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Walrasian versus Cournot behavior in an oligopoly of boundedly rational firms

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Abstract An evolutionary oligopoly game, where firms can select between the *best-reply* rule and the *Walrasian* rule, is considered. The industry is characterized by a finite number of *ex-ante* homogeneous firms that, characterized by naïve expectations, decide next-period output by employing one of the two behavioral rules. The inverse demand function is linear and all firms have the same quadratic and convex cost function (decreasing return to scale). Based upon realized profits, the distribution of behavioral rules is updated according to a replicator dynamics. The model is characterized by two equilibria: the Cournot-Nash equilibrium, where all firms adopt the best-reply rule and produce the Cournot-Nash quantity, and the Walrasian equilibrium, where all firms adopt the Walrasian rule and produce the Walrasian quantity.

The analysis reveals that the Walrasian equilibrium is globally stable as long as the rate of change of marginal cost exceeds the sum of residual market price sensitivities to output. If not, the Walrasian equilibrium loses stability and an attractor, representing complicated dynamics with *evolutionary stable heterogeneity*, arises through a bifurcation. As the propensity of firms to select the more profitable behavioral rule increases, the attractor disappears through a global bifurcation and the Cournot-Nash equilibrium can become a global Milnor attractor. To sum up, the best-reply rule can be evolutionary dominant over the Walrasian rule and this can lead an oligopoly to select the Cournot-Nash equilibrium.

Keywords Behavioral evolutionary oligopoly · Walrasian versus best reply · Bounded rationality · Nonlinear dynamics
JEL Classifications: C62 · C73 · D21 · D43 · L13

1 Introduction

Industries are often populated by a limited number of firms taking decisions about next-period production in an economic environment that can differ from the one postulated by the *economic theory of perfect competition.* In many markets, for example, firms are not simply *price takers* but their decisions about the level of production influence the selling price of the goods they produce. In this case, the choice of the optimal production level requires to solve a strategic game, better known as the Cournot oligopoly game. The Cournot-Nash equilibrium, proposed by Cournot in 1838, represents, under general assumptions, a Nash solution of this game.

Concerning firms, this equilibrium represents a more efficient solution in terms of output and profits than the one of perfect competition, also known as Walrasian equilibrium, as the output is lower and the earned profits are higher. Nevertheless, the Walrasian-equilibrium quantity can be evolutionary stable over the Cournot-Nash one. In this regard, adopting the framework of the standard evolutionary game theory, Schaffer (1989) finds that in a quantity-setting duopoly characterized by firms with identical and constant marginal costs only the Walrasian-equilibrium quantity is evolutionary stable. The same result is obtained by Rhode and Stegeman (2001) for stochastic evolutionary games and by Vega-Redondo (1997) for quantity-setting oligopolies where firms produce homogeneous commodities and determine, by imitation, their output among a finite set of choices. In the latter case, the only requirements are a downward sloping market demand and the existence of Walrasian equilibrium.

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These results find additional strength in Vallée and ldızoğlu (2009), where the Walrasian equilibrium is proved to be stable even under multiple and simultaneous homogeneous mutations, and in Apesteguia et al (2010), where experiments confirm the evolutionary stability of the Walrasian equilibrium for oligopolies populated by firms with asymmetric costs.

Schaffer (1989) underlines that these counter-intuitive results are due to an imitation process based on relative profits that characterizes the evolutionary game theory and find an explanation in terms of spiteful behaviors, i.e. a strategy is played just because it damages my competitors more than myself. A firm forgoes playing the Cournot-Nash-equilibrium quantity, acts as a price taker and thus decreases its profits, but its change of strategy harms its competitors that play the Cournot-Nash-equilibrium quantity more than itself; see, e.g. Vallée and ldızoğlu (2009) and Riechmann (2006).

The mentioned findings about the evolutionary selection of the Walrasian-equilibrium quantity in oligopolies rely on standard evolutionary games. In recent years, a new form of evolutionary game is proposed where the imitation process is combined with behavioral dynamics. Introduced for the first time to model oligopolies in Droste et al (2002), these games are inspired by the practice of relying on routines or consolidated management procedures followed by complex organizations such as firms; see, e.g., Winter (1964) and Nelson and Winter (1982). In fact, in these games, here named *behavioral evolutionary games*, agents select behavioral rules instead of outputs to produce. A behavioral rule is a heuristic, or algorithm, adopted by a firm to determine the quantity to be produced in the current period as a function of past observations; see, e.g. Nelson and Winter (1982) and Droste et al (2002). One of the most well-known behavioral rules in quantity-setting oligopoly is the so-called *best reply* with naïve expectations or Cournot behavior; see, e.g., Bischi et al (2010) and reference therein. According to this rule, the quantity to be produced maximizes the firm's profit under the belief that the output of the competitors will not change. The resulting quantity dynamics follows a dynamic equation that allows the Cournot-Nash-equilibrium output as the unique steady state. This behavioral rule has been tested in Droste et al (2002) against the so-called *rational behavioral rule*, according to which a firm, being a perfect foresight agent, always play a Nash equilibrium in a game contaminated with a number of best-reply firms. The evolutionary selection of the two rules, which is based on a replicator dynamics, reveals that the cheaper best-reply rule is successful and is adopted by some firms, although it has destabilizing effect in the output dynamics, and bifurcation routes to complicated dynamics may occur.

Here, the best-reply dynamics is compared with the *Walrasian* rule, which is a non-Cournotion behavior based on naïve expectations. The *Walrasian* rule, or Walrasian behavior, is built on the assumption that a firm acting as price taker decides next-period output maximizing its profit. The resulting quantity dynamics leads to a dynamic equation that allows the Walrasian-equilibrium output as the unique steady state.

The aim of this work is to study the evolutionary stability of the Walrasian-equilibrium quantity in the framework of *behavioral evolutionary game theory.* The analysis focuses on the stability of the Walrasian equilibrium in an evolutionary oligopoly game in which the Walrasian behavior is opposed to the best-reply rule and quantity-setting firms have naïve expectations. Every period each firm decides on the level of production according to one of the two behavioral rules. The choice about the behavioral rule is made on the basis of past profits and it is revised at each new period based on an evolutionary mechanism. Thus, the behavioral rule that was more successful in the previous period stands a higher chance to be adopted in future. The object of study is a three-dimensional nonlinear dynamic systems in discrete time that includes population dynamics in the form of a replicator equation; see, e.g., Cabrales and Sobel (1992), Hofbauer and Weibull (1996), Hofbauer and Sigmund (2003) and Kopel et al (2014), and two output dynamics arising from the two behavioral rules. The map that describes the dynamics of the model is piecewise smooth; see, e.g., di Bernardo et al (2008), Sushko and Gardini (2010) and reference therein.

In summary, the main difference between this approach and the classical approach of evolutionary game theory adopted in Schaffer (1989), Vega-Redondo (1997), Rhode and Stegeman (2001) and Vallée and ldızoğlu (2009) is the strategic set of the evolutionary game. In the present setup, according to the evolutionary mechanism, firms adopt a behavioral rule and the production pattern follows from this choice. Thus, the quantity to be produced is determined only indirectly by the evolutionary mechanism and directly by the behavioral rule.

Following this approach, the Walrasian-equilibrium quantity that is proved to be evolutionary desirable with respect to any other feasible output (see again Vega-Redondo (1997)) is not necessary selected. Two further conditions are required for this to occur. First, the set of choices should include at least a behavioral rule such that, if adopted by all firms, the Walrasian-equilibrium quantity is the fixed point of the output dynamics and attracts each out-of-equilibrium sequence of outputs. Second, this behavioral rule should be evolutionary desirable.

The analysis reveals that a behavioral rule that satisfies the first condition is evolutionarily desirable, although its existence depends on the market configuration and the rate of change of marginal cost. In particular, the proposed model is characterized by only two fixed points: a symmetric Cournot-Nash equilibrium and the Walrasian equilibrium. In the Cournot-Nash equilibrium, all firms adopt the best-reply rule and produce the Cournot-Nash quantity. In the Walrasian equilibrium, all firms adopt the Walrasian rule and produce the Walrasian quantity. For rate of change of marginal cost higher than the sum of residual market price sensitivities to output, every production pattern generated by the Walrasian rule converges to the Walrasian quantity and the Walrasian equilibrium is globally stable. However, for rate of change of marginal cost lower than the sum of residual market price sensitivities to output, the out-of-equilibrium production patterns generated by the Walrasian rule do not converge to the Walrasian quantity and the Walrasian equilibrium is unstable. The loss of stability of the Walrasian equilibrium coincides with a bifurcation through which a non-equilibrium attractor is created. The attractor represents a stable *polymorphic state* where the distribution of the two behavioral rules fluctuates over time and the choice of the firms never polarizes towards a unique heuristic. Increasing the intensity of choice, i.e. the intensity with which firms select the better performing behavioral rule, the *polymorphic state* can disappear due to a global bifurcation. After the bifurcation, the Cournot-Nash equilibrium can become a Milnor attractor. However, for a higher price reactivity to output, the Cournot-Nash equilibrium becomes unstable and a 2-cycle appears through a degenerate flip bifurcation followed by a persistence border collision. This attractor is such that all firms adopt the best-reply rule.

To sum up, as long as the Walrasian quantity can be attained, implying the existence of a behavioral rule that allows this output, the Walrasian equilibrium is stable. This result is consistent with the economic literature; see, e.g., Schaffer (1989), Vega-Redondo (1997) and Rhode and Stegeman (2001). However, the lack of a behavioral rule that generates production patterns converging to the Walrasian quantity may result in firms adopting the best-reply rule and producing the Cournot-Nash quantity. This finding provides an evolutionary justification for the existence of Cournot oligopolies and for the adoption of the best-reply rule. Moreover, it has some implications in terms of applied economics as the observation of Cournot-Nash output in an industry does not exclude that firms follow an imitation learning based on relative performances.

As a final remark, it is worth underlining that in oligopoly theory there is a growing stream of literature focused on the evolutionary competition among behavioral rules built upon different assumptions of rationality and knowledge; see, e.g., Anufriev et al (2013), Bischi et al (2015), Cerboni Baiardi et al (2015) and Cavalli et al (2015). In this regard, the present model is the only one that includes a behavioral rule that leads to the Walrasian-equilibrium quantity.

The paper is organized as follows. Sect. 2 introduces the model. Sect. 3 contains a detailed analysis of the equilibria, of their asymptotic stability and of the monomorphic dynamics. Sect. 4 discusses the condition under which a polymorphic attractor exists and shows the global bifurcation through which the Cournot-Nash equilibrium becomes a Milnor attractor. Sect. 5 provides some insights about the frequency distribution of the average output of the oligopoly and discusses the possible practical implications of the results. Sect. 6 concludes and provides some ideas for further work to be put on the agenda. All proofs are in the Appendix.

2 Model Setup

Let us consider an oligopoly with N ex-ante identical firms producing homogeneous goods. At time $t \in \mathbb{N}$, the goods are sold in a market characterized by the following inverse demand function:[1]

$$P(Q(t)) = \max[a - bQ(t), 0] \tag{1}$$

where $Q(t)$ is the total output of the industry, $a > 0$ is the *reservation price* and $b > 0$ measure price sensitivity to output. Moreover, let $q_i(t)$ denote the output of producer i, $i \in \{1, 2, 3, \ldots, N\}$, at time t and let us assume that the cost to produce that output is given by the following cost function characterized by a positive curvature, also known as rate of change of marginal cost, equal to $2c$:

$$C(q_i(t)) = cq_i(t)^2 \tag{2}$$

which implies that firms adopt a homogeneous technology of Cobb-Douglas type with decreasing-return to scale and the price of the production inputs is constant; see, e.g., Fisher (1961), Alós-Ferrer (2004) and Bischi et al (2010) for similar assumptions.

