A FRAMEWORK FOR DESCRIBING TECHNO-MATHEMATICAL FLUENCY IN BEYOND-SCHOOL PROBLEM SOLVING

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This study seeks to characterize the mathematical problem solving activity with digital tools that emerges from students' participation in an online mathematics competition. Using a qualitative approach, we elected to study the case of a participant, Jessica, aiming at understanding the ways in which she interweaves her mathematical competence and her technological fluency for solving two geometrical problems, using GeoGebra. Main results expose the role of the digital tool that permeates every stage of the problem solving process, since Jessica uses GeoGebra as a tool-to-think-with. We further propose a framework for describing the processes that may capture the interplay between mathematical knowledge and technological fluency for solving problems as amounting to techno-mathematical fluency.

Keywords: beyond school mathematics; digital literacy; mathematical competitions; problem solving; techno-mathematical fluency.

INTRODUCTION

The constant immersion in a technologically pervaded world is changing the "kind of mathematical abilities that are needed for success beyond schools" (Lesh, 2000, p. 177), especially since the new and powerful tools made available are introducing "new kinds of problem-solving situations in which mathematics is useful, (…) and they radically expand the kinds of mathematical understanding and abilities that contribute to success in these situations" (p. 178). Whilst the kinds of mathematical thinking needed outside the classroom are shifting, "the kinds of problem solving situations in which some form of mathematical thinking is needed" are also changing (English, Lesh & Fennewald, 2008, p. 5). Furthermore, little is still known about the problem solving that occurs beyond the classroom (English & Sriraman, 2010) and further research is needed specially to understand the role of digital technologies in such activity (Santos-Trigo & Barrera-Mora, 2007).

A glimpse on the context: the mathematical competition Sub14

Sub14® is a web-based mathematical problem solving competition organised by the University of Algarve. Addressing 12-13 years-old students, it is supported by a website where the problems are published, that provides tools for submitting answers, deadline reminders, lists of participants, a set of exemplary answers, and a synthesis of their accomplishment. The Qualifying consists of ten problems, each one published every two weeks. Participants may solve the problems using their favourite methods or tools, but they must send their solution and a detailed explanation of their reasoning and solving process through the website tools or their email. Every answer is assessed by the Organizing Committee who replies to each participant providing a constructive feedback. At this stage, the rules allow and encourage help seeking from friends, teachers, family members or the Organizing Committee. Participants who answer correctly to eight or more problems may attend the Final stage, which consists of a one-day tournament at the University of Algarve (see Carreira, 2012).

Our goal is to investigate mathematical problem solving with technological tools in this beyond-school competition, where participants may use their favourite digital tools but, at the same time, are required to use a mathematical stance. We report our progress on looking for a way of analysing how they merge their mathematical knowledge and their technological fluency for solving the competition's problems.

THEORETICAL BACKGROUND

In acknowledging the fundamental role that technological tools play in the development of mathematical thinking, our study is supported by a conception of inseparability between the subject and the digital tool. Thus, we consider *humans-with-media* (Borba & Villarreal, 2005) as a central unit in understanding problem solving activity with technology. This metaphor brings forth the idea that processes mediated by technologies lead to a reorganization of the human mind, and that knowledge itself is an outcome of this symbiosis between humans and the technology with which they act.

Mathematical problem solving – the mainstream view

The competition poses non-routine problems, whose context is fully and clearly expressed in the statement, and can be solved in different ways by combining several techniques, procedures or tools. As these problems are not aligned with the mathematics curriculum and a diversity of approaches and tools is encouraged, this kind of problem solving activity can be seen as the development of a productive way of thinking about a challenging situation (Lesh & Zawojewski, 2007) which involves a conception of mathematical knowledge that is not reducible to proficiency on facts, rules, techniques or computational skills. The perspectives that regard problem solving as an important source of mathematical knowledge are mirrored in current frameworks that consider mathematical literate person as someone who is *active problem solver*. Accordingly, being mathematically literate means to have the "capacity to formulate, employ, and interpret mathematics in a variety of contexts. It includes reasoning mathematically and using mathematical concepts, procedures, facts and tools to describe, explain and predict phenomena" (OECD, 2013, p. 5).

