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PERMUTUTATIONAL LABELLING OF CONSTANT WEIGHT GRAY CODES

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We prove that for positive integers n and r satisfying $1 < r < n$, with the single exception of $n = 4$ and $r = 2$, there exists a constant weight Gray code of r-sets of $X_n = \{1, 2, \ldots, n\}$ that admits an orthogonal labelling by distinct partitions, with each subsequent partition obtained from the previous one by an application of a permutation of the underlying set. Specifically, an r-set A and a partition π of X_n are said to be orthogonal if every class of π meets A in exactly one element. We prove that for all n and r as stated, and $i = 1, 2, \ldots, \binom{n}{r}$ taken modulo $\binom{n}{r}$, there exists a list $A_1, A_2, \ldots, A_{\binom{n}{r}}$ of the distinct r-sets of X_n with $|A_i \cap A_{i+1}| = r - 1$ and a list of distinct partitions $\pi_1, \pi_2, \ldots, \pi_{\binom{n}{r}}$ such that π_i is orthogonal to both A_i and A_{i+1}, and $\pi_{i+1} = \pi_i \lambda_i$ for a suitable permutation λ_i of X_n.

1. ORTHOGONALLY LABELLED HAMILTONIAN CYCLES

We prove a combinatorial result regarding labelling of constant weight Gray codes. The paper is aimed at understanding the combinatorics of subsets and partitions of finite sets and their efficient listing.

Let $X_n = \{1, 2 \ldots, n\}$. An r element subset A of X_n is referred to as an r-set. Let $G_{n,r}$ be the graph whose vertices constitute all the r-sets of X_n, with two r-sets being adjacent if their intersection has exactly $r-1$ elements. A *path* in a graph is a sequence of distinct pairwise adjacent vertices; a *cycle* is a path in which the first and the last vertices are adjacent. A *Hamiltonian* path (cycle) is one that contains every vertex of the graph. It is well-known that $G_{n,r}$ is Hamiltonian; that is, that it contains Hamiltonian cycles. Hamiltonian cycles of $G_{n,r}$ are also known as *constant weight Gray codes* and were among the earliest examples of *combinatorial Gray codes* ([**6**]).

A partition π of X_n is said to have *weight* r if π has r distinct classes. The partition π and the set A are said to be *orthogonal* if every class of π contains exactly one element of A. An *orthogonally labelled list* of r-sets in X_n is a sequence

$$A_1, \pi_1, A_2, \pi_2, \ldots, A_{\binom{n}{r}}, \pi_{\binom{n}{r}} \tag{1}$$

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alternating between distinct r-sets A_i and distinct partitions π_i of weight r, such that for $i = 1, 2, \ldots, \binom{n}{r}$ taken modulo $\binom{n}{r}$, the partition π_i is simultaneously orthogonal to A_i and A_{i+1}. The sequence $A_1, A_2, \ldots, A_{\binom{n}{r}}$ of $\binom{n}{r}$ distinct r-sets of X_n is referred to as the *set-sequence*, and is denoted by $\mathcal{A} = A_1 A_2 \ldots A_{\binom{n}{r}}$. The sequence $\pi_1, \pi_2, \ldots, \pi_{\binom{n}{r}}$ of $\binom{n}{r}$ distinct partitions is referred to as the *partition-sequence*, and is denoted by $\Pi = \pi_1 \pi_2 \ldots \pi_{\binom{n}{r}}$. We identify the orthogonally labelled list in (1) with an ordered pair $(\mathcal{A}, \Pi)$. In the sequel, we omit the commas between the elements of set-sequences and partition-sequences.

In [**1**], Howie and McFadden prove the existence of orthogonally labelled lists as stated below.

Theorem 1.1. ([**1**]) *For all positive integers n and r with $1 < r < n$ there exist an orthogonally labelled list of the r-sets of X_n.*

If the partition-sequence Π is such that for each $i = 1, 2, \ldots, \binom{n}{r}$ taken modulo $\binom{n}{r}$, there exists a permutation λ_i of X_n with $\pi_{i+1} = \pi_i \lambda_i$, the orthogonally labelled list $(\mathcal{A}, \Pi)$ is referred to as the *permutational orthogonally labelled list.* If the set sequence $\mathcal{A}$ is a Hamiltonian cycle in $G_{n,r}$, the orthogonally labelled list $(\mathcal{A}, \Pi)$ is referred to as an *orthogonally labelled Hamiltonian cycle.* Our objective in this paper is to prove the following strengthening of Theorem 1.1.