Given inverse demand function (1) and cost function (2), the profit function of firm i at time t is given by

$$\Pi(Q_{-i}(t), q_i(t)) = P(Q_{-i}(t) + q_i(t))\, q_i(t) - C(q_i(t)) \tag{3}$$

where $Q_{-i}(t) + q_i(t) = Q(t)$.

[1] A linear price function is obtained by assuming that the utility function of the representative consumer is quadratic; see, e.g., Bischi et al (2010) and Vives (2001)

By assumption, firms can choose between two behavioral rules: the *best-reply* rule and the *Walrasian* rule. Assuming that firm i selects the best-reply rule, then its next-period output $q_i(t+1)$ solves the following optimization problem:

$$\max_{q_i \geq 0} \Pi\left(Q^e_{-i}(t+1), q_i\right) \tag{4}$$

where $Q^e_{-i}(t+1)$ is the output decision of the rest of the industry as expected by firm i at time $t+1$. In contrast, if firm i adopts the Walrasian rule, it behaves as a *price taker* and its next-period output $q_i(t+1)$ solves the following optimization problem:

$$\max_{q_i \geq 0} \left[p^e_W(t+1)\, q_i - C(q_i)\right] \tag{5}$$

where $p^e_W(t+1)$ is the market price as expected by Walrasian firms at time $t+1$ based on the information at time t. By assumption, firms have naïve expectations. This implies $Q^e_{-i}(t+1) = Q_{-i}(t)$ for best-reply firms and $p^e_W(t+1) = p(t)$ for the Walrasian firms.[2] Therefore, a Walrasian firm does not take into account the effect that its own level of production and the one of the competitors (negative externalities) have on the market price of the produced goods. Nevertheless, Walrasian firms are aware that prices may change over time due to factors not related to the production side.

In summary, to determine the level of production for the next period, first of all, each firm has to select one of the two behavioral rules, i.e. either best-reply rule or Walrasian rule. Let $r(t) \in [0,1]$ denote the probability of adoption of the best-reply rule at time t and consequently $1 - r(t)$ denotes the probability of adoption of the Walrasian rule. Furthermore, let us indicate by $x(t)$ the level of production of a best-reply firm at time t and by $y(t)$ the level of production of a Walrasian firm at time t. Then, $q_i(t) \in \{x(t), y(t)\}$, for $i = 1, \ldots, N$. Assuming naïve expectations and solving the optimization problem (4), we obtain the result that a best-reply firm produces the following quantity of goods:

$$x(t+1) = T_x(x(t), y(t), r(t)) = \begin{cases} R(x(t), y(t), r(t)) & \text{if } p^e_{BR}(t+1) > 0 \\ 0 & \text{if } p^e_{BR}(t+1) = 0 \end{cases} \tag{6}$$

where R is the classical *reaction function* or *best-reply correspondence* (see, e.g. Bischi et al (2010)) given by

$$R(x(t), y(t), r(t)) = \frac{a - b(N-1)\bar{q}(t)}{2(b+c)} \ ; \quad p^e_{BR}(t+1) = P\left((N-1)\bar{q}(t) + R(x(t), y(t), r(t))\right) \tag{7}$$

and

$$\bar{q}(t) = r(t)\, x(t) + (1 - r(t))\, y(t) \tag{8}$$

is the average level of production of a single firm at time t.[3] Moreover, assuming naïve expectations and solving the optimization problem (4), a Walrasian firm produces the following quantity of goods:

$$y(t+1) = T_y(x(t), y(t), r(t)) = \begin{cases} W(x(t), y(t), r(t)) & \text{if } p(t) > 0 \\ 0 & \text{if } p(t) \leq 0 \end{cases} \tag{9}$$

where[4]

$$W(x(t), y(t), r(t)) = \frac{p(t)}{2c} = \frac{a - bN\bar{q}(t)}{2c} \tag{10}$$

For $r = 0$, function T_y depends only on the level of production of Walrasian firms, i.e. y, and equation (9) defines a quantity dynamics that is qualitatively equivalent to the one of the cobweb model (see, e.g., Hommes (1994)) with non-negativity constraint.

Firms can revise their decisions about the behavioral rule to be adopted at each time period $t \in \mathbb{N}$. By assumption, the probability of adopting a behavioral rule is profit-based, and is approximated by the following

[2] Note that $p(t)$ indicates the price at time t, while $P(\cdot)$ indicates the inverse demand function or price function.

[3] Let us note that $x(t+1) \geq 0$ if and only if $p^e_{BR}(t+1) \geq 0$.

[4] Let us note that $y(t+1) \geq 0$ if and only if $p(t) \geq 0$.

evolutionary dynamics (also known as replicator equation; see, e.g., Cabrales and Sobel (1992) and Hofbauer and Sigmund (2003)):[5]

$$r(t+1) = T_r(x(t), y(t), r(t)) = \frac{r(t)}{r(t) + (1 - r(t))\, e^{\beta \Delta\Pi(x(t), y(t), r(t))}} \tag{11}$$

where

$$\Delta\Pi(x(t), y(t), r(t)) = (P(N(x(t)r(t) + (1 - r(t))y(t))) - c(x(t) + y(t)))(y(t) - x(t)) \tag{12}$$

is the difference at time t between the profit made by a Walrasian firm and the profit made by a best-reply firm. Moreover, β takes positive values and is the so-called intensity of choice, i.e., it measures the propensity of firms to switch to the more profitable behavioral rule. A large value of β means a strong propensity to switch to the behavioral rule that performed better in the last period in terms of relative profits.[6]

By definition of the production function T_x and T_y, the levels of output are always non negative, say $x, y \geq 0$. Moreover, variable r is considered to be included in $[0,1]$, which is an invariant set for T_r. Thus, coupling the output dynamics (6) and (9) with the replicator dynamics (11), we obtain the following map $T := \mathbb{R}_+^2 \times [0,1] \to \mathbb{R}_+^2 \times [0,1]$:

$$\begin{aligned}(x(t+1), y(t+1), r(t+1)) &= T(x(t), y(t), r(t)) \\ &= (T_x(x(t), y(t), r(t)), T_y(x(t), y(t), r(t)), T_r(x(t), y(t), r(t)))\end{aligned} \tag{13}$$

where $N \in [2, +\infty)$ and $\beta, a, b, c \in (0, +\infty)$. From the definition of the map, we have the result that the phase plane of the dynamical system can be divided into several regions where the system is defined by different smooth functions. On the boundaries of the regions, the map is continuous but not differentiable. Let $\bar{q} = rx + (1 - r)y$, then the boundaries of non-differentiability are given by the curves

$$BC_1 := \bar{q} = \frac{a}{bN}, \quad \text{where} \quad p(t) = 0, \quad \text{and} \quad BC_2 := \bar{q} = \frac{a}{b(N-1)}, \quad \text{where} \quad p_{BR}^e(t+1) = 0. \tag{14}$$

Thus, the phase space is partitioned into three regions, in each of which a different smooth function is to be applied. The three regions are:

$$\begin{aligned}\Omega_1 &= \left\{(x, y, r) \,|\, 0 \leq \bar{q} < \tfrac{a}{bN}\right\} \\ \Omega_2 &= \left\{(x, y, r) \,|\, \tfrac{a}{bN} \leq \bar{q} < \tfrac{a}{b(N-1)}\right\} \\ \Omega_3 &= \left\{(x, y, r) \,|\, \tfrac{a}{b(N-1)} \leq \bar{q}\right\}\end{aligned} \tag{15}$$

and, in these regions, map T is defined as follows:

$$\begin{aligned}(x, y, r) \in \Omega_1 &: \left(x', y', r'\right) = (R(x, y, r), W(x, y, r), T_r(x, y, r)) \\ (x, y, r) \in \Omega_2 &: \left(x', y', r'\right) = (R(x, y, r), 0, T_r(x, y, r)) \\ (x, y, r) \in \Omega_3 &: \left(x', y', r'\right) = (0, 0, T_r(x, y, r))\end{aligned} \tag{16}$$

where $'$ indicates one-period step.

[5] In the modeling framework of this paper, the replicator dynamics describes the time evolution of the probability of selecting a behavioral rule. This interpretation is beyond the evolutionary one originally proposed in biology, and it is consistent with the usual random matching hypothesis adopted in game theory; see, e.g., Hauert et al (2006), van Veelen (2011) and references therein. As underlined by an anonymous referee, abandoning the probabilistic interpretation of the state variable r would require that r takes only rational numbers of type $0, 1/N, 2/N, \ldots, 1$. This modeling choice implies the loss of the mathematical tractability of the model and is employed in Vriend (2000), where a genetic algorithm confirms that firms select the Walrasian quantity whenever a learning mechanism based on relative performances, as the one implied by the replicator equation (11), is considered. The development of a computational model as the one in Vriend (2000) but based on behavioral rules would represent an interesting way of testing the results of the current paper.

[6] According to equation (11), firms can decide to switch to a less sophisticated behavioral rule, such as the Walrasian one, once they have experienced the more advanced best-reply rule. In this framework, it is a common approach to associate an additional fixed cost to the more sophisticated behavioral rule. In the current model, as long as it is small enough, such a cost would not affect the results and thus is omitted for the sake of simplicity.

3 Equilibria, asymptotic stability and monomorphic dynamics

Model (13) describes the dynamics of an evolutionary oligopoly game where firms select behavioral rules that determine the output dynamics of the industry. This section is devoted to investigating existence and stability of fixed points of the model and its global dynamics when the population of firms is monomorphic, i.e. either all best-reply firms or all Walrasian firms, by means of analytical arguments.

As a first step, let us underline the main properties of map (13). It is a three dimensional dynamical system that evolves inside the three-dimensional phase space $(x, y, r) \in \mathbb{R}_+^2 \times [0, 1]$, where $\mathbb{R}_+ = [0, +\infty)$. The boundary planes $r = 0$ and $r = 1$, where all firms agree on employing the same behavioral strategy, respectively Walrasian and best reply, are invariant sets of the map. On these planes, the dynamics is governed by the two-dimensional restrictions of map (13) on them. An attractor lying on one of these two-dimensional restrictions of the phase space may be transversely attracting so that it attracts trajectories starting outside the restriction, i.e. from $r(0) \in (0, 1)$. In this case, an attractor for the restriction is also an attractor of map T defined in (13). Otherwise, the attractor for the restriction may be transversely repelling and it may not be reached by trajectories coming from outside the invariant subspace. It is possible to demonstrate (proof in Appendix) the following result showing that all the possible equilibria[7] of the evolutionary oligopoly game belong to the two boundary planes $r = 0$ and $r = 1$.

Lemma 1 *The following monomorphic states are fixed points of model* (13)*:*

– *The Walrasian equilibrium,* $E_0 = \left(q_W^* \frac{b+2c}{2(b+c)}, q_W^*, 0\right) \in \Omega_1$*, where each firm adopts the Walrasian rule and its production, the market price and its profit are:*

$$q_W^* = \frac{a}{2c + bN}, \quad p_W^* = 2cq_W^* \quad \text{and} \quad \Pi(E_0) = c\left(q_W^*\right)^2 \tag{17}$$

– *The Cournot-Nash equilibrium,* $E_1 = \left(q_{CN}^*, q_{CN}^* \frac{b+2c}{2c}, 1\right) \in \Omega_1$*, where each firm adopts the best-reply rule and its production, the market price and its profit are:*

$$q_{CN}^* = \frac{2c + bN}{2c + b(N + 1)} q_W^*, \quad p_{CN}^* = (2c + bN)\, q_W^* \quad \text{and} \quad \Pi(E_1) = (c + K)\left(q_W^*\right)^2, \quad K > 0 \tag{18}$$

No other fixed points exist.

