Computing the survival probability in the Madan-Unal credit risk model: Application to the CDS market

Luca Vincenzo Ballestra[1], Graziella Pacelli[2,*], Davide Radi[3]

[1]Department of Statistics, Universitá di Bologna, Bologna, Italy
[2]Department of Management, Universitá Politecnica delle Marche, Ancona, Italy
[3]Department of Management, Universitá Politecnica delle Marche, Ancona, Italy

February 17, 2016

Abstract

We obtain a quasi-analytical approximation of the survival probability in the credit risk model proposed in Madan and Unal (1998). Such a formula, which by extensive numerical simulations reveals to be accurate and computationally fast, can also be employed for pricing credit default swaps (CDSs). Specifically, we derive a quasi-analytical approximate expression for CDS par spreads, and we use it to estimate the parameters of the model. The results obtained show a rather satisfactory agreement between theoretical and real market data.

Keywords: Default risk; survival probability; Madan-Unal model; credit default swap; CDS.

1 Introduction

Pricing the risk of default is a fundamental task for several financial market players such as corporate bond investors, credit derivative traders, banks, mortgage suppliers and insurance companies. To this aim, various models of credit risk have been developed which are based on two different approaches: the structural approach and the reduced-form approach.

Structural models describe the default event by means of one or more variables related to the capital structure of the firm issuing the debt. For example, according to the first proposed structural model, which has been developed in Merton (1974), a firm defaults if at the debt maturity the value of its assets is lower than the value of its obligations. An improvement of this model is presented in Black and Cox (1976), where the possible occurrence of premature bankruptcy as well as the debt seniority are taken into account. Other more sophisticated structural models are also available which incorporate variables such as the interest rate (Longstaff and Schwartz, 1995; Briys and de Varenne, 1997; Bernard et al., 2005), tax benefits (Anderson and Sundaresan, 1996), debt restructuring (Abínzano et al., 2009), liquidation costs (Leland and Toft, 1996) and downgrade-triggered termination clauses (Feng and Volkmer, 2012). As revealed by several empirical studies (see, e.g, Jones et al., 1984; Franks and Torous, 1989), such a kind of models have a major issue: if the firm's assets value is specified as a continuous-time stochastic process, then for short debt

*Corresponding author: Graziella Pacelli, Piazzale Martelli 8, 60121, Ancona, Italy. Email: g.pacelli@univpm.it Phone: +39 071 2207050

maturities the probability of default that the models predict turns out to be very close to zero, contrary to what happens in reality (see, e.g, Crouhy et al., 2000; Bäuerle, 2002). In order to account for high short-term spreads, some authors, see, e.g. (Zhou, 2001; Chen and Panjer, 2003), have proposed structural models with unexpected jumps in the firm asset value. Nevertheless, this approach lacks analytical tractability, which makes it difficult to calibrate the model parameters to observed credit spreads.

Instead, according to the reduced-form approach (see, e.g, Duffee, 1999; Duffie and Singleton, 1999; Madan and Schoutens, 2008; Schoutens and Cariboni, 2009; Fontana and Montes, 2014) the default event is modeled as the first jump of a counting process whose intensity, termed intensity of default, is not assumed to be firm-specific but is prescribed exogenously. This allows one to take into account the possible occurrence of a sudden (unpredictable) default event and henceforth the high credit spreads that are often experienced for short debt maturities can be recovered. In addition, reduced-form models are relatively simple from a mathematical standpoint and thus they usually offer a large amount of analytical tractability.

However, reduced-form models have the heavy drawback of not taking into account any information about the capital structure of the firm. Such an issue has prompted some authors to develop hybrid models of credit risk in which the reduced-form approach is combined with some structural variable (see, e.g, Madan and Unal, 1998, 2000; Duffie and Lando, 2001; Cathcart and El-Jahel, 2003, 2006; Giesecke, 2006; Ballestra and Pacelli, 2014). Among these models, the one proposed in Madan and Unal (1998) deserves a special attention as it is a parsimonious hybrid model. In particular, it is mainly developed based on the reduced-form approach, but the default intensity, instead of being prescribed exogenously, is specified as a convenient function of the firm's equity value. This allows us to recover the desirable features of both the structural and the reduced-form models, but at the same time the parameters involved are only one more than the parameters of Merton's model (see Merton, 1974). Therefore, the approach by Madan and Unal turns out to be particularly appealing and suitable for practical uses.

Nevertheless, the model in Madan and Unal (1998) does not have an analytical closed-form solution. To be precise, Madan and Unal have provided a closed-form expression for the survival probability, but, as pointed out in Grundke and Riedel (2004), the procedure used to derive this formula is not mathematically correct. Consequently, as shown in Grundke and Riedel (2004), where the default probability is computed by finite difference approximation, the closed-form solution obtained by Madan and Unal yields a survival probability that can also differ substantially from the true survival probability of the model. In order to compensate for the lack of a closed-form solution, an analytical approximation of the survival probability of the model by Madan and Unal has been proposed in Ballestra and Pacelli (2009). Such a formula is fairly accurate and computationally fast, but it is applicable only if one of the model parameters is sufficiently small (as it is based on a perturbation approach, see Ballestra and Pacelli (2009)).

In the present paper, a quasi-analytical approximation of the survival probability in the model by Madan and Unal (1998) is presented. Such a formula, which is based on a Laplace-transform approach, turns out to be very accurate and computationally fast (an error of the order of 10^{-4} or 10^{-5} is obtained in a time of the order of 10^{-4} seconds). Remarkably, this analytical expression for the survival probability allows us to price credit default swaps (CDSs) very easily. Specifically, a quasi-analytical formula to compute CDS par spreads is derived which is used in this manuscript to calibrate the Madan-Unal credit risk model by fitting realized CDS par spreads. In particular, CDS names with different Moody's ratings are considered and the agreement between theoretical and empirical data is rather satisfactory, especially if we think that the stochastic differential equations on which the model by Madan and Unal stands involve only two unknown parameters.

We stress that the contribution of the paper is as follows: first of all, we derive an (exact) analytical expression for the Laplace transform of the survival probability in the model by Madan and Unal (1998). Second, by applying a formula for Laplace transform inversion, we obtain a quasi-analytical approximation of the survival probability that is very efficient from the computational standpoint, and also easy to implement. In particular, such an approximate solution can be used with any set of parameter values (unlike the asymptotic expansion proposed in Ballestra and Pacelli (2009), which requires one of the model parameters to be small) and, as revealed by numerical experiments reported in Section 5, turns out to be significantly faster than the popular Crank-Nicolson finite difference method. Finally, we derive a quasi-analytical approximation of CDS spreads and show how it can be used to estimate the parameters of the model.

The paper is organized as follows. Section 2 introduces the model by Madan and Unal. Section 3 is devoted to the calculation of the Laplace transform of the survival probability. Section 4 shows how to recover the survival probability. Section 5 contains numerical experiments that validate the proposed approach. Section 6 presents a quasi-analytical approximate formula to compute CDS par spreads. Section 7 shows the calibration of the model and the agreement between theoretical and real market data. Section 8 concludes.

2 The mathematical model

In this section the basic facts about the model by Madan and Unal (1998) are recalled. Let us consider an indebted firm, and let the discounted (or, as called by Madan and Unal, "relativized") value of its equity at time t be defined as follows:

$$A(t) = \frac{e(t)}{B(t)} \tag{1}$$

where e is the firm's equity value and B is the money market account:

$$B(t) = e^{\int_0^t r(\theta)d\theta} \tag{2}$$

and r is the default-free spot interest rate, which may be considered either deterministic or stochastic (see, e.g., Madan and Unal, 1998). Madan and Unal assume that the discounted equity value follows the stochastic process

$$dA(t) = \sigma A(t)\, dW(t) \tag{3}$$

where $\sigma > 0$ is a constant volatility parameter and W is a standard Wiener process under the risk-neutral measure, see, e.g., Björk (2009, pp. 102-104) and Elliott and Kopp (2005, pp. 167-171). In Madan and Unal (1998), the choice of directly modeling the discounted equity value A (rather than the equity value e) aims to avoid to specify the dynamics of the interest rate, which is incorporated in the dynamics of A. Thus, as explicitly declared in Madan and Unal (1998), A serves as a sufficient statistic for assessing the risk of default.

The model by Madan and Unal bridges the gap between the structural and the reduced-form approach. In particular, as is typical of reduced-form models, it is assumed that default occurs as the first jump of a Poisson process. However, in order to take into account structural information, the intensity of this process, i.e. the so-called default intensity, is specified as a convenient function of the discounted equity value. Precisely, under the risk-neutral measure, the default intensity, which we denote with ϕ, is modeled as follows:

$$\phi(A(t)) = \frac{c}{\left(\ln(A(t)) - \ln(\delta)\right)^2} \tag{4}$$

where $c > 0$ and $\delta > 0$ are constant parameters. A sound theoretical justification for this choice can be found in (Madan and Unal, 1998), here we simply observe that according to (4) the probability of default is measured by the distance of A from the critical value δ. In particular, when A reaches the threshold level δ, the intensity ϕ becomes infinite, and default occurs with certainty.

Note that a variety of authors agree on the fact that the probability of default of a firm is related to the value of its equity. In fact, many of the structural models that are employed in the literature assume that default occurs as long as the equity value reaches the zero level (see, e.g., Merton, 1974; Sepp, 2006). In addition, some empirical studies highlight that corporate bond yields are significantly affected by the equity volatility (see, e.g., Campbell and Taksler, 2003; Kim and Stock, 2014). However, there are also other variables that can determine the default of a firm, for example government taxes or liquidity. As reported by Duffee (1999), one of the main advantages of the reduced-form approach is to take into account all these factors, which, instead of being modeled one by one (which would introduce a large amount of analytical complexity) are somehow subsumed by the Poisson process driving the unexpected default. For example, according to the model described above, we can have a significant probability of default even if the value of the equity is relatively large, due to the reduced-form component specified as in (4).

Finally, as already mentioned, both the equity value process (3) and the default intensity (4) are written under the risk-neutral measure. This is very common in reduced-form models, which are usually directly specified under a risk-neutral measure that allows for martingale pricing and that is identified by direct calibration to market prices (see, e.g., Duffee, 1999; Luciano, 2007; Pan and Singleton, 2008). Therefore, the survival probability that we are going to obtain in the following is, actually, a risk-neutral survival probability, which can be used for pricing. Instead, if the objective (real-world) probability were needed, then one should further extend the model by introducing a drift parameter in (3) and by assuming that (4) incorporates a premium for the jump risk (see, e.g., Duffee, 1999; Pan and Singleton, 2008). Nevertheless, the calculation of the objective survival probability is not taken into account by Madan and Unal (as their model is designed for pricing), and goes beyond the purpose of the present paper.

