Journal of Heat Transfer

Technical Notes

This section contains shorter technical papers. These shorter papers will be subjected to the same review process as that for full papers.

One-Dimensional Heat Conduction in a Semi-infinite Solid With the Surface Temperature a Harmonic Function of Time: A Simple Approximate Solution for the Transient Behavior

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1 Introduction

Consider the temperature distribution $\vartheta(x, t)$ in a semi-infinite solid extending from the surface $x=0$ to $+\infty$. For $t<0$ the solid has the constant temperature $\vartheta=0$. For $t\geq 0$ the temperature at the surface $x=0$ is assumed to be a periodic harmonic function of time. The temperature distribution in the medium is governed by the following equations:

$$\frac{\partial \vartheta}{\partial t} = a\frac{\partial^2 \vartheta}{\partial x^2} \qquad \text{(heat conduction equation) (1)}$$

$$t=0 \text{ and } x>0\text{: } \vartheta=0 \qquad \text{(initial condition) (2)}$$

$$t\geq 0 \text{ and } x=0\text{: } \vartheta=\vartheta_o \cos(\omega t-\epsilon) \qquad \text{(boundary condition) (3)}$$

where a is the thermal diffusivity of the solid, ϑ_o represents the amplitude, ω the angular frequency, and ϵ the phase displacement of the temperature distribution at the surface.

The solution of equation (1) with the boundary condition (3) and the initial condition (2) is discussed in many classical books dealing with the theory of heat conduction in solids, e.g., Carslaw and Jaeger (1973), Eckert and Drake (1972), Özişik (1980), and Boelter et al. (1965). The resulting solution can be split off in two parts:

$$\vartheta=\vartheta_p-\vartheta_t \qquad (4)$$

ϑ_p represents the periodic part of the solution, which is a steady oscillation with the frequency given by the boundary condition, reading

$$\vartheta_p=\vartheta_o e^{-x\sqrt{\frac{\omega}{2a}}}\cos\left[\omega t - x\sqrt{\frac{\omega}{2a}}-\epsilon\right] \qquad (5)$$

This part is treated extensively in the literature and needs no further discussion.

The second term ϑ_t in equation (4) describes the transient behavior of the temperature in the solid from the beginning of the surface temperature oscillation at time $t=0$ to the steady oscillation ϑ_p at large times ($t\rightarrow\infty$):

$$\vartheta_t=\frac{2\vartheta_o}{\sqrt{\pi}}\int_o^{\frac{x}{2\sqrt{at}}}\cos\left[\omega\left(t-\frac{x^2}{4a\,\mu^2}\right)-\epsilon\right]e^{-\mu^2}d\mu \qquad (6)$$

ϑ_t is of importance only for small times and vanishes at large times. The knowledge of the behavior of the transient part ϑ_t is necessary in some engineering problems such as the prediction of the transient behavior of recuperators or hardening furnaces, when modeling as a semi-infinite solid with a harmonic boundary condition holds. Therefore, and also to satisfy theoretical requirements, a detailed investigation of ϑ_t is presented, particularly because there exists none in the literature.

2 A Short-Time Solution for the Temperature Distribution

To simplify the considerations, the following dimensionless quantities are introduced:

$$\text{dimensionless time:} \qquad \tau=\omega t \qquad (7)$$

$$\text{dimensionless coordinate:} \qquad \eta=x\sqrt{\frac{\omega}{2a}} \qquad (8)$$

$$\text{dimensionless temperature:} \qquad \theta=\vartheta/\vartheta_o \qquad (9)$$

After inserting these variables into equations (5) and (6), the following dimensionless expressions for ϑ are obtained:

$$\theta_p=e^{-\eta}\cos(\tau-\eta-\epsilon) \qquad (10)$$

$$\theta_t=\frac{2}{\sqrt{\pi}}\int_o^{\frac{\eta}{\sqrt{2\tau}}}\cos\left[\tau-\frac{\eta^2}{2\mu^2}-\epsilon\right]e^{-\mu^2}d\mu \qquad (11)$$

In order to get a short-time solution for the temperature field, equation (11) should be integrated. However, an analytical treatment would be very difficult and a numerical integration procedure cannot show the functional dependence of the solution on the variables. Therefore, another method will be employed to find a solution that is valid for short times.