As expected, the level of production of the whole oligopoly at the Walrasian equilibrium, namely $Q_W^* = Nq_W^*$, is larger than at the Cournot-Nash equilibrium, namely $Q_{CN}^* = Nq_{CN}^*$. Then, the market price at the Walrasian equilibrium, which equals the marginal cost, is lower than the one at the Cournot-Nash equilibrium. These variables influence the economic performances of firms, the profit of which is higher at the Cournot-Nash equilibrium than at the Walrasian equilibrium, i.e. $\Pi(E_1) > \Pi(E_0) > 0$.

Another economic variable that plays a key role in the output dynamics is the *potential* level of production of a Walrasian firm at the Cournot-Nash equilibrium. Although there are not Walrasian firms, this virtual level of production, being larger than the one of a best-reply firm, indicates that, at the Cournot-Nash equilibrium, the profit of a Walrasian firm would be higher than the one of a best-reply firm, and by continuity this holds true also in a neighborhood of the equilibrium. Similar considerations indicate that a Walrasian firm outperforms a best-reply firm in a neighborhood of the Walrasian equilibrium as well. This leads to a counter intuitive stability result. Notwithstanding that firms are better off in the Cournot-Nash equilibrium, it is possible to demonstrate (proof in Appendix) the following result about the evolutionary stability of the Walrasian equilibrium and transverse instability of the Cournot-Nash equilibrium.

Proposition 1 *The Walrasian equilibrium is locally asymptotically stable if and only if* $\frac{\partial^2 C}{\partial y^2} > -\frac{\partial P(E_0)}{\partial y}$*, i.e.* $bN < 2c$*. The Cournot-Nash equilibrium is transversally unstable.*

This result confirms, in a dynamic and behavioral framework, the findings by Schaffer (1989), Vega-Redondo (1997) and Rhode and Stegeman (2001) about the evolutionary dominance of the Walrasian-equilibrium quantity over the Cournot-Nash one. Moreover, as also observed in Schaffer (1989), the profit-quantity dynamics reveals that Walrasian firms act as *spiteful* players toward best-reply firms, i.e. behaving as a Walrasian firm damages the best-reply firms more than the Walrasian firm itself. In short, in a sufficiently close neighborhood of the Cournot-Nash equilibrium, a Walrasian firm acts as a *free rider* toward a best-reply firm, i.e. it makes extra profits by producing more and taking advantage of the high prices experienced thanks to the lower level of production of best-reply firms. Thus, the Walrasian behavior represents a profitable deviation from the Cournot-Nash equilibrium, as shown in the time series of Figure 1. The figure shows that as long as the number

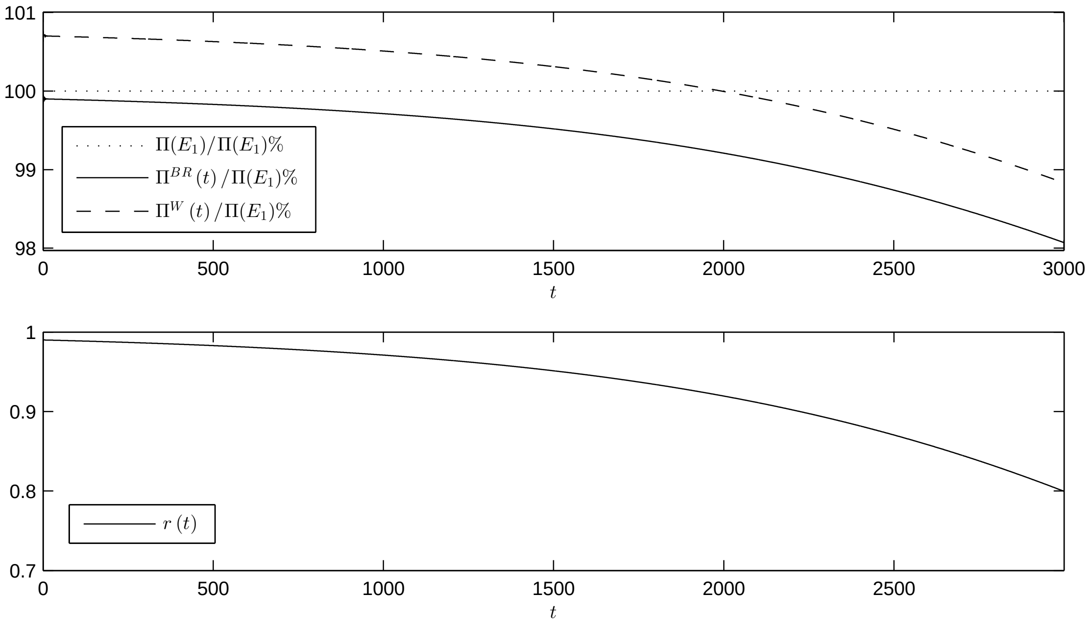

Fig. 1 Example of spiteful behavior by Walrasian firms in a neighborhood of the Cournot-Nash equilibrium. Profit dynamics (as percentage of $\Pi(E_1)$) of a Walrasian firm $\frac{\Pi^W(t)}{\Pi(E_1)}\%$, dashed line, and of a best-reply firm $\frac{\Pi^{BR}(t)}{\Pi(E_1)}\%$, solid line. $\Pi^{BR}(t) = P(Q(t))\,x(t) - C(x(t))$ and $\Pi^W(t) = P(Q(t))\,y(t) - C(y(t))$. The trajectory generated by the initial condition $E_1 - (0, 0, 0.01)$ shows that a Walrasian firm outperforms a best-reply firm. This decreases the probability of adopting the best-reply rule, see $r(t)$ depicted in the second line, as well as the profits of all firms in the long run. The parameters are: $a = 1$, $b = 0.1$, $c = 0.51$, $N = 10$ and $\beta = 1$.

of Walrasian firms is small, r is close to one, being Walrasian is more profitable than being in the Cournot-Nash equilibrium.

Note that Proposition 1 implies that the Walrasian equilibrium *may* not be locally asymptotically stable. This is an important addition to the findings in Schaffer (1989), Vega-Redondo (1997) and Rhode and Stegeman (2001). In fact, the instability of the Walrasian equilibrium is not related to the evolutionary dominance of the Walrasian-equilibrium quantity but it is caused by coordination problems in the quantity dynamics, which is a peculiarity of the behavioral approach. In particular, whenever the sum of the residual market price sensitivities to output[8], i.e. bN, exceeds the rate of change of marginal cost, i.e. $2c$, even if all firms are Walrasian the output dynamics do not converge to the Walrasian-equilibrium quantity, unless it is initially selected. Thus, for such market configurations, the strategic set of the evolutionary game, which is made of the best-reply rule and the Walrasian rule, does not include a behavioral rule that, once adopted by all firms, allow to have long-run output patterns that converge to the Walrasian-equilibrium quantity and this precludes the stability of the Walrasian equilibrium.

This aspect clearly results from the investigation of the dynamics of model (13). By eigenvalue analysis[9], the Walrasian equilibrium has always a stable manifold outside $r = 0$, which means that the evolutionary dynamics attracts trajectories towards the invariant set where the Walrasian equilibrium is. Nevertheless, this equilibrium is also characterized by an unstable manifold along the invariant set $r = 0$ whenever $bN > 2c$. This means that, once all firms are Walrasian, it is the Walrasian rule itself that prevents the convergence to the Walrasian equilibrium. This result marks the difference between a classical evolutionary game and a *behavioral evolutionary game*. In a behavioral evolutionary game, the local stability of an evolutionary dominant strategy requires a behavioral rule that allows that strategy to be played.

These results are obtained within a modeling framework such that firms make the so called *supernormal profits* by playing the Walrasian equilibrium, i.e. $\Pi(E_0) > 0$. According to the general equilibrium theory or the theory of perfect competition, this can only be a short-term situation. In the long term, new firms will enter the

[7] Steady states of model (13) are named equilibria instead of fixed points for analogy with both economic and game theory.

[8] As b measures market price sensitivity to output and the oligopoly includes N firms each characterized by a residual demand function, bN can be also seen as the sum of the residual market price sensitivities to output.

[9] Note that $0 < \lambda_3^W < 1$, where λ_3^W is the eigenvalue computed of the Walrasian equilibrium and associated to the eigenvector orthogonal to the subset $r = 0$. See proof of Proposition 1 in Appendix.

industry pushing profits to zero. Nevertheless, the theoretical results provided here are quite general. Indeed, it is enough to assume the existence of some fixed costs to have $\Pi(E_0) = 0$ (or even $\Pi(E_0) < 0$, *abnormal losses*). As the imitation dynamics is based on relative performance, these fixed costs will not influence the dynamics of the oligopoly and the validity of Proposition 1.

These results refer to the local dynamics of model (13). For more general results, a detailed analysis of the global dynamics is required. This can be done following the approach proposed in Bischi et al (2015) and Cerboni Baiardi et al (2015). The first step is to study the restriction of map (13) on its two invariant sets, i.e. $r = 0$ and $r = 1$. The restriction of the map to the invariant set $r = 0$ reduces to a two-dimensional dynamical system without evolutionary dynamics, which is easier to investigate. Adopting this method, the following Proposition (proof in Appendix) provides a global analysis of the dynamics in the invariant set $r = 0$.

Proposition 2 (Pure Walrasian behavior) *The following statements hold true for the dynamics of model* (13) *in the invariant subset* $r = 0$*:*

– *For* $\frac{\partial^2 C}{\partial y^2} > -\frac{\partial P(E_0)}{\partial y}$, *after at most two iterations the dynamics is bounded in the segment*

$$A = \left\{(x, y, r)\,|y \in \left[0, \frac{a}{2c}\right], x = \frac{c(N-1)}{N(b+c)}y + \frac{a}{2N(b+c)}, r = 0\right\} \tag{19}$$

and the transversely attracting Walrasian equilibrium, E_0*, is a global attractor with respect to the invariant set* $r = 0$*.*

– *For* $\frac{\partial^2 C}{\partial y^2} = -\frac{\partial P(E_0)}{\partial y}$, *the Walrasian equilibrium undergoes a degenerate flip bifurcation and region A is filled with 2-cycles. One of these 2-cycles, namely* $\mathcal{C}_1^W = \left\{\bar{\mathcal{C}}^W,\ \underline{\mathcal{C}}^W\right\}$*, lies on the borders of region A and undergoes a persistence border collision.* $\mathcal{C}_1^W$ *is the only 2-cycle, among those filling region A, that persists in the plane* $r = 0$ *after the bifurcation.*

– *For* $\frac{\partial^2 C}{\partial y^2} < -\frac{\partial P(E_0)}{\partial y}$, *the Walrasian equilibrium is unstable and either the 2-cycle* $\mathcal{C}_1^W$ *(if* $b(N-1) < 2c < bN$*) or the 2-cycle* $\mathcal{C}_2^W = \left\{\bar{\mathcal{C}}^W,\ \hat{\mathcal{C}}^W\right\}$ *(if* $2c < b(N-1)$*), with* $\hat{\mathcal{C}}^W = (0,0,0) \in \Omega_1$*, attracts all the points of the restriction* $r = 0$ *except for* E_0*.*

The 2-cycles $\mathcal{C}_1^W$ *and* $\mathcal{C}_2^W$ *are transversely repelling.*

The quantity dynamics of the map T in the invariant set $r = 0$ is qualitatively the same of the one implied by the price dynamics of the cobweb model with nonnegative price constraints, downward linear demand function, quadratic costs and constant expectations; see, e.g., Hommes (1994). Then, the stability condition of the Walrasian equilibrium is equivalent to the well-known stability condition of the cobweb model, i.e. the absolute value of the relative slopes of supply and demand has to be less than one.

