The Bulletin of Aichi Univ. of Education, 49 (Natural Science), pp. 23~26, March, 2000

Fuzzy Injections

Moritoshi SASAKI

(Department of Mathematical Science)

1. PRELIMINARIES

Let X be a set. A fuzzy set on X is a concept A on X which is characterized by a function $mA : X \to [0,1]$. This function mA is called the membership function of A. For a, $b \in X$, if $mA(a) \geqq mA(b)$, then a is related to A more strongly than b. For a fuzzy set on X, the set

$$\mathrm{supp}(A) = \{x \mid x \in X,\ mA(x) > 0\}$$

is called the support set of A.

Let X and Y be sets. A fuzzy set R on $X \times Y$ is called a fuzzy relation on $X \times Y$. Furthermore let S be a fuzzy relation on $Y \times Z$. The sup-$\wedge$ composition of R and S is a fuzzy relation on $X \times Z$. The membership function of RoS is defined by

$$mRoS(x,\ z) = \sup_{y \in Y} [mR(x,\ y) \wedge mS(y,z)] \quad \forall x \in X, \forall z \in Z.$$

In this paper, $\alpha \wedge \beta$ and $\alpha \vee \beta$ mean $\min[\alpha, \beta]$ and $\max[\alpha, \beta]$, respectively.

We have defined fuzzy equalities and fuzzy functions as follows.

[Definition 1. [1]]

Let X be a set. If a fuzzy relation EX on $X \times X$ satisfies

(e1) $mEX(a,b) = 1 \iff a = b \quad \forall a, \forall b \in X,$

(e2) $mEX(a,b) = mEX(b,a) \quad \forall a, \forall b \in X,$

(e3) $mEX(a,c) \geqq (mEX(a,b) + mEX(b,c) - 1) \quad \vee 0 \ \forall a, \forall b, \forall c \in X,$

then EX is called a fuzzy equality on X.■

[Definition 2. [1]]

Let X and Y be sets. Let R be a fuzzy relation on $X \times Y$. Let EX and EY be fuzzy equalities on X and Y, respectively. If R satisfies

(f1) $\forall a \in X, \exists\, c \in Y$ such that $mR(a,c) > 0,$

(f2) $mR(a,c) > 0$ and $mR(b,d) > 0$

$\Rightarrow mEY(c,d) \geqq mEX(a,b) \wedge [mR(a,c) \vee mR(b,d)],$

$\forall a, \forall b \in X, \forall c, \forall d \in Y,$

then R is called a fuzzy function and denoted by $R : X \sim\to Y$.■

2. FUZZY INJECTIONS

Let X and Y be sets and $f : X \rightarrow Y$ be a function. If f satisfies

$$\text{(in1) } f(a)=f(b) \Rightarrow a=b \quad \forall a, \forall b \in X, \tag{1}$$

then f is called an injection. Let EX and EY be fuzzy equalities on X and Y, respectively.

Then the condition (in1) is equivalent to

$$mEY(f(a), f(b))=1 \Rightarrow mEX(a,b)=1. \tag{2}$$

Furthermore, since f is a function, f satisfies

$$f(a) \neq f(b) \Rightarrow a \neq b. \tag{3}$$

This condition is equivalent to

$$mEY(f(a), f(b))=0 \Rightarrow mEX(a,b)=0. \tag{4}$$

Therefore if f is a injection, then

$$mEY(f(a), f(b))=mEX(a,b), \tag{5}$$

by (2) and (4). But this condition is too strong. We want to use a weaker condition. There exist following four cases about $mEY(f(a), f(b))$ and $mEX(a,b)$.

$$\text{(c1) } mEY(f(a), f(b))=0 \text{ and } mEX(a,b)=0 \tag{6}$$
$$\text{(c2) } mEY(f(a), f(b))=0 \text{ and } mEX(a,b)=1 \tag{7}$$
$$\text{(c3) } mEY(f(a), f(b))=1 \text{ and } mEX(a,b)=0 \tag{8}$$
$$\text{(c4) } mEY(f(a), f(b))=1 \text{ and } mEX(a,b)=1 \tag{9}$$

If f is an injection, then f satisfies (c1) and (c4). The case (c2) and the fact that f is a function contradict each other. So, we consider the condition which selects (c1) and (c4) from (c1), (c3) and (c4). In this context, we propose the condition

$$mEY(f(a), f(b)) \leqq mEX(a,b) \tag{10}$$

Now our definition of a fuzzy injection is as follows. Notice that $f(a)=c$ corresponds to $mR(a,c)=1$. Furthermore, $mR(a,c) \in [0,1]$ in the case of a fuzzy function.

[Definition 3.]

Let X and Y be sets and let $R : X \sim\rightarrow Y$ be a fuzzy function. Let EX and EY be fuzzy equalities on X and Y, respectively. If R satisfies

$$\text{(fi1) } mR(a,c)>0 \text{ and } mR(b,c)>0$$
$$\Rightarrow mEY(c,d) \leqq mEX(a,b) \quad \forall a, \forall b \in X, \forall c, \forall d \in Y, \tag{11}$$

then R is called fuzzy injection.■

[Theorem 1.]