After introducing the dimensionless quantities into equation (3) we get

$$\tau\geq 0 \text{ and } \eta=0\text{: } \theta=\cos(\tau-\epsilon) = \cos\epsilon\cos\tau+\sin\epsilon\sin\tau \qquad (12)$$

Equation (12) can be expanded into a power series in τ

$$\theta=\cos\epsilon\sum_{n=0}^{\infty}(-1)^n\frac{\tau^{2n}}{(2n)!}+\sin\epsilon\sum_{n=0}^{\infty}(-1)^n\frac{\tau^{2n+1}}{(2n+1)!} \qquad (13)$$

The solution of equations (1), (2), and (13) can be received by

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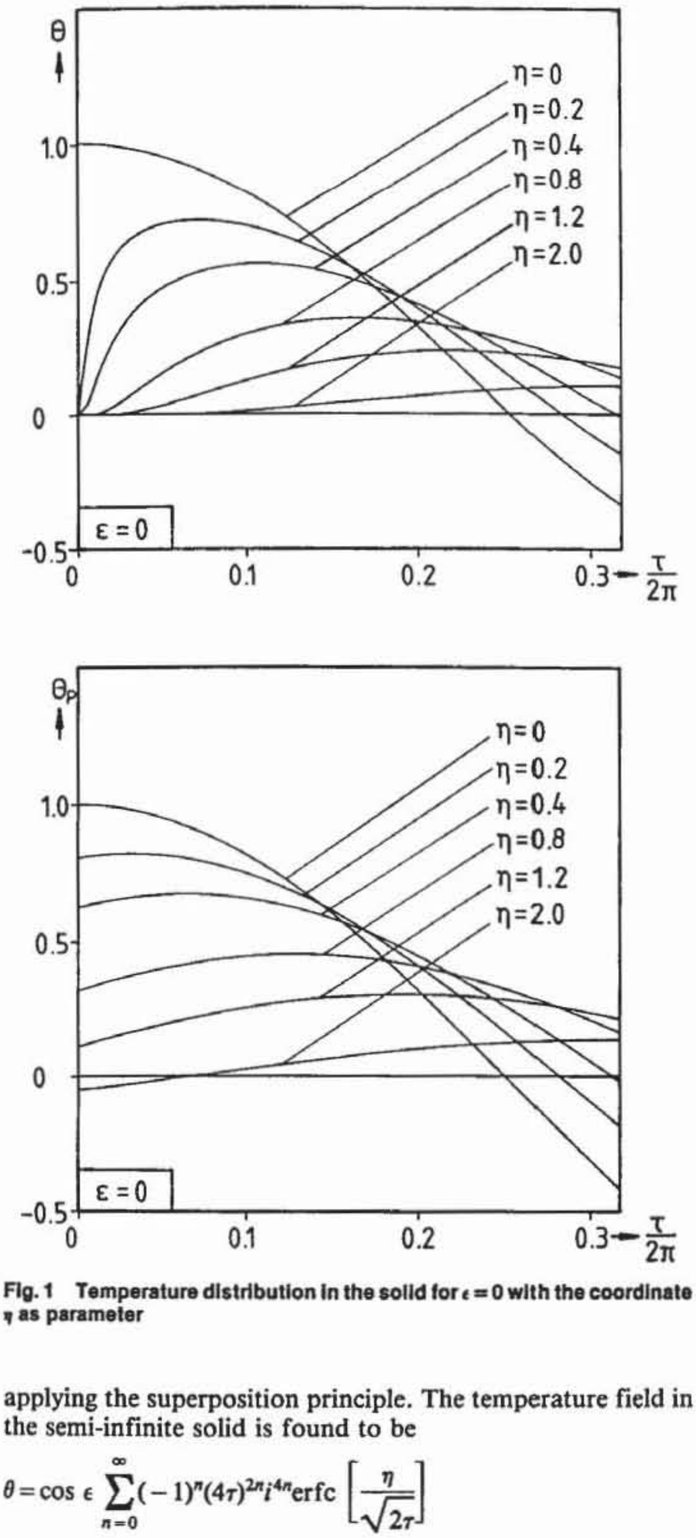