Similarly, the restriction of the map to the invariant set $r = 1$ reduces to a two-dimensional dynamical system without evolutionary dynamics, which is easier to investigate. Adopting this method, the following Proposition (proof in Appendix) provides a global analysis of the dynamics in the invariant set $r = 1$.

Proposition 3 (Pure best-reply behavior) *The following statements hold true for the dynamics of model* (13) *in the invariant subset* $r = 1$*:*

– *For* $\frac{\partial R(E_1)}{\partial x} > -1$, *after at most two iterations the dynamics is bounded in the segment*

$$B = \left\{(x, y, r)\,|x \in \left[0, \frac{a}{2(c+b)}\right], y = \max\left[\frac{N(b+c)}{c(N-1)}x - \frac{a}{2c(N-1)}, 0\right], r = 1\right\} \tag{20}$$

and the transversely repelling Cournot-Nash equilibrium, E_1*, attracts all the points of the invariant set* $r = 1$*.*

– *At* $\frac{\partial R(E_1)}{\partial x} = -1$, *the Cournot-Nash equilibrium undergoes a degenerate flip bifurcation and region B is filled with 2-cycles. One of these 2-cycles, namely* $\mathcal{C}^{CN} = \left\{\bar{\mathcal{C}}^{CN},\ \underline{\mathcal{C}}^{CN}\right\}$*, lies on the border of region B and undergoes a persistence border collision.* $\mathcal{C}^{CN}$ *is the only 2-cycle, among those filling region B, that persists after the bifurcation.*

– *For* $\frac{\partial R(E_1)}{\partial x} < -1$, *the Cournot-Nash equilibrium,* E_1*, is unstable and the 2-cycle* $\mathcal{C}^{CN}$ *attracts all the points of the restriction* $r = 1$ *except for* E_1*.*

The 2-cycle $\mathcal{C}^{CN}$ *is transversely attracting.*

The quantity dynamics in the invariant set $r = 1$ is similar to the well-known best-reply dynamics where all outputs are homogeneous; see, e.g., Fisher (1961). As indicated in Proposition 3, the convergence to the Cournot-Nash quantity depends on the slope of the reaction function, which imposing the condition of being greater than -1 leads to the Fisher's stability condition (see condition (6.1) in Fisher (1961)) $b(N-3) < 2c$. This one and the stability condition of the Walrasian equilibrium underline the conclusion that *increasing marginal costs* are a stabilizing factor. This is reasonable, since increasing marginal costs tend to inhibit large output variations and thus to make the constant rivals' output assumption a better one, as also pointed out in Fisher (1961).[10] Moreover, the condition $\frac{\partial R(E_1)}{\partial x} = -1$ implies the instability of the Walrasian equilibrium, showing that the Cournot model is always more likely to be stable whatever the number of firms if each firm takes its own effect on price into account than if it does not.

4 Evolutionary stable heterogeneity and learning to play Cournot-Nash equilibrium

The scope of this section is to determine the conditions under which a behavioral rule is evolutionary dominant over the other one, firms learn to play the Cournot-Nash equilibrium and *evolutionary stable heterogeneity*, i.e. attractors characterized by behavioral heterogeneity, emerges. To this aim, the analytical results for the dynamics of the map in the two invariant sets $r = 0$ and $r = 1$ are complemented with a numerical analysis devoted to study the structural changes in the dynamics of the model and the related economic implications as the rate of change of marginal cost decreases. This investigation reveals three dynamic scenarios that correspond to three different situations in terms of evolutionary dominance of the two behavioral rules:

- The first scenario, which refers to the case of rate of change of marginal cost larger than bN, is characterized by local asymptotic stability of the Walrasian equilibrium;

- The second scenario, which refers to the case of rate of change of marginal cost smaller than bN but larger than $b\,(N-3)$, is characterized by a unique and polymorphic attractor with the oligopoly populated by both best-reply firms and Walrasian firms. Increasing the intensity of choice, the polymorphic attractor disappears through a global bifurcation and the Cournot-Nash equilibrium *may* become the unique attractor;

- The third scenario, which refers to the case of rate of change of marginal cost smaller than $b\,(N-3)$, is characterized by a unique attractor given by a 2-cycle where all firms coordinate on the best-reply rule.

Concerning the first scenario, numerical simulations show that the Walrasian equilibrium is globally stable except for a set of at most zero measure that includes the Cournot-Nash equilibrium and its stable manifold. Then, the Walrasian behavior is evolutionarily dominant and, in the long run, the industry is populated only by Walrasian firms that produce the Walrasian-equilibrium quantity.

Decreasing the rate of change of marginal cost, the loss of stability of the Walrasian equilibrium marks the transition from the first to the second scenario. In this second dynamic scenario, the points of the invariant plane $r = 0$ are attracted by a 2-cycle, either $\mathcal{C}_1^W$ or $\mathcal{C}_2^W$, and the Cournot-Nash equilibrium attracts the points of the invariant plane $r = 1$. Numerical simulations confirm the existence of a further 2-cycle $\mathcal{C}^M = \left\{\left(\underline{x}, \underline{y}, \underline{r}\right), \left(\bar{x}, \bar{y}, \bar{r}\right)\right\}$, such that $\underline{r}, \bar{r} \in (0, 1)$. It originates when the Walrasian equilibrium loses stability and one of the 2-cycles that fills region A (see Proposition 2) undergoes a bifurcation (of transverse eigenvalue $+1$) through which $\mathcal{C}^M$ appears. Numerical simulations show that this 2-cycle is unstable and is surrounded by an attractor consisting of two disjoint closed curves; see Figure 2. Originated through a supercritical Neimark-Sacker bifurcation of the 2-cycle $\mathcal{C}_2^W$, this attractor represents a *polymorphic state* where the probability of adoption of each behavioral rule changes over time but the choice of the firms never polarizes towards a single heuristic. Numerical experiments confirm that this invariant set is globally stable, except for the fixed points and the 2-cycles, and related stable sets. This scenario is also known as *evolutionary stable heterogeneity*; see, e.g. Cerboni Baiardi et al (2015).

The presence of evolutionary stable heterogeneity underlines the fact that the two behavioral rules act in different ways and have an output-dependent level of evolutionary attractiveness. In fact, the related polymorphic state is characterized by a cyclical dynamics with large fluctuations in the level of the output whenever the industry is mainly populated by Walrasian firms and with almost constant level of outputs otherwise. Thus, Walrasian firms destabilize the output dynamics. However, as the quantity dynamics deviates from the Walrasian output, the best-reply behavior becomes evolutionarily attractive and more and more firms start to adopt this rule. The increase in the number of best-reply firms has a stabilizing effect on the quantity dynamics and the overall level of production gets closer to the Cournot-Nash quantity. At this point, the Walrasian behavior becomes evolutionarily attractive and the number of Walrasian firms increases. Then, the fluctuations in the level of production start to increase again and the loop goes on. Compare the time series of $\frac{\bar{q}(t)}{q_W^*}\%$ and $r\,(t)$ in

[10] It is worth pointing out that, despite the assumption of constant expectations as in Theocharis (1960), the well-know Theocharis' stability result that more than three firms implies instability of the Cournot-Nash equilibrium does not hold within this modeling setup due to the assumption of non constant marginal costs.

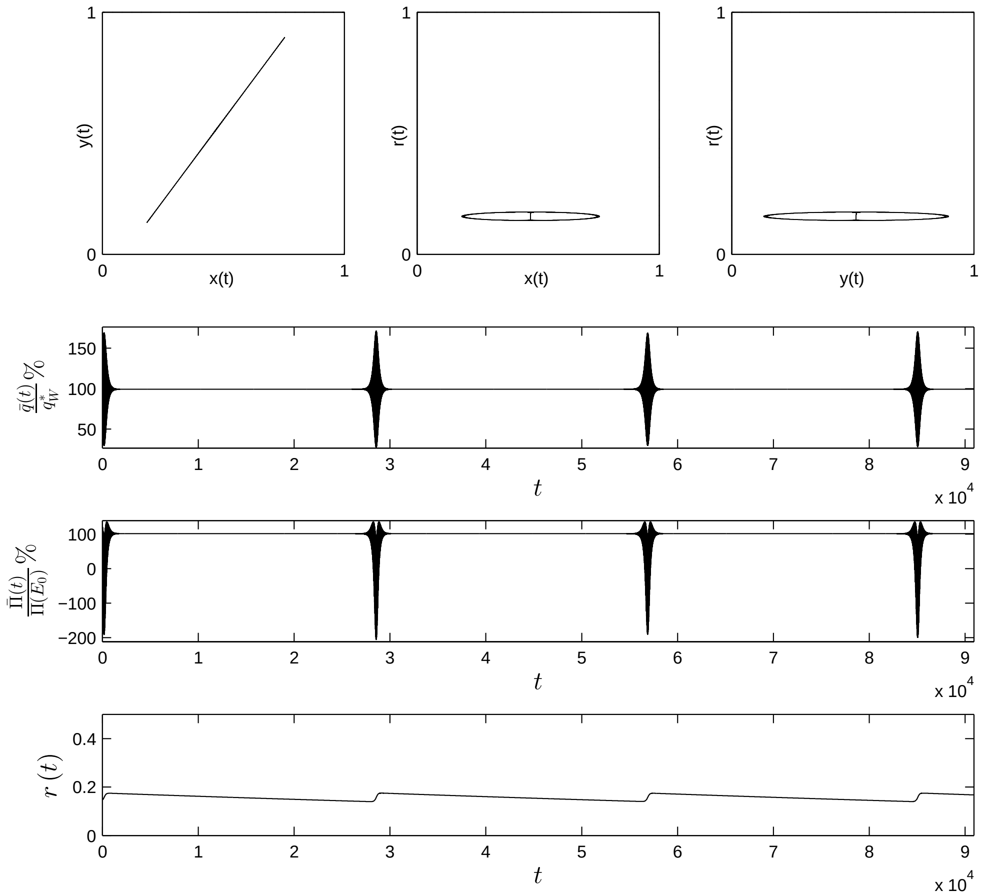

Fig. 2 Attractor of map (13) projected onto three planes. The attractor surrounds the 2-cycle $\mathcal{C}^M$ and is made of two separate closed curves, although it does not clearly result from the picture. Dynamics along the attractor of the average level of production (as percentage of the Walrasian quantity), $\frac{\bar{q}(t)}{q_W^*}\%$ in second line, of the average profit of a single firm, $\frac{\bar{\Pi}(t)}{\Pi(E_0)}\%$ in third line, and of the probability of being a best-reply firm, $r(t)$ in the third line. Let us point out that $\bar{\Pi}(t) = \Pi^{BR}(t)\, r(t) + (1 - r(t))\, \Pi^W(t)$ and the average $\frac{\bar{\Pi}(t)}{\Pi(E_0)}\%$ over the entire period is 98.86%. The parameters are: $a = 1$, $b = 0.1$, $c = 0.48$, $N = 10$ and $\beta = 0.01$. Case $b(N-3) < 2c < bN$.

Figure 2 that refer to the dynamics along the inner attractor (polymorphic state) depicted in the first line of the same Figure.

Decreasing further the rate of change of the marginal cost, a codimension two bifurcation marks the transition from the second to the third scenario. In particular, when $\frac{\partial R(E_1)}{\partial x} = -1$, the internal attractor collapses (Neimark-Sacker bifurcation) in the unstable 2-cycle $\mathcal{C}^M$, which merges (either fold or transcritical bifurcation) with one of the 2-cycles that fill region B; see Proposition 3. Among those 2-cycles there is $\mathcal{C}^{CN}$, which lies on the borders of region B. For $\frac{\partial R(E_1)}{\partial x} < -1$, the 2-cycle $\mathcal{C}^{CN}$ persists as the unique attractor of the system while $\mathcal{C}^M$ disappears. This scenario is characterized by the evolutionary dominance of the best-reply behavior and by persistent fluctuations in the quantity dynamics. Thus, the best-reply rule is evolutionarily dominant when it has a destabilizing effect in the quantity dynamics. This is consistent with the architecture of the two behavioral rules. In fact, the best-reply is based on a more sophisticated price expectation scheme, which allows it to have a substantial better prediction of the next-period price, and so higher profits, for large output variations.

