The Bulletin of Aichi Univ. of Education, 49 (Natural Science), pp. 1 ~ 3, March, 2000

Some Properties of Generalized Denominator Ideals

Mitsuo KANEMITSU, Susumu ODA* and Ken-ichi YOSHIDA**
(Department of Mathematics, Aichi University of Education,
Igaya-cho, Kariya, Aichi 448-8542 JAPAN),
(*Matsusaka Commercial High School,
Toyohara, Matsusaka, Mie 515-0205 JAPAN) and
(**Department of Applied Mathematics, Okayama University of Science,
Ridai-cho, Okayama 700-0005 JAPAN).
(*Received August 3, 1999*)

Let R be a Noetherian domain and $R[X]$ a polynomial ring. Let α be a non-zero element of an algebraic field extension L of the quotient field K of R and let π: $R[X] \longrightarrow R[\alpha]$ be the R-algebra homomorphism sending X to α. Let $\varphi_\alpha(X)$ be the monic minimal polynomial of α over K with deg $\varphi_\alpha(X)=d$ and write $\varphi_\alpha(X)=X^d+\eta_1X^{d-1}+\cdots+\eta_d$. For $\gamma \in K$, put $I_\gamma=(R:_R\gamma)=\{b \in R \mid b\gamma \in R\}$. Let $I_{[\alpha]}:=\cap_{i=1}^d I_{\eta_i}$, which is called a *generalized denominator ideal* of R (cf. [1]). Let $J_{[\alpha]}=I_{[\alpha]}c(\varphi_\alpha(X))$, where $c(\varphi_\alpha(X))$ denotes the fractional ideal of R generated by the coefficients 1, η_1, η_2, $\cdots$, η_d of $\varphi_\alpha(X)$. Note that for $\alpha \in K$, $I_{[\alpha]}=I_\alpha$ and $J_{[\alpha]}=I_{[\alpha]}+\alpha I_{[\alpha]}$. The element $\alpha \in L$ is called an *anti-integral element of degree d* over R if Ker $\pi=I_{[\alpha]}\varphi_\alpha(X)R[X]$. When α is an anti-integral element over R, we say that $R[\alpha]$ is an *anti-integral extension* of R. Also, α is called a *super-primitive element of degree d* over R if $J_{[\alpha]} \not\subseteq p$ for any $p \in \mathrm{Dp}_1(R)$, the set of depth one prime ideals of R (cf. [4], [8]).

In [5], S. Oda and K. Yoshida studied some properties of a generalized denominator ideal $I_{[\alpha]}$ and the structure of a ring extension $R[\alpha]$ of R satisfying $I_{[\alpha]}R[\alpha]=R[\alpha]$. The related topics are also seen in [3].

In this paper, we study some relations between $I_{[\alpha]}R[\alpha] \cap R$ and $R[\eta_1, \cdots, \eta_d]$, and some conditions that grade $(I_{[\alpha]}+I_{[\alpha^{-1}]})>1$.

Throughout this paper, we use the following notation unless otherwise specified:
Let R be a Noetherian domain, and let α be a non-zero element of an algebraic field extension L of the quotient field K of R. Let π: $R[X] \longrightarrow R[\alpha]$ be the R-algebra homomorphism sending X to α. Let $\varphi_\alpha(X)$ be the monic minimal polynomial of α over K with deg $\varphi_\alpha(X)=d$ and write $\varphi_\alpha(X)=X^d+\eta_1X^{d-1}+\cdots+\eta_d$, let $I_{[\alpha]}=\cap_{i=1}^d(R:_R\eta_i)$ and let $J_{[\alpha]}=I_{[\alpha]}c(\varphi_\alpha(X))$.

All rings treated in this paper are commutative rings with identity. The reference of unexplained technical terms is [2].

Definition (cf. [1] and [7]). An element $\alpha \in L$ is called an *exclusive element* over R if $R[\alpha] \cap K=R$.

