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The Expressive Power of Simple Logical Fragments over Traces

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Abstract

We compare the expressive power of some first-order fragments and of two simple temporal logics over Mazurkiewicz traces. Over words, most of these fragments have the same expressive power whereas over traces we show that the ability of formulating concurrency increases the expressive power.

We also show that over so-called dependence structures it is impossible to formulate concurrency with the first-order fragments under consideration. Although the first-order fragments $\Delta_n[<]$ and $\mathrm{FO}^2[<]$ over partial orders both can express concurrency of two actions, we show that in general they are incomparable over traces. For $\mathrm{FO}^2[<]$ we give a characterization in terms of temporal logic by allowing an operator for parallelism.

1 Introduction

Mazurkiewicz traces are a model for concurrent systems that extends the theory of words by allowing commutation between certain letters [12]. Within three decades of trace theory, many results have been obtained. For details see *The Book of Traces* [2]. Diekert and Gastin have shown that (local) temporal logic for traces is complete [1], i.e. it can express the same set of properties as first-order logic. This makes temporal logic a suitable formalism for specifying properties of concurrent systems.

This paper contributes to a better comprehension of the relation between temporal logic and first-order logic over traces. We will compare the expressivity of certain logical fragments. Over words, these fragments are known to have the same expressive power (see [15]). As a tool for this purpose, we will use Ehrenfeucht-Fraïssé (EF) games [3]. Originally, these games were applied to the first-order logic over relational structures in general. In [5] a modified version was presented in order to characterize fragments of temporal logic over words. We continue this approach and alter EF games for words in order to capture specific logical fragments over traces.

In particular, we consider first-order formulae with two variables. This is a natural restriction, because three variables are already sufficient to express all first-order properties [8]. Traces can be given either as labeled partially ordered sets or as dependence structures, i.e. labeled sets with directed arcs only between dependent letters. Dependence structures are no restriction for the full first-order fragment [1]. We explain how the distinction between partial orders and dependence structures affects the expressivity of restricted first-order fragments and show a connection to temporal logic with and without an operator for parallelism. Surprisingly, there are simple fragments that have more exhausting characterizations on dependence structures than on partial orders, although the latter representation would seem to be more natural.

We will proceed as follows: Section 2 introduces Mazurkiewicz traces as well as some fragments of first-order and temporal logic for traces. In Section 3 two types of Ehrenfeucht-Fraïssé games are presented and they are shown to characterize fragments of first-order and temporal logic. In Section 4 we first show that the properties expressible with two variables in first-order logic can also be characterized by simple fragments of temporal logic, both for the interpretation of traces as partially ordered sets and as dependence structures. Next, we show that on traces interpreted as dependence structures, first-order formulae restricted to one quantifier alternation that begin with an existential quantifier capture exactly the class of polynomials. We further give a couple of (natural) properties that are used to separate several of the fragments under consideration.

2 Preliminaries

An independence alphabet (Σ, I) consists of a finite set Σ with a symmetric and irreflexive *independence relation* $I \subset \Sigma \times \Sigma$. Whenever $a\,I\,b$ holds for two symbols $a, b \in \Sigma$ they are called independent. Otherwise they are dependent, and accordingly, $D = (\Sigma \times \Sigma) \setminus I$ is called the *dependence relation*. The congruence $\sim_I$ on Σ^* is the reflexive and transitive closure of the relation $\approx_I$ defined by

$$\forall w, v \in \Sigma^* \, \forall a, b \in \Sigma\colon\, wabv \approx_I wbav \;\Leftrightarrow\; a\, I\, b.$$

The set of equivalence classes $[w]_{\sim_I}$ with respect to that congruence forms the free partially commutative monoid generated by (Σ, I). It is denoted by $\mathbb{M}(\Sigma, I)$. Its elements are called *(Mazurkiewicz) traces* and the set itself a *trace monoid* following a convention from [12]. It is common to determine a trace monoid by the graph of its dependence relation, in which the loops resulting from the reflexivity are omitted.

Since $w \sim_I v$ implies that the words w and v contain the same letters, possibly in a different order, the *length* and the *alphabet* of w are invariant within an equivalence class. Therefore, these notions can also be applied to traces, and for $[w]_{\sim_I} = t \in \mathbb{M}(\Sigma, I)$ we denote the length of the trace t by $|t| = |w|$ and its alphabet is $\mathrm{alph}(t) = \{\, a \in \Sigma \mid a \text{ occurs in } w \,\}$.

Another point of view is the following: A *trace* for the independence alphabet (Σ, I) is a relational structure $t = (X_t, \mathrm{label}, \multimap, <)$ with a finite set of positions X_t, a mapping $\mathrm{label} : X_t \to \Sigma$, and binary relations $\multimap$ and $<$ over the set of positions X_t such that:

- $\multimap$ is acyclic and irreflexive,
- for all $\nu, \chi \in X_t$ with $\nu \neq \chi$, we have that $\mathrm{label}(\nu)\, D\, \mathrm{label}(\chi)$ holds if and only if either $\nu \multimap \chi$ or $\chi \multimap \nu$ holds,
- the relation $<$ is the transitive closure of $\multimap$.

The word $w = \mathrm{label}(\nu_1) \cdot \mathrm{label}(\nu_2) \cdots \mathrm{label}(\nu_n) \in \Sigma^*$ with $\{\, \nu_1, \ldots, \nu_n \,\} = X_t$ is a *representative* of t if and only if $\nu_m < \nu_\ell$ implies $m < \ell$ for all $1 \leq m, \ell \leq n$, i.e. if and only if w is a linearization of $(X_t, <)$. We have $t = [w]_{\sim_I} \in \mathbb{M}(\Sigma, I)$ if and only if $w = a_1 a_2 \cdots a_n$ is a representative of t. The relations $\multimap$ and $<$ contain the same information: $<$ is the transitive closure of $\multimap$, and for any two elements $\nu, \chi \in X_t$ we have

$$\nu \multimap \chi \;\Leftrightarrow\; \nu < \chi \wedge \mathrm{label}(\nu)\, D\, \mathrm{label}(\chi)$$

therefore $\multimap$ can be reconstructed if only $<$ is given. For this reason, it is equally valid to describe a trace using only one of these relations and to write $t = (X_t, \mathrm{label}, \multimap)$ or $t = (X_t, \mathrm{label}, <)$. From [1, Lemma 5] we can conclude the following lemma.

Lemma 2.1 *The partial order relation $<$ equals* $\displaystyle\bigcup_{1 \leq i < |\Sigma|} (\multimap)^i$.

The parallelism relation $\parallel$ is the complement of the symmetric and reflexive closure of $<$. When the considered object is clearly a trace, it is also common to omit the brackets and to write $t = a_1 a_2 \cdots a_n$. Let $\prec \subseteq X_t \times X_t$ be minimal such that $<$ is the transitive closure of $\prec$. The directed graph $(X_t, \prec)$ is called the *Hasse diagram* of t.

In *first order* logic, allowed logical formulae contain only quantifiers $\exists$ and $\forall$ that bind variables representing single elements. First-order logic on traces is called $\mathrm{FO}[<]$ when it is applied to the representation of a trace as $t = (X_t, \mathrm{label}, <)$, and $\mathrm{FO}[\multimap]$ if the predicate

$\twoheadrightarrow$ is used instead. In addition to the binary predicates $=$ for equality and $<$ or $\twoheadrightarrow$, the formulae may contain the unary predicates $a(\cdot)$ for $a \in \Sigma$ that hold at all positions labeled with that letter.