For problem solving to foster mathematical thinking the solver must adopt a mathematical stance that impels mathematisation, that is, to model, to symbolize, to abstract, to represent and to use mathematical language and tools (Schoenfeld, 1985, 1992). Looking for a way of explaining student's and expert's problem solving performance, Schoenfeld (1985) proposed a model comprised of five dimensions:

basic resources, heuristics, control, and belief systems. The processes followed by the solvers were organized into five stages: *read* - time spent “ingesting the problems conditions” (p. 297); *analysis* – attempt to fully understand the problem “sticking rather closely to the conditions or goals” (p. 298) that may include a selection of ways of approaching the solution; *exploration* – a “search for relevant information” (p. 298) that moves away from the context of the problem; *planning and implementation* – defining a sequence of actions and carrying them out orderly; *verification* – the solver reviews and assesses the solution.

While paper and pencil were the predominant tools used in Schoenfeld’s studies, today’s wide dissemination of powerful technological tools is raising new queries, namely if and to what extent these frameworks still account for the mathematical problem solving proficiency in the presence of digital tools (Barrera-Mora & Reyes-Rodríguez, 2013; Santos-Trigo, 2007, Santos-Trigo & Camacho-Machín, 2013).

Bringing together mathematical and technological literacies

Handling digital technologies in beyond school environments has become a focus of interest for many researchers over the last years (Barbeau & Taylor, 2009). Reporting a study that aimed at identifying the mathematical skills and competencies needed in several workplaces, Hoyles, Wolf, Molyneux-Hodgson and Kent (2002) highlighted an interrelationship of the information technology and the mathematical skills of the workers so they propose the term *Techno-mathematical Literacies* (TmL) as a notion that encapsulated both the technological and the mathematical skills needed within those workplaces. Later, this notion came to designate the functional mathematical knowledge mediated by technological tools, grounded in a specific work context (Hoyles, Noss, Kent, & Bakker, 2010).

Debates concerning the digital skills needed in our daily activities are undergoing. The European project DigEuLit developed a theoretical framework addressing the meaning and operationalization of “digital literacy” by describing the activity of a digital literate person when dealing with a digital task or problem (Martin & Grudziecki, 2006). At a first glance, those processes (Table 1) can be summarized as actions required before solving the problem (stating, identifying, accessing, evaluating, interpreting, organizing), the hands-on the problem (integrating, analysing, synthesising, creating, communicating), and actions that occur afterwards (disseminating and reflecting).

To some extent, this list of processes resembles the problem solving stages proposed by Schoenfeld (1985). Assuming that the digital task to be addressed by the solver is a mathematical problem proposed by the web-based competition Sub14 which requires a number of technological skills, we conjecture that these two frameworks can provide the necessary level of detail for describing problem-solving-with-technologies. We, therefore, ponder an association of the stages and processes: *read* – statement; *analyse* – identification, accession, evaluation, interpretation; *explore* –

organization, integration, analysis; *implement* – synthesis, creation, communication; and *verify* could be complemented by dissemination and reflection. Accordingly, the notion of *techno-mathematical fluency* stresses the need to be fluent in a language that entails mathematical and technological knowledge, promoting the skilful use of digital tools, the efficient interpretation and communication of the solution produced.

Process	**Problem or Digital Task**
Statement	State clearly the problem to be solved or task to be achieved and the actions required.
Identification	Identify the digital resources required to solve a problem or complete a task.
Accession	Locate and obtain the required digital resources.
Evaluation	Assess the objectivity, accuracy, reliability and relevance of digital resources.
Interpretation	Understand the meaning conveyed by a digital resource.
Organisation	Organise and set out digital resources in a way that will enable the solution of the problem or achievement of the task.
Integration	Bring digital resources together in combinations relevant to the problem or task.
Analysis	Examine digital resources using concepts and models which will enable solution of the problem or achievement of the task.
Synthesis	Recombine digital resources in new ways which will enable solution of the problem or achievement of the task.
Creation	Create new knowledge objects, units of information, media products or other digital outputs which will contribute to solution of the problem or achievement of the task.
Communication	Interact with relevant others whilst dealing with the problem or task.
Dissemination	Present the solutions or outputs to relevant others.
Reflection	Consider the success of the problem-solving or task-achievement process, and reflect upon one's own development as a digitally literate person.

Table 1 – Processes of digital literacy (Martin & Grudziecki, 2006)

RESEARCH METHODS

Our main goal is to develop a deeper comprehension of the interplay among mathematical knowledge and technological fluency during the development of the solving process within Sub14. Thus, this is an interpretative study where the research methods were steered by qualitative techniques for gathering, organizing and analysing empirical data (Quivy & Campenhoudt, 2008).

