frac{v^\star}{\sigma}\right)} \tag{A.85}$$

and the last derivative is given by equation (3.44).

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