Theorem 1.2. *For all positive integers n and r with $1 < r < n$, except for the $n = 4, r = 2$ case, there exists a permutational orthogonally labelled Hamiltonian cycle in $G_{n,r}$.*

We prove the theorem after providing a definition and several examples. A partition of the set X_n has type $\tau = {d_1}^{t_1} {d_2}^{t_2} \ldots {d_k}^{t_k}$ if it has t_i classes of size d_i for $i = 1, 2, \ldots, k$, where $d_1 > d_2 \ldots > d_k$. We use τ to refer to the set of all partitions of X_n of that type.

Example 1.1. We show that $G_{4,2}$ has no permutational orthogonally labelled Hamiltonian cycle (nor even a permutational orthogonally labelled list). There are seven partitions of weight two of X_4: three of these are of type 2^2 and four of type $3\,1$. There are six 2-sets in X_4; hence, any orthogonally labelled Hamiltonian list in $G_{4,2}$ must contain partitions of both types, 2^2 and $3\,1$. No permutation of X_4 can transform a partition of one type into the other; hence there exists no permutational orthogonally labelled list in $G_{4,2}$. It is somewhat surprising that $n = 4, r = 2$ turns out to be the only exceptional case as Theorem 1.2 indicates.

In the table below, we also present two permutational orthogonally labelled Hamiltonian cycles for $n = 5$, one for the case of $r = 2$, the other for the case $r = 3$.

Set	Partition	Set	Partition
A_i	π_i	B_i	γ_i
12	25|134	123	3|15|24
23	12|345	134	3|12|45
13	14|235	234	2|13|45
34	23|145	124	4|13|25
24	34|125	145	4|12|35
14	24|135	245	4|23|15
45	15|234	345	3|14|25
35	45|123	135	5|12|34
25	35|124	235	5|13|24
15	13|245	125	1|24|35

Figure 1: Permuational Orthogonally Labelled Hamiltonian cycles in $G_{5,2}$ and $G_{5,3}$

An orthogonally labelled list $(\mathcal{A}, \Pi)$ in which every partition in Π has type τ, is called an *orthogonally τ-labelled list.* If $\mathcal{A}$ is a Hamiltonian cycle, then $(\mathcal{A}, \Pi)$ is referred to as an orthogonally τ-labelled Hamiltonian cycle. For a fixed type τ, the group S_n of permutations of X_n acts transitively on the set of partitions of type τ. In particular, an orthogonally τ-labelled list is a permutational orthogonally labelled list. The following proposition is concerned with the case of partitions of weight two and begins the proof of the theorem.

PROPOSITION 1.3. *Let $d \geqslant 3$ and $\tau = d2$. There exists an orthogonally τ-labelled Hamiltonian cycle in $G_{d+2,2}$.*

PROOF: We prove inductively that for $d \geqslant 3$ there exists an orthogonally $(d2)$-labelled Hamiltonian cycle $(\mathcal{A}, \Pi)$, such that the first set in the set-sequence is $\{1, 2\}$, the last set in the set-sequence is $\{1, d+2\}$, and the last partition in the partition sequence has a doubleton class $\{1, 3\}$.

The base step with $d = 3$ is presented in the two left-most columns of Figure 1; they comprise an orthogonally labelled Hamiltonian cycle in $G_{5,2}$ with the properties described above.

Suppose that for $d \geqslant 4$ there exists an orthogonally $\big((d-1)\,2\big)$-labelled Hamiltonian cycle $(\mathcal{B}, \Gamma)$, satisfying the above inductive assumptions. Specifically, if $\mathcal{B} = B_1 B_2 \dots B_{\binom{d+1}{2}}$ then $B_1 = \{1, 2\}$, $B_{\binom{d+1}{2}} = \{1, d+1\}$, and if $\Gamma = \gamma_1 \gamma_2 \dots \gamma_{\binom{d+1}{2}}$ then the doubleton class of $\gamma_{\binom{d+1}{2}}$ is $\{1, 3\}$. Then the partition sequence $\Gamma' = \gamma'_1 \gamma'_2 \cdots \gamma'_{\binom{d+1}{2}}$, obtained from Γ by adjoining $d+2$ to the $(d-1)$-class of each partition γ_i in Γ, orthogonally labels the cycle $\mathcal{B} = B_1 B_2 \dots B_{\binom{d+1}{2}}$ in $G_{d+2,2}$.

