BEYOND-SCHOOL MATHEMATICAL PROBLEM SOLVING: A CASE OF STUDENTS-WITH-MEDIA

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This paper addresses mathematical problem solving activity within the context of a web-based beyond-school competition – SUB14. Using a qualitative approach, we aim at finding evidences of the contestants' mathematical competence and technological fluency by analysing four solutions to a particular geometry problem from participants who decided to use GeoGebra. Even though they all make use of the same tool, their approaches to the problem differ in terms of the mathematical and technological fluency they show. We interpret their different ways of dealing with the tool and with mathematical knowledge as instances of students-with-media in problem solving.

INTRODUCTION

Several authors have been stressing the need for a deeper understanding of the mathematical activities in which young people engage, particularly in technologically rich environments that can be considered extensions of the school curriculum (Barbeau & Taylor, 2009). The University of Algarve has been promoting a web-based mathematical problem solving competition, addressed to 7th and 8th graders (12-13 years-old), named SUB14®. In this beyond-school and web-based competition there is a mathematical problem published every two weeks that the participants must solve individually or in small teams. Students have to send their solutions electronically, using attachments if they wish so, but those must include a complete and detailed explanation of their reasoning and solving process. Previous results indicate that the SUB14's participants often show sophisticated technological fluency when solving the competition's problems (Jacinto, Carreira, & Amado, 2011), although we know that putting such abilities into practice in the classroom is still rare for most of them.

This study extends the research on understanding mathematical problem solving in a beyond-school technologically rich environment, by characterizing how SUB14's participants reveal their technological fluency and their mathematical competence. Moreover, we aim at understanding how the use of a technological tool, like GeoGebra, supports and shapes four different approaches to a geometry problem.

THEORETICAL FRAMEWORK

Our conceptual framework is grounded on a sociocultural view of mathematics and draws on the idea that: (i) mathematical competence comprises the ability to use mathematical knowledge, namely, for solving problems; (ii) technology is a powerful mediational means of the mathematical activity, and (iii) technological fluency is expected to be a leverage to face many of the 21st century societal challenges.

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Mathematical knowledge and problem solving

It is widely accepted that a problem is an intellectually challenging situation for an individual who is willing to solve it, but does not possess an algorithm or a procedure that leads immediately and surely to the answer (Lester, 1983).

In the past years, the Portuguese mathematics curriculum has placed problem solving at the heart of classroom activities, and the current syllabus even puts a renewed and stronger emphasis on this "cross-content skill", and it acknowledges that improving the ability to solve problems is crucial for the development of other mathematical skills (ME, 2007). Seeing problem solving as the development of a productive way of thinking (Lesh & Zawojewski, 2007) entails a conception of mathematical knowledge that is not reducible to proficiency on facts, rules, techniques, computational skills, theorems, or structures. This conception moves towards broader constructs closer to the notion of mathematical competence (Perrenoud, 1999) and regards problem solving as a source of mathematical knowledge. Considering that mathematical problem solving fosters mathematical thinking (Lesh & Zawojewski, 2007; Schoenfeld, 1992), the solver must adopt a mathematical stance, which impels mathematization, that is, to model, to symbolize, to abstract, to represent and to use mathematical language and tools.

Mathematical knowledge under the light of technological fluency

The impact of digital tools in our society has been a focal point of interest for researchers over the past decades. Changing, reshaping, and affording are some of the keywords that have been recently highlighted to describe and explain such impact. Noss (2001) speaks of the representational transformation as a central feature of post-industrial societies and discusses how computational representations are reshaping the nature of mathematical knowledge. Kaput (1989) had already suggested that the production of mathematical meaning is anchored in the ability to use various representations and stems essentially from making conversions between different representations. Lately, this representational fluency is considered a core competency in the development of mathematical thinking (Lesh & Doerr, 2003), and is acknowledged as a fundamental tool in beyond-school environments, where it mediates decision-making, the interpretation of complex systems, or the use of technologies (Dark, 2003; Lesh, Zawojewski, & Carmona, 2003). While observing that mathematics plays an increasingly significant role in society, Noss (2001) states that some mathematical concepts and processes may be concealed by technological tools. Thus, many authors choose the term *affordance* to define the set of features of a particular technological tool that invite the subject to undertake an action upon it (Artigue, 2007; Noss, 2001).

Researchers have theorised on the representational side of technology-based mathematical activity by looking at the ways students recognise the affordances of the tools to generate mathematical meaning. The semiotic dimension of mathematical knowledge has become more intertwined with the awareness of the mediational role of technological mathematical representations, as semiotic systems are changed by the introduction of digital technologies. One emergent conclusion is that mathematics and technology cannot be seen as disjoint and the role of technology cannot even be reduced to conversions between representational systems (Artigue & Bardini, 2010).