Let us consider the partial differential formulation of the stochastic model (3) and (4). To this aim, let t_1 and t_2 denote two generic times, such that $0 \leq t_1 \leq t_2$ and let $\Psi(t_1, a, t_2)$ denote the probability that the firm survives up to time t_2 given no default prior to t_1 and $A(t_1) = a$. As shown in Madan and Unal (1998), Ψ depends only on the time lag $\tau = t_2 - t_1$ and on a (the discounted value of the firm's equity at time t_1). Then, we can set $\Psi(t_1, a, t_2) = H(a, \tau)$. It can be shown (see, e.g., Madan and Unal, 1998) that for $a \in (\delta, +\infty)$ and $\tau \in (0, t_2]$, the function $H(a, \tau)$ satisfies the following partial differential equation:

$$\frac{\partial H(a,\tau)}{\partial \tau} - \frac{\sigma^2 a^2}{2}\frac{\partial^2 H(a,\tau)}{\partial a^2} + \frac{cH(a,\tau)}{(\ln(a) - \ln(\delta))^2} = 0 \tag{5}$$

with boundary conditions

$$H(\delta,\tau) = 0, \qquad \lim_{a \to +\infty} H(a,\tau) = 1 \tag{6}$$

and initial condition

$$H(a,0) = 1 \tag{7}$$

Then, using the change of variable

$$x = \ln\left(\frac{a}{\delta}\right) \tag{8}$$

and setting

$$P(x,\tau) = H(\delta e^x, \tau) \tag{9}$$

the partial differential problem (5)-(7) can be rewritten as follows:

$$\frac{\partial P(x,\tau)}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 P(x,\tau)}{\partial x^2} + \frac{\sigma^2}{2}\frac{\partial P(x,\tau)}{\partial x} + \frac{cP(x,\tau)}{x^2} = 0 \tag{10}$$

for $x \in (0, +\infty)$ and $\tau \in (0, t_2]$, with boundary conditions

$$P(0, \tau) = 0, \qquad \lim_{x \to +\infty} P(x, \tau) = 1 \tag{11}$$

and initial condition

$$P(x, 0) = 1 \tag{12}$$

The partial differential problem (10)-(12) does not admit a closed-form solution. However, as shown in the next Section, an analytical expression for the Laplace transform of P can be obtained.

3 A quasi-analytical Laplace-based solution

Let us consider the Laplace transform of $P(x, \tau)$ with respect to τ:

$$F(x, \omega) = (\mathcal{L}P)(x, \omega) \tag{13}$$

where

$$(\mathcal{L}P)(x, \omega) = \int_0^{+\infty} e^{-\omega\tau} P(x, \tau)\, d\tau \tag{14}$$

and $\omega \in \mathbb{C}$ is the so-called variable conjugate to τ. Well-known properties of the Laplace transform are as follows (see Tang, 2007, pp. 274-275, relations (6.11) and (6.17)):

$$\left(\mathcal{L}\frac{\partial P}{\partial \tau}\right)(x, \omega) = \omega (\mathcal{L}P)(x, \omega) - P(x, \tau), \quad \left(\mathcal{L}Pe^{-a\tau}\right)(x, \omega) = (\mathcal{L}P)(x, \omega + a), \;\; a \in \mathbb{R} \tag{15}$$

Moreover, if we define $\overline{P}(x, \tau) = \int_0^\tau P(x, y)dy$, from the first of (15) we obtain:

$$\left(\mathcal{L}\overline{P}\right)(x, \omega) = \frac{1}{\omega} (\mathcal{L}P)(x, \omega) \tag{16}$$

Taking the Laplace transform of (10)-(11) and using the first of (15) (together with relation (12)), we obtain the following non-homogeneous differential equation:

$$\omega F(x, \omega) - 1 - \frac{\sigma^2}{2}\frac{\partial^2 F(x, \omega)}{\partial x^2} + \frac{\sigma^2}{2}\frac{\partial F(x, \omega)}{\partial x} + \frac{cF(x, \omega)}{x^2} = 0 \tag{17}$$

with boundary conditions

$$F(0, \omega) = 0, \qquad \lim_{x \to +\infty} F(x, \omega) = \frac{1}{\omega} \tag{18}$$

The general solution of the homogeneous differential equation associated to (17) can be obtained as follows: first of all, as in Polyanin and Zaitsev (2003, p. 252, equation (2.1.2.135)), let us make the substitution

$$F(x, \omega) = g(x, \omega)\, e^{\frac{x}{2}} \tag{19}$$

so that the homogeneous differential equation associated to (17) is rewritten as follows:

$$-\left(h^2(\omega)\, x^2 + \frac{2c}{\sigma^2}\right) g(x, \omega) + x^2 \frac{\partial^2 g(x, \omega)}{\partial x^2} = 0 \tag{20}$$

where

$$h(\omega) = \sqrt{\frac{2\omega}{\sigma^2} + \frac{1}{4}} \tag{21}$$

Then, according to Polyanin and Zaitsev (2003, p. 249, equation (2.1.2.115)), let us perform a further substitution

$$g(x,\omega) = x^{\lambda} u(x,\omega) \tag{22}$$

where λ is the negative root of equation $\lambda^2 - \lambda - \frac{2c}{\sigma^2} = 0$, i.e.

$$\lambda = \frac{1}{2} - \sqrt{\frac{\sigma^2 + 8c}{4\sigma^2}} \tag{23}$$

Using (22) and (23), equation (20) is rewritten as follows:

$$x\frac{\partial^2 u(x,\omega)}{\partial x^2} + 2\lambda \frac{\partial u(x,\omega)}{\partial x} - h^2(\omega)\, x u(x,\omega) = 0 \tag{24}$$

This differential equation has the following fundamental solutions (see, Polyanin and Zaitsev, 2003, p. 248, equation (2.1.2.108) and Table 15):

$$u_1(x,\omega) = x^{\nu} I_{\nu}(h(\omega)x), \quad u_2(x,\omega) = x^{\nu} K_{\nu}(h(\omega)x) \tag{25}$$

where $I_{\nu}(\cdot)$ and $K_{\nu}(\cdot)$ are the modified Bessel functions of the first and of the second kind, respectively, and

$$\nu = \sqrt{\frac{\sigma^2 + 8c}{4\sigma^2}} \tag{26}$$

According to (19) and (22), we have $F(x,\omega) = e^{\frac{x}{2}} x^{\lambda} u(x,\omega)$. Therefore, starting from the fundamental solutions of (24), we can obtain the fundamental solutions of the homogeneous differential equation associated to (17):

$$F_1(x,\omega) = e^{\frac{x}{2}} x^{\frac{1}{2}} I_{\nu}(h(\omega)x), \quad F_2(x,\omega) = e^{\frac{x}{2}} x^{\frac{1}{2}} K_{\nu}(h(\omega)x) \tag{27}$$

Then, using the method of variation of parameters described in Appendix A, we write down the solution of problem (17)-(18):

$$F(x,\omega) = F_1(x,\omega)\frac{2}{\sigma^2}\int_x^{+\infty} e^{-\frac{z}{2}} z^{\frac{1}{2}} K_{\nu}(h(\omega)z)\,dz + F_2(x,\omega)\frac{2}{\sigma^2}\int_0^x e^{-\frac{z}{2}} z^{\frac{1}{2}} I_{\nu}(h(\omega)z)\,dz \tag{28}$$

Finally, by simple setting

$$y = h(\omega)\, z \tag{29}$$

relation (28) can be rewritten as follows:

$$F(x,\omega) = \frac{2F_1(x,\omega)}{\sigma^2\sqrt{h^3(\omega)}}\int_{h(\omega)x}^{+\infty} e^{-\frac{y}{2h(\omega)}} y^{\frac{1}{2}} K_{\nu}(y)\,dy + \frac{2F_2(x,\omega)}{\sigma^2\sqrt{h^3(\omega)}}\int_0^{h(\omega)x} e^{-\frac{y}{2h(\omega)}} y^{\frac{1}{2}} I_{\nu}(y)\,dy \tag{30}$$

Unfortunately, the integrals in (30) are not elementary integrals and cannot be evaluated using simple closed-form expressions, at least to the best of our knowledge (symbolic calculation softwares such as Mathematica or Matlab cannot find any analytical formula for these integrals). This can be easily understood if we think that the integral of the Bessel functions I_ν or K_ν alone, if performed analytically, leads to an infinite series of functions (see, e.g., Poularikas, 1999, p. 25-4).

Therefore, in the following, given that simple closed-form expressions cannot be obtained, the integrals in (30) are evaluated by series expansions, which anyway allows for a fast and direct quasi-analytical calculation.

3.1 Analytical calculation of the integrals in (30)

Let us consider the power series expansions that define the two Bessel functions inside the integrals in (30) (see Polyanin and Zaitsev, 2003, p. 772, formulae (S.2.6)):

$$I_\nu(y) = \sum_{k=0}^{\infty} \frac{\left(\frac{y}{2}\right)^{2k+\nu}}{k!\Gamma(k+1+\nu)} \tag{31}$$

$$K_\nu(y) = \frac{\pi}{2} \frac{\sum_{k=0}^{\infty} \frac{\left(\frac{y}{2}\right)^{2k-\nu}}{k!\Gamma(k+1-\nu)} - \sum_{k=0}^{\infty} \frac{\left(\frac{y}{2}\right)^{2k+\nu}}{k!\Gamma(k+1+\nu)}}{\sin(\pi\nu)} \tag{32}$$

where Γ is the well-known gamma function (see, e.g., Abramowitz and Stegun, 1972, p. 255, formula (6.1.1)). As we can observe, the first integral in (30) requires us to compute $K_\nu(y)$ when y varies in $[h(\omega)x, +\infty)$, but the Bessel series expansion (32) starts to become inaccurate if y is large. In fact, if y is large, the powers $\left(\frac{y}{2}\right)^{2k-\nu}$ are large too, and the accuracy of formula (32) is compromised by round-off and cancelation errors.

Therefore, in the present paper relation (32) is employed only for $y \leq \overline{y}$, where $\overline{y}$ is to be chosen properly (see below). By contrast, for $y > \overline{y}$, the Bessel function $K_\nu(y)$ is computed using the asymptotic expansion

$$K_\nu(y) = \sqrt{\frac{\pi}{2y}} e^{-y} \left(1 + \sum_{i=1}^{+\infty} q_i y^{-i}\right) \tag{33}$$

where

$$q_i = \frac{\prod_{k=1}^{i} \left(4\nu^2 - (2k-1)^2\right)}{i! 8^i} \tag{34}$$

which, according to Polyanin and Zaitsev (2003, p. 773, relation (S.2.6-6)) and Olver and Maximon (2010), converges well for large values of y. By trial and error we have found that the value of $\overline{y}$ that (roughly) yields the higher levels of accuracy is $\overline{y} = 7.5$, which is then used in all the numerical simulations reported in the present paper. In addition, we have also experienced that for values of $\overline{y}$ in the range $[5, 10]$ the error related to the calculation of the Bessel function K_ν is always smaller than the error due to the numerical inversion of the Laplace transform performed in Section 4 (more precisely, the survival probability is computed with an error which, at the leading order, is due to the numerical inversion of the Laplace transform and not to the calculation of K_ν). Instead, for values of $\overline{y}$ smaller than 5 or larger than 10, the error due to the the calculation of K_ν starts to be significant and in some cases can also become larger than the error due to the Laplace transform inversion.

