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Memory effect in electromagnetic radiation

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Tiivistelmä – Referat – Abstract

A memory effect is a net change in matter distribution due to radiation. It is a classically observable effect that takes place in the asymptotic region of spacetime. The study of memory effects started in gravitational physics where the effect is manifested as a permanent displacement in a configuration of test particles due to gravitational waves. Recently, analogous effects have been studied in the context of gauge theories. This thesis is focused on the memory effect present in electrodynamics.

The study starts by a discussion on the fundamental aspects of electrodynamics as U(1) gauge invariant theory. Next, the tools of conformal compactification and Penrose diagram of Minkowski space are introduced. After these preliminaries, the electromagnetic analog of gravitational-wave memory, first analyzed by L. Bieri and D. Garfinkle, is studied in detail. Starting with Maxwell's equations, a partial differential equation is derived, in which the two-sphere divergence of the memory vector depends on the total charge flux F that reaches the null infinity and the initial and final values of the radial component of the electric field. The memory vector is then found to consist of two parts: the ordinary memory vector and the null memory vector. The solution of Bieri and Garfinkle for the null memory vector is reproduced by expanding the flux F in terms of spherical harmonics.

Finally, the connection between the electromagnetic memory effect and the so-called asymptotic symmetries of U(1) gauge theory is analyzed. The memory effect is found to determine a large gauge transformation (LGT) in which the gauge parameter becomes a function of angles at null infinity. Since a LGT is a local symmetry of U(1) theory, there must be a conserved Noether current and Noether charge associated with it. As the memory effect generates a LGT, it is natural to expect a connection between the memory effect and the Noether charge. The study thus culminates in an equation that relates the conserved charge to the memory effect.

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Contents

1 Introduction

1.1 The subject and its background

The subject of this master's thesis is the electromagnetic memory effect, which has been studied recently as an analog to the memory effect present in General Relativity, Einstein's theory of gravity. Memory effects have to do with permanent changes induced by radiation on physical configurations, like a collection of test particles with certain positions and velocities. They are classical observable effects in the low-energy region of gravity and gauge theories [1].

The study of memory effects started within gravitational physics, in connection with one of the intriguing consequences of Einstein's theory: gravitational waves. The existence of gravitational waves was first proposed by Einstein himself, but the time lapse between the first prediction and the first observation was a hundred years long. Einstein's original wave solution to the Einstein equation, the central equation of General Relativity, was found in the linearized theory in which the left hand side of the equation is expanded to the first order of the metric perturbation. Einstein was able to show that these metric perturbations are plane waves that travel at the speed of light. However, it was suspected that the gravitational wave solution was only a remnant of the linearization of the theory that would disappear in the fully nonlinear theory. Einstein even gave arguments (that later turned out to be fallacious) to the effect that nonlinear gravitational waves cannot exist [2].

There were many questions that needed answers before gravitational wave search could take off [2]:

1. How should a plane gravitational wave be defined in the nonlinear Einstein theory?
2. Is a plane gravitational wave a solution of the nonlinear theory?
3. Does a plane gravitational wave carry energy?
4. How should a gravitational wave with a nonplanar front be defined in the nonlinear theory?
5. How much energy do those waves carry?
6. Are these waves solutions of the nonlinear theory?
7. Is it possible to have bounded sources emitting gravitational waves in the nonlinear theory?

It took several decades to find solutions to these fundamental problems. Due to the pioneering work of H. Bondi, F. Pirani, I. Robinson and A. Trautman, among others, at the turn of 1950s and 1960s, the theoretical existence of gravitational waves was established, which opened the door for experimental work in gravitational wave research [2]. Finally, in autumn 2015, the LIGO team managed to make an observation of a passing gravitational wave emitted from the merger of two black holes of stellar mass [3]. As an acknowledgement of the importance of this discovery, the key figures who made the observation possible, R. Weiss, B. Barish, and K. Thorne, were awarded the 2017 Nobel Prize in physics [5].

The confirmation of the existence of gravitational waves naturally calls for further examination of the properties of gravitational waves. One of these is the so-called ”gravitational-wave memory,” in which physicists have been interested recently. A passing gravitational wave periodically stretches and shrinks the relative distance of test particles that lie in the plane perpendicular to the direction of wave propagation. It can be shown, however, that after the wave has passed, the particles have no relative velocity. Instead, the wave has made a permanent change in the spacetime geometry, due to which the relative positions of the particles have changed. In the literature, this phenomenon goes under the name ”memory effect” since the ”memory” of a passing gravitational wave is left permanently in the geometry of the spacetime [4].

The first calculation of the memory effect was carried out by Y. Zel'dovich and A. Polnarev in 1974 in the context of the linearized Einstein theory [6]. They argued that the effect is too small to be detectable. Contrary to this, in 1991 the mathematician D. Christodoulou pointed out that the memory effect is in fact larger than it was previously thought – large enough to make its detection possible in principle [7]. Christodoulou made his calculations based on the nonlinear theory, which is why the effect he predicted is sometimes called the nonlinear memory effect. A possible physical explanation for Christodoulou's calculation was given by K. Thorne [8] and A. Wiseman and C. Will [9]: the nonlinear effect comes from gravitons emitted by gravitational wave and the energy contribution of these gravitons should be included in the memory effect formula.

Besides the gravitational memory effect, there has been a growing interest among physicists in analogous memory effects in gauge theories. It has been claimed that a similar kind of memory effect can be found in electrodynamics [10], [12], [13] and Yang-Mills theory [14]. Furthermore, it has been argued that memory effects are closely associated with two other facets of gravity and gauge theories: soft theorems and asymptotic symmetries. Soft theorems state

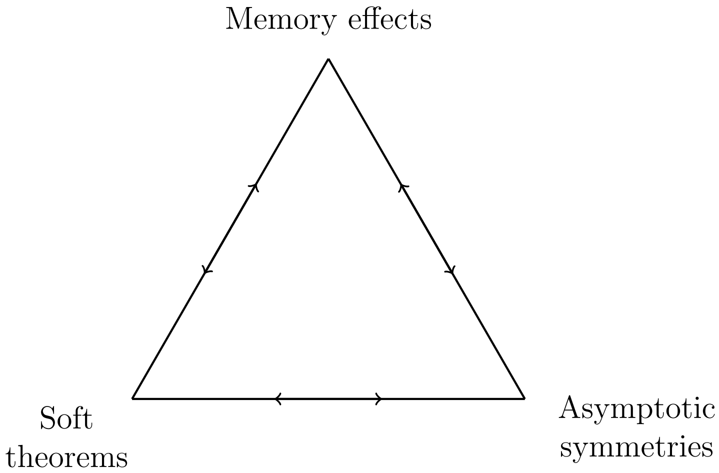

Figure 1.1. The infrared triangle. Three infrared phenomena (memory effects, asymptotic symmetries and soft theorems), turn out to be equivalent to each other. Memory effects and asymptotic symmetries are connected to each other by vacuum transitions. Memory effects and soft theorems are related by Fourier transforms. Soft theorems are Ward-Takashi identities of asymptotic symmetries.

that in a quantum field theory (QFT) scattering process, when the energy of a massless external particle tends to zero (i.e. it becomes soft), an infinite number of zero-energy particles are generated. Asymptotic symmetries, for one, are symmetries and conservation laws of a physical theory at arbitrarily faraway distances. Despite their name, these symmetries are really exact, but they hold in a region that is approached asymptotically. These three are prima facie disjoint phenomena that have been studied independently of each other for a long time. Only recently it was realized that these three subjects are in fact equivalent to each other (see Figure 1.1). This discovery has led to interesting new studies in the infrared structure of gravity and gauge theories [15–21]. The basic pedagogical text on this subject today is [1].

In this master's thesis, we will focus on the memory effect present in the electromagnetic theory. It is in and of itself an interesting fact that there is an electromagnetic analog to the gravitational-wave memory. It is always interesting to find analogous structures in different areas of physics. Moreover, studying the electromagnetic analog might shed some light on the more complex gravitational case and thus be useful for the gravitational wave research. There are thus good reasons to be interested in the electromagnetic analog of gravitational-wave memory.

The structure of this thesis is as follows: First, we will lay the foundation for the study of the memory effect by presenting the basics of classical electrodynamics. We examine the fundamental properties of electromagnetism as a U(1) invariant gauge theory. Next, as a necessary preliminary to the discussion on the electromagnetic memory effect, we will present a method called conformal compactification of spacetime and construct a Penrose diagram of Minkowski space. Finally, we will study the behavior of Maxwell's equations at the conformal boundary of Minkowski space. This will allow us to derive the electromagnetic analog of the gravitational-wave memory. In the course of this study, we assume that the fundamentals of GR, as they are presented in e.g. [22], are known to the reader.

1.2 Conventions

We will use units in which the speed of light is $c = 1$. In Minkowski metric we use the sign convention $(- + + +)$. We also set the vacuum permittivity to $\epsilon_0 = 1$, which implies that the magnetic permeability is $\mu_0 = 1$, too. We also employ the Einstein summation convention where the summation takes place over the repeated index:

$$\sum_{\mu=0}^{3} V_\mu W^\mu \equiv V_\mu W^\mu. \tag{1.1}$$

We use Greek letters to denote a sum over all spacetime components. When Latin letters are used, like in

$$V_i W^i, \tag{1.2}$$

the summation takes place over spatial components only. Since this is a study in electromagnetism, Maxwell's equations are the physical laws that we are most interested in. Maxwell's equations can be formulated in curved spacetime (see e.g. [23], [24]), but the flat space formulation is enough for the purposes of this study. In the inhomogeneous Maxwell's equation

$$\nabla_\mu F^{\mu\nu} = J^\nu \tag{1.3}$$

we use the sign convention in which the four-current $J^\nu = (\rho, J^i)$ has a positive sign when on the left hand side the contraction takes place over the first upper index.

2 Electromagnetism as U(1) gauge invariant theory

It is one of the fundamental facts of physics that symmetries of a theory are associated with conservation laws. This fact is encapsulated in the famous Noether's theorems, which state that for any continuous symmetry of a theory, either global or local, there exists a conservation law. A symmetry of a theory is its invariance under some transformation, which we call then a symmetry transformation of the theory. By a global symmetry we mean a symmetry that is generated by a constant transformation at every point of spacetime. In contrast, in a local symmetry the transformation parameter is a function of spacetime and does not have to assume the same value everywhere.

In this chapter we will study the foundations of electromagnetism as a gauge theory that is invariant under the U(1) group of transformations. We start by setting up the formalism for electromagnetism, derive the equations of motion, and finally show how charge conservation flows out of U(1) gauge invariance.

2.1 Maxwell's equations

Classical electrodynamics is governed by Maxwell's equations, which are a set of partial differential equations relating the derivatives of electric and magnetic field components to charge distribution and flow. In the usual vector representation, Maxwell's equations are

$$\nabla \cdot \vec{E} = \rho \tag{2.1}$$

$$\nabla \cdot \vec{B} = 0 \tag{2.2}$$

$$\nabla \times \vec{E} = -\partial_t \vec{B} \tag{2.3}$$

$$\nabla \times \vec{B} = \vec{J} + \partial_t \vec{E}, \tag{2.4}$$

where $\vec{E}$ is the electric field and $\vec{B}$ the magnetic field, ρ is the charge density and $\vec{J}$ the current density. The first (2.1) and the last one (2.4) of these equations are called inhomogeneous Maxwell's equations since they contain the source terms ρ and $\vec{J}$, respectively. The remaining equations lack any source terms so they are called homogeneous Maxwell's equations. These comprise a total of eight equations when all the vector components are taken into account.

As it is well known, classical electrodynamics is Lorentz covariant: Maxwell's equations retain their form under Lorentz transformations. Historically, Maxwell's electrodynamics was a crucial stepping-stone between Newtonian physics and the Special Theory of Relativity. Furthermore, Hermann Minkowski realized that Einstein's Special Relativity could be equivalently formulated using a geometric structure that unites time and space into a single entity, Minkowski spacetime. Thus, Minkowski spacetime is also the underlying structure of Maxwell theory. It is a space that has a flat geometry, yet it is non-Euclidean. In the Cartesian coordinates its metric is

$$g_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.5}$$

and the inverse metric $g^{\mu\nu}$ is given by the same matrix as the metric. Thus raising and lowering indices with the metric in the Cartesian coordinates is a trivial operation. However, in our study we will usually employ such coordinate systems that the metric takes a more complicated form and raising and lowering indices becomes a non-trivial matter, even though we operate in a flat spacetime.