For $i = 1, 2, \dots, d+1$, let $C_i = \{d+2-i, d+2\}$. Then

$$\mathcal{A} = B_1 B_2 \dots B_{\binom{d+1}{2}} C_1 C_2 \dots C_{d+1}$$

is a Hamiltonian cycle in $G_{d+2,2}$ with $B_1 = \{1,2\}$ and $C_{d+1} = \{1, d+2\}$. To label $\mathcal{A}$ with orthogonal partitions of type $d2$, define the following partitions of X_{d+2}: $\pi_1 = \{1, d+2\} \mid (X_{d+1} - \{1\})$, $\pi_2 = \{2, d+2\} \mid (X_{d+1} - \{2\})$, and for $i = 3, 4, \ldots, d+1$, $\pi_i = \{d+4-i, d+2\} \mid (X_{d+1} - \{d+4-i\})$ (note that for $i = 1, 2, \ldots, d+1$ the partitions π_i have $d+2$ in a two element class, and so they are distinct from partitions in Γ'). Let $\Pi = \gamma'_1\gamma'_2 \cdots \gamma'_{\binom{d+1}{2}-1}\pi_1\pi_2\pi_3 \ldots \pi_{d+1}\gamma'_{\binom{d+1}{2}}$, then $(\mathcal{A}, \Pi)$ is an orthogonally $(d2)$-labelled Hamiltonian cycle in $G_{d+2,2}$ with the doubleton class of $\gamma'_{\binom{d+1}{2}}$ being of the form $\{1,3\}$. ∎

Given a partition type τ on X_n, let $\tau \oplus 1$ denote a partition type on X_{n+1} obtained from τ by adjoining one singleton class. If τ has a class of size $d_s > 1$, let $\tau - d_s$ be a partition type on X_{n-1} obtained from τ by reducing the size of one of its d_s-blocks by 1.

PROPOSITION 1.4. *Let $\tau = d_1^{t_1} d_2^{t_2} \ldots d_k^{t_k}$ be a partition type on X_n of weight r having at least two distinct class sizes $d_s, d_t \geqslant 2$. Suppose that there exist Hamiltonian cycles in $G_{n,r}$ and $G_{n,r+1}$ that can be labelled by partitions of type τ and $\omega = (\tau - d_s) \oplus 1$ respectively. Then there exists a Hamiltonian cycle in $G_{n+1,r+1}$ that can be labelled by partitions of type $\tau \oplus 1$.*

PROOF: Observe that ω is a partition type on X_n of weight $r+1$. Let $\mathcal{A} = A_1 A_2 \ldots A_{\binom{n}{r+1}}$ be a Hamiltonian cycle in $G_{n,r+1}$, and let $\Omega = \sigma_1\sigma_2 \ldots \sigma_{\binom{n}{r+1}}$ be a corresponding partition sequence of partitions of type ω that orthogonally labels the cycle. For each partition σ_i in Ω, let σ'_i be a partition of X_{n+1} of type $\tau \oplus 1$ obtained from σ_i by adjoining the element $n+1$ to a $(d_s - 1)$-class.

Let $\mathcal{B} = B_1 B_2 \ldots B_{\binom{n}{r}}$ be a Hamiltonian cycle in $G_{n,r}$ and let $\Gamma = \gamma_1\gamma_2 \ldots \gamma_{\binom{n}{r}}$ be a corresponding partition sequence of partitions of type τ that orthogonally label the cycle. For each B_i in $\mathcal{B}$, let B'_i be the $(r+1)$-set $B_i \cup \{n+1\}$. For each partition γ_i in Γ, let γ'_i be a partition of X_{n+1} of type $\tau \oplus 1$ obtained from γ_i by adjoining a new class $\{n+1\}$.

Without loss of generality we may assume that $A_1 = \{1, 2, \ldots, r, r+1\}$, $A_{\binom{n}{r+1}} = \{1, 2, \ldots, r, n\}$ and $B'_1 = \{1, 2, \ldots, r, n+1\}$ and $B'_{\binom{n}{r}} = \{1, 2, \ldots, r-1, n, n+1\}$ (or else we simply can relabel the elements of X_n). Choose two partitions of X_{n+1} of type $\tau \oplus 1$ containing $n+1$ in a class of size d_t such that β is orthogonal to B'_1 and A_1, and δ is orthogonal to $A_{\binom{n}{r+1}}$ and $B'_{\binom{n}{r}}$.