Substituting (31), (32) and (33) in (30), interchanging summation and integration and using simple algebraic manipulations, we obtain:

$$
\begin{aligned}
F(x,\omega) =& g_1(x,\omega)\sum_{k=0}^{+\infty}\frac{\left(\frac{1}{2}\right)^{2k}}{k!}\left(\frac{\int_{h(\omega)x}^{7.5} y^{2k+\frac{1}{2}-\nu}e^{-\frac{y}{2h(\omega)}}dy}{2^{-\nu}\Gamma(k+1-\nu)}-\frac{\int_{h(\omega)x}^{7.5} y^{2k+\frac{1}{2}+\nu}e^{-\frac{y}{2h(\omega)}}dy}{2^{\nu}\Gamma(k+1+\nu)}\right)\\
&+g_2(x,\omega)\int_{7.5}^{+\infty}e^{-l(\omega)y}\left(1+\sum_{i=1}^{+\infty}q_i y^{-i}\right)dy+g_3(x,\omega)\sum_{k=0}^{+\infty}\frac{\int_0^{h(\omega)x} y^{2k+\frac{1}{2}+\nu}e^{-\frac{y}{2h(\omega)}}dy}{2^{2k+\nu}k!\Gamma(k+1+\nu)}
\end{aligned}
\tag{35}
$$

if $h(\omega)x \leq 7.5$, and

$$
F(x,\omega) = g_2(x,\omega)\int_{h(\omega)x}^{+\infty}e^{-l(\omega)y}\left(1+\sum_{i=1}^{+\infty}q_i y^{-i}\right)dy+g_3(x,\omega)\sum_{k=0}^{+\infty}\frac{\int_0^{h(\omega)x} y^{2k+\frac{1}{2}+\nu}e^{-\frac{y}{2h(\omega)}}dy}{2^{2k+\nu}k!\Gamma(k+1+\nu)}
\tag{36}
$$

if $h(\omega)x > 7.5$, where

$$
g_1(x,\omega)=\frac{\pi e^{\frac{x}{2}}x^{\frac{1}{2}}I_\nu(h(\omega)x)}{\sigma^2\sqrt{h^3(\omega)}\sin(\pi\nu)},\quad g_2(x,\omega)=\frac{\sqrt{2\pi}e^{\frac{x}{2}}x^{\frac{1}{2}}I_\nu(h(\omega)x)}{\sigma^2\sqrt{h^3(\omega)}},\quad g_3(x,\omega)=\frac{2e^{\frac{x}{2}}x^{\frac{1}{2}}K_\nu(h(\omega)x)}{\sigma^2\sqrt{h^3(\omega)}}
\tag{37}
$$

and

$$
l(\omega)=\frac{1+2h(\omega)}{2h(\omega)}
\tag{38}
$$

Finally, algebraic calculations shown in Appendix B allow us to rewrite relations (35) and (36) as follows:

$$
\begin{aligned}
F(x,\omega) =& g_1(x,\omega)\sum_{k=0}^{+\infty}\frac{(2h(w))^{2k+\frac{3}{2}-\nu}\left(\Gamma_{\text{inc}}\left(\frac{7.5}{2h(w)},2k+\frac{3}{2}-\nu\right)-\Gamma_{\text{inc}}\left(\frac{x}{2},2k+\frac{3}{2}-\nu\right)\right)}{2^{2k-\nu}k!\Gamma(k+1-\nu)}\\
&-g_1(x,\omega)\sum_{k=0}^{+\infty}\frac{(2h(w))^{2k+\frac{3}{2}+\nu}\left(\Gamma_{\text{inc}}\left(\frac{7.5}{2h(w)},2k+\frac{3}{2}+\nu\right)-\Gamma_{\text{inc}}\left(\frac{x}{2},2k+\frac{3}{2}+\nu\right)\right)}{2^{2k+\nu}k!\Gamma(k+1+\nu)}\\
&+\frac{g_2(x,\omega)}{l(\omega)}\left(e^{-\underline{w}}+\sum_{i=1}^{+\infty}(-1)^i l^i(\omega)\left(\frac{e^{-\underline{w}}}{\underline{w}}\sum_{\substack{j=0,\\ i\geq 2}}^{i-2}\left(\frac{j!}{(-\underline{w})^j}\right)-E_1(\underline{w})\right)\frac{q_i}{(i-1)!}\right)\\
&+g_3(x,\omega)\sum_{k=0}^{+\infty}\frac{(2h(w))^{2k+\frac{3}{2}+\nu}\Gamma_{\text{inc}}\left(\frac{x}{2},2k+\frac{3}{2}+\nu\right)}{2^{2k+\nu}k!\Gamma(k+1+\nu)}
\end{aligned}
\tag{39}
$$

if $h(\omega)x \leq 7.5$, and

$$
\begin{aligned}
F(x,\omega) =& \frac{g_2(x,\omega)}{l(\omega)}\left(e^{-\overline{w}}+\sum_{i=1}^{+\infty}\frac{(-1)^i l^i(\omega)\left(\frac{e^{-\overline{w}}}{\overline{w}}\sum_{\substack{j=0,\\ i\geq 2}}^{i-2}\left(\frac{j!}{(-\overline{w})^j}\right)-E_1(\overline{w})\right)q_i}{(i-1)!}\right)\\
&+g_3(x,\omega)\sum_{k=0}^{+\infty}\frac{(2h(w))^{2k+\frac{3}{2}+\nu}\Gamma_{\text{inc}}\left(\frac{x}{2},2k+\frac{3}{2}+\nu\right)}{2^{2k+\nu}k!\Gamma(k+1+\nu)}
\end{aligned}
\tag{40}
$$

if $h(\omega)x > 7.5$. In (39) and (40), Γ_{inc} and E_1 are the so-called incomplete gamma function and exponential integral function (see Appendix B), respectively, and

$$\underline{w} = 7.5l(\omega), \quad \overline{w} = l(\omega)\, h(\omega)\, x \tag{41}$$

It is important to observe that both the incomplete gamma function and the exponential integral function can be evaluated very quickly on a personal computer using standard numerical algorithms, see e.g. Abramowitz and Stegun (1972, Section 6.5, pp. 260-266), Amos (1980); Cody (2006), Press et al. (2007, Section 6, pp. 255-265) and Kalitkin and Panin (2009).

4 A quasi-analytical approximate solution

The survival probability (9) can be obtained as the inverse Laplace transform of function F, namely

$$P(x,\tau) = \left(\mathcal{L}^{-1}F\right)(x,\tau) \tag{42}$$

This task is accomplished using the algorithm of Gaver-Stehfest, which is a very efficient approach for numerically inverting Laplace transforms (see, e.g., Gaver Jr., 1966; Stehfest, 1970; Kuznetsov, 2013). According to this method, the survival probability is approximated as follows:

$$P_{ap}(x,\tau) = \frac{\ln(2)}{\tau}\sum_{k=1}^{2M}\alpha_k^M F\left(x, \frac{k\ln(2)}{\tau}\right), \tag{43}$$

where:

$$\alpha_k^M = \frac{(-1)^{M+k}}{M!}\sum_{j=\lfloor (k+1)/2\rfloor}^{\min(k,M)} j^{M+1}\binom{M}{j}\binom{2j}{j}\binom{j}{k-j} \tag{44}$$

$\lfloor\cdot\rfloor$ being the integer part of $\cdot$ and M is a positive integer which controls the accuracy of the approximation obtained. In particular, the optimal values of M depend on the function being antitransformed. However, small values of M are often sufficient in order to obtain very accurate estimations of the antitransform. On the other hand, large values of M are not recommended because, as k in (43) increases, the terms of the sum (43) tend to increase in magnitude (with alternating sign), and round-off instabilities may occur (Davies and Martin, 1979; Kuhlman, 2013; Kuznetsov, 2013). In the numerical experiments performed in the present paper we set $M = 6$, which, by trial and error, yields the most accurate numerical results. By direct numerical simulation, we have found that for $M = 7$ and $M = 8$ the survival probability is computed with an error which is at most one order of magnitude larger than the error obtained using $M = 6$. Instead, for $M < 6$ or $M > 8$, the accuracy of formula (43) starts to degrade and, in particular, for $M \geq 10$ the survival probability is often calculated with not even two correct digits (due to the aforementioned numerical instabilities). Finally, we have experienced that the time required in order to obtain the survival probability increases (approximately) linearly with M, this meaning that all the terms in the sum (43) take (approximately) the same time to be computed.

Finally, for computational reasons, the series in (39) must be truncated at some positive integer, which in all the numerical simulations carried out in this work is chosen equal to 30. In fact, by numerical experiments we have found that keeping only the first thirty terms of each series appearing in (39) is sufficient to compute the Laplace transform F with essentially no error.

Remark 1 *Relation (43) is actually a quasi-analytical approximate solution for the survival probability of the model* *by Madan and Unal. We would like to remark that the analytical expression of the Laplace transform of the survival probability (relations (39) and (40) in the previous section) and the quasi-analytical approximation of the survival probability (relation (43)) are the main theoretical contributions of the paper.*

5 Validation of the Laplace transform approach

Let us evaluate the performances of the quasi-analytical approximation (43). To this aim, we compute the survival probability for different values of the parameters σ and c. In particular, the set of values employed include also $\sigma = 0.36633$ and $c = 0.003419$, which have been estimated in Madan and Unal (1998) from historical series of defaultable bond prices. Note that in the present paper the parameters σ and c are expressed in annual units. Moreover, as far as the volatility is concerned, following, for example, Hull (2009) (Section 11.3, p. 222), the decimal notation (rather than the percentage notation) is employed.

As far as the time to maturity τ is concerned, following Ballestra and Pacelli (2009), we consider both $\tau = 3$ months (Test Case 1.a, Table 1), $\tau = 1$ year (Test Case 1.b, Table 2), and $\tau = 10$ years (Test Case 1.c, Table 3). Moreover, for each test case, we compute the survival probability in correspondence of different values of x, which are chosen as in Ballestra and Pacelli (2009). For the reader's convenience, we recall that, according to (8), x measures the distance between the equity value and the default barrier.

Let P_{ap} denote the approximate survival probability obtained using the proposed formula (43), and let $RelErr$ denote the relative error of P_{ap}, which is evaluated as follows:

$$RelErr = \frac{|P_{ap}(x,\tau) - P(x,\tau)|}{P(x,\tau)} \tag{45}$$

Note that P, the exact solution of problem (10)-(12), is not available. Nevertheless, in order to measure the accuracy of the quasi-analytical approximate solution obtained in this paper, we need to compute P with a sufficiently large degree of exactness. Therefore, a very accurate estimation of P is obtained by finite difference approximation (as we can rely on the fact that the finite difference method, if used in conjunction with very refined meshes, allows one to compute P with, say, any desired accuracy). In particular, problem (10)-(12) is discretized using the popular Crank-Nicolson finite difference scheme, and a very large number of collocation nodes (100000) is employed in both the x and the τ directions. By trial and error, we have found that such a procedure allows us to evaluate P with at least 6 correct digits (let us also do mention that it takes some ten of minutes in order to obtain such a very accurate approximate solution).

Moreover, the computer time required in order to obtain P_{ap} is denoted as $CPUTime$. To provide a better understanding of the performances of the proposed method, we specify that all the numerical simulations are carried out on a computer Intel Core i7 CPU (2.3 GHz) and the software programs are written using Matlab 8. The results obtained are shown in Table 1, Table 2 and Table 3. In these tables, we also report results obtained using the finite difference method (FDM). In particular, in order to give an immediate comparison between formula (43) and the FDM, we directly show the computer time which is taken by the Crank-Nicolson finite difference scheme in order to achieve (approximately) the same error $RelErr$ that is obtained using the quasi-analytical approximation (43).

We observe that P_{ap} is a very accurate approximation of the survival probability. In fact, the relative errors are always of the order of 10^{-4} or 10^{-5}. Such a level of accuracy is considerable higher than the level of accuracy that is usually required in everyday business practice. Moreover, formula (43) allows us to compute the survival probability extremely fast, as $CPUTime$ is always of the order of 10^{-4} seconds.