This geometrized approach admits of use of the powerful formalism of tensor calculus in electrodynamics. Classical electrodynamics can be formulated using a single antisymmetric rank two tensor field inhabiting in Minkowski spacetime: the electromagnetic field strength $F_{\mu\nu}$. The tensor is defined as the exterior derivative of the gauge field A_μ, i.e.

$$F_{\mu\nu} = (dA)_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.6}$$

The tensor is invariant under gauge transformations of the form

$$A_\mu \to A_\mu + \partial_\mu \alpha, \tag{2.7}$$

since partial derivatives commute. The electric and magnetic fields are defined using the field strength tensor as

$$F_{0i} = E_i, \qquad F_{ij} = -\epsilon_{ijk} B^k. \tag{2.8}$$

Note that here ϵ_{ijk} is properly understood to be the Levi-Civita tensor, not just the Levi-Civita symbol $\tilde{\epsilon}_{ijk}$, i.e.

$$\tilde{\epsilon}_{ijk} = \begin{cases} 1, & \text{for even permutations of } ijk \\ -1, & \text{for odd permutations of } ijk \\ 0 & \text{otherwise} \end{cases} \tag{2.9}$$

$$\epsilon_{ijk} = \sqrt{|g|}\tilde{\epsilon}_{ijk}, \tag{2.10}$$

where g is the determinant of the metric $g_{\mu\nu}$. Taking ϵ_{ijk} to be a proper tensor makes B^k a well-defined vector. Using the Levi-Civita tensor properties one can invert the implicit definition of B^k to find an explicit formula for the magnetic field:

$$B^l = \frac{\mathrm{sgn}(g)}{2}\epsilon^{ijl}\epsilon_{ijk}B^k = -\frac{\mathrm{sgn}(g)}{2}\epsilon^{ijl}F_{ij}, \tag{2.11}$$

where $\mathrm{sgn}(g)$ is the sign of the metric determinant. In generic form we write the field strength tensor as the matrix

$$F_{\mu\nu} = \begin{bmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -\sqrt{|g|}B^3 & \sqrt{|g|}B^2 \\ -E_2 & \sqrt{|g|}B^3 & 0 & -\sqrt{|g|}B^1 \\ -E_3 & -\sqrt{|g|}B^2 & \sqrt{|g|}B^1 & 0 \end{bmatrix}. \tag{2.12}$$

The contravariant field strength tensor is then, using the inverse metric to raise the indices,

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -\sqrt{|g|}B^3 & \sqrt{|g|}B^2 \\ E_2 & \sqrt{|g|}B^3 & 0 & -\sqrt{|g|}B^1 \\ E_3 & -\sqrt{|g|}B^2 & \sqrt{|g|}B^1 & 0 \end{bmatrix} \tag{2.13}$$

Note, however, that the form of the contravariant field tensor is dependent on the choice of coordinate system. In the Cartesian coordinates we have just some minus signs changing, but in some more exotic coordinate systems the tensor will take a more complex outlook.

It is straightforward to show that with the field tensor Maxwell's equations can be expressed as

$$\nabla_\mu F^{\mu\nu} = J^\nu \tag{2.14}$$

$$\partial_{[\alpha}F_{\mu\nu]} = 0, \tag{2.15}$$

where $J^\nu = (\rho, J^i)$ is the current density four-vector, ∇ is the covariant derivative operator, and $[\alpha\mu\nu]$ denotes the antisymmetrized sum over permutations of the indices α, μ, and ν [25]. The inhomogeneous equations are given by the covariant divergence equation (2.14), and the homogeneous equations by (2.15). (2.15) is in fact a Bianchi identity following from the fact that $F_{\mu\nu}$ was defined as the exterior derivative of the gauge potential. The homogeneous equations are thus built into the structure of the field strength tensor. The inhomogeneous equations, however, do not come that easily. They derive from the action principle applied to the Maxwell action. We return to this in the next subsection. Since the field tensor is antisymmetric, the equations can alternatively be written in the form:

$$\frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}F^{\mu\nu}\right) = J^\nu \tag{2.16}$$

and

$$\partial_\alpha F_{\mu\nu} + \partial_\mu F_{\nu\alpha} + \partial_\nu F_{\alpha\mu} = 0. \tag{2.17}$$

These forms turn out to be useful in actual calculations.

2.2 Equations of motion from the principle of least action

Typically, field theory has a Lagrangian $\hat{L}(\phi_a(x), \nabla_\mu\phi_a)$ that is a function of a set of fields $\{\phi_a\}$ and their derivatives with respect to the spacetime coordinates. In the electromagnetic field theory, the Lagrangian is of the form

$$\hat{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + L_M, \tag{2.18}$$

where L_M is the matter contribution to the Lagrangian. For the sake of generality, we need more than just the Lagrangian, namely, the Lagrangian density L, which is obtained from the Lagrangian by multiplying it with the square-root of the absolute value of the metric determinant: $L = \sqrt{|g|}\hat{L}$. The Lagrangian density is needed since in the action integral

$$S = \int d^4x L \tag{2.19}$$

d^4x is a tensor density and $\hat{L}$ is a scalar. $\hat{L}$ also becomes a density when multiplied by $\sqrt{|g|}$, thus making the product of d^4x and L a well-defined tensor quantity.

If we vary the set of fields $\{\phi_a\}$ by

$$\phi_a \to \phi_a + \delta\phi_a, \tag{2.20}$$

the Lagrangian density changes by $L \to L + \delta L$ with, to first order,

$$\begin{aligned}
\delta L &= \frac{\partial L}{\partial \phi_a}\delta\phi_a + \frac{\partial L}{\partial(\nabla_\mu \phi_a)}\delta(\nabla_\mu \phi_a) \\
&= \sqrt{|g|}\frac{\partial \hat{L}}{\partial \phi_a}\delta\phi_a + \sqrt{|g|}\frac{\partial \hat{L}}{\partial(\nabla_\mu \phi_a)}\partial_\mu(\delta\phi_a) \\
&= \left(\sqrt{|g|}\frac{\partial \hat{L}}{\partial \phi_a} - \partial_\mu\left(\sqrt{|g|}\frac{\partial \hat{L}}{\partial(\nabla_\mu \phi_a)}\right)\right)\delta\phi_a + \partial_\mu\left(\sqrt{|g|}\frac{\partial \hat{L}}{\partial(\nabla_\mu \phi_a)}\cdot\delta\phi_a\right),
\end{aligned} \tag{2.21}$$

where on the last line we applied the Leibniz rule. The variation of the action is then

$$\begin{aligned}
\delta S = &\int_{t_i}^{t_f} d^4x\sqrt{|g|}\left[\frac{\partial \hat{L}}{\partial \phi_a} - \frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}\frac{\partial \hat{L}}{\partial(\nabla_\mu \phi_a)}\right)\right]\delta\phi_a \\
&+ \int_{t_i}^{t_f} d^4x\sqrt{|g|}\frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}\frac{\partial \hat{L}}{\partial(\nabla_\mu \phi_a)}\cdot\delta\phi_a\right).
\end{aligned} \tag{2.22}$$

If the indices a only enumerate different scalar fields, then we can straightforwardly use the fact that in the Christoffel connection

$$\nabla_\mu V^\mu = \frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}V^\mu\right) \tag{2.23}$$

allowing us to write the variation of the action as

$$\begin{aligned}
\delta S = &\int_{t_i}^{t_f} d^4x\sqrt{|g|}\left[\frac{\partial \hat{L}}{\partial \phi_a} - \nabla_\mu\left(\frac{\partial \hat{L}}{\partial(\nabla_\mu \phi_a)}\right)\right]\delta\phi_a \\
&+ \int_{t_i}^{t_f} d^4x\sqrt{|g|}\nabla_\mu\left(\frac{\partial \hat{L}}{\partial(\nabla_\mu \phi_a)}\cdot\delta\phi_a\right).
\end{aligned} \tag{2.24}$$

This should vanish for the extremal configurations $\phi_a(x)$. One notices that the second integral is now explicitly in the form that admits of the use of Stokes' theorem. For first-order variations $\delta\phi_a$ vanishing at the end points, but not within, the interval, one concludes by Stokes' theorem that the second integral gives zero and thus also the first one has to vanish. This holds for all variations only if the factor in the square brackets vanishes. Thus the extremal configurations are determined by the conditions

$$\frac{\partial \hat{L}}{\partial \phi_a} - \nabla_\mu\left(\frac{\partial \hat{L}}{\partial(\nabla_\mu \phi_a)}\right) = 0. \tag{2.25}$$

These are just the Euler-Lagrange equations of motion for fields ϕ_a.

On the other hand, if some of the fields in $\{\phi_a\}$ is a vector field, then matters are not so straightforward. However, in the case of Maxwell theory we are dealing with a Lagrangian that is constructed out of an antisymmetric second rank tensor $F_{\mu\nu}$, and the derivative the Lagrangian with respect to $\nabla_\mu A_\nu$ in (2.22) yields just the contravariant tensor $F^{\mu\nu}$. For any antisymmetric $S^{\mu\nu}$, we have

$$\begin{aligned}\nabla_\mu S^{\mu\nu} &= \partial_\mu S^{\mu\nu} + \Gamma^\mu_{\mu\lambda} S^{\lambda\nu} + \Gamma^\nu_{\mu\lambda} S^{\mu\lambda} \\ &= \partial_\mu S^{\mu\nu} + \frac{1}{\sqrt{|g|}}(\partial_\lambda \sqrt{|g|}) S^{\lambda\nu} \\ &= \frac{1}{\sqrt{|g|}} \partial_\mu (\sqrt{|g|} S^{\mu\nu}),\end{aligned} \tag{2.26}$$

where on the first line we used the fact that in the Christoffel connection $\Gamma^\mu_{\mu\lambda} = \frac{1}{\sqrt{|g|}} \partial_\lambda \sqrt{|g|}$ and the lower indices in the connection coefficients commute. Thus we can apply the same reasoning as above and are free to use the result (2.25) in the Maxwell theory.

2.3 Global symmetries

Consider then global, position independent, symmetry transformations. For these $\delta L = 0$ and the equation (2.21) then implies that for on-shell configurations, those satisfying the equation of motion (2.25), there is a conserved current

$$J^\mu = \frac{\partial L}{\partial(\nabla_\mu \phi_a)} \cdot \delta\phi_a, \quad \nabla_\mu J^\mu = 0. \tag{2.27}$$

This is Noether's first theorem.

Consider this in U(1) symmetric scalar theory with the Lagrangian

$$L = L(\phi, \phi^*) = -\nabla^\mu \phi^* \nabla_\mu \phi - V(\phi^* \phi), \tag{2.28}$$

which is invariant under

$$\phi \to U\phi = e^{-i\theta}\phi \approx \phi - i\theta\phi, \tag{2.29}$$

Then for the extremal configurations, with variations vanishing at the end points, the Euler-Lagrange equation (2.25) leads to the Klein-Gordon equation

$$(\nabla^\mu \nabla_\mu - V')\phi = 0, \tag{2.30}$$

where prime denotes the derivative. Since U(1) is a symmetry of the theory, δL has to vanish under U(1) transformations. The conserved Noether current in (2.27) then is explicitly

$$\begin{aligned} J^\mu &= \frac{\partial L}{\partial(\partial_\mu \phi)} \cdot \delta\phi + \frac{\partial L}{\partial(\partial_\mu \phi^*)} \cdot \delta\phi^* \\ &= \partial^\mu \phi^* (i\theta\phi) - \partial^\mu \phi (i\theta\phi^*) \\ &= i(\partial^\mu \phi^* \cdot \phi - \phi^* \partial^\mu \phi)\theta. \end{aligned} \tag{2.31}$$

Noether charges can be defined by integrating J^0 over a volume:

$$Q_V(t) = \int_V d^3x \, J^0(t, \vec{x}). \tag{2.32}$$

Conservation $\partial_0 J^0 + \nabla \cdot \vec{J} = 0$ implies that this is time dependent so that what is created within the volume, flows out through its surface. Conserved charges can be defined if we have a vanishing surface flux

$$\int_{\partial V} dS \, n_i J^i, \tag{2.33}$$

where dS is the surface element and n^i is the unit normal vector to the surface. Usually the surface ∂V is pushed to infinity. Then charge conservation means that the total amount of charge enclosed in an infinite spatial slice of Minkowski space is conserved from moment to moment.