Fig. 1 Temperature distribution in the solid for $\epsilon = 0$ with the coordinate η as parameter

applying the superposition principle. The temperature field in the semi-infinite solid is found to be

$$\theta = \cos\epsilon \sum_{n=0}^{\infty}(-1)^n(4\tau)^{2n} i^{4n}\mathrm{erfc}\left[\frac{\eta}{\sqrt{2\tau}}\right] + \sin\epsilon \sum_{n=0}^{\infty}(-1)^n(4\tau)^{2n+1} i^{2(2n+1)}\mathrm{erfc}\left[\frac{\eta}{\sqrt{2\tau}}\right] \tag{14}$$

The repeated integrals of the error function complement appearing in equation (14) are defined according to Spanier and Oldham (1987)

$$i^n\mathrm{erfc}(\xi) = \int_\xi^\infty i^{n-1}\mathrm{erfc}(\mu)d\mu = \frac{1}{2^n}\sum_{j=0}^{\infty}\frac{(-2\xi)^j}{j!\Gamma\left(1+\dfrac{n-j}{2}\right)} \tag{15}$$

which is equivalent to

$$\frac{d}{d\xi}\left[i^n\mathrm{erfc}(\xi)\right] = -i^{n-1}\mathrm{erfc}(\xi) \tag{16}$$

where Γ denotes the gamma function.

Equation (14) represents the exact solution of equations (1), (2), and (3) in a new formulation. For large times τ the numerical evaluation of this formulation becomes disadvantageous, but it can be used advantageously as a short-time solution for the temperature field. By restricting equation (14) to terms including $n=2$ and observing that (14) is an alternating series, it follows that the truncation error in equation (14) is less than

$$F_{\theta\max} = |\cos\epsilon|(4\tau)^6 i^{12}\mathrm{erfc}\left(\frac{\eta}{\sqrt{2\tau}}\right) + |\sin\epsilon|(4\tau)^7 i^{14}\mathrm{erfc}\left(\frac{\eta}{\sqrt{2\tau}}\right) \tag{17}$$

The maximum values of the repeated integrals of the error function complement occur at $\eta/\sqrt{2\tau} = 0$ (i.e., at the surface $\eta=0$) and are given by

$$i^k\mathrm{erfc}(0) = \left[2^k\Gamma\left(1+\frac{k}{2}\right)\right]^{-1} \tag{18}$$

Introducing equation (18) into (17) yields

$$F_{\theta\max} = \frac{\tau^6}{6!}|\cos\epsilon| + \frac{\tau^7}{7!}|\sin\epsilon|$$

in accordance with equation (13). For $\tau = \pi/2$ we get

$$F_{\theta\max}\left(\tau=\frac{\pi}{2}\right) = \left(\frac{\pi}{2}\right)^6\frac{1}{6!}\left[|\cos\epsilon| + \frac{\pi}{14}|\sin\epsilon|\right] < 0.0214$$

This indicates the accuracy of the short-time solution (14) taking into account terms including $n=2$, provided we restrict our attention to the interval $0 \le \tau \le \pi/2$. Figure 1 shows the temperature distribution in the solid for $\epsilon = 0$ with the coordinate η as parameter. The two figures represent the solution given by equation (14), compared with the periodic solution given by equation (10). It can be seen that the difference between equations (10) and (14) disappears faster for smaller values of η than for greater ones. This means that the transient terms dies away more rapidly close to the surface than it does for large distances.