Moreover, the quasi-analytical approximation proposed in the present paper is significantly more efficient than the finite difference method. In fact, to obtain the survival probability with (roughly) the same accuracy, formula (43) takes an extremely smaller time than the Crank-Nicolson scheme (more than one hundredth of times smaller).

Nevertheless, to make a fair comparison between the proposed quasi-analytical approximation and the finite difference approach, we shall also acknowledge what follows: while the quasi-analytical approximation yields, in one numerical simulation, the survival probability at only one point (x, τ), the finite difference method yields, in one numerical simulation, the survival probability on an entire grid of points in the (x, τ) domain. At the same time, however, the finite difference method requires one to generate a mesh, to store large arrays of data and to solve a set of linear systems, and thus it is usually considered (for instance, by practitioners) more difficult to implement than a direct quasi analytical formula such as formula (43).

Table 1 goes here.

Table 2 goes here.

Table 3 goes here.

Finally, in Figure 1 we report the survival probability obtained using formula (43) in Test Cases 1.a, 1.b and 1.c. As we may observe, the analytical formula (43) yields a very smooth approximation of the survival probability (with no spurious oscillations or spikes), which is always contained in the interval $[0, 1]$ (note that Figure 1 refers to the case $\sigma = 0.36633$ year$^{-1/2}$ and $c = 0.003419$ year^{-1}, but the results obtained using the other values of σ and c reported in the tables above are analogous).

Figure 1 goes here.

6 Pricing credit default swaps

The survival probability of a firm plays a crucial role in the pricing of credit default swaps (CDSs). Therefore, in this section we show how the quasi-analytical approximate solution (43) can be used to value CDSs, and, in particular, to compute CDS par spreads.

A CDS is a financial derivative issued by a protection seller that, under the payment of premiums, insures a protection buyer against the risk of default of a third (underlying) entity. Specifically, let us consider a risky bond issued by either a corporate or a sovereign borrower, with face value N and maturity T. Then, a CDS can be used to transfer credit risk from the bondholder (the protection buyer) to a trader (the protection seller) willing to take on the risk. In order to enter the contract, the protection buyer makes predetermined payments to the protection seller, until the possible default of the bond issuer or the maturity date of the bond, whichever comes first.

In the present paper, following a common approach (see, e.g., Madan and Schoutens, 2008; Schoutens and Cariboni, 2009; Hao et al., 2013) the flow of payments from the protection buyer to the protection seller is assumed to be continuous. In particular, in the time interval dt prior to maturity, the protection buyer pays (if the bond has not defaulted yet) an amount equal to $sNdt$, where s is the so-called CDS spread.

The risk that the bond defaults is modeled using the approach by Madan and Unal described in Section 2. Therefore, let us consider the process (1) where A denotes the discounted value of the equity of the reference entity issuing the risky bond. Then, the current value (at time $t = 0$) of the premium payments, i.e. the so-called premium leg, is given by:

$$sN \int_0^T e^{-r\tau} P\left(x_0, \tau\right) d\tau \tag{46}$$

where r is the risk-free interest rate, which is here assumed to be constant, P is the survival probability defined as in (9) and, according to (8), we set

$$x_0 = \ln\left(\frac{a_0}{\delta}\right) \tag{47}$$

a_0 being the value of A at time $t = 0$ and δ being the default threshold (see (4)).

Following a quite common approach (see, e.g., Duffie and Singleton, 1999; Pan and Singleton, 2008) we assume that in case of default, the buyer receives a fractional recovery of par at the default date. Specifically, the protection seller must pay to the bondholder $(1 - R)\, N$ where $R \in [0, 1]$ is the so-called recovery rate, which, according, for example, to Altman and Kishore (1996), Cathcart and El-Jahel (2003), Jankowitsch et al. (2008), Hao et al. (2013) and Madan (2014), is assumed to be constant. Therefore, the current value of the sum paid by the protection seller, the so-called default leg, is (see, e.g., Schoutens and Cariboni, 2009):

$$-\left(1 - R\right) N \int_0^T e^{-r\tau} dP\left(x_0, \tau\right) \tag{48}$$

Pricing the CDS amounts to finding the par spread $s = s^*(T)$ such that the premium leg is equal to the default leg (see, e.g., Schoutens and Cariboni, 2009):

$$s^*\left(T\right) = \frac{\left(1 - R\right)\left(-\int_0^T e^{-r\tau} dP\left(x_0, \tau\right)\right)}{\int_0^T e^{-r\tau} P\left(x_0, \tau\right) d\tau} \tag{49}$$

which upon integration by parts can be rewritten as follows:

$$s^*\left(T\right) = \left(1 - R\right)\left(\frac{1 - e^{-rT} P\left(x_0, T\right)}{\int_0^T e^{-r\tau} P\left(x_0, \tau\right) d\tau} - r\right) \tag{50}$$

The above par spread can be computed using the Laplace transform derived in Section 3. Indeed, setting

$$G\left(x_0, T\right) = \int_0^T e^{-r\tau} P\left(x_0, \tau\right) d\tau \tag{51}$$

and using Laplace transform properties analogous to the second of (15) and (16) we have

$$\left(\mathcal{L}G\right)\left(x_0, \omega\right) = \frac{1}{\omega} F\left(x_0, \omega + r\right) \tag{52}$$

where F is defined in (13). Then, using (42), (51) and (52), relation (50) can be rewritten as follows:

$$s^*(T) = (1-R)\left(\frac{1-e^{-rT}P(x_0,T)}{Q(x_0,T)} - r\right) \tag{53}$$

where Q is the inverse Laplace transform of (52).

In (53) the survival probability P is replaced with the quasi-analytical approximation (43). Moreover, using again the Gaver-Stephest algorithm to antitransform (52), a quasi-analytical approximate expression for the CDS par spread can be obtained:

$$s^*(T) \approx (1-R)\left(\frac{1-e^{-rT}P_{ap}(x_0,T)}{Q_{ap}(x_0,T)} - r\right) \tag{54}$$

where

$$Q_{ap}(x_0,T) = \sum_{k=1}^{2M} \frac{1}{k}\alpha_k^M F\left(x_0, \frac{k\ln(2)}{T} + r\right) \tag{55}$$

and α_k^M is defined as in (44).

7 Calibration to CDS spreads

Let us calibrate the model by Madan and Unal by fitting relation (54) to real market data. We proceed as follows. First of all, the recovery rate R is set to 0.4, which is a very typical value in the literature (see, e.g., Altman and Kishore, 1996; Jankowitsch et al., 2008; Madan, 2014). In order to investigate wether the model parameter are stable over time, the calibration is repeated at several different times, hereafter referred to as “calibration dates” (those reported in Tables 4-9).

The volatility σ needed in (53) is obtained as the historical volatility of the equity of the firm issuing the bond underlying the CDS considered. In particular, such a value is obtained as the variance of the daily returns on equity experienced in the five years prior to the calibration date considered. This yields an estimation of the daily volatility, from which an estimation of the annual volatility is obtained by considering 1 year = 252 days, see for example, the procedure outlined in Hull (2009, Section 17.1, pp. 372-373). We remark that we have also tried to use equity returns on a time period of two years (rather than five years), but the results obtained do not change substantially (the volatility values differ only for few percentage points).

Furthermore, the interest rate r is chosen equal to the 5-year Eurirs observed at the calibration date considered (let us recall that the Eurirs, or Euro Interest Rate Swap, is the fixed rate that European banks use to trade interest rate swaps among themselves).

Once that σ, r and R are available, the remaining model parameter c and δ (see equation (4)) are obtained by fitting realized CDS spreads. This is done using a calibration procedure analogous to the one proposed in Hao et al. (2013). Specifically, let $s^*_{emp}(T)$ denote the market value of the CDS spread observed at the calibration date considered, with time left to maturity T. Following Hao et al. (2013), let us consider a set of CDSs written on the same entity, with different maturities: $T_1 = 1$ year, $T_2 = 2$ years, $T_3 = 3$ years, $T_4 = 4$ years, $T_5 = 5$ years, $T_6 = 7$ years and $T_7 = 10$ years. The parameter c and the value δ are then obtained by minimizing the root mean squared error (RMSE) between the empirical CDS spreads and the theoretical ones:

$$\mathrm{RMSE}(c,\delta) = \sqrt{\frac{1}{7}\sum_{i=1}^{7}\left(s^*_{emp}(T_i) - s^*(T_i)\right)^2} \tag{56}$$

where $s^*(T_i)$ is evaluated using (54). We remark that RMSE is a function of the model parameter δ as it depends on the initial value x_0 (through the probability $P(x_0, \tau)$) which in turn depends on δ (see (47)). Note that RMSE also varies with the calibration date (at which the empirical data are collected), even if, to keep the notation simple, we do not explicitly indicate it.

The calibration of the model by Madan and Unal is done for six CDSs traded on the European market and written on firms whose equities are listed on the Frankfurt Stock Exchange (each of these firms has a different Moody's rating). As far as the calibration dates are concerned, we consider various different times in the years 2014 and 2015, which may be thought of as years of relative financial stability. In addition, we also choose some calibration dates in the year 2008, i.e. in the midst of the past worldwide financial crisis.

The result obtained are reported in Tables 4-9. Note that in these tables, in order to provide a better understanding of the fitting error with respect to the level of CDS spreads, we show a relative error RRMSE which is defined as follows:

$$\mathrm{RRMSE}(c, \delta) = \frac{\mathrm{RMSE}(c, \delta)}{\sqrt{\frac{1}{7}\sum_{i=1}^{7}\left(s^*_{emp}(T_i)\right)^2}} \tag{57}$$

As we can observe, in the years 2014 and 2015 the model fits the term structures of the CDSs considered particularly well. In fact, RRMSE is never greater than 11.29% (value experienced for Wal-Mart Stores Inc., see Table 4), and in most of the cases, is smaller than 10%. Moreover, always in the years 2014 and 2015, the parameters typical of the model by Madan and Unal, i.e. c and δ, exhibit a rather good stability over time. Actually, for some of the CDSs considered (namely William Cos Inc., Sealed Air and Safeway Inc., see Tables 6, 7 and 8), the parameter δ approximately doubles when passing from 31/01/2014 to 31/07/2015. Nevertheless, the quantity $\ln(\delta)$, i.e. the one that truly matters in the model by Madan and Unal (according to (4), the default intensity depends on the parameter δ only through $\ln(\delta)$), is rather stable. Finally, we also observe that the rates of variation of c and δ for the years 2014 and 2015 are roughly similar, for size, to the rates of variation of parameters that are experienced when calibrating other models for CDS pricing (see, e.g., Bianchi, 2012; Das and Hanouna, 2009).

Instead, for the year 2008, the agreement between theoretical and empirical data is worse than for the years 2014 and 2015. In fact, for the year 2008, RRMSE is, on average, around 10%, and ranges up to 16.41% (value experienced for Ford Motor Company, see Table 5). Therefore, for the year 2008 the model has a smaller explanatory power, which is presumably due to the following reason: in times of crisis, besides the equity value, there are also other factors that may significantly influence the default of a firm. For example, a variable that surely plaid a relevant role during the 2008 financial breakdown is the small liquidity of the markets. Now, as already noticed, in the model by Madan and Unal these "other factors" are somehow accounted for by the default intensity (4). Nevertheless, the reduced-form approach (4) provides only a broad description of the variables other than the equity value, and thus some discrepancy in pricing the risk of default occurs.