2.4 Local symmetries

Consider then the case of local symmetries, for which the symmetry parameters depend on the space-time coordinate. A discussion closer to the spirit of the original work [26] would then start by assuming that the variation is of the form (see e.g. the appendix of [27], [28])

$$\delta\phi = f(\phi)\theta(x) + f^\mu(\phi)\partial_\mu \theta(x), \tag{2.34}$$

or even with more derivatives. Here $\theta(x)$ is the symmetry parameter and f, f^μ are a set of functions of ϕ. However, the outcome is obtained more simply by writing the action in a form in which the derivatives of the globally invariant action do not spoil the local invariance. If the symmetry is

$$\phi(x) \to U(x)\phi(x) \tag{2.35}$$

we will define a gauge covariant derivative $D_\mu = \nabla_\mu + \lambda A_\mu$, $\lambda = const.$ so that

$$D_\mu \phi(x) \to U(x) D_\mu \phi(x) \tag{2.36}$$

and we have an invariant operator

$$(D^\mu \phi)^\dagger D_\mu \phi, \tag{2.37}$$

where † denotes the Hermitean conjugate of an operator. This requires that

$$A_\mu \to U A_\mu U^{-1} + \frac{1}{\lambda} U \partial_\mu U^{-1}, \tag{2.38}$$

$$F_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu = \nabla_\mu A_\nu - \nabla_\nu A_\mu + \lambda \left[A_\mu, A_\nu\right]. \tag{2.39}$$

Conventional choices of λ are ig, $-ig, +1$. For U(1) the transformation is

$$U = e^{i\alpha(x)}, \quad A_\mu \to A_\mu - \frac{i}{\lambda} \partial_\mu \alpha. \tag{2.40}$$

Choosing $\lambda = ie$ and $\alpha(x) = -ie\theta(x)$ the Lagrangian

$$L = L(A_\mu, \phi, \phi^*) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - (D^\mu \phi)^* D_\mu \phi - V(\phi^* \phi) \tag{2.41}$$

is invariant under

$$\phi \to e^{-ie\theta} \phi \approx \phi - ie\theta\phi, \quad \phi^* \to \phi^* + ie\theta\phi^*, \quad A_\mu \to A_\mu + \partial_\mu \theta. \tag{2.42}$$

The ϕ equation of motion is (2.30) with $\nabla_\mu \to D_\mu$ and the A_μ equation of motion is

$$\frac{\partial L}{\partial A_\mu} - \nabla_\nu \left(\frac{\partial L}{\partial(\nabla_\nu A_\mu)} \right) = -J_e^\mu + \nabla_\nu F^{\nu\mu} = 0 \tag{2.43}$$

with the electric current

$$J_e^\mu = +ie[(D^\mu \phi)^* \phi - \phi^* D^\mu \phi]. \tag{2.44}$$

The Noether current is, summing over all field fluctuations in (2.27),

$$\begin{aligned} J_N^\mu &= \frac{\partial L}{\partial(\partial_\mu A_\nu)} \delta A_\nu + \frac{\partial L}{\partial(\partial_\mu \phi)} \cdot \delta\phi + \frac{\partial L}{\partial(\partial_\mu \phi^*)} \cdot \delta\phi^* \\ &= -F^{\mu\nu} \partial_\nu \theta + ie \left[(D^\mu \phi)^* \phi - \phi^* D^\mu \phi\right] \theta \\ &= F^{\nu\mu} \partial_\nu \theta + J_e^\mu \theta \\ &= F^{\nu\mu} \nabla_\nu \theta + \nabla_\nu F^{\nu\mu} \cdot \theta && (2.45) \\ &= \nabla_\nu (F^{\nu\mu} \theta). && (2.46) \end{aligned}$$

This remarkably simple result for the Noether current for local symmetry is basically Noether's 2nd theorem. Note, in particular, that in a flat spacetime we have, for any (2,0) tensor,

$$[\nabla_\mu, \nabla_\nu]\, S^{\rho\sigma} = R^\rho{}_{\lambda\mu\nu} S^{\lambda\sigma} + R^\sigma{}_{\lambda\mu\nu} S^{\rho\lambda} = 0 \tag{2.47}$$

$$\implies \nabla_\mu \nabla_\nu S^{\rho\sigma} = \nabla_\nu \nabla_\mu S^{\rho\sigma} \tag{2.48}$$

and thus due to the antisymmetry of $F^{\mu\nu}$ this current is identically conserved. One need not even require solutions of equations of motion. Note also the appearance of the gauge transformation parameter $\theta(x)$. If θ is constant, this is the electric current and one is back to the case of global symmetries. The conserved charge in this case is

$$Q_V(t) = \int_V d\Omega\, dr\, r^2 \partial_i (F^{i0} \theta(t, r, \Omega)) = \int_{\partial V} d\Omega\, r^2 n_i F^{i0} \theta. \tag{2.49}$$

The usual argument is that if $F^{i0} \sim r^2$ at large r this diverges if $\theta \sim r^\epsilon,\ \epsilon > 0$, vanishes if $\theta \sim r^{-\epsilon}$ and produces the same constant as in the global case if $\theta = const.$

We have now studied the fundamental aspects of electrodynamics as U(1) gauge theory. The invariance of the Maxwell Lagrangian under global U(1) transformations gives us, via Noether's first theorem, the conserved electric current. Local U(1) invariance yields a Noether current that is identically conserved in Minkowski space.

Our main goal, however, is to study the electromagnetic radiation memory effect, which is essentially an effect in the asymptotic domain of spacetime. In order to understand this effect, we first need to understand the asymptotic structure of spacetime. This is especially important in curved but asymptotically flat spaces, but here we only need to do this in a flat space. Therefore in the next chapter we will examine the conformal boundary of Minkowski space.

3 The conformal infinity

Minkowski space is a non-compact, boundaryless spacetime in which we can move both in time dimension and in space dimensions without limit. In order to understand the causal structure of spacetime and the behavior of physical theories arbitrarily far away from the origin, it is convenient to have a compact representation of the entire Minkowski space. In particular, we want to understand the behavior of Maxwell's theory in the region that electromagnetic radiation approaches asymptotically. In this chapter we present a tool first introduced in [29], that is constructed for these special purposes: the Penrose diagram. We will first provide the definition of a conformal transformation. Then we will construct the Penrose diagram of Minkowski space.

3.1 Conformal transformations

A conformal transformation is a local rescaling of metric of the form

$$ds^2 \rightarrow \widehat{ds}^2 = \Omega(x)^2 ds^2, \tag{3.1}$$

where $\Omega(x)$ is the so-called conformal factor. The conformal factor is a function of spacetime coordinates that gives us a new metric by scaling the old metric ds^2 in a suitable way. What is meant by "suitable" will be explained more precisely in due course. The basic idea is that the conformal factor "shrinks" the distances so fast that the infinitely far away comes to a finite distance from the origin. This enables one to have a compact representation of an infinite spacetime.

For a transformation of the metric to count as conformal, it must satisfy several conditions. We require from the conformal factor following things [30]:

1. It must be well-behaving in its domain of definition.
2. The conformal infinity lies at a finite distance from the origin:
$$\Delta s = \int_0^\infty \widehat{ds} = \int_0^\infty \Omega(r) dr < \infty. \tag{3.2}$$
3. The conformal boundary must be compact, that is to say, if $\widehat{M}$ is the new manifold, then
$$\int_{\partial \widehat{M}} \widehat{d\gamma} < \infty, \tag{3.3}$$

where $\widehat{d\gamma}$ is the induced metric on the boundary $\partial\widehat{M}$.

4. $\Omega(x^\mu) > 0$ for all original coordinate values.
5. Given that we can consistently expand the new manifold to include the conformal infinity, the conformal factor must vanish in that region: $\Omega(\infty) = 0$. What consistency here amounts to is that the manifold must be smooth at infinity and have a finite curvature at every point.

An important feature of conformal transformations is that they preserve the timelikeness, nullness and spacelikeness of vectors. It is easy to see this just by noting that

$$\begin{cases} \widehat{g}_{\mu\nu}V^\mu V^\nu < 0 \iff \Omega^2 g_{\mu\nu}V^\mu V^\nu < 0 \\ \widehat{g}_{\mu\nu}V^\mu V^\nu = 0 \iff \Omega^2 g_{\mu\nu}V^\mu V^\nu = 0 \\ \widehat{g}_{\mu\nu}V^\mu V^\nu > 0 \iff \Omega^2 g_{\mu\nu}V^\mu V^\nu > 0 \end{cases} \tag{3.4}$$

From this it follows that a curve that is timelike, null, or spacelike in the original spacetime remains timelike, null, or spacelike, respectively, under a conformal transformation. That is to say, the transformation does not change the causal structure of spacetime. Furthermore, null geodesics are left invariant under conformal mappings (for a proof of this statement, see Appendix A). This does not hold for geodesics in general. Therefore conformal mappings are especially well-suited for studying the behavior of electromagnetic and gravitational waves.

3.2 Penrose diagram of Minkowski space

Let us now construct a conformal compactification and a Penrose diagram of a spacetime that is important to our later discussion, namely Minkowski space. Minkowski space metric in spherical coordinates is

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2, \tag{3.5}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the unit two-sphere metric.

First, we introduce retarded time

$$u = t - r \tag{3.6}$$

and advanced time

$$v = t + r. \tag{3.7}$$

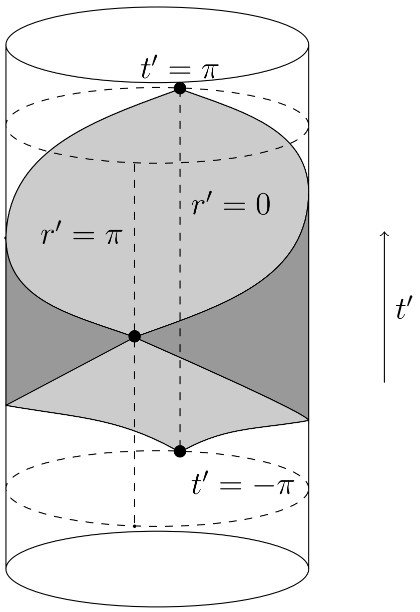

Figure 2.1. Conformally compactified Minkowski spacetime embedded in the Einstein static universe. The interior of the diamond wrapped around the cylinder represent the physical Minkowski space, whereas the boundaries of the diamond are the conformal infinity. The two end-points meet at the spatial infinity point on the other side of the cylinder.

Both of these coordinates have the range $-\infty < u, v < \infty$, $u \leq v$. Incoming lightrays travel along $v =$ const. lines and outgoing lightrays along $u =$ const. lines. We can now write the metric in the so-called null coordinates (u, v, θ, ϕ):

$$ds^2 = -dudv + \frac{1}{4}(v - u)^2 d\Omega^2. \tag{3.8}$$

To put it informally, what we want to do is to bring the infinitely far away region to a finite distance in such a way that preserves the angles of lightrays. This can be done by a choice of new coordinates:

$$u' = \arctan u \tag{3.9}$$

$$v' = \arctan v. \tag{3.10}$$

These new coordinates have a finite range: $-\frac{\pi}{2} < u', v' < \frac{\pi}{2}$, $u' \leq v'$. The metric now becomes

$$ds^2 = \frac{1}{\cos^2 u' \cos^2 v'} \left[-du'dv' + \frac{1}{4} \sin^2(v' - u') d\Omega^2 \right]. \tag{3.11}$$

Finally, we return to timelike and spacelike coordinates

$$t' = v' + u' \tag{3.12}$$

and

$$r' = v' - u', \tag{3.13}$$

which have ranges $0 \leq r' < \pi$, $|t'| + r' < \pi$. As a result the metric takes the form

$$\begin{aligned} ds^2 &= \frac{1}{4\cos^2\left[\frac{1}{2}(t'-r')\right]\cos^2\left[\frac{1}{2}(t'+r')\right]}\left[-dt'^2 + dr'^2 + \sin^2 r' d\Omega^2\right] && (3.14)\\ &= \left[\cos t' + \cos r'\right]^{-2}\left[-dt'^2 + dr'^2 + \sin^2 r' d\Omega^2\right] && (3.15)\\ &\equiv \Omega(t', r')^{-2}\left[-dt'^2 + dr'^2 + \sin^2 r' d\Omega^2\right]. && (3.16)\end{aligned}$$

The $dr'^2 + \sin^2 r' d\Omega^2$ part is the metric of a spacelike three-sphere. Since $t' \in (-\pi, \pi)$, the whole metric lives in a limited region of the Einstein static universe, which has the metric $-dt'^2 + dr'^2 + \sin^2 r' d\Omega^2$ with t' running from $-\infty$ to ∞ and which has correspondingly the topology $\mathbb{R} \times \mathbb{S}^3$. Thus, we have found a conformal factor such that

$$\begin{aligned} \widehat{ds}^2 &= -dt'^2 + dr'^2 + \sin^2 r' d\Omega^2 && (3.17)\\ &= \Omega(t', r')^2 ds^2. && (3.18)\end{aligned}$$

The conformal transformation we just constructed maps the Minkowski metric to a new metric that describes the geometry of the Einstein universe. Minkowski spacetime is flat, whereas the Einstein universe has a non-zero curvature, so clearly the transformed metric is not a physical one.

The boundaries of Minkowski space embedded in the Einstein universe are called ”conformal infinity”. The result of uniting Minkowski space with conformal infinity is referred to as the ”conformal compactification”. Diagrammatically, the conformally compactified Minkowski space can be represented as a diamond wrapped around a cylinder as in Figure 2.1. To get a two-dimensional representation of the Minkowski space where the entire space is confined within the conformal boundary and lightrays travel in a fixed 45 degrees angle, one has to construct a Penrose diagram, which is done in Figure 2.2.