Let a first-order formula φ contain the free variables free(φ). Then, it can only be assigned a truth value on a trace t when an interpretation $g : \text{free}(\varphi) \to X_t$ of these variables is indicated. Whenever free(φ) is $\{ x_1, \ldots, x_n \}$, interpretations will be denoted by sequences $w \in X_t^n$. We write $(t, w) \models \varphi$ if φ using the interpretation w is true on t. We will discuss the following fragments of first-order logic on traces:

- $\mathrm{FO}^m[<]$ contains all properties that can be expressed with m variables.
- $\Sigma_n^m[<]$ consists of the properties expressible by $\Phi(n, m)$ formulae without free variables. Intuitively, n describes the number of quantifier blocks and m the number of nested variables. More formally: the formulae without quantifiers constitute $\Phi(0, 0)$. A formula φ with free variables $x_1, \ldots, x_j$ is in $\Phi(n, m)$ if and only if for some $k \in \mathbb{N}$ it can be written as $\bigvee_{1 \leq i \leq k} \exists x_{j+1} \cdots \exists x_{j+\ell(i)} \neg\psi_i$ with $\ell(i) \in \mathbb{N}$, $\psi_i \in \Phi(n-1, m-\ell(i))$ for all $1 \leq i \leq k$.
- $\Pi_n^m[<]$ contains all properties expressible as $\neg\varphi$ with $\varphi \in \Phi(n, m)$.

  We also write $\Sigma_n[<] = \bigcup_{i \in \mathbb{N}} \Sigma_n^i[<]$ and $\Pi_n[<] = \bigcup_{i \in \mathbb{N}} \Pi_n^i[<]$.
- $\Delta_n[<]$ is defined as the intersection $\Sigma_n[<] \cap \Pi_n[<]$.

By applying the same restrictions on $\mathrm{FO}[\twoheadrightarrow]$, analogous logical fragments such as $\mathrm{FO}^n[\twoheadrightarrow]$ and $\Sigma_n[\twoheadrightarrow]$ are obtained. Particular attention will be paid to the special cases with $n = 2$.

A different way to formulate logical properties of traces is (local) temporal logic TL, which generalizes linear temporal logic LTL for words [1]. In a TL formula, the quantifiers of first-order logic appear again, but only implicitly by the means of *temporal operators* that can be seen as macros representing FO subformulae. We will use four operators: *ne*X*t* F*uture* (XF), Y*esterday* P*ast* (YP), PAR*allel* (PAR) and *so*M*ewhere* (M). Although these operators are well-known, there is no uniform way of referring to them, e.g. the XF operator can also be called 'strict future' or 'next eventually'. The operator PAR has also been defined as Eco or *co* by some authors, e.g. [7, 9]. The role of M is only auxiliary. It will enable us to switch to another connected component within a trace.

The syntax of TL is given as follows: every $a \in \Sigma$ is a temporal formula. Let φ and ψ be temporal formulae and Op a temporal operator. Then $\neg\varphi$, $(\varphi \vee \psi)$ and $\mathsf{Op}\,\varphi$ are temporal formulae as well.

Let $t \in \mathbb{M}(\Sigma, I)$ be a trace. Then the semantics of formulae in temporal logic with the operators XF, YP, PAR and M is defined inductively as follows, where φ and ψ are subformulae and $\nu \in X_t$ is a position of the trace:

$$\begin{aligned}
& t,\nu \models \top \\
\forall a \in \Sigma\colon\ & t,\nu \models a && \Leftrightarrow\ \text{label}(\nu) = a \\
& t,\nu \models \neg\varphi && \Leftrightarrow\ t,\nu \not\models \varphi \\
& t,\nu \models \varphi \vee \psi && \Leftrightarrow\ t,\nu \models \varphi \text{ or } t,\nu \models \psi \\
& t,\nu \models \mathsf{XF}\,\varphi && \Leftrightarrow\ \exists \chi \in X_t\colon \nu < \chi \,\wedge\, t,\chi \models \varphi \\
& t,\nu \models \mathsf{YP}\,\varphi && \Leftrightarrow\ \exists \chi \in X_t\colon \chi < \nu \,\wedge\, t,\chi \models \varphi \\
& t,\nu \models \mathsf{PAR}\,\varphi && \Leftrightarrow\ \exists \chi \in X_t\colon \nu \parallel \chi \,\wedge\, t,\chi \models \varphi \\
& t,\nu \models \mathsf{M}\,\varphi && \Leftrightarrow\ \exists \chi \in X_t\colon t,\chi \models \varphi
\end{aligned}$$

This can be applied to a trace according to the rules

$$\begin{aligned}
t &\models \mathsf{Op}\,\varphi && \Leftrightarrow\ \exists \nu \in X_t\colon t,\nu \models \varphi \\
t &\models \neg\mathsf{Op}\,\varphi && \Leftrightarrow\ \forall \nu \in X_t\colon t,\nu \not\models \varphi \\
t &\models \varphi \vee \psi && \Leftrightarrow\ t \models \varphi \text{ or } t \models \psi \\
t &\models \varphi \wedge \psi && \Leftrightarrow\ t \models \varphi \text{ and } t \models \psi
\end{aligned}$$

where Op represents any of the temporal operators XF, YP, PAR and M. We denote the set of all properties $L(\varphi) = \{\, t \in \mathbb{M}(\Sigma, I) \mid t \models \varphi \,\}$ where φ is a temporal formula that contains no temporal operators apart from $\mathsf{Op}_1, \ldots, \mathsf{Op}_n$ by $\mathrm{TL}[\mathsf{Op}_1, \ldots, \mathsf{Op}_n]$.

Remark 2.2 *It is known from [11] that* $\mathrm{TL}[\mathsf{XF}, \mathsf{YP}] = \mathrm{TL}[\mathsf{XF}, \mathsf{YP}, \mathsf{M}]$*, hence the* M *operator does not add any expressivity to this fragment.*

3 Ehrenfeucht-Fraïssé games for traces

Ehrenfeucht-Fraïssé (EF) games [3, 6] are often used to characterize logical fragments. For the boolean closure of $\Sigma_n[<]$, Thomas presented an EF game [16] that can be modified in order to describe $\Sigma_n[<]$. The main difference consists in the fact that $\Sigma_n[<]$ is not closed under complementation. If we want to capture this fragment, it is therefore insufficient to determine whether two traces t and s are equivalent or not. Instead, we will ask if t models *at least the same* $\Sigma_n[<]$ properties as s does. By limiting the number of pebbles to $m \in \mathbb{N}$, an idea introduced in [10], it is possible to characterize the $\Sigma_n^m[<]$ fragments.