The conducted qualitative analysis of the dynamics of the evolutionary model reveals that attractors characterized by fluctuations on quantity dynamics and distribution of behavioral rules can arise when the Walrasian equilibrium loses stability. Thus, a further economic consideration regards the profitability of these dynamic configurations. Let us start by pointing out that the industry experiences average profits along the 2-cycle $\mathcal{C}^{CN}$ that are not positive. This is an unsustainable economic situation in the long run as firms prefer to exit the market than making non-positive profits. In contrast, in case of a stable polymorphic state, the average profits are more similar to the ones observed at the Walrasian equilibrium, as shown, for example, in Figure 2. Here, long periods of time in which the oligopolistic industry is populated by a large fraction of best-reply

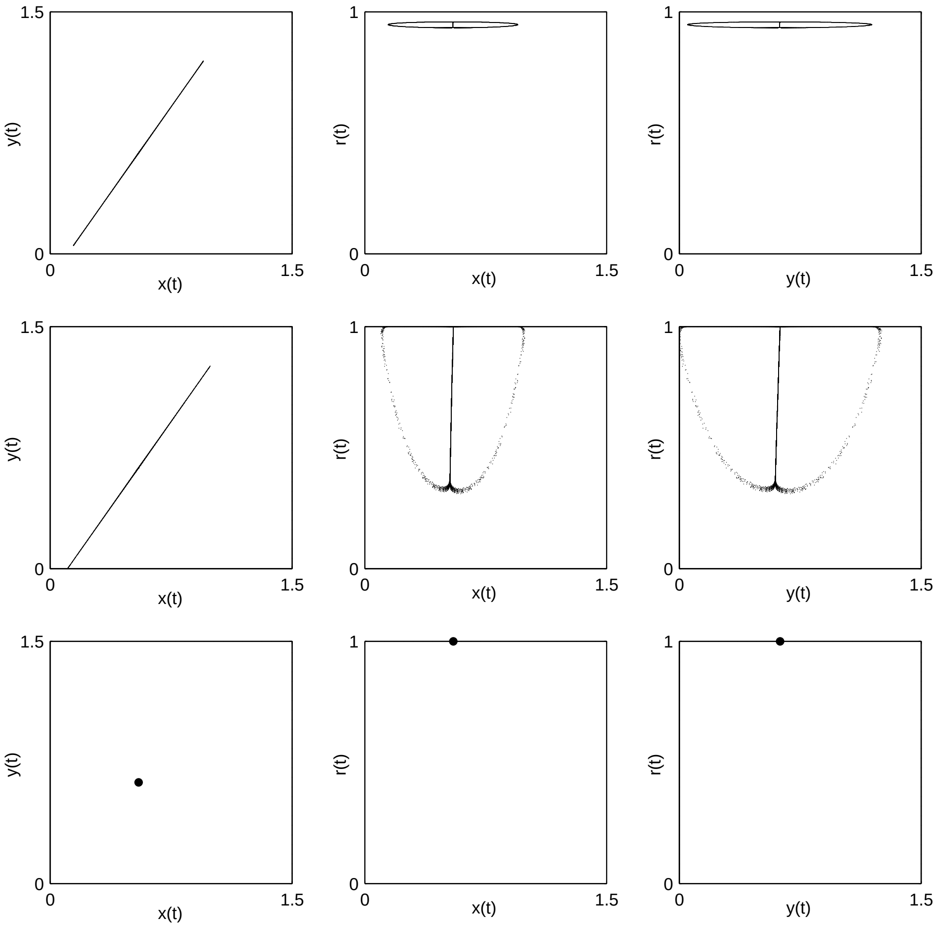

Fig. 3 Asymptotic dynamics of map (13) projected onto three planes. First line $\beta = 0.01$. Second line $\beta = 6.3$. Third line $\beta = 10$. The other parameters are: $a = 1$, $b = 0.1$, $c = 0.36$, $N = 10$. Initial condition $E_0 + (0.01, 0.01, 0.5)$. Case $b(N-3) < 2c < bN$.

firms and the average profit is slightly higher than the one at the Walrasian equilibrium are alternated with short periods of fluctuating and sometimes negative profits in which the oligopolistic industry is populated by a larger fraction of Walrasian firms. Overall, the average profit of a single firm in the polymorphic state is about 98.86% of the average profit made by a single firm in the Walrasian equilibrium. The polymorphic state can be even more profitable, on average, than the Walrasian equilibrium. This occurs when the polymorphic state is characterized by a large fraction of best-reply firms that reduce the total production level of the industry increasing the single profit of each firm. For example, when the ratio between market price and marginal cost variation rises, the attractor representing the polymorphic state gets closer to the invariant set $r = 1$; see Figure 3. In this case, the average profit made by a firm along the polymorphic state is 110% the profit a firm would make in the Walrasian equilibrium; see Figure 4.

Concerning the profitability of the oligopoly, the intensity of choice is another element worth being mentioned. This parameter plays an important role when the polymorphic state, or inner attractor, exists. Numerical simulations show that the inner attractor has contact with one of the two invariant sets, either $r = 0$ or $r = 1$, when the intensity of choice increases. As observed through numerical analyses, for small differences between the sum of residual market price sensitivities to output and rate of change of marginal cost, an increase in the intensity of choice makes the inner attractor and the plane $r = 0$ collide. As a consequence, all the trajectories starting with $r_{ic} \neq 1$ end up on the invariant plane $r = 0$ and converge to the unique attractor of this plane, namely, either the 2-cycle $\mathcal{C}_1^W$ or the 2-cycle $\mathcal{C}_2^W$. Along these cycles, the industry is populated by Walrasian firms that experience only non-positive profits. Thus, as the intensity of choice increases, firms move from a profitable situation, the polymorphic state, to a non-profitable situation. In this case, the *evolutionary frenzy* in selecting the better performing behavioral rule, caused by a higher intensity of choice, turns out to be harmful for the firms. In contrast, for larger differences between the sum of residual market price sensitivities to output

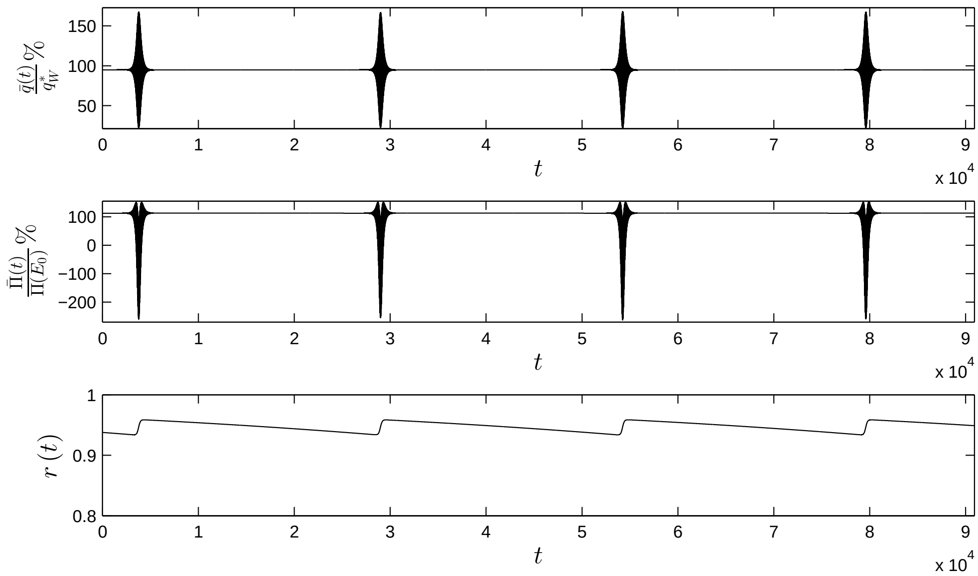

Fig. 4 Dynamics along the attractor in Figure 3, first line, of the average level of production (as percentage of the Walrasian quantity), $\frac{\bar{q}(t)}{q_W^*}\%$ in second line, of the average profit of a single firm, $\frac{\bar{\Pi}(t)}{\Pi(E_0)}\%$ in third line, and of the probability of being a best-reply firm, $r(t)$ in the third line. The average $\frac{\bar{\Pi}(t)}{\Pi(E_0)}\%$ over the entire period is 110%.

and rate of change of marginal cost, an increase in the intensity of choice makes the inner attractor and the invariant sets $r = 1$ collide. As a consequence, all the trajectories starting with $r_{ic} \neq 0$ end up on the invariant plane $r = 1$ and converge to the unique attractor of this invariant plane, i.e. the Cournot-Nash equilibrium (see in Figure 3 an example of the metamorphosis of the inner attractor as β increases). This further dynamic scenario originates from a global bifurcation and represents the more profitable equilibrium situation of the industry. Surprisingly, in the long run firms can learn to play the Cournot-Nash equilibrium despite its local instability. Moreover, these examples show that the Cournot-Nash equilibrium and the 2-cycle $\mathcal{C}_1^W$ can be stable in Milnor's sense.[11]

To sum up, the analysis of the evolutionary selection between a best-reply rule and a Walrasian rule underlines the fundamental role played by the quantity dynamics. The Walrasian behavior is evolutionarily dominant for model (13) and the Walrasian equilibrium can be asymptotically stable, confirming the findings in Schaffer (1989), Vega-Redondo (1997) and Rhode and Stegeman (2001). Nevertheless, certain market configurations induce fluctuations in the output dynamics and the Walrasian equilibrium becomes unstable. Instability of the Walrasian equilibrium and *overshooting* in the quantity dynamics cause the occurrence of a polymorphic state or, alternatively, the global stability of the Cournot-Nash equilibrium. These findings represent a novelty unobserved in previous contributions and an evolutionary justification for the existence of Cournot oligopolies and for the adoption of the best-reply behavioral rule.

5 Quantitative dynamics and economic remarks

The results in the previous section provide a detailed overview of the local and global dynamics of the oligopoly game, with special emphasis on the description of the different qualitative scenarios that can occur for different constellations of the parameters of model (13). In this section, the focus lies on the quantitative aspect of the dynamics, and on the frequency distribution of the average level of production of the oligopoly industry. The aim is to analyze how often the oligopoly output is close to the Walrasian quantity or to the Cournot-Nash quantity and how this is affected by the different market configurations. In doing so, the price function (1) is considered to be affected by random noise, as follows:

$$P(Q(t)) = \max\left[a - bQ(t) + \varepsilon(t), 0\right], \quad \varepsilon(t) \sim \mathcal{U}[-\epsilon, \epsilon] \tag{21}$$

[11] A Milnor attractor is an invariant set with a stable set of positive measure but not attracting in the topological sense (i.e. without an attracting neighborhood). In fact, initial conditions arbitrarily close to a Milnor attractor can generate trajectories that are locally repelled out from the Milnor attractor itself; see, e.g. Milnor (1985) and Bischi and Lamantia (2005).

where $\varepsilon(t)$ is a uniform random variable that represents exogenous price variations. In the following analysis, the parameter ϵ is considered to be both zero and positive. If ϵ is positive, the dynamics is affected by random noise and model (13) takes the form of random dynamical systems; see, e.g., Arnold (1998). In this case, the market price of the goods is influenced by two forces: an endogenous one, which is represented by the output choice of firms, and an exogenous one, represented by a random process that is independently uniformly distributed. The presence of random noise helps to test the robustness of the deterministic results with respect to possible external disturbances.