We shall also acknowledge, however, that the fitting errors obtained for the year 2008 can be deemed quite satisfactory anyway, due to the following reasons: first of all, the model by Madan and Unal is not so complex from the mathematical standpoint, and thus we have a nice trade-off between analytical simplicity and goodness-of-fit; second, the parameters which we have tuned in order to fit CDS spreads are only two. Note, indeed, that a better agreement could be achieved if not only c and δ, but also the recovery rate R (or even the volatility σ) were chosen such to minimize the error (56), see, e.g., Das and Hanouna (2009). Nevertheless, this is left as a future work as it goes beyond the scope of the present paper. Third, the errors that are often experienced when fitting CDS spreads are often as large as those which we obtain for the year 2008, or even much larger. For example, the errors in Bianchi (2012) are, on average, around 10%, whereas Koziol et al. (2015) reports that "...more than one half of CDS premia are mispriced by less than 20%, which is tolerable for various applications. However, the top 5% of deviations take very large values that are even above 100%.".

Furthermore, the value of c and δ obtained for the year 2008 exhibit, in some cases, significant variations and they are not, on the overall, as flat as those estimated for the years 2014 and 2015 (for example, for Sealed Air, at 31/01/2008 $c = 0.01118$ whereas at 31/10/2008 $c = 0.07490$, see Table 7). This clearly reflects the shock that hit the European economy in 2007-2008.

Table 4 goes here.

Table 5 goes here.

Table 6 goes here.

Table 7 goes here.

Table 8 goes here.

Table 9 goes here.

8 Conclusions

A quasi-analytical approximation of the survival probability in the model by Madan and Unal (1998) is presented. This formula, which is obtained using a Laplace-transform approach, is easy to implement and, as shown by several numerical experiments, also very accurate and fast. In addition, it is shown that the analytical expression obtained can also be used for pricing credit default swaps. Specifically, a quasi-analytical approximation of CDS par spreads is derived which can be used to estimate the parameters of the model by fitting term structures of empirical credit spreads. This is done in this paper and the results obtained reveal that the model by Madan and Unal provides a rather satisfactory description of real market data.

A Deriving formula (28)

According to Polyanin and Zaitsev (2003, Section 0.2.1-6, p. 46), the general solution of the non-homogeneous equation (17) is as follows:

$$F(x,\omega) = C_1 F_1(x,\omega) + C_2 F_2(x,\omega) + F_1(x,\omega) \int \frac{2F_2(x,\omega)\,dx}{\sigma^2 W(F_1(x,\omega), F_2(x,\omega))} - F_2(x,\omega) \int \frac{2F_1(x,\omega)\,dx}{\sigma^2 W(F_1(x,\omega), F_2(x,\omega))} \tag{58}$$

where F_1 and F_2 are given by (27) and C_1 and C_2 are arbitrary constants and

$$W(F_1, F_2) = F_1 \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial x} F_2 \tag{59}$$

is the so-called Wronskian determinant. According to Polyanin and Zaitsev (2003, Section S.2.6-4, p. 772), we have

$$W(F_1(x,\omega), F_2(x,\omega)) = -e^x \tag{60}$$

Substituting (27) and (60) in (58) we obtain

$$F(x,\omega) = F_1(x,\omega)\left(C_1 + \frac{2}{\sigma^2}\int e^{-\frac{x}{2}} x^{\frac{1}{2}} K_\nu(h(\omega)x)\,dx\right) + F_2(x,\omega)\left(C_2 - \frac{2}{\sigma^2}\int e^{-\frac{x}{2}} x^{\frac{1}{2}} I_\nu(h(\omega)x)\,dx\right) \tag{61}$$

so that, using elementary integral calculus, we can write

$$F(x,\omega) = F_1(x,\omega)\left(C_1 - \frac{2}{\sigma^2}\int_1^x e^{-\frac{z}{2}} z^{\frac{1}{2}} K_\nu(h(\omega)z)\,dz\right) + F_2(x,\omega)\left(C_2 + \frac{2}{\sigma^2}\int_1^x e^{-\frac{z}{2}} z^{\frac{1}{2}} I_\nu(h(\omega)z)\,dz\right) \tag{62}$$

Then, by imposing the boundary conditions (18), we obtain

$$C_1 = \frac{2}{\sigma^2}\int_1^{+\infty} e^{-\frac{z}{2}} z^{\frac{1}{2}} K_\nu(h(\omega)z)\,dz, \quad C_2 = \frac{2}{\sigma^2}\int_1^{0} e^{-\frac{z}{2}} z^{\frac{1}{2}} I_\nu(h(\omega)z)\,dz \tag{63}$$

so that, upon substitution of (63) in (62), relation (28) easily follows.

B Deriving formula (39)

Let us consider relation (35):

$$F(x,\omega) = g_1(x,\omega)\sum_{k=0}^{+\infty}\frac{\left(\frac{1}{2}\right)^{2k}}{k!}\left(\frac{I_1(k)}{2^{-\nu}\Gamma(k+1-\nu)} - \frac{I_2(k)}{2^{\nu}\Gamma(k+1+\nu)}\right) + g_2(x,\omega) I_4 + g_3(x,\omega)\sum_{k=0}^{+\infty}\frac{I_3(k)}{2^{2k+\nu}k!\Gamma(k+1+\nu)} \tag{64}$$

where

$$I_1\left(k\right)=\int_{h(\omega)x}^{7.5}y^{2k+\frac{1}{2}-\nu}e^{-\frac{y}{2h(\omega)}}dy,\quad I_2\left(k\right)=\int_{h(\omega)x}^{7.5}y^{2k+\frac{1}{2}+\nu}e^{-\frac{y}{2h(\omega)}}dy \tag{65}$$

$$I_3\left(k\right)=\int_{0}^{h(\omega)x}y^{2k+\frac{1}{2}+\nu}e^{-\frac{y}{2h(\omega)}}dy,\quad I_4=\int_{7.5}^{+\infty}e^{-l(\omega)y}\left(1+\sum_{i=1}^{+\infty}q_iy^{-i}\right)dy \tag{66}$$

Then, using the change of variable

$$m=\frac{y}{2h\left(\omega\right)} \tag{67}$$

the integrals $I_1\left(k\right)$ can be rewritten as follows:

$$I_1\left(k\right)=\left(2h\left(\omega\right)\right)^{2k+\frac{3}{2}-\nu}\int_{\frac{x}{2}}^{\frac{7.5}{2h(\omega)}}m^{2k+\frac{1}{2}-\nu}e^{-m}dm \tag{68}$$

so that

$$I_1\left(k\right)=\left(2h\left(\omega\right)\right)^{2k+\frac{3}{2}-\nu}\left(\Gamma_{\text{inc}}\left(\frac{7.5}{2h\left(\omega\right)},2k+\frac{3}{2}-\nu\right)-\Gamma_{\text{inc}}\left(\frac{x}{2},2k+\frac{3}{2}-\nu\right)\right) \tag{69}$$

where Γ_{inc} is the incomplete gamma function:

$$\Gamma_{\text{inc}}\left(x,\gamma\right)=\int_0^x m^{\gamma-1}e^{-m}dm \tag{70}$$

The integrals $I_2\left(k\right)$ and $I_3\left(k\right)$ can be treated analogously, which yields (the details are left to the reader):

$$\begin{aligned}F\left(x,\omega\right)=&g_1\left(x,\omega\right)\sum_{k=0}^{+\infty}\frac{\left(2h\left(w\right)\right)^{2k+\frac{3}{2}-\nu}\left(\Gamma_{\text{inc}}\left(\frac{7.5}{2h(\omega)},2k+\frac{3}{2}-\nu\right)-\Gamma_{\text{inc}}\left(\frac{x}{2},2k+\frac{3}{2}-\nu\right)\right)}{2^{2k-\nu}k!\Gamma\left(k+1-\nu\right)}\\&-g_1\left(x,\omega\right)\sum_{k=0}^{+\infty}\frac{\left(2h\left(w\right)\right)^{2k+\frac{3}{2}+\nu}\left(\Gamma_{\text{inc}}\left(\frac{7.5}{2h(\omega)},2k+\frac{3}{2}+\nu\right)-\Gamma_{\text{inc}}\left(\frac{x}{2},2k+\frac{3}{2}+\nu\right)\right)}{2^{2k+\nu}k!\Gamma\left(k+1+\nu\right)}\\&+g_2\left(x,\omega\right)I_4+g_3\left(x,\omega\right)\sum_{k=0}^{+\infty}\frac{\left(2h\left(w\right)\right)^{2k+\frac{3}{2}+\nu}\Gamma_{\text{inc}}\left(\frac{x}{2},2k+\frac{3}{2}+\nu\right)}{2^{2k+\nu}k!\Gamma\left(k+1+\nu\right)}\end{aligned} \tag{71}$$

The integral I_4 in (66) can be rewritten as follows:

$$I_4=\int_{7.5}^{+\infty}e^{-l(\omega)y}dy+\sum_{i=1}^{+\infty}q_i\int_{7.5}^{+\infty}y^{-i}e^{-l(\omega)y}dy \tag{72}$$

Using the change of variable

$$w=l\left(\omega\right)y \tag{73}$$

we obtain:

$$I_4=\frac{1}{l\left(\omega\right)}\int_{\underline{w}}^{+\infty}e^{-w}dw+\sum_{i=1}^{+\infty}q_il^{i-1}\left(\omega\right)\int_{\underline{w}}^{+\infty}w^{-i}e^{-w}dw \tag{74}$$

where $\underline{w}=7.5l\left(\omega\right)$. Using integration by parts, it is easy to show that

$$\int_{\underline{w}}^{+\infty}w^{-i}e^{-w}dw=\frac{\left(-1\right)^{i-1}}{\left(i-1\right)!}E_1\left(\underline{w}\right)+\frac{e^{-\underline{w}}}{\underline{w}^i\left(i-1\right)!}\sum_{\substack{j=1,\\ i\geq 2}}^{i-1}\left(-1\right)^{j+1}\underline{w}^j\left(i-1-j\right)! \tag{75}$$

where

$$E_1\left(\underline{w}\right) = \int_{\underline{w}}^{+\infty} \frac{e^{-w}}{w} dw \tag{76}$$

is the exponential integral function. Substitution of (75) in (74) yields:

$$I_4 = \frac{1}{l\left(\omega\right)} e^{-\underline{w}} + \sum_{i=1}^{+\infty} q_i l^{i-1}\left(\omega\right) \left(\frac{\left(-1\right)^{i-1}}{\left(i-1\right)!} E_1\left(\underline{w}\right) + \frac{e^{-\underline{w}}}{\underline{w}^i \left(i-1\right)!} \sum_{\substack{j=1,\\ i\geq 2}}^{i-1} \left(-1\right)^{j+1} \underline{w}^j \left(i-1-j\right)! \right) \tag{77}$$

Using simple algebra, the integral (77) can be rewritten as follows

$$I_4 = \frac{1}{l\left(\omega\right)} \left(e^{-\underline{w}} + \sum_{i=1}^{+\infty} \frac{\left(-1\right)^i q_i l^i\left(\omega\right) \left(\frac{e^{-\underline{w}}}{\underline{w}} \sum_{\substack{j=0,\\ i\geq 2}}^{i-2} \frac{j!}{\left(-\underline{w}\right)^j} - E_1\left(\underline{w}\right) \right)}{\left(i-1\right)!} \right) \tag{78}$$

and thus by substituting (78) in (71) we obtain (39). Analogous calculations allow us to obtain (40).