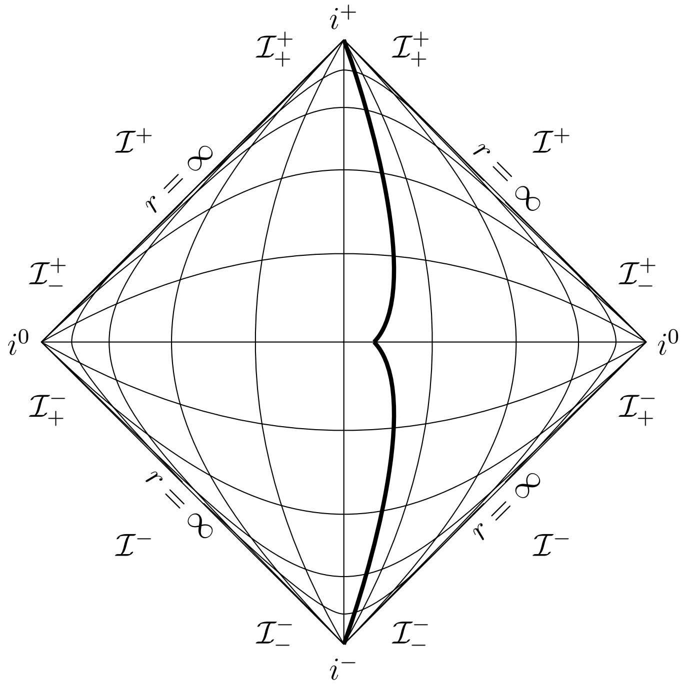

Figure 2.2. Penrose diagram of Minkowski space. The future null infinity is depicted as the two lines labeled by $\mathcal{I}^+$, the past null infinity by $\mathcal{I}^-$. The distant future of $\mathcal{I}^\pm$ is denoted by $\mathcal{I}^\pm_+$ and the distant past by $\mathcal{I}^\pm_-$. Lightrays come from the past null infinity and travel to the future null infinity, always maintaining a 45 degree angle. Massive bodies start from the past timelike infinity i^- and end up in the future timelike infinity i^+, as represented by the thick line in the diagram. The left and right end-points of the diamond represent one and the same point, the spatial infinity i^0. Every point in the diagram, except the origin, i^0, and $i^\pm$, represents a two-sphere. The points on the left and right halves are antipodally related to each other. On the curves connecting the left and the right corners we have $t = const.$, and on the curves connecting i^- and i^+ we have $r = const.$

4 Electromagnetic memory effect

In this chapter, we will derive the electromagnetic analog of the gravitational memory effect following the procedure of [10]. The memory effect is essentially a phenomenon at null infinity, so understanding the effect requires that we first understand the large distance asymptotics of electrodynamics. After this, we will proceed to analyze the memory effect. The analysis yields a formula for the electromagnetic memory effect in general, but we will focus particularly on the analog of the gravitational Christodoulou memory. Finally, we will study the relation between the memory effect and the so-called large gauge transformations of U(1) theory, and derive an equation that relates the conserved Noether charge of U(1) symmetry to the memory effect.

4.1 Maxwell's equations in spherical coordinates

We examine a situation in which we have a source that emits radiation in the radial direction. Thus it is most convenient to work in the spherical coordinate system. Recall from Chapter 1 that Maxwell's equations can be written in the form:

$$\frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}F^{\mu\nu}\right) = J^\nu \tag{4.1}$$

and

$$\partial_\alpha F_{\mu\nu} + \partial_\mu F_{\nu\alpha} + \partial_\nu F_{\alpha\mu} = 0. \tag{4.2}$$

Starting with these basic forms, we now want to write Maxwell's equations in spherical coordinates (t, r, θ, ϕ). First, we simplify our notation by denoting the angular components by a single capital Latin letter: (t, r, θ^A). Minkowski metric and its inverse in spherical coordinates read

$$g_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r^2 h_{AB} \end{bmatrix}, \; g^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r^{-2} h^{AB} \end{bmatrix}, \tag{4.3}$$

where h_{AB} and h^{AB} are the unit two-sphere metric and its inverse, respectively. In spherical coordinates the Levi-Civita tensor is $\epsilon_{rAB} = r^2\epsilon_{AB}$, where ϵ_{AB} is the unit two-sphere Levi-Civita tensor. In these coordinates the field strength thus reads

$$F_{\mu\nu} = \begin{bmatrix} 0 & E_r & E_A \\ -E_r & 0 & r^2 B^C \epsilon_{CA} \\ -E_A & -r^2 B^C \epsilon_{CA} & -r^2 \epsilon_{AB} B^r \end{bmatrix}. \tag{4.4}$$

Raising the indices of the field tensor with (4.3), we obtain the contravariant field tensor:

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_r & -r^{-2}h^{AB}E_A \\ E_r & 0 & B^C\epsilon_{CA}h^{AB} \\ r^{-2}h^{AB}E_A & -\epsilon_{CA}h^{AB} & -r^{-2}\epsilon_{AC}h^{AB}h^{CD} \end{bmatrix}. \tag{4.5}$$

Now we are in a position to derive the spherical coordinate representation of Maxwell's equations. We denote the covariant derivative with respect to the unit two-sphere with D_A. Plugging first $\nu = t$ to equation (4.1) yields:

$$\frac{1}{\sqrt{|g|}}\partial_r\left(\sqrt{|g|}F^{rt}\right) + \frac{1}{\sqrt{|g|}}\partial_A\left(\sqrt{|g|}F^{At}\right) = J^t$$
$$\implies \partial_r E_r + \frac{2}{r}E_r + \frac{1}{r^2}D_A E^A = \rho. \tag{4.6}$$

Choosing then $\alpha = r, \mu = A, \nu = B$ in equation (4.2) gives Gauss' law for magnetism in spherical coordinates:

$$\partial_r F_{AB} + \partial_A F_{Br} + \partial_B F_{rA} = 0$$
$$\implies \partial_r B_r + \frac{2}{r}B_r + \frac{1}{r^2}D_A B^A = 0. \tag{4.7}$$

After plugging $\alpha = t, \mu = A, \nu = B$ in equation (4.2), it is straightforward to show that

$$\partial_t F_{AB} + \partial_A F_{Bt} + \partial_B F_{tA} = 0$$
$$\implies \partial_t B_r + \frac{1}{r^2}\epsilon^{AB}D_A E_B = 0. \tag{4.8}$$

Then $\nu = r$ in (4.1), $\alpha\mu\nu = t\,r A$ in (4.2) and $\nu = A$ in (4.1) yield, respectively:

$$-\partial_t E_r + D_A\left(\epsilon^{AB}B_B\right) = -J^r \tag{4.9}$$

and

$$\partial_t B_A - h^{CB}\epsilon_{AC}\left(\partial_B E_r - \partial_r E_B\right) = 0 \tag{4.10}$$

and

$$-\partial_t E_A + h^{BC}\epsilon_{BA}\partial_r B_C - h^{CD}D_B\left(\epsilon_{CA}B_r\right) = J_A. \tag{4.11}$$

Since the geometry of a sphere is of paramount importance in the calculations to come, we now switch to a convention where the indices are raised and lowered with the unit 2-sphere metric. Using the fact that $g^{AB} = r^{-2}h^{AB}$ conveniently enables us to separate the r-dependence from angular components and derivatives. Recalling also that the Levi-Civita tensor satisfies $D_A \epsilon_{BC} = 0$ and $\epsilon_{AB}\epsilon^{AC} = \delta_B^C$, we finally obtain the following set of equations:

$$\begin{cases} \partial_r E_r + \dfrac{2}{r} E_r + \dfrac{1}{r^2} D_A E^A = \rho & (4.12) \\ \partial_r B_r + \dfrac{2}{r} B_r + \dfrac{1}{r^2} D_A B^A = 0 & (4.13) \\ \partial_t B_r + \dfrac{1}{r^2} \epsilon^{AB} D_A E_B = 0 & (4.14) \\ \partial_t E_r - \dfrac{1}{r^2} \epsilon^{AB} D_A B_B = -J_r & (4.15) \\ \partial_t B_A + \epsilon_A{}^B \left(D_B E_r - \partial_r E_B \right) = 0 & (4.16) \\ \partial_t E_A - \epsilon_A{}^B \left(D_B B_r - \partial_r B_B \right) = -J_A. & (4.17) \end{cases}$$

Equations are grouped so that the first two (4.12) and (4.13) correspond to J^t and the corresponding homogeneous equation, the one corresponding to a cyclic permutation of r, A, and B. These equations contain no time derivatives. The next two (4.14) and (4.15) correspond to J^r and the corresponding homogeneous Maxwell equation, the radial component of the Faraday law $\partial_t \vec{B} + \nabla \times \vec{E} = 0$. For the memory analysis these two amount to the same information as the first two. The final two (4.16) and (4.17) (actually four, since the capital A is standing for the two angular coordinates) correspond to J^A and the angular components of the Faraday law. In the memory analysis these last ones will basically give us radiative transverse and orthogonal electric and magnetic fields.

The equations contain partial derivatives with respect to t and r. In the memory analysis a crucial approximation is $\partial_t \approx \partial_u$ and $\partial_r \approx -\partial_u$ with $u = t - r$.

4.2 Large distance asymptotics of electrodynamics

Now that we have found the representation of Maxwell's equations in spherical coordinates, next we want to study what happens to the electromagnetic field and charged matter when $r \to \infty$. This will finally enable us to understand the behavior of the Maxwell theory at null infinity.

4.2.1 The behavior of electric and magnetic fields

As r becomes very large, it is useful to express the electromagnetic field in terms of asymptotic expansions of the components of electric and magnetic fields:

$$E_r = \sum_{n=0}^{\infty} \frac{E_r^{(n)}}{r^n} \qquad B_r = \sum_{n=0}^{\infty} \frac{B_r^{(n)}}{r^n} \tag{4.18}$$

$$E_A = \sum_{n=0}^{\infty} \frac{E_A^{(n)}}{r^n} \qquad B_A = \sum_{n=0}^{\infty} \frac{B_A^{(n)}}{r^n}, \tag{4.19}$$

where the $E_a^{(n)} = E_a^{(n)}(u, A)$ and $B_a^{(n)} = B_a^{(n)}(u, A)$ are the nth coefficients in the expansions. In order to operate consistently with the Maxwell theory at null infinity, we require that the electromagnetic field is smooth at $\mathcal{I}^{\pm}$. [1]

Plugging the expansions of the radial components in equations (4.12) and (4.13) and requiring that the equations hold at all times at any $r > 0$ entails that the radial components behave as $E_r \sim 1/r^2 \sim B_r$. Let us show this in detail for the electric field. We assume that the total charge enclosed in the entire space is finite. Thus when we integrate the equation (4.12) over $\mathbb{R}^3$, the both sides of

$$\int_{\mathbb{R}^3} \nabla_a E^a dV = \int_{\mathbb{R}^3} \rho \, dV \tag{4.20}$$

must be finite. By the divergence theorem, the left hand side can be written as

$$\lim_{r\to\infty} \int_{S^2(r)} n_a E^a r^2 d\Omega = \lim_{r\to\infty} \int_{S^2(r)} r^2 E_r \, d\Omega, \tag{4.21}$$

where $S^2(r)$ is a sphere with radius r centered at the origin and $d\Omega$ is the two-sphere surface element. Now plugging the asymptotic expansion in the integrand, we find that the first two factors in the expansion have to be zero for the integral to converge. The magnetic field case is handled by a similar argument. Thus, we have shown that

$$E_r^{(0)} = E_r^{(1)} = B_r^{(0)} = B_r^{(1)} = 0. \tag{4.22}$$

Next, we want to determine the r-dependency of angular components. Consider the Poynting vector $\vec{S}$, which tells us the directional energy flux of the electromagnetic field and which in our unit convention reads:

$$\vec{S} = \vec{E} \times \vec{B}. \tag{4.23}$$

[1]Note that the field tensor is neither smooth nor even continuous in the neighbourhood of spatial infinity. This can be seen by considering the Lienard-Wiechert potential of particles moving at constant velocity and taking the limit to the spatial infinity via $\mathcal{I}^-$ and $\mathcal{I}^+$. One then finds that the electromagnetic field takes different but antipodally related values depending on the direction from which one took the limit [1].

In an orthonormal coordinate system, this can be written in index notation as

$$S_a = \epsilon_{abc} E^b B^c. \tag{4.24}$$

In order to keep the energy flux of the radiation field through the sphere $S^2(r)$ finite but non-zero, i.e. to have

$$0 < \int S_r \, r^2 d\Omega < \infty \tag{4.25}$$

when $r \to \infty$, we require that $S_r \sim 1/r^2$. On the other hand, the radial component of the Poynting vector is

$$\begin{aligned} S_r &= \epsilon_{rAB} E^A B^B = r^2 \sqrt{h}\, \tilde{\epsilon}_{rAB} E^A B^B \\ &= r^2 \sqrt{h}\, \tilde{\epsilon}_{rAB}\, g^{AC} g^{BD} E_C B_D \\ &= \frac{1}{r^2} \sqrt{h}\, \tilde{\epsilon}_{rAB}\, h^{AC} h^{BD} E_C B_D, \end{aligned} \tag{4.26}$$

where, you recall, $g^{AB} = r^{-2} h^{AB}$ which explains the $1/r^2$ factor in the last expression. From this we conclude that, since $S_r \sim 1/r^2$, the angular components of the electric and magnetic fields behave as

$$E_A, B_A \sim 1. \tag{4.27}$$

Then the fact that

$$E = \sqrt{E_a E^a} = \sqrt{E_r^2 + g^{AB} E_A E_B} \tag{4.28}$$

implies that $E \sim 1/r$ since g^{AB} brings a factor of $1/r^2$ with it, which again comes from the geometry of the sphere. The same goes for the magnetic field, i.e. $B \sim 1/r$.