Let us start the investigation turning to the *evolutionary stable heterogeneity* depicted in Figure 2 and the related frequency distribution of the average level of production $\bar{q}$. For a sample of two million observations, the deterministic dynamics shows strong concentration of $\bar{q}$ around the Walrasian quantity. Over 90% of the observed average levels of production of the oligopoly are either equal to or close to the Walrasian quantity; see Figure 5, panel (a). This indicates that, despite the heterogeneity in the choice of the behavioral rule, the total level of production of the oligopoly is substantially similar to the one observed in the Walrasian equilibrium. This result is not surprising as it is sufficient to increase slightly the rate of change of marginal cost so that the stability of the Walrasian equilibrium is attained. In presence of random noise, specifically for $\epsilon = 0.01$, the frequency distribution of the average quantities changes as is characterized by a higher level of dispersion (see panel (b) and panel (a) in Figure 5). In this case, only 20% of the observed average levels of production of the oligopoly are equal to or close to the Walrasian quantity. Nevertheless, the Walrasian quantity is still the modal observation and the output levels are concentrated in a small neighborhood of this quantity. Despite this difference due to the random noise that acts as a destabilizing force in the quantity dynamics, the two frequency distributions, the one with noise and the one without, are qualitatively similar.

Reducing the rate of change of marginal cost from 0.96 to 0.72, we move away from the set of values of the parameters that ensures the stability of the Walrasian equilibrium. In this case, the frequency distribution of average outputs for the model without noise (see Figure 6, panel (a)) shows a frequency of more than 90% of average levels of production equal or almost equal to the Cournot-Nash quantity. This frequency distribution refers to the quantity dynamics of Figure 3, first row. The shift of the modal observation is due to a substantial increase in the probability of adoption of the best-reply rule. Let us observe that an increase in the rate of change of marginal cost causes the unique attractor of model (13) to move from a region characterized by a large concentration of Walrasian firm to a region characterized by a large concentration of best-reply firms; compare the dynamics in the first rows of Figure 2 and Figure 3. Even in this case, the presence of a random force has the only effect of increasing the dispersion of the average level of production around the modal observation, which remains the Cournot-Nash quantity. This indicates that, although the Cournot-Nash equilibrium is not evolutionarily stable, the average production of the oligopoly is close to the Cournot-Nash quantity.

The distribution of average outputs provides also further insights about the role played by the intensity-of-choice parameter β. This parameter represents the intensity with which the firms choose the strategy with higher relative performance. Therefore, as the Walrasian quantity offers better performance in terms of relative profits in contrast to the Cournot-Nash quantity, it is reasonable to expect that the higher the β, the higher the frequency with which the Walrasian output is observed. Nevertheless, the frequency distributions of Figure 6, obtained for $\beta = 0.01$, and of Figure 7, obtained for $\beta = 10$, indicate the opposite. The explanation for this counterintuitive result relies again on the strategic set of the evolutionary oligopoly game. For $bN > 2c$, neither the best reply nor the Walrasian rule generates sequences of out-of-equilibrium quantities that converge to the Walrasian output. Therefore, when the intensity of choice rises, firms start to adopt the Walrasian rule but not to produce the Walrasian output. In particular, the Walrasian rule is evolutionarily desirable and will be adopted by firms as long as their output is similar to the one at the Walrasian equilibrium. This choice destabilizes the output dynamics of the Oligopoly and makes firms produce an amount of goods that is different from the Walrasian quantity. This output dynamics makes the best-reply rule evolutionarily desirable. Then, an increase in the intensity of choice results in faster selection of the Cournot-Nash quantity. This typical nonlinear dynamics explains the observation of frequency distributions with Cournot-Nash output as the modal value.

This game-theoretical evidence has some implications for applied economics. In fact, the tendency of output levels to converge to the Walrasian competitive equilibrium highlighted by theoretical results on Cournot oligopoly with imitation of successful firms, e.g., Schaffer (1989) and Vega-Redondo (1997), allows us to use market outcomes as a proxy to infer the presence of imitation, as pointed out in Bosch-Domènech and Vriend (2003). Our findings suggest caution in doing so as they show that a market characterized by non-Walrasian output, specifically Cournot-Nash equilibrium, can be populated by imitators. This can occur in complex economic environments where firms have to rely on routine or managerial practices, such as behavioral rules, to determine next-period output.

A similar result is obtained in Alós-Ferrer (2004) but starting from different assumptions. In particular, Alós-Ferrer (2004) shows that keeping record of past performance, instead of only last-period performance,

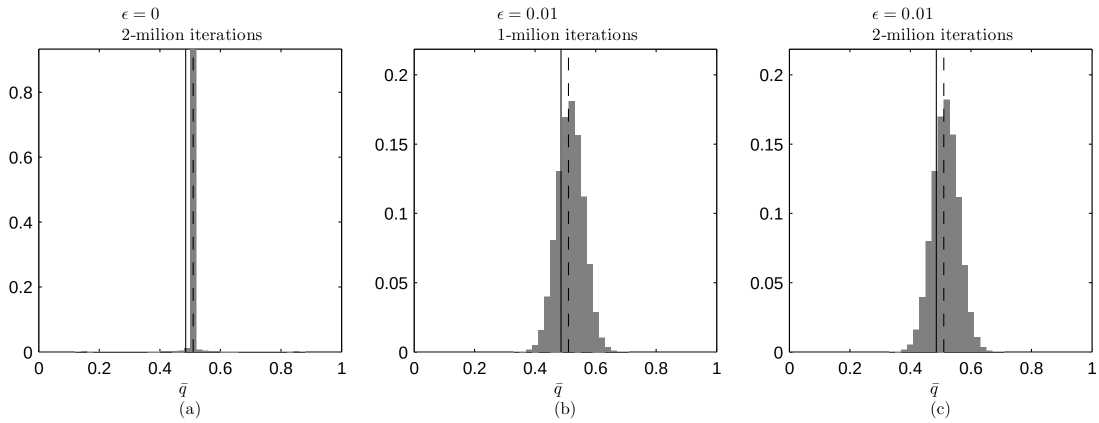

Fig. 5 Frequency distribution of the individual average output $\bar{q}$. Values of the parameters as in Fig. 2. The dashed line indicates the Walrasian quantity, the solid line the Cournot-Nash quantity.

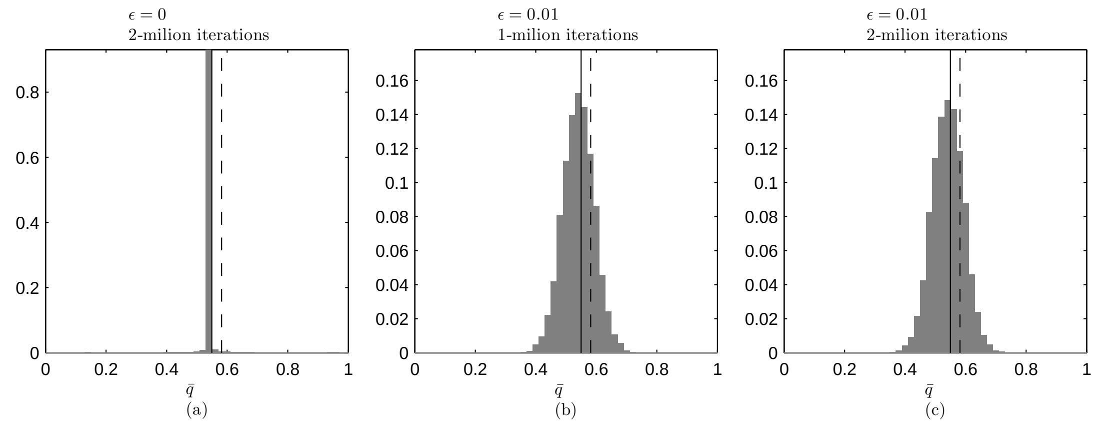

Fig. 6 Frequency distribution of the individual average output $\bar{q}$. Values of the parameters as in Fig.s 3 and 4, i.e. $\beta = 0.01$. The dashed line indicates the Walrasian quantity, the solid line the Cournot-Nash quantity.

may result in the attainment of the Cournot-Nash equilibrium. Thus, the present findings and the one in Alós-Ferrer (2004) provide two different possible game-theoretical justifications for profits commonly observed in many oligopolistic markets, which are more consistent with the Cournot-Nash equilibrium than the Walrasian one. In this respect, the game-theoretical literature offers other explanations that justifies the phenomenon. For example, the capability of firms to select the Cournot-Nash equilibrium is also discussed in Riechmann (2006), Vriend (2000) and Anufriev-Kopanyi (2017). However, the arguments used in these contributions are based on a learning mechanism that differs from imitation based on *social learning*, i.e. chasing the best relative performance, as it is in this work. In particular, Vriend (2000) underlines that a learning mechanism based on *individual learning*, i.e. chasing the best absolute performance, makes firms play the Cournot-Nash equilibrium.

To sum up, learning through imitation based on relative performance can generate spiteful behavior. Nevertheless, in a complex environment where firms have to rely on routine or managerial practices, such as behavioral rules, to determine next period production, the presence of spiteful behavior does not always prevail. In fact, the combined effects of nonlinear dynamics can lead to surprising results.

As a final remark, it is worth noting that the frequency distributions of the average output of a single firm do not depend on the size of the observations; compare panels (b) and (c) in Figures 5, 6 and 7. This suggests the presence of an invariant distribution as steady state of our random dynamical system. This distribution has different shapes for different constellations of the parameters.

6 Final remarks and conclusions

The proposed model represents a quantity-setting oligopoly characterized by evolutionary selection of two behavioral rules, namely, best-reply rule and Walrasian rule. Two types of equilibria are possible in this model. The Cournot-Nash equilibrium, where all firms adopt the best-reply rule and produce the Cournot-Nash quantity,

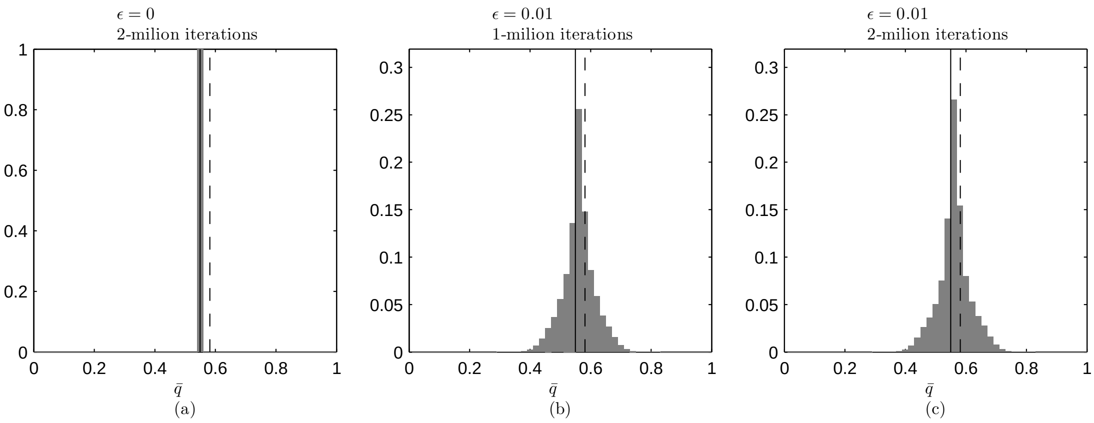

Fig. 7 Frequency distribution of the individual average output $\bar{q}$. Values of the parameters as in Fig. 3 with $\beta = 10$. The dashed line indicates the Walrasian quantity, the solid line the Cournot-Nash quantity.

and the Walrasian equilibrium, where all firms adopt the Walrasian rule and produce the Walrasian quantity. Despite the better performance of each firm in the Cournot-Nash equilibrium, the Walrasian equilibrium is the only locally asymptotically stable equilibrium of the oligopoly. Nevertheless, it becomes unstable for certain configurations of the parameters. In this case, there are three possible dynamic scenarios: A polymorphic state, where each behavioral rule is adopted by a positive fraction of firms; a stable 2-cycle, where each firm of the industry adopts the best-reply rule; a Milnor attractor, where each firm ends up producing the Cournot-Nash quantity. This last scenario reveals that the Cournot-Nash equilibrium can be a stable equilibrium of the behavioral evolutionary game.