Definition 3.1 (Ehrenfeucht-Fraïssé game for $\Sigma_n^m[<]$) *The set of* configurations *for the EF game corresponding to the fragment $\Sigma_n^m[<]$ played on the traces t_0 and t_1 with position sets X_0 and X_1 is $X_0^* \times X_1^* \times \{0, 1\}$ with the restriction that the size of the first two components is equal and does not exceed m. The first two components of the configuration are interpreted as a distribution of pebbles on the two traces: a pebble labeled with x_i lies at position $\nu \in X_j$ whenever ν is the i-th character of the word corresponding to $j \in \{0, 1\}$. The third component contains the number of the trace where Spoiler will carry out his next move.*

Let (w_0, w_1, σ) with $|w_0| = |w_1| = i \leq m$ be the current configuration, then the next turn is carried out as follows:

- *Spoiler takes $j \leq m - i$ pebbles labeled with $x_{i+1}, \ldots, x_{i+j}$ and distributes them on trace t_σ by assigning a position $\nu_\sigma(k) \in X_\sigma$ to each x_{i+k}.*
- *Duplicator places identically labeled pebbles on nodes of the other trace, such that every x_{i+k} is assigned some $\nu_{1-\sigma}(k) \in X_{1-\sigma}$.*
- *The new configuration is $(w_0\nu_0(1)\cdots\nu_0(j), w_1\nu_1(1)\cdots\nu_1(j), 1-\sigma)$.*

The game for $\Sigma_n^m[<]$ consists of n rounds. Duplicator wins if and only if initially and after each of these rounds, the partial mapping $X_0 \to X_1 : w_0(k) \mapsto w_1(k)$ with $1 \leq k \leq |w_0| = |w_1|$ induces an isomorphism with respect to labels and the relation $<$.

For $t_0, t_1 \in \mathbb{M}(\Sigma, I)$ we write $t_0 \preceq^{\mathbf{\Sigma}[<]}_{(n,\,m)} t_1$ if and only if Duplicator has a winning strategy in the EF game for the logical fragment $\Sigma_n^m[<]$, played on t_0 and t_1 and starting with the initial configuration $(\varepsilon,\, \varepsilon,\, 0)$. This relation is extended to intermediate configurations of the game by replacing the initial configuration, i.e. $(t_0, w_0) \preceq^{\mathbf{\Sigma}[<]}_{(n,\,m)} (t_1, w_1)$ whenever Duplicator has a winning strategy in the game on t_0 and t_1 starting with the configuration $(w_0, w_1, 0)$.

Lemma 3.2 *Let $n, m, j \in \mathbb{N}$ and $t, s \in \mathbb{M}(\Sigma, I)$ with sequences $w_t \in X_t^j$ and $w_s \in X_s^j$. Then $(t, w_t) \preceq^{\mathbf{\Sigma}[<]}_{(n,\,m)} (s, w_s)$ holds if and only if for all $\varphi \in \Phi(n, m)$ with j free variables we have that $(t, w_t) \models \varphi$ implies $(s, v_s) \models \varphi$. In particular:*

$$t \preceq^{\mathbf{\Sigma}[<]}_{(n,\,m)} s \;\Leftrightarrow\; \big(\forall L \in \Sigma_n^m[<] \colon t \in L \;\Rightarrow\; s \in L\big).$$

Proof: The lemma holds for $n = 0$, as without any rounds Duplicator wins the game if and only if in the initial configuration, the j pebbles are isomorphically distributed on both traces, which amounts to saying that the same $\Sigma_0^j[<]$ formulae hold for (t, w_t) and (s, w_s). Suppose $n > 0$. Let $w_t \in X_t^j$, $w_s \in X_s^j$ be interpretations such that the partial mapping $X_t \to X_s : w_t(i) \mapsto w_s(i)$, $1 \leq i \leq j$ induces an isomorphism with respect to the order relation and the label function.

($\Rightarrow$) Let $\varphi \in \Phi(n, m)$ be a formula with the free variables $x_1, \ldots, x_j$. Without loss of generality, we assume that $\exists$ (and not $\vee$) is the outermost junctor, i.e. $\varphi = \exists x_{j+1} \cdots \exists x_{j+\ell}\, \neg\psi$ with $\ell \leq m$ and $\psi \in \Phi(n-1, m-\ell)$. Suppose that Duplicator has a winning strategy in the EF game with n rounds starting from the configuration $(w_t, w_s, 0)$. Let (w_t, t) be a model of φ. Now let Spoiler distribute ℓ pebbles on positions $\nu_1, \ldots, \nu_\ell$ of X_t such that $\neg\psi$ holds on t with the interpretation $v_t = w_t\nu_1\nu_2\cdots\nu_\ell$. If Duplicator proceeds according to his winning strategy, he obtains positions χ_i for all $1 \leq i \leq \ell$. We set $v_s = w_s\chi_1\chi_2\cdots\chi_\ell$. We have

$$(s, v_s) \;\preceq^{\mathbf{\Sigma}[<]}_{(n-1,\,m-\ell)}\; (t, v_t).$$

By induction hypothesis the implication $(s, v_s) \models \psi \;\Rightarrow\; (t, v_t) \models \psi$ holds for all $\psi \in \Phi(n-1, m-\ell)$. Hence, from $(t, v_t) \models \neg\psi$ we can conclude $(s, v_s) \models \neg\psi$ and therefore $(s, w_s) \models \varphi$.

($\Leftarrow$) Assume that Spoiler has a winning strategy starting from the configuration $(w_t, w_s, 0)$. Let his first move according to this strategy consist in placing $\ell \leq m$ pebbles on

t. Let v_t be the new configuration on t. Now, after every possible response of Duplicator, Spoiler has a winning strategy with at most $n-1$ rounds starting on trace s. By induction, for each $v \in w_s X_s^\ell$ there exists a formula $\psi_v \in \Phi(n, m-\ell)$ such that $(s, v) \models \psi_v$ and $(t, v_t) \not\models \psi_v$. Since the range of possible values for v is finite, we can construct $\psi^* = \bigvee_v \psi_v$, which in turn is a $\Phi(n, m-\ell)$ formula. By construction it follows that

$$(t, w_t) \models \exists x_1 \cdots \exists x_\ell \neg\psi^* \quad \text{whereas} \quad (s, w_s) \not\models \exists x_1 \cdots \exists x_\ell \neg\psi^*.$$

The lemma now follows by contraposition. $\square$

Corollary 3.3 *By replacing $<$ with $\multimap$ we obtain relations $\preceq_{(n,m)}^{\Sigma[\multimap]}$ and EF games for $\Sigma_n^m[\multimap]$. An analogous proof shows that for the game $\Sigma_n^m[\multimap]$ starting with $(w_t, w_s, 0)$, Duplicator has a winning strategy if and only if $(t, w_t) \preceq_{(n,j)}^{\Sigma[\multimap]} (s, w_s)$ holds.*

In [5] an EF game is defined for the linear temporal logic on words with three operators: until, eventually and next. We will adapt this game in order to characterize simple fragments of temporal logic on traces.