Proposition 1. *Assume that α is a super-primitive element of degree d over R and that α is an exclusive element over R. If $I_{[\alpha]}R[\alpha]=R[\alpha]$, then α is integral over R and $R[\alpha]$ is a free R-module of rank d.*

Proof. By Theorem 15 in [5], if $I_{[\alpha]}R[\alpha]=R[\alpha]$ then $R[\eta_1, \cdots, \eta_d] \subseteq R[\alpha]$. Since α is exclusive, it follows that $A \cap K=R$. Therefore $R[\eta_1, \cdots, \eta_d] \subseteq R$, and so $\eta_1, \cdots, \eta_d \in R$. Since $\alpha^d+\eta_1\alpha^{d-1}+\cdots+\eta_d=0$, we have that α is an integral element over R and $R[\alpha]$ is a free R-module of rank d. □

Corollary 2. *If p is a prime ideal of R such that $I_{[\alpha]} \subseteq p$, then $I_{[\alpha]}R[\alpha] \cap R \subseteq p$.*

Proof. Suppose that $I_{[\alpha]}R[\alpha] \cap R \not\subseteq p$. Then there exists an element $s \in I_{[\alpha]}R[\alpha] \cap R$ and $s \notin p$. Hence $I_{[\alpha]}R[\alpha]_p=R[\alpha]_p$. We have that $I_{[\alpha]} \not\subseteq p$ by Proposition 1. This is a contradiction. Thus $I_{[\alpha]}R[\alpha] \cap R \subseteq p$. □

Theorem 3. *Let R be a Noetherian domain with quotient field K. If α is a super-primitive element of degree* *d over R and α is an exclusive element, then we have that* $\sqrt{I_{[\alpha]}R[\alpha]} \cap R = \sqrt{I_{[\alpha]}}$.

Proof. It is clear that $\sqrt{I_{[\alpha]}R[\alpha]} \cap R \supseteq \sqrt{I_{[\alpha]}}$. The converse inclusion follows from Corollary 2. □

Remark. It is known that $I_{[\alpha]}R[\alpha]=R[\alpha]$ if and only if $R[\eta_1, \cdots, \eta_d]$ is a flat extension over R and $R[\eta_1, \cdots, \eta_d] \subseteq R[\alpha]$ (cf. [5, (15.1)]).

Theorem 4. *Assume that α is a super-primitive element of degree d over R. Then*

$$\sqrt{I_{[\alpha]}R[\alpha]} \cap R = \sqrt{I_{[\alpha]}R[\eta_1, \cdots, \eta_d]} \cap R \cap \sqrt{\bigcap_{i=1}^{d}(R[\alpha]:_R \eta_i)}.$$

Proof. By Remark above, $I_{[\alpha]}R[\alpha]=R[\alpha]$ if and only if $R[\eta_1, \cdots, \eta_d]$ is a flat extension over R and $R[\eta_1, \cdots, \eta_d] \subseteq R[\alpha]$. Let $p \in \mathrm{Spec}(R)$. If $p \not\supseteq I_{[\alpha]}R[\alpha] \cap R$, then we have that $I_{[\alpha]}R[\alpha]_p=R[\alpha]_p$. Therefore $R[\eta_1, \cdots, \eta_d]_p$ is a flat extension over R_p and $R[\eta_1, \cdots, \eta_d]_p \subseteq R[\alpha]_p$. Since $R[\eta_1, \cdots, \eta_d]_p= R_p[\eta_1, \cdots, \eta_d]$, $R[\eta_1, \cdots, \eta_d]_p$ is a flat extension over R_p and $R[\eta_1, \cdots, \eta_d]_p$ is a birational extension over R_p, we have that $I_{[\alpha]}R[\eta_1, \cdots, \eta_d]_p=R[\eta_1, \cdots, \eta_d]_p$ because $I_{[\alpha]}=\cap_{i=1}^{d}I_{\eta_i}$. Thus $p \not\supseteq I_{[\alpha]}R[\eta_1, \cdots, \eta_d] \cap R$. Since $R[\eta_1, \cdots, \eta_d]_p \subseteq R[\alpha]_p$, we have that $p \not\supseteq \cap_{i=1}^{d}(R[\alpha]:_R \eta_i)$. Conversely, if $p \not\supseteq I_{[\alpha]}R[\eta_1, \cdots, \eta_d] \cap R$ and $p \not\supseteq \cap_{i=1}^{d}(R[\alpha]:_R \eta_i)$ then $R[\eta_1, \cdots, \eta_d]_p$ is a flat extension of R_p and $R[\eta_1, \cdots, \eta_d]_p \subseteq R[\alpha]_p$. Therefore $I_{[\alpha]}R[\alpha]_p=R[\alpha]_p$, and have $p \not\supseteq I_{[\alpha]}R[\alpha] \cap R$. □

Considering Remark above, we have the following corollaries.