			$x = 0.1$	$x = 0.3$	$x = 0.5$
$\sigma = 0.2$	$c = 0.0001$	$P_{ap}(x,t)$	0.664179	0.996519	0.999885
		$RelErr$	2.28×10^{-6}	2.96×10^{-5}	5.98×10^{-6}
		$CPUTime$	$7.02 \times 10^{-4}\ s$	$7.09 \times 10^{-4}\ s$	$7.09 \times 10^{-4}\ s$
		$CPUTime$(FDM)	$1.5 \times 10^{-1}\ s$	$1.5 \times 10^{-1}\ s$	$1.5 \times 10^{-1}\ s$
	$c = 0.003419$	$P_{ap}(x,t)$	0.593848	0.984269	0.996297
		$RelErr$	1.30×10^{-4}	2.62×10^{-5}	7.77×10^{-6}
		$CPUTime$	$7.02 \times 10^{-4}\ s$	$7.09 \times 10^{-4}\ s$	$7.09 \times 10^{-4}\ s$
		$CPUTime$(FDM)	$1.5 \times 10^{-1}\ s$	$1.5 \times 10^{-1}\ s$	$1.5 \times 10^{-1}\ s$
	$c = 0.01$	$P_{ap}(x,t)$	0.493836	0.960973	0.989223
		$RelErr$	5.41×10^{-5}	1.66×10^{-5}	1.01×10^{-5}
		$CPUTime$	$7.02 \times 10^{-4}\ s$	$7.09 \times 10^{-4}\ s$	$7.09 \times 10^{-4}\ s$
		$CPUTime$(FDM)	$1.5 \times 10^{-1}\ s$	$1.5 \times 10^{-1}\ s$	$1.5 \times 10^{-1}\ s$
$\sigma = 0.36633$	$c = 0.0001$	$P_{ap}(x,t)$	0.385026	0.882216	0.991760
		$RelErr$	1.57×10^{-5}	2.43×10^{-4}	1.97×10^{-5}
		$CPUTime$	$7.02 \times 10^{-4}\ s$	$7.09 \times 10^{-4}\ s$	$7.09 \times 10^{-4}\ s$
		$CPUTime$(FDM)	$1.5 \times 10^{-1}\ s$	$1.5 \times 10^{-1}\ s$	$1.5 \times 10^{-1}\ s$
	$c = 0.003419$	$P_{ap}(x,t)$	0.361489	0.866956	0.986917
		$RelErr$	1.58×10^{-4}	3.87×10^{-5}	8.85×10^{-5}
		$CPUTime$	$7.02 \times 10^{-4}\ s$	$7.09 \times 10^{-4}\ s$	$7.09 \times 10^{-4}\ s$
		$CPUTime$(FDM)	$1.5 \times 10^{-1}\ s$	$1.5 \times 10^{-1}\ s$	$1.5 \times 10^{-1}\ s$
	$c = 0.01$	$P_{ap}(x,t)$	0.322611	0.839607	0.977228
		$RelErr$	1.77×10^{-4}	2.54×10^{-4}	1.05×10^{-4}
		$CPUTime$	$7.02 \times 10^{-4}\ s$	$7.09 \times 10^{-4}\ s$	$7.09 \times 10^{-4}\ s$
		$CPUTime$(FDM)	$1.5 \times 10^{-1}\ s$	$1.5 \times 10^{-1}\ s$	$1.5 \times 10^{-1}\ s$
$\sigma = 0.5$	$c = 0.0001$	$P_{ap}(x,t)$	0.276656	0.733495	0.941621
		$RelErr$	8.23×10^{-6}	2.25×10^{-5}	1.23×10^{-4}
		$CPUTime$	$7.02 \times 10^{-4}\ s$	$7.09 \times 10^{-4}\ s$	$7.09 \times 10^{-4}\ s$
		$CPUTime$(FDM)	$1.5 \times 10^{-1}\ s$	$1.5 \times 10^{-1}\ s$	$1.5 \times 10^{-1}\ s$
	$c = 0.003419$	$P_{ap}(x,t)$	0.265110	0.722009	0.936146
		$RelErr$	9.01×10^{-4}	1.10×10^{-3}	2.81×10^{-4}
		$CPUTime$	$7.02 \times 10^{-4}\ s$	$7.09 \times 10^{-4}\ s$	$7.09 \times 10^{-4}\ s$
		$CPUTime$(FDM)	$1.5 \times 10^{-1}\ s$	$1.5 \times 10^{-1}\ s$	$1.5 \times 10^{-1}\ s$
	$c = 0.01$	$P_{ap}(x,t)$	0.244788	0.700548	0.924142
		$RelErr$	1.68×10^{-4}	4.39×10^{-5}	5.87×10^{-4}
		$CPUTime$	$7.02 \times 10^{-4}\ s$	$7.09 \times 10^{-4}\ s$	$7.09 \times 10^{-4}\ s$
		$CPUTime$(FDM)	$1.5 \times 10^{-1}\ s$	$1.5 \times 10^{-1}\ s$	$1.5 \times 10^{-1}\ s$

Table 1: Test Case 1.a ($\tau = 3$ months).

			$x = 0.1$	$x = 0.3$	$x = 1.0$
$\sigma = 0.2$	$c = 0.0001$	$P_{ap}(x,t)$	0.349843	0.843351	0.999882
		$RelErr$	4.51×10^{-5}	1.68×10^{-4}	7.71×10^{-6}
		$CPUTime$	$7.02 \times 10^{-4}\ s$	$7.09 \times 10^{-4}\ s$	$7.09 \times 10^{-4}\ s$
		$CPUTime$(FDM)	$1.5 \times 10^{-1}\ s$	$1.5 \times 10^{-1}\ s$	$1.5 \times 10^{-1}\ s$
	$c = 0.003419$	$P_{ap}(x,t)$	0.283917	0.790570	0.996251
		$RelErr$	1.71×10^{-4}	7.43×10^{-5}	9.94×10^{-6}
		$CPUTime$	$7.12 \times 10^{-4}\ s$	$7.21 \times 10^{-4}\ s$	$6.21 \times 10^{-4}\ s$
		$CPUTime$(FDM)	$1.5 \times 10^{-1}\ s$	$1.5 \times 10^{-1}\ s$	$1.5 \times 10^{-1}\ s$
	$c = 0.01$	$P_{ap}(x,t)$	0.178015	0.739602	0.980558
		$RelErr$	8.30×10^{-5}	2.49×10^{-4}	1.28×10^{-5}
		$CPUTime$	$7.02 \times 10^{-4}\ s$	$7.09 \times 10^{-4}\ s$	$7.09 \times 10^{-4}\ s$
		$CPUTime$(FDM)	$1.5 \times 10^{-1}\ s$	$1.5 \times 10^{-1}\ s$	$1.5 \times 10^{-1}\ s$
$\sigma = 0.36633$	$c = 0.0001$	$P_{ap}(x,t)$	0.177009	0.523283	0.989540
		$RelErr$	9.39×10^{-6}	3.90×10^{-5}	3.69×10^{-6}
		$CPUTime$	$7.02 \times 10^{-4}\ s$	$7.09 \times 10^{-4}\ s$	$7.09 \times 10^{-4}\ s$
		$CPUTime$(FDM)	$1.5 \times 10^{-1}\ s$	$1.5 \times 10^{-1}\ s$	$1.5 \times 10^{-1}\ s$
	$c = 0.003419$	$P_{ap}(x,t)$	0.160673	0.499600	0.984303
		$RelErr$	1.55×10^{-4}	9.92×10^{-5}	3.61×10^{-5}
		$CPUTime$	$7.12 \times 10^{-4}\ s$	$7.21 \times 10^{-4}\ s$	$6.21 \times 10^{-4}\ s$
		$CPUTime$(FDM)	$1.5 \times 10^{-1}\ s$	$1.5 \times 10^{-1}\ s$	$1.5 \times 10^{-1}\ s$
	$c = 0.01$	$P_{ap}(x,t)$	0.135103	0.459339	0.974208
		$RelErr$	1.62×10^{-4}	7.59×10^{-5}	1.09×10^{-4}
		$CPUTime$	$7.02 \times 10^{-4}\ s$	$7.09 \times 10^{-4}\ s$	$7.09 \times 10^{-4}\ s$
		$CPUTime$(FDM)	$1.5 \times 10^{-1}\ s$	$1.5 \times 10^{-1}\ s$	$1.5 \times 10^{-1}\ s$
$\sigma = 0.5$	$c = 0.0001$	$P_{ap}(x,t)$	0.119156	0.369622	0.926503
		$RelErr$	5.08×10^{-6}	2.89×10^{-5}	1.79×10^{-5}
		$CPUTime$	$7.02 \times 10^{-4}\ s$	$7.09 \times 10^{-4}\ s$	$7.09 \times 10^{-4}\ s$
		$CPUTime$(FDM)	$1.5 \times 10^{-1}\ s$	$1.5 \times 10^{-1}\ s$	$1.5 \times 10^{-1}\ s$
	$c = 0.003419$	$P_{ap}(x,t)$	0.112036	0.357429	0.882173
		$RelErr$	9.69×10^{-5}	4.94×10^{-5}	8.10×10^{-4}
		$CPUTime$	$7.12 \times 10^{-4}\ s$	$7.21 \times 10^{-4}\ s$	$6.21 \times 10^{-4}\ s$
		$CPUTime$(FDM)	$1.5 \times 10^{-1}\ s$	$1.5 \times 10^{-1}\ s$	$1.5 \times 10^{-1}\ s$
	$c = 0.01$	$P_{ap}(x,t)$	0.099880	0.335637	0.907967
		$RelErr$	1.71×10^{-4}	7.85×10^{-5}	3.93×10^{-4}
		$CPUTime$	$7.02 \times 10^{-4}\ s$	$7.09 \times 10^{-4}\ s$	$7.09 \times 10^{-4}\ s$
		$CPUTime$(FDM)	$1.5 \times 10^{-1}\ s$	$1.5 \times 10^{-1}\ s$	$1.5 \times 10^{-1}\ s$

Table 2: Test Case 1.b ($\tau = 1$ year).