Definition 3.4 *(Ehrenfeucht-Fraïssé game for* $\mathrm{TL}[\mathsf{XF}, \mathsf{YP}, \mathsf{PAR}]$*) The EF game for the fragment* $\mathrm{TL}[\mathsf{XF}, \mathsf{YP}, \mathsf{PAR}]$ *with n rounds is played on two traces $t_0, t_1 \in \mathbb{M}(\Sigma, I)$ using one pebble per trace. A configuration of the game is a pair of positions $(\nu_0, \nu_1) \in X_{t_0} \times X_{t_1}$ currently occupied by the pebbles. In each round, Spoiler selects a side $\sigma \in \{0, 1\}$ and one of the moves* XF*,* YP *and* PAR*.*

- XF*: From a position ν the pebble is moved to a position χ such that $\chi > \nu$.*
- YP*: From a position ν the pebble is moved to a position χ such that $\chi < \nu$.*
- PAR*: From a position ν the pebble is moved to a position χ such that $\chi \parallel \nu$.*

First, Spoiler moves the pebble on t_σ and then, Duplicator carries out the same type of move on $t_{1-\sigma}$. If no other starting configuration is indicated we assume that initially, both pebbles are placed beside the board at an unlabeled position we will refer to as $\iota \notin X_0 \cup X_1$. Any move starting from ι consist in placing the pebble on an arbitrary position of the respective trace.

Spoiler wins if Duplicator cannot move his pebble to the indicated direction or if, initially or after the move, the pebbles lie on differently labeled nodes. Duplicator wins if this never occurs.

The *depth* of a formula φ in temporal logic is the maximal number of nested temporal operators occurring within φ (cf. [4]).

Lemma 3.5 *The following propositions are equivalent:*

1. *Duplicator has a winning strategy for the game from Definition 3.4 with $n \in \mathbb{N}$ rounds played on the traces $t, s \in \mathbb{M}(\Sigma, I)$.*

2. *The traces t and s are models of exactly the same formulae in* $\mathrm{TL}[\mathsf{XF}, \mathsf{YP}, \mathsf{PAR}]$ *with a maximal operator depth of n.*

Proof: We first show that t, ν_t and s, ν_s are models of the same n-depth formulae if and only if Duplicator has a winning strategy for the n-round game starting at the configuration $(\nu_t, \nu_s) \in X_t \times X_s$. For $n = 0$ this is true, because at two positions the same 0-depth formulae hold if and only if they are equally labeled. Let $n > 0$.

Let $\mathsf{XF}\varphi$ be an n-depth formula such that (without loss of generality) $t, \nu_t \models \mathsf{XF}\varphi$, whereas $s, \nu_s \not\models \mathsf{XF}\varphi$. In the game with n rounds starting with the configuration (ν_t, ν_s), Spoiler can select the XF move and a position $\chi_t > \nu_t$ with $t, \chi_t \models \psi$, as opposed to Duplicator, who will find no analogous position $\chi_s > \nu_s$. By induction, Spoiler wins the game. The other temporal operators are analogous.

Now suppose that Spoiler wins the game starting with the configuration (ν_t, ν_s) within n rounds. If the positions have different labels, then with $\varphi = \text{label}(\nu_t)$ it follows that $t, \nu_t \models \varphi$ and $s, \nu_s \not\models \varphi$. Otherwise Spoiler does his first move. Without loss of generality let this first move be XF on t. Spoiler moves his pebble to χ_t. By induction, for every position $\chi \in X_s$ with $\chi > \nu_s$ there exists an $(n-1)$-depth formula φ_χ such that $t, \chi_t \models \varphi_\chi$ and $s, \chi \not\models \varphi_\chi$. Let $\varphi = \bigwedge_{\chi > \nu_s} \varphi_\chi$. Then by construction $t, \nu_t \models \mathsf{XF}\varphi$ and $s, \nu_s \not\models \mathsf{XF}\varphi$.

Let $t, \iota \models \varphi$ be equivalent to $t \models \varphi$. Now, the case that the game starts at (ι, ι) works similarly. $\square$

By omitting the PAR operator we obtain:

Corollary 3.6 *The following propositions are equivalent:*

1. *Duplicator has a winning strategy for the EF game on the traces t, s for* TL *with $n \in \mathbb{N}$ rounds and in which the move corresponding to* PAR *is not allowed.*

2. *The traces t and s are models of exactly the same formulae in* $\mathrm{TL}[\mathsf{XF}, \mathsf{YP}]$ *with a maximal operator depth of n.*

4 Comparison of logical fragments

In [4] it is proven that over words, LTL with the operators XF and YP is equally expressive as $\mathrm{FO}^2[<]$. The proof from that paper can be adapted in order to show analogous results for FO^2 over traces. In contrast to words, in the case of $\mathrm{FO}^2[<]$ two positions might be parallel. This case can be covered using the PAR operator.

Lemma 4.1 $\mathrm{FO}^2[<] \subseteq \mathrm{TL}[\mathsf{XF}, \mathsf{YP}, \mathsf{PAR}]$.

Proof: Let $\varphi(x) \in \mathrm{FO}^2[<]$ be a formula with one free variable and without universal quantifiers. The only variables of φ are x and y. We show by induction on the quantifier depth and the size of the formula that there exist a formula $\widetilde{\varphi}$ in $\mathrm{TL}[\mathsf{XF}, \mathsf{YP}, \mathsf{PAR}]$ such that

$$\forall \nu \in X_t \colon (t, \nu) \models \varphi(x) \;\Leftrightarrow\; t, \nu \models \widetilde{\varphi}.$$

If $\varphi(x) \equiv \top$ then we set $\widetilde{\varphi} \equiv a \vee \neg a$. Constant formulae will arise later in the construction. The atomic formula are translated as follows:

$$\begin{array}{lll} \varphi(x) \equiv\ a(x) \text{ for } a \in \Sigma & \rightsquigarrow & \widetilde{\varphi} \equiv a \\ \varphi(x) \equiv\ x < x & \rightsquigarrow & \widetilde{\varphi} \equiv a \wedge \neg a \\ \varphi(x) \equiv\ x = x & \rightsquigarrow & \widetilde{\varphi} \equiv a \vee \neg a. \end{array}$$

Inductively, boolean operators are translated as follows:

$$\begin{array}{lll} \varphi(x) \equiv\ \psi_1(x) \vee \psi_2(x) & \rightsquigarrow & \widetilde{\varphi} \equiv \widetilde{\psi_1} \vee \widetilde{\psi_2} \\ \varphi(x) \equiv\ \neg\psi(x) & \rightsquigarrow & \widetilde{\varphi} \equiv \neg\widetilde{\psi}. \end{array}$$

If $\varphi(x)$ is of the form $\exists x\colon \psi(x)$, in an intermediate step it is transformed into $\varphi'(x) \equiv \exists y\colon \psi(y)$ by interchanging x and y. If $\varphi(x) \equiv \exists y\colon \psi(y)$ then it can be interpreted as $\varphi(x) \equiv \exists y\colon \psi(x,y)$ where x is a dummy variable in ψ. We now consider the general case $\varphi(x) \equiv \exists y\colon \psi(x,y)$. First, we will transform $\varphi(x)$ into an equivalent formula $\varphi''(x)$ of the same quantifier depth. Let

$$\psi(x,y) \ \equiv\ \beta(\, x = y, x < y, y < x,\ \xi_1(x), \ldots, \xi_n(x),\ \zeta_1(y), \ldots, \zeta_m(y)\,)$$

where β is a propositional formula and $\xi_i(x)$, $\zeta_j(y)$ are atomic formulae or existential formulae with smaller quantifier depth. The first step in the transformation of $\varphi(x)$ is to *guess* the values of $\xi_i(x)$ before the quantification of y. We set $\varphi'(x) \equiv$

$$\bigvee_{\overline{\gamma} \in \{\top,\bot\}^n} \Bigg(\bigwedge_{1 \le i \le n} (\xi_i(x) \leftrightarrow \gamma_i) \ \wedge \ \exists y\colon \beta(x = y, x < y, y < x,\ \gamma_1, \ldots, \gamma_n,\ \zeta_1(y), \ldots, \zeta_m(y)\,) \Bigg).$$