Corollary 5. *If* $R[\eta_1, \cdots, \eta_d] \subseteq R[\alpha]$, *then*

$$\sqrt{I_{[\alpha]}R[\alpha]} \cap R = \sqrt{I_{[\alpha]}R[\eta_1, \cdots, \eta_d]} \cap R.$$

Therefore $I_{[\alpha]}R[\alpha]=R[\alpha]$ *if and only if* $R[\eta_1, \cdots, \eta_d]$ *is a flat extension of R.*

Corollary 6. *If* $R[\eta_1, \cdots, \eta_d]$ *is a flat extension over R, then* $\sqrt{I_{[\alpha]}R[\alpha]} \cap R = \sqrt{\cap_{i=1}^{d}(R[\alpha]:_R \eta_i)}$. *Therefore* $I_{[\alpha]}R[\alpha]=R[\alpha]$ *if and only if* $R[\eta_1, \cdots, \eta_d] \subseteq R[\alpha]$.

In Proposition 11 in [6], S. Oda and K. Yoshida proved that if grade $(I_{[\alpha]}+I_{[\alpha^{-1}]})>1$, then $\alpha R[\alpha] \cap R= I_{[\alpha^{-1}]}$.

We consider a condition that grade$(I_{[\alpha]}+ I_{[\alpha^{-1}]})>1$.

Proposition 7. *Then the following conditions are equivalent:*

(1) grade$(I_{[\alpha]}+ I_{[\alpha^{-1}]})>1$;

(2) $I_{[\alpha]}=I_{\eta_d}$ *and* η_d *is a super-primitive element over R.*

Moreover, if these conditions are satisfied, then α is a super-primitive element of degree d over R.

Proof. (2) ⇒ (1). Since $I_{[\alpha^{-1}]}=\eta_d I_{[\alpha]}=\eta_d I_{[\eta_d]}$, we have that $I_{[\alpha]}+I_{[\alpha^{-1}]}=I_{\eta_d}+\eta_d I_{\eta_d}=J_{\eta_d}$. Since η_d is a super-primitive over R, we have that grade $(I_{[\alpha]}+I_{[\alpha^{-1}]})>1$.

(1) ⇒ (2). Since $I_{[\alpha]}=\cap_{i=1}^{d}I_{\eta_i}$, we shall show that $I_{[\alpha]} \supseteq I_{\eta_d}$. Let p be a prime divisor of $I_{[\alpha]}$. Since $I_{[\alpha]}$ is a divisorial ideal of R, we have that depth $R_p=1$. Therefore $I_{[\alpha]}+I_{[\alpha^{-1}]} \not\subseteq p$. Since $I_{[\alpha^{-1}]}=\eta_d I_{[\alpha]}$, we see that $I_{[\alpha]}(1, \eta_d)R_p=R_p$. Consequently, $I_{[\alpha]}R_p$ is an invertible ideal of R_p and $I_{[\alpha]}^{-1}=(1, \eta_d)R_p$. Since $I_{[\alpha]}R_p= I_{\eta_d}R_p$, we have that $I_{\eta_d} \subseteq I_{[\alpha]}R_p$. Therefore $I_{\eta_d} \subseteq I_{[\alpha]}$ and have $I_{[\alpha]}=I_{\eta_d}$. □

Corollary 8. *If the conditions of Proposition 7 hold, then* $\alpha R[\alpha] \cap R=I_{\eta_d^{-1}}=\eta_d I_{\eta_d}$ *and in particular,* $\eta_d I_{\eta_d} \subseteq \alpha R[\alpha]$.

Remark (cf. [8]). Assume that α is an anti-integral element over R. Then

(1) $I_{[\alpha]}=R$ if and only if α is integral over R.