4.2.2 The behavior of charged matter

Having found the asymptotic behavior of the electromagnetic field, we now turn to the charge-current-terms of Maxwell's equations. Since the ultimate goal here is to find the electromagnetic analog of gravitational-wave memory, we will consider a situation in which the current reaches the future null infinity $\mathcal{I}^+$. At a later point we will see that in order for the memory effect analogous to Christodoulou's gravitational-wave memory to take place, this kind of charge-current behavior is needed. Of course, this kind of situation is unphysical, since there are no massless electric charges in Nature. Nevertheless, it is completely consistent to examine this kind of charged radiation on a theoretical level. Thus, it is meaningful to take this setup as a thought experiment that reveals analogous structures between different physical theories.

Since we are considering a situation in which there are electric charges at null infinity, we get non-trivial asymptotic expansions for charge and current densities. Furthermore, we assume that the amount of electric charge is finite and that all the charges at null infinity get there by being radiated along with the outgoing light rays, i.e. there are no sources at the spatial infinity i^0 that emit similar kind of charged radiation to the direction of null generators. This also means that there are no electric currents circulating along the celestial sphere in the neighbourhood of spatial infinity. In other words, we assume that at large distances the angular components J_A are much smaller than the radial ones.

Let us consider these assumptions a bit more in detail. First, we introduce again retarded and advanced times $u = t - r$ and $v = t + r$, recall (3.6) and (3.7). These relations can be inverted to get $t = (u + v)/2$, $r = (v - u)/2$. Expand then the current four-vector components in the neighborhood of $r = \infty$:

$$J_u = \sum_{n=0}^{\infty} \frac{J_u^{(n)}}{r^n}, \quad J_v = \sum_{n=0}^{\infty} \frac{J_v^{(n)}}{r^n}, \quad J_A = \sum_{n=0}^{\infty} \frac{J_A^{(n)}}{r^n}. \tag{4.29}$$

Our assumption that a finite amount of charge gets to null infinity along $u = const.$ lines can be formulated as follows:

$$\begin{cases} J_u = -\dfrac{J_u^{(2)}}{r^2} + \mathcal{O}(1/r^3) & (4.30) \\ J_v = \mathcal{O}(1/r^3) & (4.31) \\ J_A = \mathcal{O}(1/r^3). & (4.32) \end{cases}$$

Let us then apply the vector and dual vector transformation formulae

$$V'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} V^{\nu}, \quad V'_{\mu} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} V_{\nu}, \tag{4.33}$$

where $\partial x'^{\mu}/\partial x^{\nu}$ is the coordinate transformation matrix and $\partial x^{\nu}/\partial x'^{\mu}$ its inverse, to find a relation between vectors in tr basis and vectors in uv basis. First, the transformation matrix between tr and uv bases and its inverse matrix are

$$\frac{\partial x'^{\mu}}{\partial x^{\nu}} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad \frac{\partial x^{\mu}}{\partial x'^{\nu}} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \tag{4.34}$$

where we have omitted the angular coordinates, for which the transformation is trivial. A generic dual vector then takes the form

$$\begin{bmatrix} V_u & V_v \end{bmatrix} = \begin{bmatrix} V_t & V_r \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} V_t - V_r & V_t + V_r \end{bmatrix}. \tag{4.35}$$

In particular, we therefore have

$$J_u = -\frac{1}{2}(\rho + J_r) \tag{4.36}$$

$$J_v = \frac{1}{2}(J_r - \rho). \tag{4.37}$$

Applying the constraints (4.30)–(4.32) one then finds

$$J_r - \rho = \mathcal{O}(1/r^3) \tag{4.38}$$

$$\implies \rho \approx J_r = \frac{J_r^{(2)}}{r^2} + \mathcal{O}(1/r^3), \tag{4.39}$$

where the equality between ρ and J_r holds to leading order. Note that the electric current through a sphere of radius r is given by $I = \int d\Omega\, r^2 J_r$, and thus the condition above guarantees that the amount of charge is finite.

Let us sum up the r-dependencies we have found for these physical quantities:

$$\begin{cases} E_r = \dfrac{E_r^{(2)}}{r^2} + \mathcal{O}(1/r^3) \equiv \dfrac{\mathcal{E}_r}{r^2} + \mathcal{O}(1/r^3) & (4.40) \\ B_r = \dfrac{B_r^{(2)}}{r^2} + \mathcal{O}(1/r^3) \equiv \dfrac{\mathcal{B}_r}{r^2} + \mathcal{O}(1/r^3) & (4.41) \\ E_A = E_A^{(0)} + \mathcal{O}(1/r) \equiv \mathcal{E}_A + \mathcal{O}(1/r) & (4.42) \\ B_A = B_A^{(0)} + \mathcal{O}(1/r) \equiv \mathcal{B}_A + \mathcal{O}(1/r) & (4.43) \\ \rho = J_r = \dfrac{J_r^{(2)}}{r^2} + \mathcal{O}(1/r^3)) \equiv \dfrac{L}{r^2} + \mathcal{O}(1/r^3), & (4.44) \end{cases}$$

where we have renamed the leading term coefficients for the sake of notational convenience. Since the unit of J_r is [charge]/[distance]^2/[time], the unit of L must be [charge]/[time]. Thus, the choice of the letter L is not a coincidence, since physically it represents the amount of radiated charge per time, which is an analogue of luminosity for the charged radiation. Moreover, L is a function of angular coordinates so it should not be interpreted as the luminosity of the source without qualification, but as the "directional" luminosity. Thus, it is the luminosity per unit solid angle, and integrating it over all angles gives the absolute luminosity of the source.

4.3 The memory effect as a "kick"

Recall that in the Einstein theory, the memory effect is nothing but a permanent change in the relative displacement of test particles. A passing gravitational wave periodically changes the relative distances of test particles, but after the wave has gone, there is no relative velocity between the particles. In the electromagnetic case, however, the memory effect is manifested as a "kick", a residual velocity imparted on a charge by the electromagnetic field. Next we give a derivation of the kick formula. By Newton's second law and the Lorentz force formula, we have

$$m\frac{d^2\vec{x}}{dt^2} = q\vec{E} + q\vec{v} \times \vec{B}, \tag{4.45}$$

where m is the mass of the particle, q is its charge, and $\vec{v}$ is its velocity. In the general case, since the magnetic field contributes to the total force through a cross product, the calculation is very complicated. However, we assume that we are in the slow motion limit, which makes the magnetic field term small. Furthermore, we found the asymptotic behavior of the field components to be $E_r, B_r \sim 1/r^2$ and $E_A, B_A \sim 1$. Thus, we can approximate that the magnetic field contribution is negligible, i.e.

$$m\frac{d^2\vec{x}}{dt^2} \approx q\vec{E}, \tag{4.46}$$

and far from the source the electric field effectively points to the angular direction. Then we integrate over time to get:

$$\Delta\vec{v} = \vec{v}(\infty) - \vec{v}(-\infty) \approx \frac{q}{m}\int_{-\infty}^{\infty} \vec{E}\,dt. \tag{4.47}$$

Thus, to find the change of velocity of the particle, the only thing to do, essentially, is to find the behavior of the electric field over time and to calculate the integral. We want to do this at $\mathcal{I}^+$, so next we will see how this happens.

4.4 Maxwell at $\mathcal{I}^+$

The source emits radiation over some time-interval and the radiation propagates to null infinity. We change the time coordinate to retarded time u, and plug the asymptotic expansions (4.18) and (4.19) to Maxwell's equations

(4.12)–(4.17) and take the limit to $\mathcal{I}^+$. The equations now take the forms

$$\begin{cases} -\partial_u \mathcal{E}_r + D_A \mathcal{E}^A = L & (4.48) \\ -\partial_u \mathcal{B}_r + D_A \mathcal{B}^A = 0 & (4.49) \\ \partial_u \mathcal{B}_r + \epsilon^{AB} D_A \mathcal{E}_B = 0 & (4.50) \\ \partial_u \mathcal{E}_r - \epsilon^{AB} D_A \mathcal{B}_B = -L & (4.51) \\ \partial_u \mathcal{B}_A + {\epsilon_A}^B \partial_u \mathcal{E}_B = 0 & (4.52) \\ \partial_u \mathcal{E}_A - {\epsilon_A}^B \partial_u \mathcal{B}_B = 0. & (4.53) \end{cases}$$

Not all of these equations are independent of each other. This can be seen by first integrating the equation (4.52) over u, which yields the solution

$$\mathcal{B}_A = -{\epsilon_A}^B \mathcal{E}_B + C_A, \tag{4.54}$$

where C_A is constant with respect to u. Then after using this result in other equations, we notice that the equation (4.53) reduces to identity, the equation (4.51) just becomes the equation (4.48) and (4.49) becomes (4.50). Hence the only independent equations are

$$\begin{cases} -\partial_u \mathcal{E}_r + D_A \mathcal{E}^A = L & (4.55) \\ \partial_u \mathcal{B}_r + \epsilon^{AB} D_A \mathcal{E}_B = 0. & (4.56) \end{cases}$$

With our boundary conditions, the Maxwell theory thus got simplified to a system of two equations at $\mathcal{I}^+$. These are just Gauss' law and Faraday's at null infinity.

Now integrating (4.55) and (4.56) over all the values of u, one obtains the following pair of equations:

$$\begin{cases} D_A \int_{-\infty}^{\infty} \mathcal{E}^A du = \int_{-\infty}^{\infty} L du + \mathcal{E}_r(\infty) - \mathcal{E}_r(-\infty) & (4.57) \\ \epsilon^{AB} D_A \int_{-\infty}^{\infty} \mathcal{E}_B du = -\mathcal{B}_r(\infty) + \mathcal{B}_r(-\infty). & (4.58) \end{cases}$$

$\int_{-\infty}^{\infty} \mathcal{E}^A du$ is an important quantity here since it gives us the memory effect; it is thus appropriate to call it the "memory vector". The direction and magnitude of the memory field depends on the luminosity integral and the radial component of the electric field at $\mathcal{I}^+_+$ and $\mathcal{I}^+_-$. We denote

$$\begin{cases} M^A \equiv \int_{-\infty}^{\infty} \mathcal{E}^A du & (4.59) \\ F \equiv \int_{-\infty}^{\infty} L du. & (4.60) \end{cases}$$

Integration over u removes the retarded time dependence, so M^A and F only depend on the angles. Since L is the directional luminosity of radiation, F naturally represents the total charge the source has radiated over time per unit solid angle, which depends on the direction. With these shorthand notations, we rewrite the equations (4.57) and (4.58) as

$$\begin{cases} D_A M^A = F + \mathcal{E}_r(\infty) - \mathcal{E}_r(-\infty) & (4.61) \\ \epsilon^{AB} D_A M_B = -\mathcal{B}_r(\infty) + \mathcal{B}_r(-\infty). & (4.62) \end{cases}$$

Taking a closer look at the equation (4.61), we notice that the left-hand-side is the divergence of a vector defined on a two-sphere. Hence the left-hand-side integrated over a two-sphere gives a zero, the proof of which is given in Appendix C. Thus the equation (4.61) is consistent only if the right-hand-side integrated over a two-sphere also gives a zero. The physical reason behind this mathematical constraint is Gauss' law and the conservation of electric charge. On the one hand, $\int F d\Omega$ is the total amount of charge radiated away to null infinity. On the other hand, the integral of $\mathcal{E}_r$ over a two-sphere is, by Gauss' law, Q where Q is the total charge inside the sphere. Thus, the integral of $\mathcal{E}_r(\infty) - \mathcal{E}_r(-\infty)$ over S^2 gives the change of total charge inside the sphere. Therefore, we have that

$$\int_{S^2} d\Omega \left[\mathcal{E}_r(\infty) - \mathcal{E}_r(-\infty)\right] = -\int_{S^2} d\Omega F, \tag{4.63}$$

which accounts for the constraint we got.

4.5 Separating the ordinary and null memory equations

Consider first the magnetic field in the equation (4.62). The magnetic field of a point charge moving at constant velocity is given by

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{q\vec{v} \times \hat{r}}{r^2}, \tag{4.64}$$

where μ_0 is the vacuum permeability, $\vec{v}$ is the velocity of the charge and $\hat{r} = \vec{r}/|\vec{r}|$ is the unit radial vector. Thus, in our situation where the electric currents in the angular direction die off faster than $1/r^2$, we conclude that $\mathcal{B}_r$ must vanish. This is consistent with the approximation we made in (4.46). Hence the equation (4.62) implies that

$$\epsilon^{AB} D_A M_B = 0. \tag{4.65}$$

We can therefore only focus on the memory effect of the electric kind arising from (4.61).