The next step is to *guess* the relation τ that holds between x and y in advance. The possible relations are $x = y$, $x < y$, $x > y$ or none them and then x and y correspond to parallel positions, $x \parallel y$. Hence we choose τ from the set $\{=, <, >, \parallel\}$. We set $\varphi''(x) \equiv$

$$\bigvee_{\overline{\gamma} \in \{\top,\bot\}^n} \Bigg(\bigwedge_{1 \le i \le n} (\xi_i(x) \leftrightarrow \gamma_i) \ \wedge \bigvee_{\tau \in \{=,<,>,\parallel\}} \exists y\colon (x \tau y \ \wedge\ \beta(y) \Bigg)$$

with $\beta(y) = \beta(x = y^\tau, x < y^\tau, y < x^\tau, \overline{\gamma}, \zeta_1(y), \ldots, \zeta_m(y)\,)$. Note that the first $3+n$ arguments are constant boolean values at this point. After this transformation of $\varphi(x)$ it remains to show how to translate formulae of the form $\exists y\colon (x\tau y \wedge \beta(y))$ with $\tau \in \{=, <, >, \parallel\}$:

$$\begin{array}{lll} \exists y\colon (x = y \ \wedge\ \beta(y)) & \rightsquigarrow & \widetilde{\beta} \\ \exists y\colon (x < y \ \wedge\ \beta(y)) & \rightsquigarrow & \mathsf{XF}\widetilde{\beta} \\ \exists y\colon (x > y \ \wedge\ \beta(y)) & \rightsquigarrow & \mathsf{YP}\widetilde{\beta} \\ \exists y\colon (x \parallel y \ \wedge\ \beta(y)) & \rightsquigarrow & \mathsf{PAR}\widetilde{\beta}. \end{array}$$

Let now $\varphi \in \mathrm{FO}^2[<]$ be a formula without free variables. Next, we show how to construct a formula $\widehat{\varphi} \in \mathrm{TL}[\mathsf{XF}, \mathsf{YP}, \mathsf{PAR}]$ such that

$$t \models \varphi \;\Leftrightarrow\; t \models \widehat{\varphi}.$$

For this we use the following transformation scheme:

$$\begin{array}{lll} \varphi \equiv \psi_1 \vee \psi_2 & \rightsquigarrow & \widehat{\varphi} \equiv \widehat{\psi_1} \vee \widehat{\psi_2} \\ \varphi \equiv \neg\psi & \rightsquigarrow & \widetilde{\varphi} \equiv \neg\widehat{\psi} \\ \varphi \equiv \exists x \colon \psi(x) & \rightsquigarrow & \widetilde{\varphi} \equiv \mathsf{XF}\widetilde{\psi}. \end{array}$$

Formulæ of the form $\varphi \equiv \exists y \colon \psi(y)$ are in an intermediate step transformed into $\exists x \colon \psi(x)$ by interchanging x and y. $\square$

Lemma 4.2 $\mathrm{FO}^2[\multimap] \subseteq \mathrm{TL}[\mathsf{XF}, \mathsf{YP}]$.

The proof is analogous to Lemma 4.1. The only difference is that we use the operator M if none of $x = y$, $x \multimap y$ or $y \multimap x$ holds. In this case, we additionally have to check that the labels of x and y are independent. Afterwards the M operator can be removed by Remark 2.2.

We say that a language L is a *polynomial* if it is a finite union of languages of the form

$$A_0^* a_1 A_1^* \cdots a_n A_n^*,$$

where $n \in \mathbb{N}$ and $A_i \subseteq \Sigma$ for all $0 \leq i \leq n$. By Pol we denote the class of such languages and by coPol we denote the class of complements of such languages.

Lemma 4.3 $\mathrm{Pol} \subseteq \Sigma_2[\multimap]$.

Proof: It is easy to see that languages of the form A^* for $A \subseteq \Sigma$ are in $\Sigma_2[\multimap]$. Since Pol and $\Sigma_2[\multimap]$ are both closed under union, it suffices to show that $L_0 a L_1 \in \Sigma_2[\multimap]$ if L_0 and L_1 are in $\Sigma_2[\multimap]$. Suppose L_0 and L_1 are expressed by the $\Sigma_2[\multimap]$ formulae φ_0 and φ_1, respectively. We will construct a $\Sigma_2[\multimap]$ formula φ expressing $L = L_0 a L_1$. We have $t \in L$ if and only if there exists a factorization $t = t_0 a t_1$ with $t_0 \in L_0$ and $t_1 \in L_1$. Our aim is to to check whether there exists such a factorization. This will be done by determining for each letter in t_0 (resp. t_1) its last occurrence (resp. its first occurrence). We will then use these positions to restrict φ_0 (resp. φ_1) to the left (resp. right) factor by means of the relation $\multimap$. We introduce some macro formulae:

$$\begin{array}{rcl} \mathrm{label}(x) = \mathrm{label}(y) & \equiv & \displaystyle\bigwedge_{a \in \Sigma} a(x) \leftrightarrow a(y) \\ \mathrm{path}(x_1, \ldots, x_n) & \equiv & \displaystyle\bigwedge_{1 \leq i < n} (x_i = x_{i+1} \vee x_i \multimap x_{i+1}) \\ x \parallel y & \equiv & \forall x_1 \ldots \forall x_{|\Sigma|} \colon \\ & & \big(\neg\mathrm{path}(x_1, \ldots, x_{|\Sigma|}) \vee x \neq x_1 \vee y \neq x_{|\Sigma|}\big) \wedge \\ & & \big(\neg\mathrm{path}(x_1, \ldots, x_{|\Sigma|}) \vee x \neq x_{|\Sigma|} \vee y \neq x_1\big) \end{array}$$

$$x < y \equiv x \neq y \wedge \exists x_2 \ldots \exists x_{|\Sigma|-1}\colon \mathrm{path}(x, x_2, \ldots, x_{|\Sigma|-1}, y).$$