This paper confirms and extends the results about the evolutionary stability of the Walrasian equilibrium to a *behavioral evolutionary oligopoly* with quantity dynamics and firms characterized by naïve expectations. Despite the peculiarity of the expectations, a different assumption should not lead to substantial different conclusions. For example, both Walrasian equilibrium and Cournot-Nash equilibrium do not change in case of the more sophisticated adaptive expectations (see, e.g., Hommes (1994) and Fisher (1961)) and so neither does their evolutionary stability since it does not depend on expectations. The only foreseeable difference is a less stringent stability condition of the Walrasian equilibrium as underlined in Hommes (1994), where a classical cobweb model with adaptive expectations is analyzed, which implies less possibility to have a global bifurcation through which the Cournot-Nash equilibrium gains global stability. This should not be a surprise, as naïve expectations are the main driving force of output fluctuations which, paradoxically, are essential to a stable Cournot-Nash equilibrium.

For future research, preference for outperforming opponents (classical example of rivalistic behavior in oligopolies; see, e.g., Fouraker and Siegel (1963)) can be integrated into the model. For example, a firm could be interested determining the next-period production in order to maximize its own profit and, at the same time, to minimize the profits of the competitors. The resulting behavioral rule could be another example of spiteful behavior that could be evolutionary dominant over both Walrasian and best-reply rule, as suggested in Bosch-Domènech and Vriend (2003). Moreover, some form of individual learning combined with learning through imitation, or social learning, can be considered; see, e.g., Vriend (2000). For example, the evolutionary dynamics used to model social learning in this work can be modified to include individual learning.

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Appendix

Proof (of Lemma 1) From (13), it is straightforward to see that the steady states of the evolutionary model are obtained for $r = 0$, $r = 1$ or any $r^* \in (0, 1)$ for which $\Delta\Pi = 0$. With $r = 0$ (all Walrasian firms), map (13) becomes:

$$T^0 := \begin{cases} x(t+1) = \max\left[\frac{a-b(N-1)y(t)}{2(b+c)}, 0\right] \\ \\ y(t+1) = \max\left[\frac{a-bNy(t)}{2c}, 0\right] \end{cases} \tag{22}$$

where the second component is uncoupled from the first one, i.e. it is a one-dimensional difference equation (master equation) and it is piecewise linear, whereas the first component depends only on the second variable (slave equation). Thus, by straightforward algebra, E_0 is the only equilibrium when $r = 0$ and by simple calculations p_W^* and $\Pi(E_0)$ are obtained.

With $r = 1$ (all best-reply firms), map (13) becomes:

$$T^1 := \begin{cases} x(t+1) = \max\left[\frac{a-b(N-1)x(t)}{2(b+c)}, 0\right] \\ \\ y(t+1) = \max\left[\frac{a-bNx(t)}{2c}, 0\right] \end{cases} \tag{23}$$

where the first component is uncoupled from the second one, i.e. it is a one-dimensional difference equation (master equation) and it is piecewise linear, whereas the second component depends only on the first variable (slave equation). Thus, by straightforward algebra, E_1 is the only equilibrium when $r = 1$ and defining

$$K = \frac{bN^2 + 2c(N-1)}{(2c + b(N+1))^2} b^2 > 0 \tag{24}$$

by simple calculation p_{CN}^* and $\Pi(E_1)$ are obtained.

Let us now investigate the existence of other equilibria (x^*, y^*, r^*) with $r^* \in (0,1)$. If $(x^*, y^*, r^*) \in \Omega_1$, then

$$\begin{cases} x^* = R(x^*, y^*, r^*) \\ y^* = W(x^*, y^*, r^*) \\ \Delta\Pi(x^*, y^*, r^*) = 0 \end{cases} \tag{25}$$

From the third equation of the system we obtain either $x^* = y^*$ or $r^* = \frac{c(x^*+y^*)+bNy^*-a}{bN(y^*-x^*)}$. Solving the first and the second equation for $x^* = y^*$, we obtain $x^* \neq y^*$, which is a contradiction. Analogously, substituting $r^* = \frac{c(x^*+y^*)+bNy^*-a}{bN(y^*-x^*)}$ in the first and second equation we obtain $x^* = -y^* - \frac{a}{(N-1)c}$ and $x^* = -y^*$, respectively. Thus, a further equilibrium in Ω_1 cannot exist. An equilibrium (x^*, y^*, r^*) in Ω_2 implies $y^* = 0$, $x^* = R(x^*, 0, r^*)$ and $\Delta\Pi(x^*, y^*, r^*) = 0$. Solving, we obtain $x^* = \frac{a}{2Nb+cN+c}$ and $r^* = \frac{2b+c}{b}$. Since $r^* > 1$, an equilibrium in Ω_2 cannot exist. Moreover, the only possible equilibrium in Ω_3 is $(0, 0, r^*)$ which does not belong to such region. Thus, E_0 and E_1 are the only possible equilibria of the model (13). ■

Proof (of Proposition 1) A straightforward computation allows us to obtain the Jacobian matrix of model (13), when $r = 0$ and $(x, y, r) \in \Omega_1$, given by

$$J(x,y,0) = \begin{bmatrix} 0 & \frac{-b(N-1)}{2(b+c)} & \frac{b(N-1)(y-x)}{2(b+c)} \\ 0 & \frac{-bN}{2c} & \frac{bN(y-x)}{2c} \\ 0 & 0 & J_{33}(x,y,0) \end{bmatrix} \quad \text{where} \quad J_{33}(x,y,0) = e^{-\beta(a-bNy-c(x+y))(y-x)} \tag{26}$$

which is singular and upper triangular. Thus, the associated eigenvalues are the diagonal entries and the eigenvalues for the Jacobian matrix computed at the Walrasian equilibrium, i.e. $J(E_0)$, are:

$$\lambda_1^W = 0, \quad \lambda_2^W = \frac{-bN}{2c} = \frac{\frac{\partial P(E_0)}{\partial y}}{\frac{\partial^2 C}{\partial y^2}}, \quad \lambda_3^W = J_{33}(E_0) = e^{-\beta c\left(\frac{ab}{(2c+bN)2(b+c)}\right)^2} \tag{27}$$

Imposing the well-known conditions for the local asymptotic stability of a fixed point, i.e. all eigenvalues of $J(E_0)$ inside the unit circle, we obtain the first part of the proposition.

Similar computation allows us to obtain the Jacobian matrix of model (13), when $r = 1$ and $(x, y, r) \in \Omega_1$, given by

$$J(x,y,1) = \begin{bmatrix} \frac{\partial R(x,y,1)}{\partial x} & 0 & \frac{b(N-1)(y-x)}{2(b+c)} \\ \frac{-bN}{2c} & 0 & \frac{bN(y-x)}{2c} \\ 0 & 0 & J_{33}(x,y,1) \end{bmatrix} \quad \text{where} \quad J_{33}(x,y,1) = e^{-\beta(a-bNx-c(x+y))(y-x)} \tag{28}$$

Thus, the eigenvalues for the Jacobian matrix computed at the Cournot-Nash equilibrium, i.e. $J(E_1)$, are:

$$\lambda_1^{CN} = \frac{\partial R(E_1)}{\partial x} = \frac{-b(N-1)}{2(b+c)}, \quad \lambda_2^{CN} = 0, \quad \lambda_3^{CN} = e^{\beta \frac{a^2 b^2}{4c(2c+b(N+1))^2}} \tag{29}$$

Noting that $\lambda_3^{CN} > 1$, the transverse instability of the Cournot-Nash equilibrium follows. ■

Proof (of Proposition 2) The restriction of map (13) to the invariant plane $r = 0$ reduces to the two-dimensional map $T^0 : \mathbb{R}_+^2 \to \mathbb{R}_+^2$ defined in (22). Let us divide the plane $\mathbb{R}_+^2$ into the following three regions:

$$\begin{aligned} \Omega_1^0 &= \left\{(x,y) \,|\, 0 \le y < \tfrac{a}{bN}\right\} \\ \Omega_2^0 &= \left\{(x,y) \,|\, \tfrac{a}{bN} \le y < \tfrac{a}{b(N-1)}\right\} \\ \Omega_3^0 &= \left\{(x,y) \,|\, \tfrac{a}{b(N-1)} \le y\right\} \end{aligned} \tag{30}$$

The lines

$$\tilde{BC}_1 := y = \frac{a}{bN} \quad \text{and} \quad \tilde{BC}_2 := y = \frac{a}{b(N-1)} \tag{31}$$

mark the borders of non differentiability between regions Ω_1^0 and Ω_2^0 and between Ω_2^0 and Ω_3^0, respectively.

By straightforward considerations, see Lemma (1), it follows that $\tilde{E}_0 = \left(q_W^* \frac{b+2c}{2(b+c)}, q_W^*\right) \in \Omega_1^0$, with $q_W^* = \frac{a}{2c+bN}$, is the unique fixed point of T^0.

For $bN < 2c$, we have that all points of region Ω_2^0 and Ω_3^0 are mapped in one iteration in Ω_1^0. In fact, all points of region Ω_2^0 are mapped in $\{(y,x) \,|\, y = 0,\, x \ge 0\} \in \Omega_1^0$ and all points of Ω_3^0 are mapped in $(0,0) \in \Omega_1^0$. Moreover, the map T^0 is linear in Ω_1^0 with a unique inner fixed point $\tilde{E}_0$. The eigenvalues associated to $\tilde{E}_0$ are $\tilde{\lambda}_1^W = 0$ and $\tilde{\lambda}_2^W = \frac{-bN}{2c}$ with $-1 < \tilde{\lambda}_2^W < 0$. Thus, $\tilde{E}_0$ is locally asymptotically stable and the image of region Ω_1^0 is the manifold spanned by the eigenvector $v = \left[1, \frac{N(b+c)}{c(N-1)}\right]'$ associated to $\tilde{\lambda}_2^W$:

$$x = \frac{c(N-1)}{N(b+c)} y + \frac{a}{2N(b+c)} \tag{32}$$

Noting that after one iteration $y \in \left[0, \frac{a}{2c}\right]$, it follows that the following region:

$$\tilde{A} = \left\{(x,y) \,|\, y \in \left[0, \frac{a}{2c}\right],\; x = \frac{c(N-1)}{N(b+c)} y + \frac{a}{2N(b+c)}\right\} \tag{33}$$

is invariant and attracts all points of $\mathbb{R}_+^2$ in at most two iterations. In addition, $\tilde{E}_0 \in \tilde{A}$.

For $bN = 2c$, the eigenvalues associated to $\tilde{E}_0 \in \tilde{A}$ are $\tilde{\lambda}_1^W = 0$ and $\tilde{\lambda}_2^W = -1$. Then, the linearity of T^0 in $\tilde{A}$ implies a degenerate flip bifurcation through which an infinity of 2-cycles that fill region $\tilde{A}$ are originated. It follows that $\tilde{\mathcal{C}}_1^W = \{\bar{\mathcal{C}}_0^W, \underline{\mathcal{C}}_0^W\}$, where

$$\bar{\mathcal{C}}_0^W = \left(\frac{a}{2(b+c)}, \frac{a}{2c}\right) \quad \text{and} \quad \underline{\mathcal{C}}_0^W = \left(a\frac{2c - b(N-1)}{4c(b+c)}, 0\right) \tag{34}$$

are the borders of region $\tilde{A}$, is a 2-cycle. Since $\bar{\mathcal{C}}_0^W$ lies on the border of non differentiability $\tilde{BC}_1$, we have that $\tilde{\mathcal{C}}_1^W$ undergoes a persistence border collision.