Then $A_1 A_2 \ldots A_{\binom{n}{r+1}} B'_{\binom{n}{r}} \ldots B'_2 B'_1$ is a Hamiltonian cycle in $G_{n+1,r+1}$ which is $\tau \oplus 1$-labelled by partitions in the sequence $\sigma'_1\sigma'_2 \ldots \sigma'_{\binom{n}{r+1}-1}\delta\gamma'_{\binom{n}{r}-1} \cdots \gamma'_2\gamma'_1\beta$. The partitions in this sequence are distinct, as partitions σ'_i contain the element $n+1$ in a d_s-class, partitions γ'_i contain $n+1$ in a singleton class, and β, δ contain $n+1$ in a d_t-class. ∎

The following theorem appears in [2].

THEOREM 1.5. *For $r \geqslant 2$ and $1 \leqslant s < r$, there exist orthgonally $2^s 1^{r-s}$-labelled Hamiltonian cycles in $G_{s+r,r}$.*

So that the work here is self-contained, we prove the aspects of Theorem 1.5 that will be used to prove the main theorem (Theorem 1.2).

LEMMA 1.6. *For $r \geqslant 2$, there exist orthogonally $2\,1^{r-1}$ and $2^2\,1^{r-2}$-labelled Hamiltonian cycles.*

PROOF: We prove the existence of stated Hamiltonian cycles with an additional condition, namely that the first set of the set-sequence is $\{1, 2, \ldots, r\}$ and the last set of the set-sequence is $\{1, 2, \ldots, r-1, n\}$, where $n = r+1$ for the $2\,1^{r-1}$-labelled cycle, and $n = r+2$ for the $2^2\,1^{r-2}$-labelled cycle.

Let $\mathcal{A} = A_1 \ldots A_{r+1}$ be any Hamiltonian cycle in $G_{r+1,r}$ with $A_1 = \{1, 2, \ldots, r\}$ and $A_{r+1} = \{1, 2, \ldots, r-1, r+1\}$. Let $\Pi = \pi_1\pi_2\ldots\pi_{r+1}$ be the sequence of partitions of the type $2\,1^{r-1}$ such that the only non-singleton class of π_i is the symmetric difference of A_i and A_{i+1}, where $i = 1, 2, \ldots, r+1$, calculated $\mod (r+1)$. Then $(\mathcal{A}, \Pi)$ is an orthogonally $2\,1^{r-1}$-labelled Hamiltonian cycle in $G_{r+1,r}$ satisfying the stated conditions on the first and the last set.

Now we prove inductively that for $r \geqslant 3$ there exists an orthogonally $2^2\,1^{r-2}$-labelled Hamiltonian cycle $(\mathcal{B}, \Gamma)$ in $G_{r+2,r}$ satisfying the stated conditions on the first and the last set. The base step with $r = 3$ is presented in the two right-most columns of Figure 1: they comprise an orthogonally $2^2\,1$-labelled Hamiltonian cycle in $G_{5,3}$ such that the first set is $\{1, 2, 3\}$ and the last set is $\{1, 2, 5\}$.

Suppose that for $r \geqslant 4$ there exists an orthogonally $2^2\,1^{r-3}$-labelled Hamiltonian cycle $(\mathcal{C}, \Psi)$ with the partition-sequence $\mathcal{C} = C_1C_2\ldots C_{\binom{r+1}{r-1}}$ satisfying the following conditions: $C_1 = \{1, 2, \ldots, r-1\}$ and $C_{\binom{r+1}{r-1}} = \{1, 2, \ldots, r-2, r+1\}$. Note that $\mathcal{C}$ is a Hamiltonian cycle in $G_{r+1,r-1}$, and for each C_i in $\mathcal{C}$ let $C_i' = C_i \cup \{r+2\}$ be an r-set in X_{r+2}. For each partition ψ_i in Ψ let ψ_i' be a partition of weight r of X_{r+2} obtained from ψ_i by adjoining a new singleton class $\{r+2\}$. Then the partition sequence $\Psi' = \psi_1'\psi_2'\ldots\psi'_{\binom{r+1}{r-1}}$ orthogonally labels the cycle $\mathcal{C} = C_1'C_2'\ldots C'_{\binom{r+1}{r-1}}$ in $G_{r+2,r}$.