We report the case of Jessica (fictitious name), a participant whose productions stood out due to the sophisticated use of technology, namely GeoGebra, for solving the competition's problems (Jacinto & Carreira, 2013). Data collection included the solutions sent by Jessica to two early editions of Sub14, as well as the electronic messages sent. We conducted a semi-structured interview with Jessica, audio and video recorded, focusing aspects of her problem solving activity in her mathematics class and while participating in Sub14, asking her to remember and retrace some solutions submitted to the competition. Two solutions sent to Sub14, where she used GeoGebra, were also selected for a deeper analysis. Whilst the data from the interview provide a view of Jessica as a student, a problem solver and a technology user, the GeoGebra's construction protocols and the written explanations shed light upon the interactions between mathematical and technological knowledge.

The several types of data were organized using NVivo®, where audio and video data were transcribed. The analysis followed an interpretative perspective providing a holistic description of the case, by combining Jessica's perception of her own problem solving activity (interview), with the analysis of the participant's productions (GeoGebra file), enlightened by the theoretical ideas discussed above.

THE CASE OF JESSICA

Jessica is a 13 years-old girl who engaged in Sub14 during her 7th and 8th grades. Her answers to the problems are always on time, she describes her processes using a clear language, with proper justifications. She developed a particular interest on GeoGebra that stemmed out from her experience at school, since her teacher used it quite often as a way to illustrate geometrical contents. Despite being teacher-centred, this frequent use motivated Jessica to download, install and explore GeoGebra at home, independently.

Jessica: As I said, we use technology a lot. We have a board… a white board, and we also have an interactive board. We used GeoGebra very often when we were studying geometry and geometric transformations.

Researcher: When you say "we used", you mean the teacher?

Jessica: Precisely. And we watched it.

When asked to recall and retrace her solution to the problem "United and Cropped" (Figure 1), she claimed to enjoy solving geometry problems because of the possibility of improving the solutions' graphical display, afforded by GeoGebra.

Consider a sequence of squares of sides 1, 2, 3, 4,… cm, arranged so that they are connected to each other, as illustrated on the right. Once together, cut up all the squares by a line drawn from the bottom left corner of the smaller square to the upper right corner of the larger square. What is the area above the cut line, if the sequence has 8 squares?

Don't forget to explain your reasoning process!

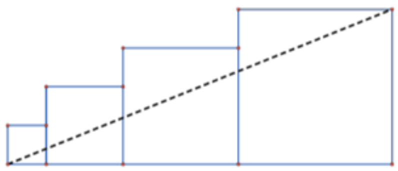


Figure 1: The problem "United and Cropped"

Jessica: I think I went straight to GeoGebra. I knew it had something to do with geometry. That was it! (…) I realise it was a triangle, that this was a triangle here, and that by rearranging it in a simpler manner all I had to do was calculate the whole area and then subtracting the area of this triangle, which is easy: base times height divided by two. And then I thought… «Oh, great! Geometry! I'm getting it neat!»

Jessica usually resorts to a notepad, coloured pens, a calculator and the computer. Initially, she thinks that GeoGebra only affords "dressing up" the solution that she finds by using paper-and-pencil but, later, she acknowledges that manipulating the constructions also led her to a powerful understanding of the problem.

Jessica: Hum… usually I look for the notepad and a pen, then the [text editor] and then I always… well I always use GeoGebra or some other software to add something to the text, for presenting a more complete work.

Researcher: So… you only use [GeoGebra] after you solve the problem?

Jessica: Yes, but… it depends. If GeoGebra or some other tools would help me understand the problem, then I'd use it firstly and afterwards I'd go to the [text editor].

Researcher: Ok, so you also use them while you're still looking for the solution…

Jessica: Yes, for instance, in this case [the problem United and Cropped] I started by going to GeoGebra to understand it properly, and then I discovered "Oh, that is a triangle right there, hence I have to subtract the area of that triangle". In that case, I started with GeoGebra for a better understanding.

Her solving activity starts outside of the computer screen but she easily recognizes that technological tools afford powerful approaches to the competition's problems. The following section reports Jessica's work with GeoGebra while solving another geometry problem.

The problem "A divided square"

The image shows a large square divided into 14 smaller squares, coloured in yellow, of different but integer dimensions, and 1 white rectangle, also of integer dimensions. The white rectangle has an area of 30 464 cm^2.
Which is the area of the larger square?

Don't forget to explain your reasoning process!