3 Heat Flux

The dimensionless heat flux is defined as

$$\psi = \frac{\dot{q}}{\lambda\vartheta_o\sqrt{\dfrac{\omega}{a}}} = \frac{-\lambda\dfrac{\partial\vartheta}{\partial x}}{\lambda\vartheta_o\sqrt{\dfrac{\omega}{a}}} = -\frac{1}{\sqrt{2}}\frac{\partial\theta}{\partial\eta} \tag{19}$$

where $\dot{q}$ denotes the dimensional heat flux and λ the thermal conductivity. Introducing equation (14) into equation (19) and using equation (16) gives

$$\psi = \frac{\cos\epsilon}{2\sqrt{\tau}}\sum_{n=0}^{\infty}(-1)^n(4\tau)^{2n} i^{4n-1}\mathrm{erfc}\left(\frac{\eta}{\sqrt{2\tau}}\right) + \frac{\sin\epsilon}{2\sqrt{\tau}}\sum_{n=0}^{\infty}(-1)^n(4\tau)^{2n+1} i^{4n+1}\mathrm{erfc}\left(\frac{\eta}{\sqrt{2\tau}}\right) \tag{20}$$

Again restricting to terms $n \le 2$, equation (20) represents a short-time solution. An error analysis analogous to that performed for the temperature field results in a truncation error less than

$$F_{\psi\max} = \frac{|\cos\epsilon|}{1\cdot3\cdot5\cdot7\cdot9\cdot11}\frac{(2\tau)^6}{\sqrt{\pi\tau}} + \frac{|\sin\epsilon|}{1\cdot3\cdot5\cdot7\cdot9\cdot11\cdot13}\frac{(2\tau)^7}{\sqrt{\pi\tau}}$$

where the maximal values occur at the surface $\eta=0$. For $\tau = \pi/2$ we get

$$F_{\psi_{max}}<0.043$$

In many technical problems the main interest is focused on the heat flux ψ^0 at the surface. Therefore, this case will be analyzed in detail. Introducing equation (18) into equation (20) yields

$$\psi^0=\frac{\cos\epsilon}{\sqrt{\pi\tau}}\sum_{n=0}^{\infty}(-1)^n\frac{(2\tau)^{2n}}{(4n-1)!!}+\frac{\sin\epsilon}{\sqrt{\pi\tau}}\sum_{n=0}^{\infty}(-1)^n\frac{(2\tau)^{2n+1}}{(4n+1)!!} \tag{21}$$

with $k!! = 1$ for $k = -1$ and $k!! = k(k-2)(k-4)\ldots\cdot5\cdot3\cdot1$ for $k = 1, 3, 5, \ldots$ resulting from relations for special arguments of the gamma function. Similar to equation (14), the numerical evaluation of equation (21) becomes disadvantageous for large times τ, but it is useful as a short-time solution. Truncating the series at $n = 2$ gives the forementioned error bound $F_{\psi_{max}}$.

In order to find a formulation for the heat flux showing favorable numerical properties also for large times, the periodic part ψ_p and the transient part ψ_t are treated separately. Introducing equations (10) and (11), respectively, into equation (19) yields

$$\psi_p=-\frac{1}{\sqrt{2}}\frac{\partial\theta_p}{\partial\eta}=e^{-\eta}\cos\left(\tau-\eta-\epsilon+\frac{\pi}{4}\right) \tag{22}$$

$$\psi_t=-\frac{1}{\sqrt{2}}\frac{\partial\theta_t}{\partial\eta}=$$

$$-\sqrt{\frac{2}{\pi}}\left\{\cos(\tau-\epsilon)\frac{\partial}{\partial\eta}\int_o^{\frac{\eta}{\sqrt{2\tau}}}\cos\left(\frac{\eta^2}{2\mu^2}\right)e^{-\mu^2}d\mu\right.$$
$$\left.+\sin(\tau-\epsilon)\frac{\partial}{\partial\eta}\int_o^{\frac{\eta}{\sqrt{2\tau}}}\sin\left(\frac{\eta^2}{2\mu^2}\right)e^{-\mu^2}d\mu\right\} \tag{23}$$

Partial differentiation and using the substitution $z = \eta/\sqrt{2\mu}$ leads to

$$\psi_t=\cos(\tau-\epsilon)\left[-\frac{2}{\sqrt{\pi}}\int_\infty^{\sqrt{\tau}}(\sin z^2)e^{-\frac{\eta^2}{2z^2}}dz-\frac{\cos\tau}{\sqrt{\pi\tau}}e^{-\frac{\eta^2}{2\tau}}\right]$$
$$+\sin(\tau-\epsilon)\left[\frac{2}{\sqrt{\pi}}\int_\infty^{\sqrt{\tau}}(\cos z^2)e^{-\frac{\eta^2}{2z^2}}dz-\frac{\sin\tau}{\sqrt{\pi\tau}}e^{-\frac{\eta^2}{2\tau}}\right] \tag{24}$$