For this, recall that every vector field on a two-sphere can be decomposed into a sum of a surface gradient term and a surface curl term:

$$M^A = D^A\phi + \epsilon^{AB} D_B\psi, \tag{4.66}$$

for some scalars ϕ and ψ. From this it follows that

$$\epsilon^{AB} D_A D_B\phi + \epsilon^{AB} D_A \epsilon_B{}^C D_C\psi = 0. \tag{4.67}$$

The first term on the left-hand-side vanishes since two covariant derivatives acting on a scalar commute. The Levi-Civita tensor commutes with the covariant derivative, so we get

$$\epsilon^{AB}\epsilon_{BC} D_A D^C\psi = 0 \tag{4.68}$$

Using then Levi-Civita tensor identities, one obtains

$$\delta^A_C D_A D^C\psi = 0 \implies D_C D^C\psi = 0. \tag{4.69}$$

We see that this is nothing but a two-dimensional Laplace equation on S^2. The operator acting on ψ is a linear differential operator; in fact, it is nothing but $-\vec{L}^2$, where $\vec{L}$ is the impulse moment operator.

Since we are operating on S^2, it is natural to solve the equation using spherical harmonic analysis. Recall that spherical harmonics are defined by

$$Y_{lm}(\theta,\phi) = (-1)^m \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\phi}, \tag{4.70}$$

where $P_l^m(x)$ are the associated Legendre functions. Spherical harmonics form an orthonormal basis of the space of square-integrable functions on the two-sphere so they satisfy the following orthonormality property:

$$\int_{S^2} d\Omega\, Y^*_{lm}\, Y_{l'm'} = \delta_{ll'}\delta_{mm'}. \tag{4.71}$$

Thus, we can expand ψ in terms of spherical harmonics as

$$\psi = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \psi_{lm} Y_{lm}, \tag{4.72}$$

where ψ_{lm} are the angle-independent expansion coefficients. From now on we will write explicitly only the first summation symbol. Using the orthonormality property, we can use the complex conjugates of spherical harmonics to find the expansion coefficients:

$$\psi_{lm} = \int_{S^2} d\Omega\, \psi\, Y^*_{lm}. \tag{4.73}$$

We know that the spherical harmonics are eigenfunctions of the operator $D_A D^A$ so that

$$D_A D^A \psi = \sum_l \psi_{lm} D_A D^A Y_{lm} = -\sum_l \psi_{lm} l(l+1) Y_{lm}. \tag{4.74}$$

Thus, using (4.69), we find

$$\sum_{l=0}^{\infty} \psi_{lm} l(l+1) Y_{lm} = 0. \tag{4.75}$$

The first term is zero because of the factor l. One also finds that, by the orthonormality property, for all $l > 0$ the expansion coefficient is $\psi_{lm} = 0$. This leaves only the first term, so we have that

$$\psi = \psi_{00} Y_{00} = const. \tag{4.76}$$

$$\implies D_A \psi = 0. \tag{4.77}$$

Thus, the solenoidal part of the arbitrary vector M_A vanishes, and we have a scalar field ϕ that satisfies

$$M_A = D_A \phi, \tag{4.78}$$

at any point of the sphere.

We can now go back to plug (4.78) to equation (4.61). We then obtain

$$D_A D^A \phi = F + \mathcal{E}_r(\infty) - \mathcal{E}_r(-\infty), \tag{4.79}$$

which is the two-dimensional Poisson equation on a sphere. By a similar argument as in (4.74) we see that the zero mode of the spherical harmonic expansion of the left hand side vanishes. Hence, also the zero mode of the right hand side has to vanish, i.e.

$$F_{avg} + [\mathcal{E}_r(\infty) - \mathcal{E}_r(-\infty)]_{avg} = 0, \tag{4.80}$$

and we can write

$$D_A D^A \phi = F + F_{avg} + \mathcal{E}_r(\infty) - \mathcal{E}_r(-\infty) + \left[\mathcal{E}_r(\infty) + \mathcal{E}_r(-\infty)\right]_{avg}. \quad (4.81)$$

The reason to write the equation in this way is that we now proceed to separate it into two equations and want to deduct the zero modes from F and $\mathcal{E}_r(\infty) - \mathcal{E}_r(-\infty)$ in these new equations. If ψ and ξ are any scalar functions defined on S^2, $D_A D^A \phi = \psi + \xi$ and G is a Green's function for $D_A D^A$, then

$$\phi = \int G(\psi + \xi) = \underbrace{\int G\psi}_{\equiv \phi_1} + \underbrace{\int G\xi}_{\equiv \phi_2} \quad (4.82)$$

Moreover, by the definition of Green's function we have that

$$\begin{cases} D_A D^A \phi_1 = D_A D^A \int G\psi = \psi \\ D_A D^A \phi_2 = D_A D^A \int G\xi = \xi. \end{cases} \quad (4.83)$$

Thus, in particular, it follows from equations (4.81) and (4.80) that

$$\begin{cases} D_A D^A \phi_1 = F - F_{avg} \\ D_A D^A \phi_2 = \mathcal{E}_r(\infty) - \mathcal{E}_r(-\infty) - \left[\mathcal{E}_r(\infty) - \mathcal{E}_r(-\infty)\right]_{avg} \end{cases} \quad (4.84)$$

for some scalars ϕ_1, ϕ_2 such that $\phi = \phi_1 + \phi_2$. Correspondingly, denote the two parts of the memory vector as $M_{1,A} \equiv D_A \phi_1$ and $M_{2,A} \equiv D_A \phi_2$.

Let us summarize our technical discussion. We managed to split the original differential equation in two parts. It was important to do this since F and $\mathcal{E}_r(\infty) - \mathcal{E}_r(-\infty)$ give two different memory effects. Of these two, we are especially interested in the first one, which is responsible for the nonlinear memory effect, analogous to the Christodoulou memory in gravity. Following the terminology in the literature, we call the first part (the upper one in (4.84)) the "null" kick and the second part (the lower one in (4.84)) the "ordinary" kick. It has been emphasized that in the context of memory effects the choice of term "null" refers to the fact that nonlinearity is not needed in order to get this part of the memory effect; what really is needed is that a flux of energy reaches null infinity. In Christodoulou memory, the effect takes place due to the flux of gravitational radiation propagating to null infinity. Here F represents an analogous flux of charged radiation that reaches $\mathcal{I}^+$ [10, 18].

4.6 The ordinary kick

The ordinary kick comes from fields radiation fields by a collection of massive charges. As a simple example, the radiation field generated by a massive moving charge along the path $\vec{r}_q(t)$ (see Figure 2.2) are [31]

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi}\left[\frac{\vec{R}-\vec{v}R}{\gamma^2(R-\vec{v}\cdot\vec{R})^3}\right]_{ret} + \frac{q}{4\pi}\left[\frac{\vec{R}/R\times((\vec{R}/R-\vec{v})\times\dot{\vec{v}})}{(1-\vec{v}\cdot\vec{R}/R)^3 R}\right]_{ret} \tag{4.85}$$

$$\vec{B} = \left[\frac{\vec{R}}{R}\times\vec{E}\right]_{ret}. \tag{4.86}$$

Here $\vec{R}(t) = \vec{r}-\vec{r}_q(t)$, q is the charge of the particle, $\vec{v}$ is its velocity, and

$$\gamma = \frac{1}{\sqrt{1-v^2}} \tag{4.87}$$

is the Lorentz factor of the particle. The subscript ret means that the vector inside the brackets should be calculated using the retarded time

$$t_r = t - |\vec{r}-\vec{r}_q(t_r)|. \tag{4.88}$$

The first term in the solution for the electric field represents the part of the field that is independent of the acceleration of the particle, whereas the second term describes the radiation field due to acceleration. Since the charges relax to constant velocity at past and future timelike infinity, the radiation field due to acceleration does not contribute to the ordinary kick so we omit it from now on. The origin of the coordinate system can be chosen in such a way that $\vec{r}_q(t) = \vec{v}t$. The radial component of the electric field is obtained from (4.85) by multiplying with the unit radial vector $\hat{r}$, which yields

$$E_r = \frac{q}{4\pi}\frac{r-\hat{r}\cdot\vec{v}t_r-\hat{r}\cdot\vec{v}|\vec{r}-t_r\vec{v}|}{\gamma^2(|\vec{r}-\vec{v}t_r|-\vec{v}\cdot\vec{r}+v^2t_r)^3}. \tag{4.89}$$

It is straightforward to show that in the limit $r\to\infty$, $u=t-r=const.$ the retarded time t_r goes to zero, and the solution simplifies to

$$r^2E_r = \mathcal{E}_r = \frac{q}{4\pi}\frac{1}{\gamma^2(1-\hat{r}\cdot\vec{v})^2}. \tag{4.90}$$

This expression does not depend on the value of u, so the radial electric field in the direction $\hat{r}$ only depends on the initial and final velocities $\vec{v}\,(u=\pm\infty)$. The ordinary kick can be determined from this formula when these velocities are known. For a charge with $\vec{v}(u=-\infty)=0$ and with a final velocity $\vec{v}(u=\infty)$ in the z-direction, the kick is [11]

$$M_{2,\theta}(\phi,\theta) = \frac{q}{4\pi}\frac{v\sin\theta}{1-v\cos\theta}, \qquad M_{2,\phi}=0. \tag{4.91}$$

The functional form of $M_{2,\theta}$ with different values of parameter v can be seen in Figure 4.1. The azimuthal component is everywhere zero. The vector field is symmetric with respect to rotations around the z-axis.

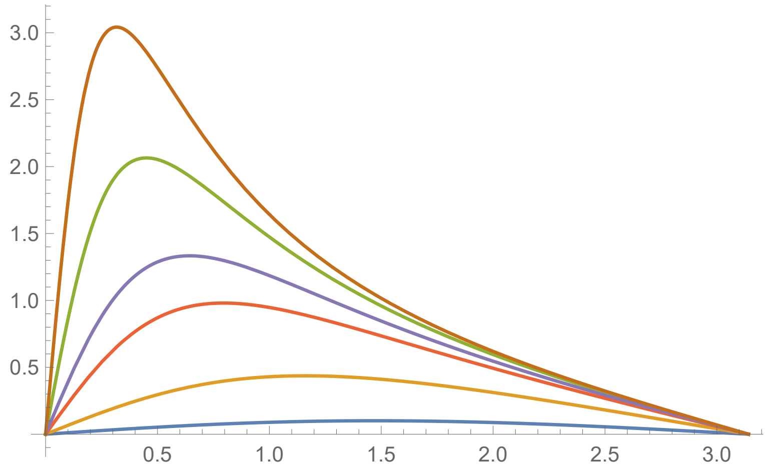

Figure 4.1. The ordinary memory vector $M_{2,\theta}$ as a function of θ, where $\theta \in [0, \pi]$. The different plots correspond to the following parameter values (starting from the lowest one): $v = 0.1, v = 0.4, v = 0.7, v = 0.8, v = 0.9, v = 0.95$. The prefactor $q/(4\pi)$ is set to unity.

4.7 Solving for the null memory

Focusing now on the null kick, we notice that F is a function of angles only, so we can expand it in spherical harmonics. Analysing it in terms of spherical harmonics will enable us to find a series expression for the memory vector. We will carry out this procedure explicitly for the null kick. The ordinary kick part, which also is a function of angles, can be handled in a similar manner. We thus write

$$F = \sum_l f_{lm} Y_{lm}. \tag{4.92}$$

The average value of F over the two-sphere can be calculated as

$$\begin{aligned} F_{avg} &\equiv \frac{1}{4\pi} \int_{S^2} d\Omega \, F \\ &= \frac{1}{\sqrt{4\pi}} \int_{S^2} d\Omega \, Y_{00}^* \left[f_{00} Y_{00} + \sum_{l>0} f_{lm} Y_{lm} \right] \\ &= \frac{f_{00}}{\sqrt{4\pi}}. \end{aligned} \tag{4.93}$$

Then we express ϕ_1 in terms of spherical harmonics:

$$\phi_1 = \sum_{l=0}^{\infty} \phi_{lm} Y_{lm}. \tag{4.94}$$

We can use again the fact that spherical harmonics are eigenfunctions of the operator $D_A D^A$, so that

$$D_A D^A \phi_1 = -\sum_{l>0} \phi_{lm} l(l+1) Y_{lm}. \tag{4.95}$$

Hence it follows from the orthonormality property that

$$\phi_{lm} = -\frac{f_{lm}}{l(l+1)}, \tag{4.96}$$

when $l > 0$. The memory vector for the null kick is now given by

$$\boxed{M_{1,A}(\theta,\phi) = -\sum_{l>0} \frac{f_{lm}}{l(l+1)} D_A Y_{lm}(\theta,\phi).} \tag{4.97}$$

Given that we know the behavior of the radiation source, we can apply this formula to calculate the memory vector. Another way to find a formula for the memory vector is to use the Green's function method as in [7] and [10, Appendix]. Using this method, the null memory vector is given by

$$\vec{M}_1 \cdot \hat{T} = \int d\Omega' \left(F_{avg} - F(\hat{r}')\right) \frac{\hat{T} \cdot \hat{r}'}{1 - \hat{r} \cdot \hat{r}'}, \tag{4.98}$$

where $\hat{r}$ and $\hat{r}'$ are unit position vectors on S^2, $\hat{T}$ is a unit vector tangent to S^2, and the integration takes place over the primed variables. This gives us the null kick projected to the direction of $\hat{T}$ at point $\hat{r}$ on the sphere.