For $b(N-1) < 2c < bN$, we have that $T^0\left(\underline{\mathcal{C}}_0^W\right) = \bar{\mathcal{C}}_0^W \in \Omega_2^0$ and all points of subregion Ω_2^0 are mapped in $\underline{\mathcal{C}}_0^W$ in one iteration. Thus, $\tilde{\mathcal{C}}_1^W$ is a 2-cycle for T^0. Moreover, we have that $\tilde{\lambda}_1^W = 0$ and $\tilde{\lambda}_2^W < -1$. Thus, $\tilde{E}_0$ is a repellor and by the linearity of map T^0 is possible to exclude the existence of 2-cycles in Ω_1^0. This implies that, except for $\tilde{\mathcal{C}}_1^W$, all 2-cycles that filled region $\tilde{A}$ have disappeared. Since all points of Ω_2^0 are mapped in $\underline{\mathcal{C}}_0^W$, all points of Ω_3^0 are mapped in $(0,0)$, $T^0(0,0) = \bar{\mathcal{C}}_0^W$ and in Ω_3^0 the map T^0 is linear with inside a unique repellor $\tilde{E}_0$, it follows that $\tilde{\mathcal{C}}_1^W$ attracts all the point of $\mathbb{R}_+^2$ except for $\tilde{E}_0$.

For $b(N-1) = 2c$, $\bar{\mathcal{C}}_0^W$ collides with the border of non-differentiability $\tilde{BC}_2$. Thus $\tilde{\mathcal{C}}_1^W$ undergoes a border collision bifurcation and disappears. Moreover, let us define $\hat{\mathcal{C}}_0^W = (0,0)$. Since $T^0\left(\hat{\mathcal{C}}_0^W\right) = \bar{\mathcal{C}}_0^W$, $T^0\left(\bar{\mathcal{C}}_0^W\right) = \hat{\mathcal{C}}_0^W$ and $\bar{\mathcal{C}}_0^W \in \tilde{BC}_2$, it follows that the 2-cycle $\tilde{\mathcal{C}}_2^W = \left\{\bar{\mathcal{C}}_0^W, \hat{\mathcal{C}}_0^W\right\}$ appears through a border collision bifurcation.

For $2c < b(N-1)$, as $T^0\left(\hat{\mathcal{C}}_0^W\right) = \bar{\mathcal{C}}_0^W$ and $T^0\left(\bar{\mathcal{C}}_0^W\right) = \hat{\mathcal{C}}_0^W$, it follows that $\tilde{\mathcal{C}}_2^W$ is a 2-cycle for T^0. In addition, since all points of Ω_3^0 are plotted in $\hat{\mathcal{C}}_0^W$, all points of Ω_2^0 are mapped in Ω_1^0, and in this latter region the map is linear with a unique inner repellor $\tilde{E}_0$, it follows that $\tilde{\mathcal{C}}_2^W$ attracts all the points of $\mathbb{R}_+^2$ except for $\tilde{E}_0$.

Since $\frac{\partial^2 C}{\partial y^2} = 2c$, $\frac{\partial P(E_0)}{\partial y} = -bN$ and T^0 is the restriction of map (13) on the invariant plane $r = 0$, the results on E_0, $\mathcal{C}_1^W$, $\mathcal{C}_2^W$ and region A, defined as in the statement of the Proposition, follow by the results on $\tilde{E}_0$, $\tilde{\mathcal{C}}_1^W$, $\tilde{\mathcal{C}}_2^W$ and $\tilde{A}$.

By the smoothness of map (13) in a neighborhood of $\bar{\mathcal{C}}^W$ and $\underline{\mathcal{C}}^W$, and from the Jacobian matrix of this map when $r = 0$ (see, e.g., (26)) follows that the transverse eigenvalue of $\mathcal{C}_1^W$ is given by $J_{33}\left(\bar{\mathcal{C}}^W\right) J_{33}\left(\underline{\mathcal{C}}^W\right)$. Since $J_{33}\left(\bar{\mathcal{C}}^W\right) > 1$ and $J_{33}\left(\underline{\mathcal{C}}^W\right) \geq 1$ the transverse instability of $\mathcal{C}_1^W$ follows. By similar considerations and calculations we have that $J_{33}\left(\bar{\mathcal{C}}^W\right) J_{33}\left(\hat{\mathcal{C}}^W\right) > 1$, from which the transverse instability of the 2-cycle $\mathcal{C}_2^W$ follows. ■

Proof (of Proposition 3) The restriction of map (13) to the invariant plane $r = 1$ reduces to the two-dimensional map $T^1 : \mathbb{R}_+^2 \to \mathbb{R}_+^2$ defined in (23). Let us divide the plane $\mathbb{R}_+^2$ into the following three regions:

$$\begin{aligned} \Omega_1^1 &= \left\{(x, y)\,|0 \leq x < \tfrac{a}{bN}\right\} \\ \Omega_2^1 &= \left\{(x, y)\,|\tfrac{a}{bN} \leq x < \tfrac{a}{b(N-1)}\right\} \\ \Omega_3^1 &= \left\{(x, y)\,|\tfrac{a}{b(N-1)} \leq x\right\} \end{aligned} \tag{35}$$

The lines

$$\bar{BC}_1 := x = \frac{a}{bN} \quad \text{and} \quad \bar{BC}_2 := x = \frac{a}{b\,(N-1)} \tag{36}$$

mark the borders of non differentiability between regions Ω_1^1 and Ω_2^1 and between Ω_2^1 and Ω_3^1, respectively.

By straightforward considerations, see Lemma (1), it follows that $\tilde{E}_1 = \left(q_{CN}^*, q_{CN}^* \frac{b+2c}{2c}\right) \in \Omega_1^1$, with $q_{CN}^* = \frac{a}{2c+b(N+1)}$, is the unique fixed point of T^1.

For $b\,(N-3) < 2c$, we have that all points of region Ω_2^1 and Ω_3^1 are mapped in one iteration in Ω_1^1. In fact, all points of region Ω_2^1 are mapped in $\{(y, x)\,|y = 0,\, x \geq 0\} \in \Omega_1^1$ and all points of Ω_3^1 are mapped in $(0, 0) \in \Omega_1^1$. Moreover, the map T^1 is linear in Ω_1^1 with a unique inner fixed point $\tilde{E}_1$. The eigenvalues associated to $\tilde{E}_1$ are $\tilde{\lambda}_1^{CN} = \frac{-b(N-1)}{2(b+c)}$ and $\tilde{\lambda}_2^{CN} = 0$ with $-1 < \tilde{\lambda}_1^{CN} < 0$. Thus, $\tilde{E}_1$ is locally asymptotically stable and the image of region Ω_1^1 is the manifold spanned by the eigenvector $v = \left[1, \frac{N(b+c)}{c(N-1)}\right]'$ associated to $\tilde{\lambda}_1^{CN}$:

$$y = \frac{N\,(b+c)}{c\,(N-1)}x - \frac{a}{2c\,(N-1)} \tag{37}$$

Noting that, after one iteration, $x \in \left[0, \frac{a}{2(b+c)}\right]$ and y cannot be negative, it follows that the region:

$$\tilde{B} = \left\{(x, y)\,|x \in \left[0, \frac{a}{2\,(b+c)}\right],\, y = \max\left[\frac{N\,(b+c)}{c\,(N-1)}x - \frac{a}{2c\,(N-1)}, 0\right]\right\} \tag{38}$$

is invariant and attracts all points of $\mathbb{R}_+^2$ in at most two iterations. In addition, $\tilde{E}_1 \in \tilde{B}$.

For $b\,(N-3) = 2c$, the eigenvalues associated to $\tilde{E}_1 \in \tilde{B}$ are $\tilde{\lambda}_1^{CN} = -1$ and $\tilde{\lambda}_2^{CN} = 0$. Then, the linearity (with respect to x) of T^1 in $\tilde{B}$ implies a degenerate flip bifurcation through which an infinity of 2-cycles that fill region $\tilde{B}$ are originated. It follows that $\tilde{\mathcal{C}}^{CN} = \left\{\bar{\mathcal{C}}_1^{CN}, \underline{\mathcal{C}}_1^{CN}\right\}$, where

$$\bar{\mathcal{C}}_1^{CN} = \left(\frac{a}{2\,(b+c)}, \frac{a}{2c}\right) \quad \text{and} \quad \underline{\mathcal{C}}_1^{CN} = (0, 0) \tag{39}$$

are the borders of region $\tilde{B}$, is a 2-cycle. Since $\bar{\mathcal{C}}_1^{CN}$ lies on the border of non differentiability $\bar{BC}_2$, we have that $\tilde{\mathcal{C}}^{CN}$ undergoes a persistence border collision.

For $2c < b\,(N-3)$, we have that $T^1\left(\underline{\mathcal{C}}_1^{CN}\right) = \bar{\mathcal{C}}_1^{CN} \in \Omega_3^1$ and all points of subregion Ω_3^1 are mapped in $\underline{\mathcal{C}}_1^{CN}$ in one iteration. Thus, $\tilde{\mathcal{C}}^{CN}$ is a 2-cycle for T^1. Moreover, we have that $\tilde{\lambda}_1^{CN} < -1$ and $\tilde{\lambda}_2^{CN} = 0$. Thus, $\tilde{E}_1$ is a repellor and by the linearity of map T^1 (with respect to x) it is possible to exclude the existence of 2-cycles in Ω_1^1 and Ω_2^1. This implies that, except for $\tilde{\mathcal{C}}_1^{CN}$, all 2-cycles that filled region $\tilde{B}$ have disappeared. Moreover, since all points of Ω_2^1 are mapped in Ω_1^1, all points of Ω_3^1 are mapped in $\underline{\mathcal{C}}_1^{CN}$, $T^1\left(\underline{\mathcal{C}}_1^{CN}\right) = \bar{\mathcal{C}}_1^{CN}$ and in Ω_1^1 the map T^1 is linear with inside a unique repellor $\tilde{E}_1$, it follows that $\tilde{\mathcal{C}}^{CN}$ attracts all the point of $\mathbb{R}_+^2$ except for $\tilde{E}_1$.

Since $\frac{\partial R(E_1)}{\partial x} = \frac{-b(N-1)}{2(b+c)}$ and T^1 is the restriction of map (13) on the invariant plane $r = 1$, the results on E_1, $\mathcal{C}^{CN}$ and region B, defined as in the statement of the Proposition, follow by the results on $\tilde{E}_1$, $\tilde{\mathcal{C}}^{CN}$ and $\tilde{B}$.

By the smoothness of map (13) in a neighborhood $\bar{\mathcal{C}}^{CN}$ and $\underline{\mathcal{C}}^{CN}$, and from the Jacobian matrix of this map when $r = 1$, see, e.g., (28), follows that the transverse eigenvalue of $\mathcal{C}^{CN}$ is given by $J_{33}\left(\bar{\mathcal{C}}^{CN}\right) J_{33}\left(\underline{\mathcal{C}}^{CN}\right)$. Since $J_{33}\left(\bar{\mathcal{C}}^{CN}\right) < 1$ and $J_{33}\left(\underline{\mathcal{C}}^{CN}\right) \leq 1$, the transverse attractiveness of $\mathcal{C}_1^{CN}$ follows. ■

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