That this definition of $<$ is equivalent with the transitive closure of $\rightarrowtriangle$ follows by Lemma 2.1. For alphabets $A_0 = \{ a_1, \ldots, a_n \}$ and $A_1 = \{ b_1, \ldots, b_m \}$ we define a $\Sigma_2[\rightarrowtriangle]$ formula $\psi_{A_0,a,A_1}(x_1, \ldots, x_n, z_1, y_1, \ldots, y_m)$ with $n + 1 + m$ free variables. Consider a factorization $t = t_0 a t_1$. The formula ψ_{A_0,a,A_1} is true if each variable x_i, $1 \leq i \leq n$ is interpreted at the last occurrence of the letter a_i in t_0 and the alphabet of t_0 is A_0. The last m variables y_i represent the first positions of letters of t_1 whose alphabet is A_1 and z_1 corresponds to the position of a in this factorization. We set $\psi_{A_0,a,A_1}(x_1, \ldots, x_n, z_1, y_1, \ldots, y_m) =$

$$\bigwedge_{1 \leq i \leq n} a_i(x_i) \ \wedge\ a(z_1) \ \wedge \bigwedge_{1 \leq i \leq m} b_i(y_i) \ \wedge$$
$$\bigwedge_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} \left\{ \begin{array}{l} (x_i \parallel z_1 \ \vee\ x_i < z_1) \ \wedge \\ (x_i \parallel y_j \ \vee\ x_i < y_j) \ \wedge \\ (z_1 \parallel y_j \ \vee\ z_1 < y_j) \end{array} \right\} \wedge$$
$$\forall z\colon \left\{ \begin{array}{l} z = z_1 \ \vee \\ \displaystyle\bigvee_{1 \leq i \leq n} \big(\mathrm{label}(z) = \mathrm{label}(x_i) \ \wedge (z = x_i \ \vee\ z \rightarrowtriangle x_i)\big) \ \vee \\ \displaystyle\bigvee_{1 \leq j \leq m} \big(\mathrm{label}(z) = \mathrm{label}(y_j) \ \wedge (y_j = z \ \vee\ y_j \rightarrowtriangle z)\big) \end{array} \right\}.$$

Let $x_1, \ldots, x_n$ be variables. The restriction of $\varphi_0 = \exists \overline{y}\, \forall \overline{z}\, \psi_0(\overline{y}, \overline{z})$ to the past of $x_1, \ldots, x_n$ is

$$\begin{aligned} \varphi_0^- = {} & \exists y_1 \ldots \exists y_m \forall z_1 \ldots \forall z_\ell\colon \\ & \bigwedge_{1 \leq j \leq m} \left(\bigvee_{1 \leq i \leq n} (y_j = x_i \ \vee\ y_j \rightarrowtriangle x_i) \right) \ \wedge \\ & \left(\bigvee_{1 \leq k \leq \ell} \left(\bigwedge_{1 \leq i \leq n} \neg\, (z_k = x_i \ \vee\ z_j \rightarrowtriangle x_i) \right) \ \vee\ \psi_0(\overline{y}, \overline{z}) \right). \end{aligned}$$

Similarly, we can define the restriction φ_1^+ of φ_1 to the future of $y_1, \ldots, y_m$. Using these restrictions, we define

$$\varphi = \bigvee_{A_0, A_1 \subseteq \Sigma} \exists x_1 \ldots \exists x_{|A_0|} \exists z_1 \exists y_1 \ldots \exists y_{|A_1|}\colon \big(\psi_{A_0,a,A_1} \ \wedge\ \varphi_0^- \ \wedge\ \varphi_1^+\big).$$

Note that φ is a $\Sigma_2[\rightarrowtriangle]$ formula. It expresses the language L. $\square$

Lemma 4.4 $\Sigma_2[\rightarrowtriangle] \subseteq \mathrm{Pol}$.

Proof: Let $L \in \Sigma_2[\to\!\!\triangleright]$. An algebraic characterization in terms of Mal'cev products of Pol is given in [11]. In order to show $L \in$ Pol we will prove that L has this algebraic property. Therefore we will use some methods from algebraic language theory, see e.g. [13, 14]. It suffices to show that for each $t \in \mathbb{M}(\Sigma, I)$ there is an $n' \in \mathbb{N}$ such that, for all $n \geq n'$ and $s \in \mathbb{M}(\Sigma, I)$ with $\mathrm{alph}(t) = \mathrm{alph}(s)$, the following implication holds:

$$\forall p, q \in \mathbb{M}(\Sigma, I)\colon pt^n q \in L \;\Rightarrow\; pt^n st^n q \in L.$$

Let $m \in \mathbb{N}$ with $L \in \Sigma_2^m[\to\!\!\triangleright]$. Consider the EF game for this fragment, played on the traces $r = pt^n q$ and $u = pt^n st^n q$ with $n \geq (m+1)^2$ starting from the initial configuration $(\varepsilon, \varepsilon, 0)$. The interior sections of both traces, i.e. t^n on r and $t^n st^n$ on u, are composed of factors t and s. We refer to the n factors t of r as blocks 1 to n.

In the first round, Spoiler places up to m pebbles on positions of r. Due to the large number of blocks in comparison to the number of pebbles, he must necessarily leave a big gap: there is an $\ell < n - m$ such that on trace r, Spoiler places no pebbles on any of the blocks $\ell + 1, \ldots, \ell + m$. We decompose r into three factors

$$r \;=\; pt^\ell \cdot t^m \cdot t^{n-\ell-m} q.$$

All pebbles lie on positions of the left and of the right factor. We analogously factorize u:

$$u \;=\; pt^\ell \cdot t^{n-\ell} st^{\ell+m} \cdot t^{n-\ell-m} q.$$

Duplicator responds by copying the distribution of the pebbles on the two identical outer factors.

In the second round, Spoiler has to distribute his remaining pebbles on u. Duplicator can carry over the distribution of the new pebbles on the outer factors of u to the corresponding positions of the outer factors of r. It remains to show how Duplicator can answer the j pebbles, $j \leq m$, on the intermediate factor $t^{n-\ell} st^{\ell+m}$ of u. Let $\nu_1, \nu_2, \ldots, \nu_j \in X_u$ be a linearization of the positions with pebbles on this intermediate factor such that $\nu_i \to\!\!\triangleright \nu_k$ implies $i < k$. Now, Duplicator uses the gap of size m beginning at block $\ell + 1$ of r and the assumption that $\mathrm{alph}(t) = \mathrm{alph}(s)$. For each position ν_i, $1 \leq i \leq j$, Duplicator places a pebble of the same type on an arbitrary position χ_i of block $\ell + i$ in trace r such that $\mathrm{label}(\nu_i) = \mathrm{label}(\chi_i)$. By construction, we have $\nu_i \to\!\!\triangleright \nu_k$ if and only if $\chi_i \to\!\!\triangleright \chi_k$. Hence, Duplicator has a winning strategy for this EF game. Therefore,

$$pt^n q \;\preceq^{\Sigma[\to\!\!\triangleright]}_{(2,\,m)}\; pt^n st^n q$$

holds due to Corollary 3.3, which shows that $pt^n q \in L$ implies $pt^n st^n q \in L$. □

Lemma 4.5 *It is not expressible in* $\mathrm{TL}[\mathsf{XF}, \mathsf{YP}]$ *whether two actions occur in parallel, whereas this property is expressible in* $\Sigma_1^2[<]$.

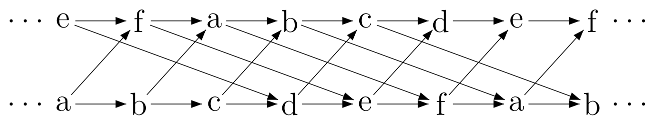

Figure 1: Hasse diagram of trace p (see proof of Lemma 4.6)

Proof: Let $\mathbb{M}(\Sigma, I)$ be the trace monoid given by the dependence relation $D = a - b - c$. We formulate the property L = 'the symbols a and c occur in parallel' that coincides with the semantics of $\varphi = \exists x \exists y \colon x \parallel y$. Consider the EF game for TL restricted to $n \in \mathbb{N}$ rounds and the moves XF and YP, i.e. without the PAR move, played on the traces $t = (babc)^n bacb (babc)^n \in L$ and $s = (babc)^{2n+1} \notin L$. We enumerate the blocks $bacb$ and $babc$ that constitute both traces with integers running from $-n$ to n such that both traces are identical apart from block 0. We sketch a simple winning strategy for Duplicator on this game.