Again we restrict the considerations to the surface $\eta=0$ and introduce the Fresnel integrals according to Spanier and Oldham (1987)

$$S(\xi)=\sqrt{\frac{2}{\pi}}\int_o^\xi(\sin z^2)dz \text{ and}$$
$$C(\xi)=\sqrt{\frac{2}{\pi}}\int_o^\xi(\cos z^2)dz \tag{25}$$

with the limiting values $S(\xi\to\infty) = C(\xi\to\infty) = 1/2$ into equation (25). This results in

$$\psi_t^o=\cos(\tau-\epsilon)\left[\sqrt{2}\left(\frac{1}{2}-S(\sqrt{\tau})\right)-\frac{\cos\tau}{\sqrt{\pi\tau}}\right]$$
$$+\sin(\tau-\epsilon)\left[\sqrt{2}\left(-\frac{1}{2}+C(\sqrt{\tau})\right)-\frac{\sin\tau}{\sqrt{\pi\tau}}\right] \tag{26}$$

Equation (26) represents the transient part of the dimensionless heat flux ψ_t^o at the surface and may be simplified with the help of the auxiliary Fresnel integrals

$$\text{Fres}(\xi)=\left[\frac{1}{2}-S(\xi)\right]\cos(\xi^2)-\left[\frac{1}{2}-C(\xi)\right]\sin(\xi^2)$$

and

$$\text{Gres}(\xi)=\left[\frac{1}{2}-S(\xi)\right]\sin(\xi^2)+\left[\frac{1}{2}-C(\xi)\right]\cos(\xi^2) \tag{27}$$

according to Spanier and Oldham (1987). This results in

$$\psi_t^o=F(\tau)\cos\epsilon+G(\tau)\sin\epsilon \tag{28}$$

where the functions $F(\tau)$ and $G(\tau)$ stand for

$$F(\tau)=\sqrt{2}\,\text{Fres}(\sqrt{\tau})-\frac{1}{\sqrt{\pi\tau}} \text{ and}$$
$$G(\tau)=\sqrt{2}\,\text{Gres}(\sqrt{\tau}) \tag{29}$$

At the surface $\eta=0$ the periodic part of the heat flux from equation (22) simplifies to

$$\psi_p^o=\cos\left(\tau-\epsilon+\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}[\cos(\tau-\epsilon)-\sin(\tau-\epsilon)] \tag{30}$$

The total dimensionless heat flux at the surface is found from equations (30) and (28), analogously to equation (4) as

$$\psi^o=\psi_p^o-\psi_t^o \tag{31}$$

The comparison of equation (31) with equation (21) leads to power series expansions for the functions $\text{Fres}(\sqrt{\tau})$ and $\text{Gres}(\sqrt{\tau})$, which are also given by Spanier and Oldham (1987). This may be considered as a proof for the correctness of our method.

For engineering applications simple approximations of the functions $F(\tau)$ and $G(\tau)$ might be helpful in using equation (28). The following approximations are recommended:

$$F(\tau)=-\left[\sqrt{\pi\tau}\left(\frac{1+\sqrt{\frac{\pi}{2}\tau+2\tau}}{1+0.1\,\tau}+\frac{4}{3}\tau^2\right)\right]^{-1} \tag{32}$$

$$G(\tau)=\left[\sqrt{2}\,\frac{1+2\sqrt{\frac{2}{\pi}\tau+3\tau}}{1+\tau}+\sqrt{\pi\tau}\,2\tau\right]^{-1}$$

These approximations show the same asymptotic behavior as equation (29). For short times this can be shown from equation (21), whereas at large times the relations

$$F(\tau)=-\frac{3}{\sqrt{\pi\tau}\,4\tau^2} \text{ and } G(\tau)=\frac{1}{\sqrt{\pi\tau}\,2\tau} \tag{33}$$

hold (see Spanier and Oldham, 1987).