(2) $J_{[\alpha]}=R$ if and only if $R[\alpha]$ is flat over R.

Theorem 9. *The following statements are equivalent*

(1) $I_{[\alpha]}+ I_{[\alpha^{-1}]}=R$;

(2) $I_{[\alpha]}=I_{\eta_d}$ *and* $J_{\eta_d}=R$.

Moreover, if one of these equivalent conditions is satisfied then all $\eta_1, \cdots, \eta_{d-1}$ *are elements of* $R[\eta_d]$.

Proof. (1) ⇒ (2). Suppose that $I_{[\alpha]}+I_{[\alpha^{-1}]}=R$. Then $\mathrm{grade}(I_{[\alpha]}+I_{[\alpha^{-1}]})>1$. By Proposition 7, we have that $I_{[\alpha]}=I_{\eta_d}$, and so $R=I_{[\alpha]}+I_{[\alpha^{-1}]}=I_{[\alpha]}+\eta_d I_{[\alpha]}=I_{\eta_d}+\eta_d I_{\eta_d}=J_{[\eta_d]}$.

(2) ⇒ (1). It follows that $R=J_{[\eta_d]}=I_{\eta_d}+\eta_d I_{\eta_d}=I_{[\alpha]}+\eta_d I_{[\alpha]}=I_{[\alpha]}+I_{[\alpha^{-1}]}$.

Next, we shall prove that $\eta_1, \cdots, \eta_{d-1} \in R[\eta_d]$. For any $P \in \mathrm{Spec}(R)$, put $p=P \cap R$. Since $J_{[\eta_d]}=R$, it follows that $R[\eta_d]$ is a flat extension over R (cf. [8]), and we have that $R[\eta_d]$ is a birational extension over R. Therefore $R[\eta_d]_p=R_p$. Since $R_p \ni \eta_d$, we have that $p \not\supseteq I_{\eta_d}=I_{[\alpha]}$, and so $p \not\supseteq I_{\eta_i}$ for any i. Therefore $\eta_1, \cdots, \eta_{d-1} \in R[\eta_d]_P$ for all $P \in \mathrm{Spec}(R[\eta_d])$. Hence $\eta_1, \cdots, \eta_{d-1} \in R[\eta_d]$. □

In the rest of this paper, we treat the case $I_{[\alpha]}=I_{\eta_1}$.

Remark. If $I_{[\alpha]}=I_{\eta_1}$ and $J_{[\eta_1]}=R$, then $R[\alpha]$ is a flat extension over R. For, it holds that $J_{[\alpha]}=I_{[\alpha]}(1, \eta_1, \cdots, \eta_d)=I_{\eta_1}(1, \eta_1, \cdots, \eta_d) \supseteq I_{\eta_1}(1, \eta_1)=J_{[\eta_1]}=R$.

References

[1] M. Kanemitsu, S. Oda and K. Yoshida, On finite generation of an intersection $R[\alpha] \cap K$, *Bull. Aichi Univ. Edu.* **48** (1999), 1-4.

[2] H. Matsumura, *Commutative Algebra* (2nd ed.), Benjamin, New York, 1980.

[3] S. Oda, M. Kanemitsu and K. Yoshida, On rings of certain type associated with simple ring-extensions, *Math. J. Okayama Univ.* **39** (1997), 81-87.

[4] S. Oda and K. Yoshida, Anti-integral extensions of Noetherian domains, *Kobe J. Math.* **5** (1988), 43-56.

[5] S. Oda and K. Yoshida, The behaviour of generalized denominator ideals in anti-integral extensions, *Math. J. Ibaraki Univ.* **29** (1997), 1-7.

[6] S. Oda and K. Yoshida, On constructions of principal ideals and denominator ideals in simple algebraic extensions, *Algebra Colloquium* **5** (1998), 355-360.

[7] S. Oda and K. Yoshida, Remarks on an exclusive extension generated by a super-primitive element, *Osaka J. Math.*, **32** (1995), 495-499.

[8] S. Oda, J. Sato and K. Yoshida, High degree anti-integral extensions of Noetherian domains, *Osaka J. Math.* **30** (1993), 119-135.