			$x = 0.6$	$x = 1.0$	$x = 5.0$
$\sigma = 0.2$	$c = 0.0001$	$P_{ap}(x,t)$	0.546453	0.816622	0.999957
		$RelErr$	3.40×10^{-5}	1.47×10^{-4}	4.16×10^{-7}
		$CPUTime$	7.02×10^{-4} s	7.09×10^{-4} s	7.09×10^{-4} s
		$CPUTime$(FDM)	1.5×10^{-1} s	1.5×10^{-1} s	1.5×10^{-1} s
	$c = 0.003419$	$P_{ap}(x,t)$	0.479364	0.764244	0.998538
		$RelErr$	3.28×10^{-5}	1.24×10^{-4}	2.77×10^{-7}
		$CPUTime$	7.12×10^{-4} s	7.21×10^{-4} s	6.21×10^{-4} s
		$CPUTime$(FDM)	1.5×10^{-1} s	1.5×10^{-1} s	1.5×10^{-1} s
	$c = 0.01$	$P_{ap}(x,t)$	0.387399	0.682195	0.995730
		$RelErr$	9.34×10^{-5}	2.12×10^{-4}	1.46×10^{-6}
		$CPUTime$	7.02×10^{-4} s	7.09×10^{-4} s	7.09×10^{-4} s
		$CPUTime$(FDM)	1.5×10^{-1} s	1.5×10^{-1} s	1.5×10^{-1} s
$\sigma = 0.36633$	$c = 0.0001$	$P_{ap}(x,t)$	0.226708	0.408389	0.999702
		$RelErr$	2.95×10^{-5}	4.97×10^{-5}	7.88×10^{-5}
		$CPUTime$	7.02×10^{-4} s	7.09×10^{-4} s	7.09×10^{-4} s
		$CPUTime$(FDM)	1.5×10^{-1} s	1.5×10^{-1} s	1.5×10^{-1} s
	$c = 0.003419$	$P_{ap}(x,t)$	0.210616	0.388423	0.997948
		$RelErr$	4.61×10^{-5}	5.40×10^{-5}	7.31×10^{-5}
		$CPUTime$	7.12×10^{-4} s	7.21×10^{-4} s	6.21×10^{-4} s
		$CPUTime$(FDM)	1.5×10^{-1} s	1.5×10^{-1} s	1.5×10^{-1} s
	$c = 0.01$	$P_{ap}(x,t)$	0.184530	0.354745	0.994485
		$RelErr$	4.11×10^{-5}	6.59×10^{-5}	6.15×10^{-5}
		$CPUTime$	7.02×10^{-4} s	7.09×10^{-4} s	7.09×10^{-4} s
		$CPUTime$(FDM)	1.5×10^{-1} s	1.5×10^{-1} s	1.5×10^{-1} s
$\sigma = 0.5$	$c = 0.0001$	$P_{ap}(x,t)$	0.119854	0.226610	0.985602
		$RelErr$	9.21×10^{-7}	7.40×10^{-5}	2.72×10^{-4}
		$CPUTime$	7.02×10^{-4} s	7.09×10^{-4} s	7.09×10^{-4} s
		$CPUTime$(FDM)	1.5×10^{-1} s	1.5×10^{-1} s	1.5×10^{-1} s
	$c = 0.003419$	$P_{ap}(x,t)$	0.113995	0.218369	0.982908
		$RelErr$	1.78×10^{-5}	8.97×10^{-5}	2.66×10^{-4}
		$CPUTime$	7.12×10^{-4} s	7.21×10^{-4} s	6.21×10^{-4} s
		$CPUTime$(FDM)	1.5×10^{-1} s	1.5×10^{-1} s	1.5×10^{-1} s
	$c = 0.015$	$P_{ap}(x,t)$	0.103820	0.203756	0.977661
		$RelErr$	1.38×10^{-5}	8.38×10^{-5}	2.55×10^{-4}
		$CPUTime$	7.02×10^{-4} s	7.09×10^{-4} s	7.09×10^{-4} s
		$CPUTime$(FDM)	1.5×10^{-1} s	1.5×10^{-1} s	1.5×10^{-1} s

Table 3: Test Case 1.c ($\tau = 10$ years).

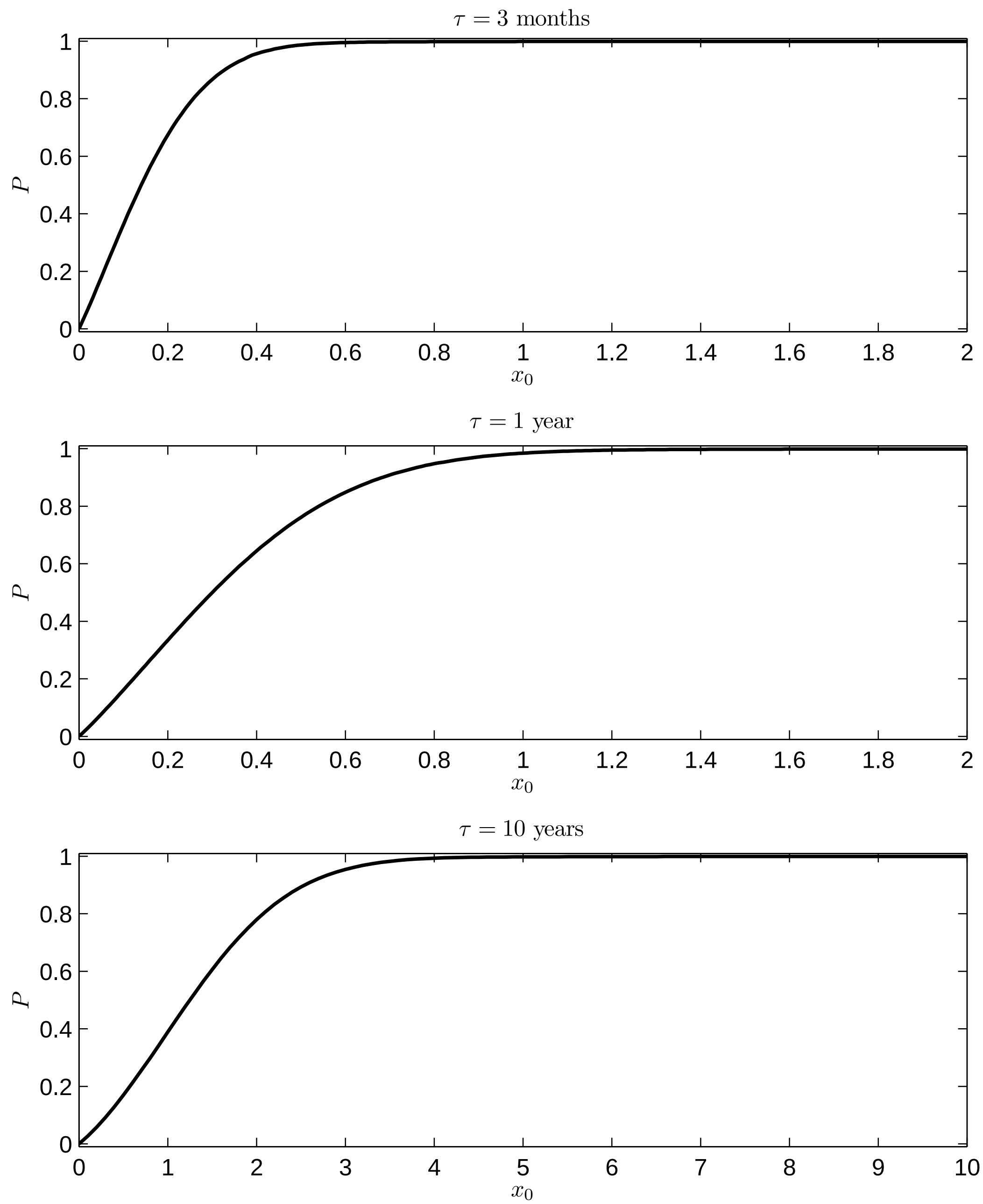

Figure 1: Survival Probability, $c = 0.003419$, $\sigma = 0.36633$.

Wal-Mart Stores Inc. (Moody's rating: Aa2)						
Calibration date	σ	a_0	δ	$\ln(\delta)$	c	RRMSE
31/07/2015	0.1790	64.78	19.32	2.96	0.00184	7.61%
30/04/2015	0.1762	69.87	21.11	3.05	0.00173	3.73%
30/01/2015	0.1752	76.53	22.64	3.12	0.00153	6.04%
31/10/2014	0.1665	61.45	19.75	2.98	0.00146	7.96%
31/07/2014	0.1678	55.16	17.91	2.88	0.00151	8.59%
30/04/2014	0.1719	57.43	18.51	2.91	0.00149	11.29%
31/01/2014	0.1728	54.70	17.92	2.89	0.00160	10.66%
31/10/2008	0.2023	44.44	7.28	1.99	0.04080	6.95%
31/07/2008	0.1972	37.88	6.48	1.87	0.01459	9.79%
30/04/2008	0.2014	37.80	6.74	1.90	0.01164	11.16%
31/01/2008	0.2069	33.97	5.52	1.71	0.01658	8.91%

Table 4: Calibration results.

Ford Motor Company (Moody's rating: Baa3)						
Calibration date	σ	a_0	δ	$\ln(\delta)$	c	RRMSE
31/07/2015	0.2171	13.41	3.80	1.33	0.00650	8.95%
30/04/2015	0.2241	14.35	4.02	1.38	0.00600	9.01%
30/01/2015	0.2318	13.13	4.58	1.52	0.00650	7.49%
31/10/2014	0.2320	10.89	3.73	1.32	0.00700	7.31%
31/07/2014	0.2376	12.99	3.41	1.23	0.00780	10.60%
30/04/2014	0.2498	11.80	3.26	1.18	0.00794	9.54%
31/01/2014	0.2578	11.16	3.09	1.13	0.00851	4.95%
31/10/2008	0.4684	18.20	1.21	0.19	0.02060	1.10%
31/07/2008	0.3868	24.31	1.44	0.37	0.05730	3.79%
30/04/2008	0.3498	29.75	1.36	0.30	0.05723	16.41%
31/01/2008	0.3466	32.00	1.23	0.21	0.05939	16.04%

Table 5: Calibration results.

Williams Cos Inc. (Moody's rating: Ba1)						
Calibration date	σ	a_0	δ	$\ln(\delta)$	c	RRMSE
31/07/2015	0.2673	48.13	15.85	2.76	0.00408	7.32%
30/04/2015	0.2684	45.44	14.16	2.65	0.00427	6.64%
30/01/2015	0.2629	38.2	14.77	2.69	0.00419	5.21%
31/10/2014	0.2506	43.50	14.44	2.67	0.00358	3.89%
31/07/2014	0.2498	41.43	13.01	2.57	0.00351	3.53%
30/04/2014	0.2522	30.35	9.79	2.28	0.00358	4.08%
31/01/2014	0.2904	29.46	8.27	2.11	0.00419	6.02%
31/10/2008	0.3074	12.21	2.14	0.76	0.09400	13.74%
31/07/2008	0.2784	17.15	2.61	0.95	0.02800	12.66%
30/04/2008	0.2950	18.77	2.59	0.95	0.03270	13.46%
31/01/2008	0.3462	17.30	1.83	0.61	0.04120	8.04%

Table 6: Calibration results.

Sealed Air (Moody's rating: Ba3)						
Calibration date	σ	a_0	δ	$\ln(\delta)$	c	RRMSE
31/07/2015	0.2959	48.72	13.49	2.59	0.00430	4.17%
30/04/2015	0.2953	39.30	11.05	2.39	0.00432	3.58%
30/01/2015	0.2911	35.57	10.75	2.36	0.00453	5.99%
31/10/2014	0.2820	27.75	8.79	2.15	0.00443	7.19%
31/07/2014	0.2776	24.18	7.73	2.02	0.00478	4.08%
30/04/2014	0.3049	23.37	6.16	1.81	0.00434	4.85%
31/01/2014	0.3274	22.96	5.87	1.75	0.00409	4.64%
31/10/2008	0.2739	12.66	2.27	0.99	0.07490	13.99%
31/07/2008	0.2505	13.34	3.62	1.28	0.06250	13.06%
30/04/2008	0.2343	17.27	3.95	1.29	0.02050	12.27%
31/01/2008	0.2238	16.46	4.49	1.50	0.01118	13.15%

Table 7: Calibration results.