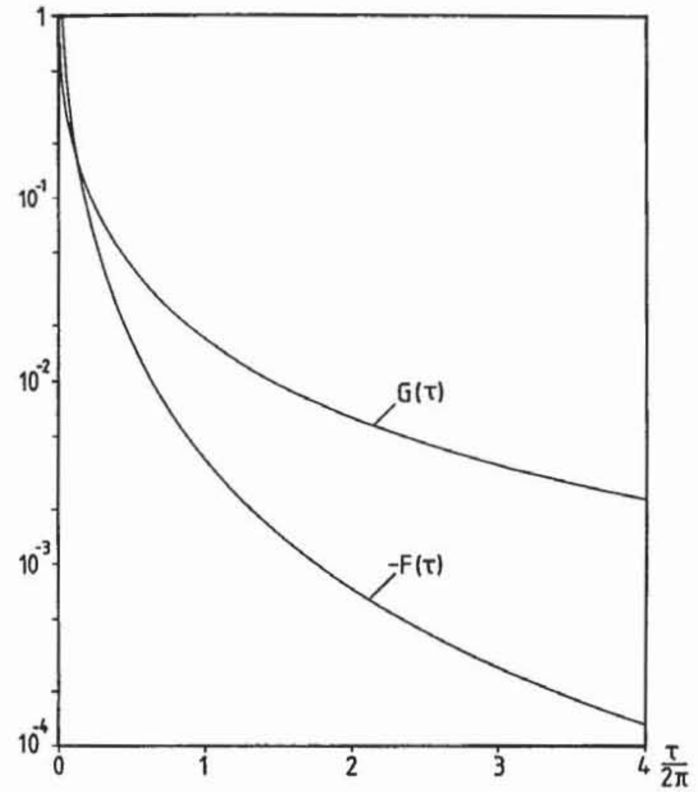

Fig. 2 Functions $F(\tau)$ and $G(\tau)$ from equation (29) representing the transient part of the heat flux ψ_t^o at the surface corresponding to equation (28)

In Fig. 2 the functions $F(\tau)$ and $G(\tau)$ are plotted according to equation (29). After the first period of the oscillation has passed ($\tau/2\pi = 1$), the values $F(2\pi) = -0.0036\ldots$ and $G(2\pi) = 0.0166\ldots$ are found, i.e., the transient part of the heat flux ψ_t^o at the surface from equation (28) has decreased to less than 2.4 percent of the periodic part ψ_p^o. In most cases, the transient part can be neglected for times larger than the first period.

The relative deviation of the approximation (32) from the exact expression (29) is less than 2.1 percent. So equation (32) represents a rather simple approximation for calculating the heat flux at the surface for arbitrary times.

Finally, it should be mentioned that the problem treated before can be adapted to the analogous problem of fluid flow in the vicinity of an oscillating flat plate (second Stokes problem). If ϑ is replaced by the flow velocity parallel to the plate, and λ is replaced by the kinematic viscosity, the heat flux at the wall $\dot{q}^o$ changes to the wall shear stress. So equation (32) represents a simple approximation to calculate the wall shear stress at any time. The periodic part of the second Stokes problem is discussed extensively by Schlichting (1982).

References

Boelter, L. M., Cherry, V. H., Johnson, V. A., and Martinelli, K. C., 1965, *Heat Transfer Notes*, McGraw-Hill, New York, pp. 218–229.

Carslaw, H. S., and Jaeger, J. C., 1973, *Conduction of Heat in Solids*, Clarendon Press, Oxford, United Kingdom.

Eckert, E. R., and Drake, R. M., 1972, *Analysis of Heat and Mass Transfer*, McGraw-Hill, New York.

Özişik, M. N., 1980, *Heat Conduction*, Wiley, New York.

Schlichting, H., 1982, *Grenzschicht-Theorie*, Braun, Karlsruhe, Federal Republic of Germany.

Spanier, J., and Oldham, K. B., 1987, *An Atlas of Functions*, Springer, Berlin.