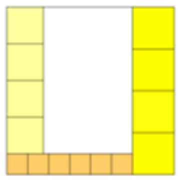
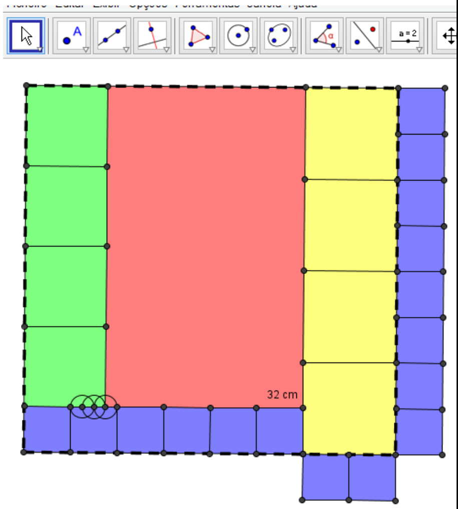


Figure 2: Statement of the problem "A divided square"

Jessica's solution (Figure 3) combines a construction, simulating the figure presented in the statement, and a written explanation where she presents a "label" that helps in interpreting the image and the processes of solving the problem, including the determination of the area. A closer analysis of the "construction protocol", which allows showing and redoing the construction step by step, reveals that GeoGebra's role goes beyond "embellishment". She starts by representing the larger square that supports the whole construction: draws two perpendicular lines and a circle centred at their intersection point and a radius of length defined by the segment CD.

Then, she constructs four squares on the right by finding midpoints, using parallel lines, perpendicular lines and their intersections. Finally, she builds four squares and colours them in yellow (Figure 4). As for the lower squares (Figure 5), Jessica marks the midpoint R, then uses a reflection of the point I over the vertical line that passes through F'_1 obtaining I', and designates S as the midpoint of the segment $I'F'_1$. She then uses circles with given centre and radius, finds intersections and midpoints, and traces parallel lines to complete the representation of the lower squares. Similarly, she constructs the remaining squares on the left side (Figure 6).



The boundaries of the larger square are dashed. I changed the colours of the smaller squares for an easy differentiation. Same colour squares have the same area.

yellow square side = 1/4 larger square side
blue square side = 1/8 larger square side
blue square side = 1/2 yellow square side
green square side = 1,75 blue square side

Let us name x to the length of the side of the blue square.
Length of the red rectangle = $4,25\,x$
Height of the red rectangle = $7\,x$
Area of the red rectangle = $4,25\,x \times 7x = 30\ 464cm^2$
$2975\,x^2$=30464
$x^2 = 30\ 464 \div 2975 = 1024$
$x = \sqrt{1024} = 32 =$ length of the blue square side

Length of the red rectangle $= 32 \times 7 = 224$
Larger square's side $= 32 + 224 = 256cm$
Larger squares' area $= 256cm \times 256cm = 65\ 536cm^2$
Answer: The area of the smaller square is 65536 square centimetres.

Figure 3: Solution of the problem "A divided square"

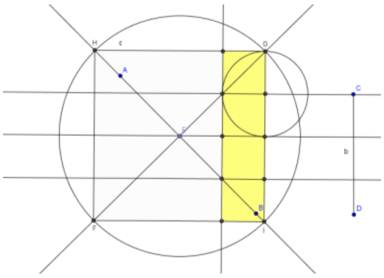


Figure 4: Initial construction

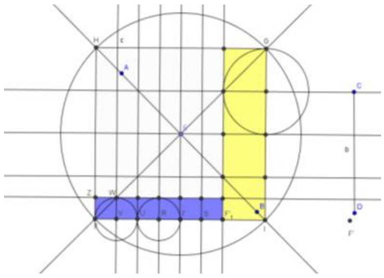


Figure 5: Constructing the lower squares

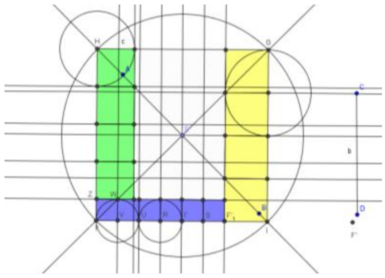


Figure 6: Constructing the left squares

Finally, she colours polygons, adds several squares along the exterior of the larger initial square and some circumferences whose centres divide the side of a smaller square in four parts (see Figure 3). These items emphasise a visual perception of the existing relations between several lengths of the geometrical figures. On the right side, a label helps interpreting the construction and establishing numerical relations between lengths of the sides of the squares. The unknown is defined as the length of the blue square and, by using those relations, she formulates an equation that will provide the measurement that is missing: $4,25x \times 7x = 30264$. With this value she determines the length of the side of the larger square and, then, its area.