Safeway Inc. (Moody's rating: B1)						
Calibration date	σ	a_0	δ	$\ln(\delta)$	c	RRMSE
31/07/2015	0.2576	30.92	8.32	2.11	0.00877	8.60%
30/04/2015	0.2680	30.92	8.09	2.09	0.00857	9.89%
30/01/2015	0.2742	30.92	8.29	2.12	0.00869	6.08%
31/10/2014	0.2758	27.66	7.69	2.04	0.01220	6.38%
31/07/2014	0.2812	25.79	6.28	1.90	0.01057	5.32%
30/04/2014	0.2868	24.42	6.34	1.85	0.01118	6.11%
31/01/2014	0.2872	20.67	4.52	1.51	0.01468	11.18%
31/10/2008	0.2548	14.91	1.32	0.27	0.06300	11.21%
31/07/2008	0.2713	15.09	1.70	0.53	0.03587	11.98%
30/04/2008	0.2685	17.90	1.93	0.66	0.03082	12.79%
31/01/2008	0.2864	18.44	1.86	0.62	0.03960	12.25%

Table 8: Calibration results.

MGM Resorts International (Moody's rating: B2)						
Calibration date	σ	a_0	δ	$\ln(\delta)$	c	RRMSE
31/07/2015	0.3885	17.85	3.70	1.31	0.03345	7.16%
30/04/2015	0.4075	18.93	3.54	1.26	0.03219	5.93%
30/01/2015	0.4210	17.43	3.41	1.23	0.03342	6.72%
31/10/2014	0.4325	18.50	3.03	1.11	0.03049	3.62%
31/07/2014	0.4738	20.09	2.81	1.03	0.02729	6.03%
30/04/2014	0.5109	17.99	1.99	0.69	0.02930	5.52%
31/01/2014	0.5632	17.76	5.32	1.67	0.03371	6.05%
31/10/2008	0.5184	11.80	4.47	1.50	0.09978	2.69%
31/07/2008	0.4095	18.90	5.33	1.67	0.06503	6.42%
30/04/2008	0.3543	32.80	7.92	2.07	0.06118	7.94%
31/01/2008	0.3490	48.00	7.15	1.97	0.12075	12.90%

Table 9: Calibration results.

References

Abínzano, I., Seco, L., Escobar, M. and Olivares, P., Single and double Black-Cox: Two approaches for modelling debt restructuring. *Econ. Model.*, 2009, **26**(5):910–917.

Abramowitz, M. and Stegun, A., *Handbook of Mathematical Functions*, 1972 (Dover Publications: New York, NY).

Altman, E.I. and Kishore, V.M., Almost everything you wanted to know about recoveries on defaulted bonds. *Financ. Anal. J.*, 1996, **52**(6):57–64.

Amos, D.E., Computation of exponential integrals. *ACM T. Math. Software*, 1980, **6**(3):365–377.

Anderson, R.W. and Sundaresan, S., Design and valuation of debt contracts. *Rev. Financ. Stud.*, 1996, **9**(1):37–68.

Ballestra, L.V. and Pacelli, G., A numerical method to price defaultable bonds based on the Madan and Unal credit risk model. *Appl. Math. Finance*, 2009, **16**(1):17–36.

Ballestra, L.V. and Pacelli, G., Valuing risky debt: A new model combining structural information with the reduced-form approach. *Ins.: Mathematics Econ.*, 2014, **55**(1):261–271.

Bäuerle, N., Risk management in credit risk portfolios with correlated assets. *Ins.: Mathematics Econ.*, 2002, **30**(2): 187–198.

Bernard, C., Courtois, O.L. and Quittard-Pinon F., Market value of life insurance contracts under stochastic interest rates and default risk. *Ins.: Mathematics Econ.*, 2005, **36**(3):499–516.

Bianchi, M.L., An empirical comparison of alternative credit default swap pricing models. Working Paper n. 882, Banca d'Italia, Rome, Italy, 2012.

Björk, T., *Arbitrage Theory in Continuous Time*, 2009 (Oxford University Press: New York, NY).

Black, F. and Cox, J.C., Valuing corporate securities: Some effects of bond indenture provisions. *J. Finance*, 1976, **31**(2):351–367.

Briys, E. and de Varenne, F., Valuing risky fixed rate debt: An extension. *J. Finan. Quant. Anal.*, 1997, **32**(2): 239–248.

Campbell, J.Y. and Taksler, G.B., Equity volatility and corporate bond yields. *J. Finance*, 2003, **58**(6):2321–2349.

Cathcart, L. and El-Jahel L., Semi-analytical pricing of defaultable bonds in a signaling jump-default model. *J. of Comp. Finan.*, 2003, **6**(3):91–108.

Cathcart, L. and El-Jahel L., Pricing defaultable bonds: A middle-way approach between structural and reduced-form models. *Quant. Finance*, 2006, **6**(3):243–253.

Chen, C.-J. and Panjer, H., Unifying discrete structural models and reduced-form models in credit risk using a jump-diffusion process. *Ins.: Mathematics Econ.*, 2003, **33**(2):357–380.

Cody, W.J., An overview of software development for special functions. In *Lecture Notes in Mathematics, 506*, edited by G. A. Watson, pp. 38–48, 2006, (Springer: Berlin).

Crouhy, M., Galai, D. and Mark, R., A comparative analysis of current credit risk models. *J. Bank. Finance*, 2000, **24**(1):59–117.

Das, S.R. and Hanouna P., Implied recovery. *J. Econ. Dyn. Control*, 2009, **33**(11):1837–1857.

Davies, B. and Martin, B., Numerical inversion of the Laplace transform: A survey and comparison of methods. *J. of Comp. Physics*, 1979, **33**(1):1–32.

Duffee, G.R., Estimating the price of default risk. *Rev. Financ. Stud.*, 1999, **12**(1):197–226.

Duffie, D. and Lando, D., Term structures of credit spreads with incomplete accounting information. *Econometrica*, 2001, **69**(3):633–664.

Duffie, D. and Singleton, K.J., Modeling term structures of defaultable bonds. *Rev. of Finan. Studies*, 1999, **12**(4): 687–720.

Elliott, R.J. and Kopp, P.E., *Mathematics of Financial Markets*, 2005 (Springer: Berlin).

Feng, R. and Volkmer, H.W., Modeling credit value adjustment with downgrade-triggered termination clause using a ruin theoretic approach. *Ins.: Mathematics Econ.*, 2012, **51**(2):409–421.

Fontana, C. and Montes, J.M.A., A unified approach to pricing and risk management of equity and credit risk. *J. Comput. Appl. Math.*, 2014, **259**:350–361.

Franks, J.R., and Torous, W., An empirical investigation of U.S. firms in reorganization. *J. Finance*, 1989, **44**(3): 747–769.

Giesecke, K., Default and information. *J. Econ. Dyn. Control*, 2006, **30**(11):2281–2303.

Grundke, P. and Riedel, K.O., Pricing the risks of default: A note on Madan and Unal. *Rev. Deriv. Res.*, 2004, **7**(2): 169–173.

Hao, X., Li, X. and Shimizu, Y., Finite-time survival probability and credit default swaps pricing under geometric Lévy markets. *Ins.: Mathematics Econ.*, 2013, **53**(1):14–23.

Hull, J., *Options, Futures and Other Derivatives*, 2009 (Pearson Prentice Hall: Upper Saddle River, NJ).

Jankowitsch, R., Pullirsch, R. and Veža, T., The delivery option in credit default swaps. *J. Bank. Finance*, 2008, **32** (7):1269–1285.

Jones, E.P., Mason, S.P. and Rosenfeld, E., Contingent claims analysis of corporate capital structures: An empirical investigation. *J. Finance*, 1984, **39**(3):611–625.

Gaver Jr., D.P., Observing stochastic processes, and approximate transform inversion. *Oper. Res.*, 1966, **14**(3): 444–459.

Kalitkin, N.N. and Panin, I.A., On the computation of the exponential integral. *Math. Mod. Comp. Simulations*, 2009, **1**(1):88–90.

Kim D.H. and Stock, D., The effect of interest rate volatility and equity volatility on corporate bond yield spreads: A comparison of noncallables and callables. *J. Corp. Finance*, 2014, **26**(June):20–35.

Koziol, C., Koziol, P., and Schön, T., Do correlated defaults matter for CDS premia? An empirical analysis. *Rev. Deriv. Res.*, 2015, **18**(3):191–224.

Kuhlman, K.L., Review of inverse Laplace transform algorithms for Laplace-space numerical approaches. *Numer. Algorithms*, 2013, **63**(2):339–355.

Kuznetsov, A., On the convergence of the Gaver-Stehfest algorithm. *SIAM J. Numer. Analysis*, 2013, **51**(6):2984–2998.

Leland, H.E. and Toft, K.B., Optimal capital structure, endogenous bankruptcy, and the term structure of credit spreads. *J. Finance*, 1996, **51**(3):987–1019.

Longstaff, F.A. and Schwartz, E.S., A simple approach to valuing risky fixed and floating rate debt. *J. Finance*, 1995, **50**(3):789–819.

Luciano, E., Calibrating risk-neutral default correlation. *J. Risk Finance*, 2007, **8**(5):450–464.

Madan, D.B., Modeling and monitoring risk acceptability in markets: The case of the credit default swap market. *J. Bank. Finance*, 2014, **47**:63–73.

Madan, D.B. and Schoutens, W., Break on through to the single side. *J. Credit Risk*, 2008, **4**(3):3–20.

Madan, D.B. and Unal H., Pricing the risk of default. *Rev. Deriv. Res.*, 1998, **2**(2):121–160.

Madan, D.B. and Unal H., A two-factor hazard rate model for pricing risky debt and the term structure of credit spreads. *J. Financ. Quant. Analysis*, 2000, **35**(1):43–65.

Merton, R.C., On the pricing of corporate debt: The risk structure of interest rates. *J. Finance*, 1974, **29**(2):449–470.

Olver, F.W.J. and Maximon, L.C., Bessel functions. In *NIST Handbook of Mathematical Functions*, edited by F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, pp. 217–275, 2010 (Cambridge University Press: New York, NY).

Pan, J. and Singleton, K.J., Default and recovery implicit in the term structure of sovereign CDS spreads. *J. Finance*, 2008, **63**(5):2345–2384.

Polyanin, A.D. and Zaitsev, V.F., *Handbook of Exact Solutions for Ordinary Differential Equations*, 2003 (Chapman & Hall/CRC press: Boca Raton, Florida).

Poularikas, A.D., *The Handbook of Formulas and Tables for Signal Processing*, 1999 (CRC Press, Boca Raton, Florida).

Press, W.H., Flannery, B.P., Teukolsky, S.A. and Vetterling, W.T., *Numerical Recipes: The Art of Scientific Computing*, 2007 (Cambridge University Press: New York, NY).

Schoutens, W. and Cariboni, J., *Lévy Processes in Credit Risk*, 2009, (Wiley: West Sussex, UK).

Sepp, A., Extended creditgrades model with stochastic volatility and jumps. *Wilmott Magazine*, 2006, September: 50–62.

Stehfest, H., Algorithm 368: numerical inversion of Laplace transform. *Commun. ACM*, 1970, **13**(1):47–49.

Tang, K.T., *Mathematical Methods for Engineers and Scientists 2 - Vector Analysis, Ordinary Differential Equations and Laplace Transforms*, 2007, (Springer: Berlin).

Zhou, C., The term structure of credit spreads with jump risk. *J. Bank. Finance*, 2001, **25**(11):2015–2040.