In the same vein, Borba and Villarreal (2005) argue that the processes mediated by technologies lead to a reorganization of the human mind itself: knowledge is an outcome of a symbiosis of humans and technology – a new entity they named humans-with-media. This concept also discloses a sociocultural perspective of the human mind, in the sense proposed by Wertsch (1991) when assuming that every "action is mediated and (…) cannot be separated from the milieu in which it is carried out" (p. 18). The notion of humans-with-media is supported by two main ideas: (i) cognition has a social and collective nature that (ii) comprises tools which mediate the production of knowledge. The key issue is that media are considered a constitutive part of the subject and cannot be seen as auxiliary or supplementary. The media that are used to communicate, to produce or represent mathematical ideas, influence the kind of mathematics as well as of mathematical thinking that is developed. This means that different collectives of humans-with-media originate different thinking: for instance, the mathematics produced by humans-with-paper-and-pencil is qualitatively different from that produced by humans-with-computers (Borba & Villarreal, 2005).

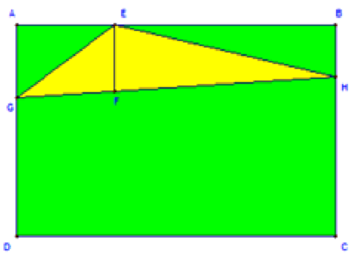
RESEARCH METHODS

The broader research project, into which this study is anchored, follows a naturalistic approach, involving qualitative techniques for data collection and analysis (Quivy & Campenhoudt, 2008). In this particular study, we are looking for evidences of technological fluency and mathematical problem solving fluency of particular, distinctive cases that illustrate variety and do not seek generalization.

Firstly, we gathered all the answers submitted by the 7th graders to a particular geometry problem (Figure 1), from the 2011 edition of SUB14. We then selected four productions of participants who have used GeoGebra at some point of their solving process which include their electronic messages and attachments.

Building a flowerbed

Rose explained to her gardener that she wanted a triangular area of flowers in her rectangular garden grass. The gardener took a 2-meter stick, held it perpendicularly to one side of the garden, at a random point (E). Then, with a rope, he drew a line through the end of the stick (F) and joining the two opposite sides of the rectangle, thus getting the yellow triangle EGH. On the next day, Rose looked at the triangle and did not like it, moved the same stick to another random point of the garden edge and she got a different triangle EGH.



When the gardener arrived he complained, saying that the area for the new flowerbed was smaller than before. But Rose assured him that it didn't change. Who is right and why?

Figure 1 – Problem #6 of the SUB14's 2011 edition

We conducted a descriptive and inductive analysis, considering the theoretical background, specifically aiming at illustrating the features related to the technological fluency and mathematical competence of the participants, namely in terms of the effective use of a digital tool – GeoGebra – to organize, expand, and sustain mathematical thinking, meaning and knowledge in their problem solving activity.

FOUR GEOGEBRA-BASED SOLUTIONS

In this section we analyse four solutions of a geometry problem, all using GeoGebra, to emphasize different mediational aspects that mathematical and technological representations enhance, specifically to: 1) obtain the solution, 2) interpret the solution, 3) confirm the solution, and 4) explore the solution.

Using GeoGebra to obtain the solution

Marta and Miguel submitted their solution along with a GeoGebra file (Figure 2). They represented the rectangular lawn as well as the three conditions of the statement: a stick with length 2 (segment FG) is perpendicular to the side AD of the rectangle, and the "rope" (segment JI) passes through the end of the stick, intersecting it at point G. Next, they determined the areas of two triangles, obtained by dividing the triangle FJI through the stick (segment FG). By dragging F they verified that the total area did not change and therefore they concluded that Rose was right.

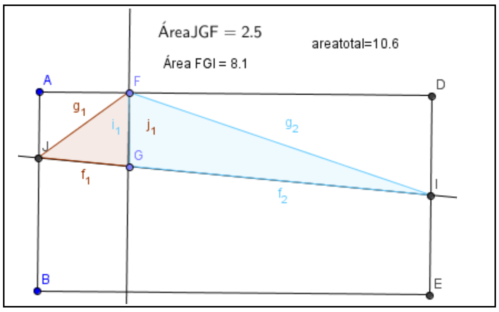


Figure 2 – Marta and Miguel's GeoGebra construction

This solution reveals Marta and Miguel's technological fluency, particularly when handling GeoGebra: they perform constructions that strictly meet the initial conditions and determine areas using the measuring tools. As to their mathematical fluency, and analysing the construction protocol, they seem to be familiar with geometrical concepts such as "perpendicular line" and "parallel line", "polygon" and "area of a polygon". Nevertheless, they fail to submit a mathematical reason for the invariance of the areas, which may result from the "certainty" they seem to get from dragging F.