As long as no pebble is placed on a or c of block 0, Duplicator can imitate all moves made by Spoiler. If at some point the pebbles are placed on such positions and Spoiler moves one of them forward or backward within block 0 then Duplicator can respond by moving the other pebble to an identically labeled position of either block -1, 0 or 1. If Spoiler moves one of the pebbles to a non-adjacent block then Duplicator can return to a configuration where both pebbles are placed in equally numbered blocks. Otherwise, he can always maintain the distance of at most one block, which means that he can respond in all of the maximally $n-1$ rounds that remain. By Corollary 3.6, L cannot be expressed in TL[XF, YP]. $\square$

Lemma 4.6 *It is not expressible in* TL[$\mathsf{XF}, \mathsf{YP}, \mathsf{PAR}$] *whether three actions occur in parallel, whereas this property is expressible in* $\Sigma_1^3[<]$.

Proof: Let $\mathbb{M}(\Sigma, I)$ be the trace monoid with $\Sigma = \{a, b, c, d, e, f, \#\}$ such that among the letters $\{a, b, c, d, e, f\}$ we have the circular dependencies $a - b - c - d - e - f - a$ and all these letters are dependent of $\#$. The maximal sets of independent letters are $\{a, c, e\}$ and $\{b, d, f\}$. In $\Sigma_1^3[<]$, we can express whether three actions occur in parallel by the formula

$$\varphi = \exists x \exists y \exists z \colon (x \parallel y \wedge y \parallel z \wedge z \parallel x).$$

Next, we will show that $L(\varphi)$ is not in TL[$\mathsf{XF}, \mathsf{YP}, \mathsf{PAR}$]. Let $r = acbdcedfeafb$. For an arbitrarily chosen $n \in \mathbb{N}$ consider the traces $p = r^{2n+1}$ and $q = acbedf$. We combine them in order to build the larger traces $t = (\#p)^{2n+1} \not\models \varphi$ and $s = (\#p)^n \# q (\#p)^n \models \varphi$, see Figures 1 and 2. We say that p and q are *segments* of t and s. We numerate the segments of t and s with numbers running from $-n$ to n from left to right. All segments of t and s consist of p, except for segment 0 of trace s, which is q. Every segment that correspond to p is further subdivided into $2n+1$ *blocks* with numbers running from $-n$ to n such that all blocks consist of r.

Consider the n-round EF game for TL[$\mathsf{XF}, \mathsf{YP}, \mathsf{PAR}$] played on the traces t and s. In all blocks, except the outermost ones $-n$ and n, every position has parallel occurrences of all letters that are independent of its own label. Although there do not exist three parallel

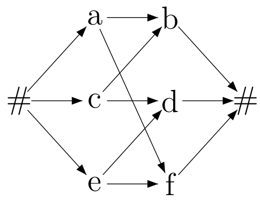

Figure 2: Hasse diagram of trace $\#q\#$ (see proof of Lemma 4.6)

positions in t, we will use this fact to mimic three parallel positions in order to construct a winning strategy for Duplicator.

The main strategy of Duplicator is to copy all moves of Spoiler. Since only the behavior on the segments 0 of t and s is not evident, we will describe the strategy of Duplicator for this case. Whenever Spoiler accesses segment 0 of trace t, Duplicator avoids placing the pebble on segment 0 of trace s by putting his pebble to the corresponding position of either segment -1 or 1 of trace s. In the up to $n-1$ remaining rounds it is possible for Duplicator to keep a maximal difference of 1 between the numbers of the segments. If Spoiler moves his pebble on segment 0 of trace s, Duplicator responds by placing the other pebble on an identically labeled position of block 0 of either segment -1, 0 or 1 of trace s. From this position he can mimic all possible PAR-moves of Spoiler on s. Each PAR-move of Spoiler could force Duplicator to increase or decrease the block number by 1. Since there occur at most $n-1$ of these moves and there are n blocks to the left as well as to the right, Duplicator wins the game. It follows that $L(\varphi) \notin \mathrm{TL}[\mathsf{XF}, \mathsf{YP}, \mathsf{PAR}]$. $\square$

Lemma 4.7 *For all $n \in \mathbb{N}$, we have* $\mathrm{TL}[\mathsf{XF}, \mathsf{YP}, \mathsf{PAR}] \not\subseteq \Sigma_n[<]$.

Proof: Let $m \in \mathbb{N}$ be arbitrary, and let a series of trace monoids $\mathbb{M}(\Gamma_i, I_i,)$, $i \geq 1$, be given inductively by the alphabets $\Gamma_1 = \{a_1\}$ and $\Gamma_i = \Gamma_{i-1} \cup \{\, a_i, b_i \,\}$ with the dependence graph $D_i = D_{i-1} - b_i - a_i$ or, more formally, $D_1 = \{(a_1, a_1)\}$ and $D_i = D_{i-1} \cup \Gamma_{i-1} \times \{b_i\} \cup \{b_i\} \times \Gamma_{i-1} \cup \{\, a_i, b_i \,\}^2$. This means that each Γ_i with $i \geq 2$ introduces a letter a_i, which is independent of all preceding letters, and a letter b_i that depends on all letters. For $i \in \mathbb{N}$, let $\ell_i = (m+1)^i$. By induction we also define the traces $t_1 = \varepsilon$, $s_1 = a_1$, and

$$\begin{aligned} t_i &= (b_i a_i s_{i-1})^{\ell_i} \\ s_i &= (b_i a_i s_{i-1})^{\ell_i} \cdot b_i a_i t_{i-1} \cdot (b_i a_i s_{i-1})^{\ell_i} \end{aligned}$$

such that $t_i, s_i \in \mathbb{M}(\Gamma_i, I_i,)$. The b_i's partition these traces into *blocks*. We number these blocks from 1 to ℓ_i on t_i and from $-\ell_i$ to ℓ_i on s_i. Within each block, the position labeled with a_i is parallel to all other positions.

We define the formulae $\varphi_1 = a_1$, $\psi_i = \neg\mathsf{PAR}\varphi_i$ and $\varphi_i = a_i \wedge \psi_{i-1}$. By induction, φ_i holds at the position of a_i in $b_i a_i t_{i-1}$ but not in $b_i a_i s_{i-1}$. It follows that s_i, which contains a factor t_{i-1}, does not model ψ_i, as opposed to t_i, which models ψ_i. Hence for all $n \in \mathbb{N}$, there is a $\mathrm{TL}[\mathsf{XF}, \mathsf{YP}, \mathsf{PAR}]$ property modeled by t_n but not by s_n.