4.8 Conserved charges associated with the memory effect

As we already mentioned in the introduction, memory effects of a theory are associated with its asymptotic symmetries, which are the symmetries of a theory at the asymptotic boundary of spacetime. In gravity, the asymptotic symmetries of spacetime are the elements of the Bondi-Metzner-Sachs (BMS) group, which was discovered in the 1960s by H. Bondi, M. van der Burg, A. Metzner, and R. Sachs. Bondi and others were studying asymptotically flat spacetimes in GR and expected to find the Poincaré group of special relativity as the isometry group of a spacetime whose curvature goes to zero in the asymptotic region. However, what they found out was that the symmetry group of an asymptotically flat spacetime is much larger, in fact infinite-dimensional, and includes along with the Poincaré group the so-called supertranslations, which are generalizations of the four spacetime translations of special relativity. These transformations generate diffeomorphisms of an asymptotically flat spacetime at null infinity [32–36].

So the question is: What are the asymptotic symmetries of electrodynamics? In analogy with the gravitational case, we can find as the asymptotic symmetries of the Maxwell theory the so called "large gauge transformations" (LGT's) that have a non-vanishing value at null infinity. Due to Noether's theorems, it is natural to expect that these asymptotic symmetries have corresponding conserved quantities. Indeed, it has been argued that there are in fact an uncountably infinite number of conserved charges that go with the asymptotic symmetries of U(1) theory [1]. On the one hand, it is possible to start with the conserved charges and derive the corresponding asymptotic symmetries. In [1] the derivation is carried out in this order. Briefly outlined, the method here is to develop a canonical Hamiltonian formalism for electromagnetism, where the phase space is given by the allowed initial data on any Cauchy surface. Using this formalism one can then identify the asymptotic symmetries with the Dirac bracket action of the conserved charges on the phase space.

On the other hand, it is also possible to begin with the asymptotic symmetries and derive the conserved charges. The derivation in this direction can be done using the Noether method. Since we started with the memory effect that is connected to asymptotic symmetries, the obvious thing to do now is to find the corresponding conserved charges. We begin by a characterization of LGT's and show the connection between them and the memory effect. Then we proceed to study the conserved charges.

4.8.1 Large gauge transformations

In U(1) gauge field theory, the Lagrangian is invariant under a transformation of the form

$$A_\mu \to A_\mu + \partial_\mu \chi. \tag{4.99}$$

LGT's are characterized by the large r fall-off conditions [37]

$$A_r = \mathcal{O}(1/r^2), \quad A_u = \mathcal{O}(1/r), \quad A_B = \mathcal{O}(1). \tag{4.100}$$

We can use our freedom to choose the temporal gauge:

$$A_u = 0, \tag{4.101}$$

which is obtained by setting

$$\chi(u, r, \theta^B) = -\int_0^u du' A_u(u', r, \theta^B). \tag{4.102}$$

With this particular gauge choice, there is a simple relation between the physical field $\vec{E}$ and gauge field transformations:

$$\begin{aligned}\int_{-\infty}^{\infty} E_B du = \int_{-\infty}^{\infty} F_{uB} du &= \int_{-\infty}^{\infty} \partial_u A_B du \\ &= A_B(\infty) - A_B(-\infty) \\ &\equiv \Delta A_B. \end{aligned} \tag{4.103}$$

What this tells us is that the memory effect taking place is equivalent to the change of the gauge potential by a finite transformation at null infinity. That is to say, for any memory effect we can find a scalar $\chi = \chi(\theta^B)$, which is a function of angles but constant with respect to u, such that the gauge transformation

$$A_B \to A_B + \partial_B \chi, \tag{4.104}$$

gives the net change in the gauge field resulting from the memory effect. This result is interesting since the gauge transformation is directly related to the kick, which is a physical effect. Thus we have a physically determined gauge transformation even though a gauge transformation is a transformation between two physically identical states. Moreover, we have not required the gauge transformation to be constant. For all we know, the gauge parameter may be any differentiable function of angular coordinates insofar as the transformation has a non-vanishing value at null infinity.

The scalar χ is determined by the difference ΔA_B up to an integration constant. Thus assuming that we are given a field E_B, we have a family of functions χ that give the corresponding LGT's at null infinity. On the other hand, with a large gauge transformation given, there are a lot of different field configurations that yield the same memory vector and hence the same gauge transformation.

4.8.2 The charges induced by LGT's

Since in the case of LGT's the gauge parameter is a function of spacetime coordinates, the gauge transformation is a local symmetry of the U(1) theory.

Thus the Noether current associated with a LGT can be formulated in such a way that it is conserved identically. The Noether current associated with the gauge parameter χ is

$$J_\chi^\nu = \nabla_\mu(\chi F^{\mu\nu}) \tag{4.105}$$

and the corresponding charge is given by

$$\begin{aligned} Q_\chi &= \lim_{\substack{r\to\infty \\ t=const.}} \int d\Omega\, r^2 n_i F^{i0}\chi = \lim_{\substack{r\to\infty \\ t=const.}} \int d\Omega\, r^2 F^{r0}\chi \\ &= \lim_{\substack{r\to\infty \\ t=const.}} \int d\Omega\, r^2 F_{0r}\chi. \end{aligned} \tag{4.106}$$

In the special case $\chi = 1$ this is just the conserved electric charge and the Noether current is the ordinary four-current. However, a memory effect is related to a non-trivial LGT, and having a non-trivial LGT requires that χ is non-constant at null infinity. Hence, the conserved charge is something different from the ordinary electric charge, and one would also expect that the memory effect is connected with this conserved charge. Thus we now start to derive an equation that relates the memory effect to the conserved Noether charge. With the formalism we constructed above, we write the equation (4.106) as

$$Q_\chi = \int_{\mathcal{I}^+_-} d\Omega\, \mathcal{E}_r\, \chi \tag{4.107}$$

$$= -\int_{S^2} d\Omega \int_{-\infty}^{\infty} du\, \chi \partial_u \mathcal{E}_r + \int_{\mathcal{I}^+_+} d\Omega\, \mathcal{E}_r \chi. \tag{4.108}$$

The second term on the right hand side is the final charge determined by the charge distribution when $t \to \infty$, whereas on the left hand side we have the initial charge determined at $t = const.$ timeslice. Move the final charge to the left hand side and denote the difference between the initial charge and final charge by

$$\Delta Q_\chi \equiv Q_\chi - \int_{\mathcal{I}^+_+} d\Omega\, \mathcal{E}_r \chi. \tag{4.109}$$

Then we use the equation of motion (4.48) to get

$$\Delta Q_\chi = \int_{S^2} d\Omega \int_{-\infty}^{\infty} du\, \chi \left(L - D_A \mathcal{E}^A\right) = \int_{S^2} d\Omega\, \chi \left(F - D_A M^A\right). \tag{4.110}$$

Recall that

$$D_A M^A = D_A M_1^A + D_A M_2^A, \tag{4.111}$$

and from (4.84) we get

$$D_A M_1^A = F - F_{avg}. \tag{4.112}$$

Plugging these into (4.110) yields

$$\Delta Q_\chi = \int d\Omega \, \chi F_{avg} - \int d\Omega \, \chi D_A M_2^A. \tag{4.113}$$

Consider now the second integral a bit more in detail. We can write it as

$$-\int d\Omega \, D_A(\chi M_2^A) + \int d\Omega \, (D_A\chi) M_2^A. \tag{4.114}$$

In the first term we have the divergence of a vector over a two-sphere, so the first term vanishes by the lemma of Appendix C. In the second term we have a derivative of the gauge parameter, and from the equations (4.59), (4.103) and (4.104), we see that this is nothing but the memory field we found earlier, i.e.

$$M_A = D_A \chi. \tag{4.115}$$

Hence we have found a relation between the Noether charge and the memory effect:

$$\boxed{\Delta Q_\chi = F_{avg} \int d\Omega \, \chi + \int d\Omega \, M_A M_2^A,} \tag{4.116}$$

where

$$\Delta Q_\chi = \int_{\mathcal{I}^+_-} d\Omega \, \mathcal{E}_r \, \chi - \int_{\mathcal{I}^+_+} d\Omega \, \mathcal{E}_r \, \chi, \qquad F_{avg} = \frac{1}{4\pi} \int d\Omega \, F, \tag{4.117}$$

i.e., ΔQ_χ is the difference between the initial and final charges and F_{avg} is the average value of flux F over the two-sphere. In the first term on the right hand side F_{avg} is multiplied by the integral of the gauge parameter over the two-sphere. In the second term we have an integral of the inner product of the entire memory field and the ordinary part of the memory field.

We have now derived an equation relating the electromagnetic memory effect and the conserved charge associated with a LGT. Related equations have been derived in [16, 37, 38]. Evaluating this relation between the conserved charge and the memory effect concretely by developing models for M_1 and M_2 would be an interesting future project, but beyond the scope of this thesis.

5 Conclusions

The main aim of this thesis was to study the electromagnetic analog of gravitational wave memory effect. After preliminary discussions on the U(1) invariance of electrodynamics and the conformal structure of Minkowski space, we proceeded to analyze a situation in which a flux F of charged radiation propagates to the future null infinity and generates the analog of Christodoulou memory of gravitational physics. Starting with Maxwell's equations, a partial differential equation was derived, in which the S^2 divergence of the memory vector depends on the total flux of charge that reaches the null infinity and the initial and final values of the radial component of the electric field. The memory vector was then found to consist of two parts: the ordinary memory vector and the null memory vector. We thus reproduced the solution of Bieri and Garfinkle [10] for the null memory vector by expanding the flux F in terms of spherical harmonics. The same procedure applies to the ordinary memory vector, even though we did not carry this out explicitly.

After this, we analyzed the connection between the electromagnetic memory effect and the asymptotic symmetries of U(1) gauge theory. The memory effect was found to determine a large gauge transformation (LGT) in which the gauge parameter χ becomes a function of angles at null infinity. Since a LGT is a local symmetry of U(1) theory, we concluded that there is a conserved Noether current and Noether charge associated with it. As the memory effect generates a LGT, it is natural to expect a connection between the memory effect and the Noether charge. Our study thus culminated in an equation in which the difference between the initial and final Noether charges equals the sum of two terms: the product of the S^2 surface average of flux F and the integral of χ over S^2, on the one hand, and the integral of the inner product of the whole memory vector and the ordinary memory vector over S^2, on the other hand.

Although related equations have been derived in recent literature, it seems that, before now, an explicit relation between the conserved Noether charge and the memory effect has not been presented. More research is needed in order to get a better understanding of this relation. The next step to this direction would be to build concrete models for the ordinary and null memories by choosing a suitable flux F that generates the null memory effect and a configuration of ordinary charges with subluminal velocity. This would probably require a numerical computation of the spherical harmonics expansion of the memory vector.

As the main motivation for studying the electromagnetic memory effect is to gain a better grasp of the analogous effect in gravity, it would be a natural continuation to this project to examine, whether the analog between the electromagnetic and gravitational-wave memory effects also covers the conserved charge we found. The relation between conserved charges associated with BMS symmetries of GR and the gravitational-wave memory effect has already been studied, see for example [41]. It would be interesting to see whether the covariant analysis of gravitational memory in [18] could form the basis of BMS conserved charges, in analogy to the relation between the LGT-induced Noether charge and the electromagnetic memory effect. This is a problem that needs further research.

Appendices

A Invariance of null geodesics under conformal mappings

Claim. Null geodesics are invariant under conformal transformations.

Proof. Let $x^\mu(\lambda)$ be a null geodesic with respect to the metric $g_{\mu\nu}$ and denote its tangent vector as $k^\mu = dx^\mu/d\lambda$. Let $\widehat{\nabla}$ be the derivative operator compatible with the transformed metric $\widehat{g}_{\mu\nu}$. Then

$$\begin{aligned}\widehat{\Gamma}^\nu_{\mu\lambda} &= \frac{1}{2}\widehat{g}^{\nu\rho}\left(\partial_\mu \widehat{g}_{\lambda\rho} + \partial_\lambda \widehat{g}_{\mu\rho} - \partial_\rho \widehat{g}_{\mu\lambda}\right)\\ &= \frac{1}{2}\Omega^{-2} g^{\nu\rho}\left(\partial_\mu\left(\Omega^2 g_{\lambda\rho}\right) + \partial_\lambda\left(\Omega^2 g_{\mu\rho}\right) - \partial_\rho\left(\Omega^2 g_{\mu\lambda}\right)\right)\\ &= \Gamma^\nu_{\mu\lambda} + \Omega^{-1}\left(\delta^\nu_\lambda \partial_\mu \Omega + \delta^\nu_\mu \partial_\lambda \Omega - g_{\mu\lambda} g^{\nu\rho}\partial_\rho \Omega\right). \end{aligned} \tag{A.1}$$

This allows us to write

$$\begin{aligned} k^\mu \widehat{\nabla}_\mu k^\nu &= \frac{d^2 x^\nu}{d\lambda^2} + \widehat{\Gamma}^\nu_{\mu\lambda} k^\mu k^\lambda \\ &= \frac{d^2 x^\nu}{d\lambda^2} + \Gamma^\nu_{\mu\lambda} k^\mu k^\lambda + \Omega^{-1}\left(\delta^\nu_\lambda \partial_\mu \Omega + \delta^\nu_\mu \partial_\lambda \Omega - g_{\mu\lambda} g^{\nu\rho}\partial_\rho \Omega\right) k^\mu k^\lambda. \end{aligned} \tag{A.2}$$

Since $x^\mu(\lambda)$ is a null geodesic, we get

$$k^\mu \widehat{\nabla}_\mu k^\nu = 2k^\nu k^\mu \partial_\mu \ln \Omega. \tag{A.3}$$

Thus, k^ν satisfies the general geodesic equation with respect to $\widehat{\nabla}$, where the right hand side is of the form αk^ν with $\alpha = 2k^\mu \partial_\mu \ln \Omega$. We can put the geodesic equation to a more familiar form, where the right-hand-side is just zero, by a reparametrization of the curve [39]. □

From (A.2) we see that, in general, geodesics are not invariant under conformal transformations. We needed the assumption of nullness to get the invariance.