Using GeoGebra to interpret the solution

Andreia, Lucas and José also sent a GeoGebra file and a brief text that seeks to validate the conclusion obtained by manipulating their construction (Figure 3). They built a rigorous and robust representation of the garden and the flowerbed, and added two measures using the tools: the length of the segment GH and the area of the triangle EGH. It seems that the manipulation of the points G and E, the observation of the invariance of the area, and the length of the bottom side of the triangle convince them that the areas do not change, whatever triangle they represent under those conditions.

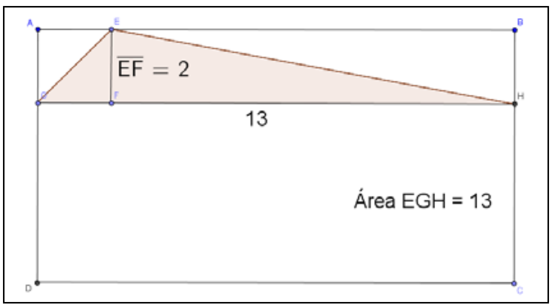


Figure 3 – GeoGebra construction, sent by Andreia, Lucas and José

In written, they try to explain the invariance of the area: "*triangles with the same base and the same height have equal areas*". This conclusion arises from the manipulation of vertices "*E and G under the conditions described in the problem*". However, moving E and G, we observe that the segment GH isn't always parallel to the side AB of the rectangle but GeoGebra indicates that the length of this segment is always "13". This may be a consequence of the default rounding – in this case, round to the unit – which, most likely, was overlooked by the participants.

The team revealed its technology fluency in their flexible use of GeoGebra, not only to represent the situation posed, but to obtain the solution and to attempt an interpretation. Although they are somewhat fluent in terms of mathematical knowledge, they seem to be convinced that the length of the segment GH is also invariant, which is not.

Using GeoGebra to confirm the solution

Sara acknowledged to have felt some difficulty in "*explaining with words*" how she thought about this problem; therefore, she decided to send a screen capture containing her construction in GeoGebra (Figure 4). According to her words, Sara "imagined" that the rectangle had a length of 12 cm and then built a representation of the rectangular garden and the triangular flowerbed, thoroughly following the statement. Therefore she determined the area of the triangle (on the left) and recognized that it matched the length of the rectangle that she initially chose. By making a second construction (on the right) she was already aiming at justifying the earlier result by dividing the flowerbed into two triangles, ONM and OMK. However, Sara explained that the 2m stick corresponded to the base of those smaller triangles, and she represented their heights using two segments, a_1 and b_1. Finally, she noted that "adding" two segments, i.e., the heights of the smaller triangles, it gives the length of the rectangular garden.

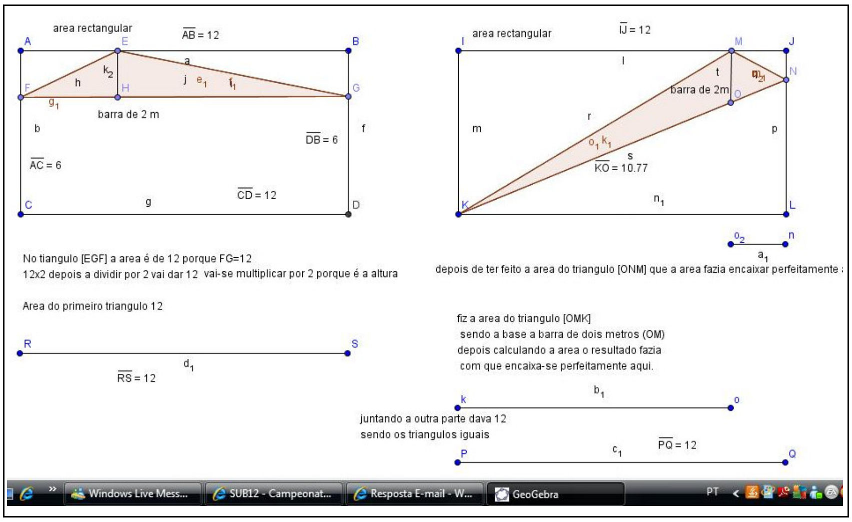


Figure 4 – Screen capture sent by Sara

Sara's technological fluency is quite evident in terms of the effective use of GeoGebra. It is also quite obvious in the diversity of tasks that she was engaged in while solving this problem, as revealed by her desktop's taskbar: she was also "chatting" online, checking the SUB12's webpage, and already drafting her answer. Considering mathematical fluency, we highlight the language she used: aside the email limitations regarding symbolic writing, Sara was clearly concerned with making herself clear and she correctly presented formulas and calculations.