Now consider the EF game for $\Sigma_n^m[<]$ played on the traces t_n and s_n, i.e. there are n rounds, m pebbles, and Spoiler places his first pebbles on t_n. Using induction, we describe a winning strategy of Duplicator for this game. The case for $n = 1$ is trivial: Spoiler cannot place any pebbles and Duplicator responds by doing the same. Assume $n > 1$, then in the first round, Spoiler places $m' \leq m$ pebbles on t_n. Because this trace consists of ℓ_n blocks, there must remain a big continuous gap of $\ell' = (m+1)^{n-1}$ blocks without any pebbles. Let such a gap start after the k-th block and consider the following factorizations of t_n and s_n:

$$\begin{aligned} t_n &= p^k \cdot p^{\ell'} && \cdot p^{\ell_n - k - \ell'} \\ s_n &= p^k \cdot p^{\ell_n - k}\, q\, p^{k+\ell'} && \cdot p^{\ell_n - k - \ell'} \end{aligned}$$

where $p = b_n a_n s_{n-1}$ and $q = b_n a_n t_{n-1}$. Duplicator can react by placing corresponding pebbles on the respective positions of the left and right factor of s_n. On both traces, the factor in the middle contains no pebbles. For the remaining rounds, we can ignore the outermost factors of both traces because they are identical. By induction we know

$$t_{n-1} \preceq^{\mathbf{\Sigma}[<]}_{(n-1,\,m)} s_{n-1},$$

i.e. for the rest of the game, the blocks p and q cannot be distinguished. Both middle factors consist of at least $(m+1)^{n-1}$ blocks, and there are $n-1$ rounds to play. This allows Duplicator to win the game. We conclude that

$$t_n \preceq^{\mathbf{\Sigma}[<]}_{(n,\,m)} s_n.$$

Hence for all $n \in \mathbb{N}$, no $\Sigma_n^m[<]$ property is modeled by t_n but not by s_n. As this holds for all $m \in \mathbb{N}$, it follows that $\mathrm{TL}[\mathsf{XF}, \mathsf{YP}, \mathsf{PAR}]$ is not a subset of $\Sigma_n[<]$. $\square$

Theorem 4.8 *We have the following relations:*

a. $\mathrm{TL}[\mathsf{XF}, \mathsf{YP}] = \mathrm{FO}^2[\rightarrowtriangle] = \Delta_2[\rightarrowtriangle]$.

b. $\mathrm{TL}[\mathsf{XF}, \mathsf{YP}, \mathsf{PAR}] = \mathrm{FO}^2[<]$.

c. $\mathrm{FO}^2[\rightarrowtriangle] \subsetneq \mathrm{FO}^2[<]$.

d. $\Delta_2[\rightarrowtriangle] \subsetneq \Delta_2[<]$.

e. For all $n \geq 2$, the fragments $\Delta_n[<]$ and $\mathrm{FO}^2[<]$ are not comparable.

f. $\mathrm{Pol} = \Sigma_2[\rightarrowtriangle] \subsetneq \Sigma_2[<]$.

g. $\mathrm{coPol} = \Pi_2[\rightarrowtriangle] \subsetneq \Pi_2[<]$.

Proof: The relation $\mathrm{FO}^2[\multimapdotbothA] \subseteq \mathrm{TL}[\mathsf{XF}, \mathsf{YP}]$ is Lemma 4.2 and we have $\mathrm{TL}[\mathsf{XF}, \mathsf{YP}] \subseteq \mathrm{FO}^2[\multimapdotbothA]$ since the definition of the temporal operators can be seen as macros using only two variables. By Lemma 2.1, the relation $<$ in those macros can be simulated by $\multimapdotbothA$. The relation $\mathrm{TL}[\mathsf{XF}, \mathsf{YP}, \mathsf{PAR}] \subseteq \mathrm{FO}^2[<]$ follows similarly and $\mathrm{FO}^2[<] \subseteq \mathrm{TL}[\mathsf{XF}, \mathsf{YP}, \mathsf{PAR}]$ is Lemma 4.1. The relations $\Sigma_2[\multimapdotbothA] = \mathrm{Pol}$ and $\Pi_2[\multimapdotbothA] = \mathrm{coPol}$ follow from Lemma 4.3 and Lemma 4.4, the latter one by complementation. Now, $\mathrm{Pol} \cap \mathrm{coPol} = \mathrm{TL}[\mathsf{XF}, \mathsf{YP}]$, see [11], implies $\mathrm{TL}[\mathsf{XF}, \mathsf{YP}] = \Delta_2[\multimapdotbothA]$. That there exists a trace monoid such that the subset relations in $\mathrm{FO}^2[\multimapdotbothA] \subsetneq \mathrm{FO}^2[<]$ and $\Delta_2[\multimapdotbothA] \subsetneq \Delta_2[<]$ and $\Sigma_2[\multimapdotbothA] \subsetneq \Sigma_2[<]$ and $\Pi_2[\multimapdotbothA] \subsetneq \Pi_2[<]$ are strict follows from Lemma 4.5 and that for all $n \geq 2$, neither $\Delta_n[<] \subseteq \mathrm{FO}^2[<]$ nor $\mathrm{FO}^2[<] \subseteq \Delta_n[<]$ holds follows from Lemma 4.6 and Lemma 4.7, respectively. □

Conclusion

The main contribution of this document is algebraic, although few methods from algebra were used. Over traces the variety $\mathrm{Pol} \cap \mathrm{coPol}$ was already known to be identical with $\mathrm{TL}[\mathsf{XF}, \mathsf{YP}]$, and over words also with $\Delta_2[<] = \mathrm{FO}^2[<]$. With the results from Theorem 4.8 we can see that the identity

$$\mathrm{Pol} \cap \mathrm{coPol} \;=\; \mathrm{TL}[\mathsf{XF}, \mathsf{YP}] \;=\; \Delta_2[\multimapdotbothA] \;=\; \mathrm{FO}_2[\multimapdotbothA]$$

remains valid in the more general setting of traces understood as dependence structures. However, this does not extend to the partial order interpretation of traces. We have shown how the difference between $\mathrm{FO}^2[<]$ and $\mathrm{FO}^2[\multimapdotbothA]$ coincides with the difference between fragments of temporal logic with and without the operator PAR. For all $n > 1$ we have seen that none of the fragments $\mathrm{FO}^2[<]$ and $\Delta_n[<]$ includes the other.

As a tool for carrying out the proofs, we mostly applied straightforward Ehrenfeucht-Fraïssé games. The rules we defined for the game that characterizes fragments of the Σ_n^m type combine the approaches from [16] and [10]. The properties of that game are valid for first-order theories in general, not only for Mazurkiewicz traces. The games for temporal logic are adaptations of the version from [5].

The examples that were given in order to prove differences between logical fragments are very natural in the sense that they are among the simplest properties that describe concurrency between actions. In particular, for the temporal formulae

$$\varphi_n \;=\; \neg\mathsf{PAR}(a_n \wedge \varphi_{n-1})$$

of linear size in n with $\varphi_1 = a_1$ there is no equivalent in $\Sigma_n[<]$. Within $\mathrm{FO}^2[\multimapdotbothA]$ we cannot distinguish whether *two* concurrent actions occur or not. That there are *three* concurrent actions cannot be expressed in $\mathrm{FO}^2[<]$.

Using the characterization for $\mathrm{FO}^2[\multimapdotbothA]$ and $\Delta_2[\multimapdotbothA]$ and the algebraic properties of $\mathrm{Pol} \cap \mathrm{coPol}$ we can conclude that membership in these fragments is decidable However, it remains an open problem to obtain algebraic characterizations of the first-order fragments $\mathrm{FO}^2[<]$ and $\Delta_2[<]$. This would be desirable in order to establish decidability results for these fragments as well.

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