This case illustrates how a digital tool, GeoGebra, is indispensable at several stages of the problem solving activity: while it stimulates a deeper understanding of the problem and fosters the devising of a strategy and its execution, it also supports the communication of the entire process. The constructions become part of the reasoning, of the process and the solution itself. This exemplifies the complexity of the symbioses that Borba and Villarreal (2005) describe and can be interpreted an instance of the problem solving activity of a human-with-GeoGebra.

REFRAMING TECHNO-MATHEMATICAL FLUENCY

As conjectured, the episodes encompass either the problem solving stages proposed by Schoenfeld (1985) or the several processes suggested by Martin and Grudziecki (2006) to describe digital literacy in accomplishing a task with technological tools.

Jessica starts by skimming the mathematical topic enclosed in the problem, realising that it refers to geometrical notions, rules and procedures, and recognizing GeoGebra as a key digital resource (*read*/*statement*). The following stage (*analyse*) is patent through the *identification* of a mathematical repertoire and a technological repertoire (geometry and GeoGebra) that are only possible because she knows how to reach them (*accession*). Moreover, Jessica's choice seems grounded on her belief about the accuracy and the reliability of GeoGebra's affordances, as well as her mathematical knowledge (*assessment* of techno-mathematical resources). Those options are also associated with Jessica's perception of the procedures that she feels able to perform and understand within the context (*interpreting* the techno-mathematical outcomes).

She then explores the possibilities for action organizing different resources – notepad, coloured pens, calculator, GeoGebra, text and image editor, e-mail and several mathematical resources, such as properties of parallel or perpendicular lines, circumferences and their representations, areas, algebraic expressions – and combining them in a relevant way to the development of her strategy (*organisation*, *integration*, and *analysis*). Based on the constructions and their manipulation, she *implements* her strategy recombining the techno-mathematical resources (*synthesis*) in order to produce new knowledge objects: strategies, representations, conceptual models (*creation*). During these processes, she may ask for the assistance of her teacher, her mother, the Sub14, to proceed in finding the solution (*communication*). It is important to note that the activity reported by Jessica and the analysis of the construction protocol suggest that the understanding of the problem and the decision on the actions necessary to solve it are not limited to the initial stage (read/statement) but it develops throughout the analysis and exploration stages and it is deepened during the construction and manipulation of the geometrical figures.

The last stage consists of reviewing the process and the solution (*verify*), but this particular activity also includes the presentation of the solution to relevant others, in this case, the GeoGebra constructions and a detailed explanation of the procedure: a small caption, the representation of the relations between the length of the sides of each square, certain computations and algebraic work (*dissemination*). As for the personal evaluation of the success accomplished during the problem solving activity (*reflexion*), there are no other concrete evidences to support it than the fact that Jessica has decided to present this solution to the judges of the competition.

CONCLUDING REMARKS

The two frameworks selected were meant to characterize the problem solving stages and the processes of digital literacy and, as such, their combination seems to offer powerful tools to approach a description of the latent processes underlying the notion of techno-mathematical fluency (TmF). However, there are some refinements that must be conveniently considered. Firstly, the processes of digital literacy are a set of actions that occur in a relatively ordered sequence, unlike the stages of problem solving that, as Schoenfeld (1985) showed in his work, are flexible enough to describe failed attempts or new appropriations of the problem. So, the descriptors of the TmF involved in solving the problems within this competition must comply with this flexibility. There may also be an overstatement of the digital literacy processes, particularly in what concerns the level of detail included in the original framework. Our comprehensive knowledge about the competition and the participants allow us to assume that: i) they often choose the tools they are most familiar with, namely everyday digital tools available in their home environment, hence accession, interpretation and evaluation could result in some benefit if they were agglutinated in a broader process, bringing together knowledge and decisions about the digital resources; ii) the communication process, which relates to possible help seeking, permeates other stages of the problem solving activity, namely in understanding the problem or in devising a path; iii) the verification of the solution is not clearly addressed in the digital literacy processes, but it is a very important metacognitive process for assuring the completeness of the solution; iv) the dissemination process, not considered in Schoenfeld's model, is extremely important given the competitive nature of this activity and the unavoidable fact of having to submit a solution to those who are responsible for their acceptance and from whom a return is expected; v) solving and expressing are inter-related activities that are often inseparable (Jacinto, Nobre, Carreira & Amado, 2014).

In light of the data and the theory, the notion of TmF that emerges from the 'problem solving with technologies' activity is a useful way of accounting for the intertwining of mathematical and technological knowledge. Future developments will concentrate on the refinement of the framework descriptors based on further analysis of other participants' problem solving activities within the same informal learning context.

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