B Helmholtz decomposition on S^2

The fundamental theorem of vector calculus is *Helmholtz theorem*: Let $\vec{V}$ be a twice continuously differentiable vector field in $\mathbb{R}^3$ that vanishes sufficiently fast as $r \to \infty$. Then it can be decomposed into a sum of an irrotational component and a solenoidal component, i.e.

$$\vec{V} = \nabla\phi + \nabla \times \vec{W}, \tag{B.1}$$

for some scalar function ϕ and vector field $\vec{W}$. For a proof, see e.g. [40].

There is a special case of Helmholtz decomposition that is important to this master's thesis:

Claim. Let V^A be a smooth vector field on S^2. Then

$$V^A = D^A\phi + \epsilon^{AB} D_B \psi \tag{B.2}$$

for some scalar functions ϕ and ψ that are unique up to a constant.

Proof. Since this is a relation between tensors, it follows that if we manage to show this in some specific coordinate system, the relation is satisfied in all coordinate systems. A convenient choice of coordinates in this case is the stereographic coordinate system $(z, \bar{z})$ of the two-sphere. That is to say, we define a complex coordinate z and its complex conjugate $\bar{z}$ with

$$z = \frac{1}{\tan\frac{1}{2}\theta} e^{i\phi} = \frac{x^1 + ix^2}{r + x^3}. \tag{B.3}$$

A vector living on the two-sphere then is

$$V^A = (V^z, V^{\bar{z}}) \tag{B.4}$$

and the S^2 metric is

$$h_{AB} = \begin{bmatrix} 0 & \dfrac{2}{(1+z\bar{z})^2} \\ \dfrac{2}{(1+z\bar{z})^2} & 0 \end{bmatrix}. \tag{B.5}$$

Note that the new coordinates change the metric into an anti-diagonal matrix. This means that

$$V^z = \frac{1}{\sqrt{|h|}} V_{\bar{z}}, \quad V^{\bar{z}} = \frac{1}{\sqrt{|h|}} V_z, \tag{B.6}$$

where h is the metric determinant. Showing that the relation (B.2) holds for some ϕ and ψ then amounts to proving the existence of solutions to a pair of partial differential equations in the complex plane:

$$V_z = \partial_z \phi + \partial_z \psi \tag{B.7}$$
$$V_{\bar{z}} = \partial_{\bar{z}} \phi - \partial_{\bar{z}} \psi. \tag{B.8}$$

Since the vector field is smooth, the complex-valued components V_A are analytic functions of both z and $\bar{z}$. Then Cauchy's Integral Theorem implies that V_z and $V_{\bar{z}}$ have integral functions $\int V_z \, dz$ and $\int V_{\bar{z}} \, d\bar{z}$. Thus, we get from (B.7) and (B.8) that

$$\phi + \psi = \int V_z \, dz \tag{B.9}$$
$$\phi - \psi = \int V_{\bar{z}} \, d\bar{z}. \tag{B.10}$$

This implies that the solutions are

$$\phi = \frac{1}{2} \left(\int V_z \, dz + \int V_{\bar{z}} \, d\bar{z} \right) \tag{B.11}$$
$$\psi = \frac{1}{2} \left(\int V_z \, dz - \int V_{\bar{z}} \, d\bar{z} \right), \tag{B.12}$$

and these are unique up to an integration constant. □

C Integral of divergence over S^2

In this appendix we give a proof of a lemma that is found useful in the calculations above.

Claim. Let V^A be a vector field on a two-sphere such that $D_A V^A$ is well-defined everywhere on the sphere. Then

$$\int_{S^2} d\Omega\, D_A V^A = 0. \tag{C.1}$$

Proof. Let V^A be a vector on S^2. Then it can be decomposed into a sum of a irrotational component and a solenoidal component, as we already showed in Appendix B :

$$V^A = D^A \phi + \epsilon^{AB} D_B \psi, \tag{C.2}$$

for some scalar functions ϕ and ψ that depend on the angles. Consider then the divergence of V^A. The divergence of the solenoidal component vanishes:

$$D_A \epsilon^{AB} D_B \psi = \epsilon^{AB} D_A D_B \psi = 0, \tag{C.3}$$

since covariant derivatives commute when acting on a scalar. Thus, we have that

$$D_A V^A = D_A D^A \phi \equiv D^2 \phi. \tag{C.4}$$

Then we expand ϕ in spherical harmonics:

$$\phi = \sum_{l,m} \phi_{lm} Y_{lm}. \tag{C.5}$$

In terms of spherical harmonics, the divergence of V^A becomes

$$D_A V^A = \sum_{l,m} \phi_{lm} D^2 Y_{lm} = -\sum_{l,m} l(l+1) \phi_{lm} Y_{lm}. \tag{C.6}$$

Now integrate both sides over all angles. Consider the right-hand-side, for all $l > 0$,

$$\int_{S^2} d\Omega\, Y_{lm} = 0, \tag{C.7}$$

and the factor $l(l+1)$ kills the first term in (C.6). Thus, the integral of the right-hand-side vanishes, and we get

$$\int_{S^2} d\Omega\, D_A V^A = 0. \tag{C.8}$$

□

References

[1] A. Strominger, "Lectures on the Infrared Structure of Gravity and Gauge theories," (2017) arXiv:1703.05448.

[2] C. D. Hill and P. Nurowski, "Part one: How the green light was given for gravitational wave search," in "Mathematics of Gravitational Waves: A Two-part Feature," Notices of the AMS **64** (2017) 686, DOI: http://dx.doi.org/10.1090/noti1551.

[3] B. P. Abbott et al., "Observation of Gravitational Waves from a Binary Black Hole Merger," Phys. Rev. Lett. **116** (2016) 061102, DOI: 10.1103/PhysRevLett.116.061102.

[4] L. Bieri, D. Garfinkle and N. Yunes, "Part two: Gravitational waves and their mathematics," in "Mathematics of Gravitational Waves: A Two-part Feature," Notices of the AMS **64**, (2017) 693, DOI: http://dx.doi.org/10.1090/noti1551.

[5] `https://www.nobelprize.org/nobel_prizes/physics/laureates/2017/`.

[6] Y. Zeldovich and A. Polnarev, "Radiation of gravitational waves by a cluster of superdense stars," Sov. Astron. **18**, (1974) 17.

[7] D. Christodoulou, "Nonlinear Nature of Gravitation and Gravitational-wave experiments," Phys. Rev. Lett. **67** (1991) 1486, DOI: 10.1103/PhysRevLett.67.1486.

[8] K. Thorne, "Gravitational-wave bursts with memory: The Christodoulou effect," Phys. Rev. **D 45** (1992) 520, DOI: 10.1103/PhysRevD.45.

[9] A. G. Wiseman and C. M. Will, "Christodoulou's nonlinear gravitational-wave memory: Evaluation in the quadrupole approximation," Phys. Rev. **D 44** (1991) R2945, DOI: 10.1103/PhysRevD.44.R2945.

[10] L. Bieri and D. Garfinkle, "An electromagnetic analog of gravitational wave memory," Class. Quantum Grav. **30** (2013) 195009, DOI: 10.1088/0264-9381/30/19/195009.

[11] J. Winicour, "Global aspects of radiation memory," (2014) arXiv:1407.0259.

[12] S. Pasterski, "Asymptotic Symmetries and Electromagnetic Memory," (2015) arXiv:1505.00716.

[13] L. Susskind, ”Electromagnetic memory,” (2015) arXiv:1507.02584.

[14] M. Pate, A.-M. Raclariu and A. Strominger, ”Color memory,” (2017) arXiv:1707.08016.

[15] A. Strominger, ”Asymptotic symmetries of Yang-Mills theory,” (2013) arXiv:1308.0589.

[16] D. Kapec, V. Lysov and A. Strominger, ”Asymptotic symmetries of massless QED in even dimensions,” (2014) arXiv:1412.2763.

[17] T. He et al., ”New symmetries of massless QED,” (2016) arXiv:1407.3789.

[18] L. Bieri and D. Garfinkle, ”A perturbative and gauge invariant treatment of gravitational wave memory,” Phys. Rev. **D 89** (2014) 084039, DOI: 10.1103/PhysRevD.89.084039.

[19] A. Strominger and A. Zhiboedov, ”Gravitational Memory, BMS Supertranslations and Soft Theorems,” (2014) arXiv:1411.5745.

[20] T. Mädler and J. Winicour, ”Radiation Memory, Boosted Schwarzschild Spacetimes and Supertranslations,” (2017) arXiv:1701.02556.

[21] S. Hawking, M. Perry, and A. Strominger, ”Soft hair on black holes,” (2016) arXiv:1601.00921.

[22] S. M. Carroll, *Spacetime and Geometry: An Introduction to General Relativity*, Pearson (2013).

[23] D. MacDonald and K. S. Thorne, ”Black-hole electrodynamics - an absolutespace/ universal-time formulation,” Mon. Not. Roy. Astron. Soc. **198** (1982) 345.

[24] F. Cabral and F. S. N. Lobo, ”Electrodynamics and Spacetime Geometry: Foundations,” Found. Phys. **47** (2017) 208, DOI: 10.1007/s10701-016-0051-6.

[25] C. Misner, K. Thorne and J. Wheeler, *Gravitation*, W.H. Freeman and Company (1973).

[26] E. Noether, “Invariant Variation Problems,” Gott. Nachr. **1918** (1918) 235 [Transp. Theory Statist. Phys. **1** (1971) 186] DOI: 10.1080/00411457108231446.

[27] D. Bak, D. Cangemi and R. Jackiw, “Energy momentum conservation in general relativity,” Phys. Rev. D **49** (1994) 5173 Erratum: [Phys.

Rev. D **52** (1995) 3753] DOI: 10.1103/PhysRevD.52.3753, 10.1103/PhysRevD.49.5173.

[28] S. G. Avery and B. U. W. Schwab, “Noether's second theorem and Ward identities for gauge symmetries,” JHEP **1602** (2016) 031 DOI: 10.1007/JHEP02(2016)031.

[29] R. Penrose, ”Conformal treatment of infinity,” in *Relativity, groups and topology*, edited by B. deWitt and C. deWitt, Gordon and Breach (1964), pp. 565–584.

[30] V. Di Carlo, *Conformal compactification and anti-de Sitter space*, Master's thesis, University of Stockholm, `http://www.fysik.su.se/~ingemar/relteori/valentinamaster.pdf`

[31] J.D. Jackson, *Classical Electrodynamics*, 3rd edition, John Wiley & Sons (1999).

[32] H. Bondi, M. van der Burg, and A. Metzner, “Gravitational waves in general relativity. 7. Waves from axisymmetric isolated systems,” Proc.Roy.Soc.Lond. **A269** (1962) 21, DOI: 10.1098/rspa.1962.0161.

[33] R. Sachs, “Gravitational waves in general relativity. 8. Waves in asymptotically flat space-times,” Proc.Roy.Soc.Lond. **A270** (1962) 103, DOI: 10.1098/rspa.1962.0206.

[34] L. Tamburino and J. Winicour, ”Gravitational Fields in Finite and Conformal Bondi Frames,” Phys. Rev. **150** (1966) 1039, DOI: 10.1103/PhysRev.150.1039.

[35] T. Mädler and J. Winicour, ”Bondi-Sachs formalism,” (2016) arXiv:1609.01731.

[36] F. Alessio and G. Esposito, ”On the structure and applications of the Bondi-Metzner-Sachs Group,” (2017) arXiv:1709.05134.

[37] B. Gabai and A. Sever, ”Large Gauge Symmetries and Asymptotic States in QED,” (2016) sect. 2.3, arXiv:1607.08599.

[38] Y. Hamada, M.-S. Seo, and G. Shiu, ”Electromagnetic Duality and the Electric Memory Effect,” (2017) Eq. 2.15, arXiv:1711.09968.

[39] R. Wald, *General Relativity*, University of Chicago Press (1984).

[40] G. Arfken, H. Weber, and F. Harris, *Mathematical methods for physicists: a comprehensive guide*, 7th edition, Elsevier Academic Press (2013).

[41] É. É. Flanagan and D. A. Nichols, "Conserved charges of the extended Bondi-Metzner-Sachs algebra," Phys. Rev. D **95** (2017) 044002, DOI:10.1103/PhysRevD.95.044002.