Let X and Y be sets and $R : X \sim\rightarrow Y$ be a fuzzy injection. Let EX and EY be fuzzy equalities on X and Y, respectively. And let $(a,c) \in \text{supp}(R)$ and $(b,d) \in \text{supp}(R)$ be arbitrary. Then the folowing propositions hold.

$$\text{(t1) } mEX(a,b) \leqq [mR(a,c) \vee mR(b,d)]$$
$$\Rightarrow mEY(c,d)=mEX(a,b) \tag{12}$$

$$(t2)\ mEX(a,b) \geqq [mR(a,c) \vee mR(b,d)] \Rightarrow [mR(a,c) \vee mR(b,d)] \leqq mEY(c,d) \leqq mEX(a,b) \quad (13)$$

(proof)

Since R is a fuzzy function, by (f2) in [Definition 2.],

$$mEY(c,d) \geqq mEX(a,b) \wedge [mR(a,c) \vee mR(b,d)]. \quad (14)$$

Furthermore, since R is a fuzzy injection, by (fi1) in [Definition 3.],

$$mEY(c,d) \leqq mEX(a,b). \quad (15)$$

Therefore, by (14) and (15),

$$mEX(a,b) \wedge [mR(a,c) \vee mR(b,d)] \leqq mEY(c,d) \leqq mEX(a,b). \quad (16)$$

holds.

Firstly, let $mEX(a,b) \leqq [mR(a,c) \vee mR(b,d)]$. Then by (16),

$$mEX(a,b) \leqq mEY(c,d) \leqq mEX(a,b) \Leftrightarrow mEY(c,d) = mEX(a,b). \quad (17)$$

Therefore (t1) holds.

Secondly, let $mEX(a,b) \geqq [mR(a,c) \vee mR(b,d)]$. Then by (16),

$$[mR(a,c) \vee mR(b,d)] \leqq mEY(c,d) \leqq mEX(a,b). \quad (18)$$

Thus (t2) holds.

(Q.E.D.)

In [Theorem 1.], (t1) means that if the grade of that a is transfered to c or b is transfered to d is high, then the property of R is similar to an ordinary injection. In this case, $mEY(c,d) = mEX(a,b) = 1$ is not always true, but $mEY(c,d) = mEX(a,b)$ holds.

On the other hand, (t2) means that if both of the grades of that a is transfered to c and b is transfered to d are low, then the grade of "the image of a is nearly equal to the image of b" is allowed to become rather smaller value than the grade of "a is nealy equal to b". Which reflects the lack of the confidence about the transfer.

[Theorem 2.]

Let X, Y and Z be sets. Let $R: X \sim\rightarrow Y$ and $S: Y \sim\rightarrow Z$ be fuzzy injections. Let EX, EY and EZ be fuzzy equalities on X, Y and Z, respectively. Then the sup-$\wedge$composition RoS is a fuzzy injection.

(proof)

Let $(a,s) \in \mathrm{supp}(RoS)$ and $(b,t) \in \mathrm{supp}(RoS)$ be arbitrary. It suffices to show that $mEZ(s,t) \leqq mEX(a,b)$.

By the definition of sup-$\wedge$composition,

$$mRoS(a,s) = \sup_{y \in Y}[mR(a,y) \wedge mS(y,s)], \quad (19)$$

$$mRoS(b,t) = \sup_{y \in Y}[mR(b,y) \wedge mS(y,t)], \quad (20)$$

Therefor,

$$\exists c \in Y, mR(a,c) \wedge mS(c,s) > 0, \quad (21)$$

$$\exists d \in Y, mR(b,d) \wedge mS(d,t) > 0. \quad (22)$$

Since R and S are fuzzy injections, by (21) and (22)

$$mEZ(s,t) \leqq mEY(c,d) \text{ and } mEY(c,d) \leqq mEX(a,b). \quad (23)$$

Thus,

$$mEZ(s,t) \leqq mEX(a,b).$$

Therefore RoS is a fuzzy injection.

(Q.E.D.)

3. CONCLUSIONS

We have defined a fuzzy injection as a special fuzzy function. It includes an ordinary injection. Furthermore the sup-$\wedge$ composition of two fuzzy injections becomes a fuzzy injection, too.

In this paper, we use binary operations $\vee$(max) and $\wedge$(min), but in the future, other t-conorms and t-norms must be considered.

REFERENCES

[1] Sasaki, M. Fuzzy functions, Fuzzy Sets and Systems 55 (1993) 295-301.
[2] Dubois, D. and Prade, H. Fuzzy Sets and Systems : Theory and Applications, Academic Press (1980).
[3] Negoita, C. V. and Ralescu, D. A. Applications of Fuzzy Sets to Systems Analysis, Interdisciplinary Systems Research, Vol. 11 (1975)
[4] 水本雅晴，ファジィ理論とその応用，サイエンス社 (1988)
[5] 日本ファジィ学会編，講座ファジィ第2巻ファジィ集合，日刊工業新聞社 (1992)
[6] 寺野寿郎，浅居喜代治，菅野道夫共編，ファジィシステム入門，オーム社 (1997)
[7] 小寺平治，入門=ファジィ数学，遊星社 (1995)

(平成11年9月10日受理)