The use of GeoGebra to explore the solution

Jessica used GeoGebra to simulate the construction of the rectangular lawn and the triangular flowerbed (Figure 5), but the text that she sent allows a clear understanding of her reasoning. She recognized that the area of the triangular flowerbed equals the value chosen for the length of the rectangle. However, this conclusion arose from the manipulation of the variable "height" of each of the coloured triangles:

> The yellow triangle is divided by the 2m stick in two triangles. The base of each triangle measures 2m – the length of the stick. To determine the area of a triangle, we have to calculate: height × base/2. In order to measure the area of those two triangles, we have: height × 2/2. But it is clear that 2/2=1, so the area of these triangles equals their height.

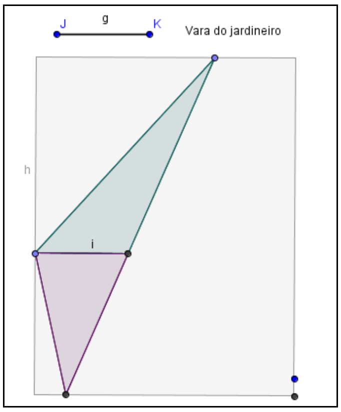


Figure 5 – Jessica's construction

Although Jessica's construction satisfies the three conditions, similarly to the previous solutions, it reveals distinct features in terms of manipulation. Such differences show that Jessica's thinking process is also distinctive: the absence of measurements or calculations stands out, the construction of a slider allows changing the stick's size; and moving the free point on the right side changes the size of the rectangle.

This file reveals Jessica's mathematical and technological fluency in that the GeoGebra construction is built under the perspective of geometrical properties and relations, rather than aiming at measuring or calculating. The quantitative relationship that she explains appears embedded in a geometric representation which is very powerful since it invites at manipulating and therefore generalizing. Adding a slider that controls the length of the stick involves analysing a variable that is not explicit in the statement of the problem; hence Jessica's exploration goes far beyond what was requested to solve the problem.

DISCUSSION AND CONCLUDING REMARKS

The data presented illustrate the diversity of ways of thinking and modes of action: four groups of solvers, who certainly have very different learning experiences, attend different schools and live in different places, realize and recognize the potential relevance of a single tool, GeoGebra, in solving this problem. These four solutions exemplify the kind of symbiosis described by Borba and Villarreal (2005) since the problem solving strategies and representations they use are revealing of subjects in action with a technological tool; so they can be identified as "students-with-media" or perhaps more accurately as "students-with-GeoGebra".

Still, it is possible to identify common aspects of their problem solving activity: they all represent the rectangular lawn and the triangular flowerbed, they all use "dragging" to check or verify, and they all analyse and conclude. But what each one takes out of that activity is not entirely the same and seems to be closely related to their ability to use, simultaneously, their mathematical competence and their technological fluency.

All participants demonstrated the ability to recognize the affordances of the tool, while their mathematical and technological activity ranged from an elementary and less powerful to an advanced and more sophisticated activity. The data suggest that the differences found are strongly related to the dynamic nature of the mathematical representations afforded by the tool, in depicting the problem conditions. For example, the introduction of additional free elements to the figure led to powerful understandings of the problem, and to generalization. In one production, the invariance of the area is not only numerically recognised but also geometrically explained; in another situation the free elements allow seeing the answer as a particular case of a more general statement; yet another case makes the problem even wider by extending the several conditions stated and allowing the exploration of a more general problem.

The "invisibility" of mathematical ideas is noticeable in the second production. The competitors naively accepted the result given by GeoGebra, and used it for attempting a mathematical justification, without a critical evaluation of such outcome. They lack critical sense in their analysis of the digital representations, which influenced their ability to transform information into knowledge (Noss, 2001).

The link between the solving strategy and the type of GeoGebra usage is clear. In particular, the understanding of the degree of generalisation of the problem and the consciousness of the affordances of the tool to achieve such generalisation are strongly interconnected. These are solid evidences of how the spontaneous use of technology changes and reshapes mathematical problem solving. The spectrum of the problem solutions also highlight the effectiveness of the use of digital tools to structure, support and extend mathematical thinking, meaning and knowledge in students' problem solving. Further research will focus on studying the mediational role of digital technologies in youngsters' problem solving activity, in light of what can be called techno-mathematical fluency (Hoyles, Noss, Kent, & Bakker, 2010).

Acknowledgments

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