By the first paragraph of this proof, there exists an orthogonally $2\,1^{r-1}$-labelled Hamiltonian cycle $(\mathcal{A}, \Pi)$ in $G_{r+1,r}$ with the partition-sequence $\mathcal{A} = A_1A_2\ldots A_{r+1}$ satisfying the following conditions: $A_1 = \{1, 2, \ldots, r\}$ and $A_{r+1} = \{1, 2, \ldots, r-1, r+1\}$. For each partition π_i in Π let π_i' be a partition of the type $2^2\,1^{r-2}$ of X_{r+2} obtained from π_i by adjoining the element $r+2$ to a singleton class of π_i not of the form $\{r-1\}$ or $\{r\}$ (such a singleton class may be selected since $r \geqslant 4$, so each π_i has at least three singleton classes). Then the partition sequence $\Pi' = \pi_1'\pi_2'\ldots\pi_{r+1}'$ orthogonally labels the cycle $\mathcal{A}$ in $G_{r+2,r}$.

Observe that

$$\mathcal{B} = A_1A_2\ldots A_{r+1}C'_{\binom{r+1}{r-1}}\ldots C_2'C_1'$$

is a Hamiltonian cycle in $G_{r+2,r}$ with the first set $A_1 = \{1, 2, \ldots, r\}$ and the last set $C_1' = \{1, 2, \ldots, r-1, r+2\}$. Let α be any partition of the type $2^2\,1^{r-2}$ which is simultaneously orthogonal to A_{r+1} and $C'_{\binom{r+1}{r-1}}$. Such α has a doubleton class $\{r-1, r+2\}$, and so it is not an element of either Ψ' or Π'. Let β be any partition of the type $2^2\,1^{r-2}$ simultaneously orthogonal to C_1' and A_1. Such a β has a doubleton class $\{r, r+2\}$, and so it is also not an element of either Ψ' or Π'. Since Ψ' or Π' have no elements in common, the sequence

$$\Gamma = \pi_1'\pi_2' \dots \pi_r'\alpha\psi'_{\binom{r+1}{r-1}-1} \dots \psi_2'\psi_1'\beta$$

consists of distinct partitions of type $2^2\,1^{r-2}$, and $(\mathcal{B}, \Gamma)$ is an orthogonally $2^2\,1^{r-2}$-labelled Hamiltonian cycle in $G_{r+2,r}$ satisfying the stated conditions on the first and the last set. □

The result below follows from Proposition 1.3, Proposition 1.4, and Lemma 1.6.

COROLLARY 1.7.

1. *For $n \geqslant 5$, $d \geqslant 2$ and $r \geqslant 2$, there exists an orthogonally $d\,2\,1^{r-2}$-labelled Hamiltonian cycle in $G_{n,r}$.*
2. *There exist orthogonally $2\,1$ and $2\,1^2$ labelled Hamiltonian cycles in $G_{3,2}$ and $G_{4,3}$ respectively.*

PROOF OF THEOREM 1.2: Let n and r be positive integers with $2 \leqslant r < n$, such that $n \neq 4$ if $r = 2$. Using Corollary 1.7, we show that there exists a Hamiltonian cycle in $G_{n,r}$ orthogonally labelled by partitions of a given fixed type τ.

If $n \geqslant 3$ and $r = n-1$ and we let $\tau = 2\,1^{r-1}$. This allows us to assume that $n \geqslant 5$ and $2 \leqslant r \leqslant n-2$. If $r = 2$ let $\tau = (n-2)\,2$. If $r = n-2$ let $\tau = 2^2\,1^{r-2}$. If $2 < r < n-2$ we let $\tau = d\,2\,1^{r-2}$, where $d \geqslant 3$. □

1.1. HAMILTONIAN CYCLES $H_{n,r}$. For given n and r with $1 \leqslant r < n$, we present the definition of the Hamiltonian cycle $H_{n,r}$. The cycles $H_{n,r}$ arise in the context of *reflected Gray codes*, certain widely studied recursively defined codes that list the subsets of X_n so that successive sets have a singleton symmetric difference. Numerous algorithms for the efficient output of $H_{n,r}$ appear in the literature ([**7**, **5**, **8**]). Below we shall outline an argument that supports the following refinement of Theorem 1.2.

THEOREM 1.8. *For all positive integers n and r with $1 < r < n$, except for the $n = 4, r = 2$ case, there exists a permutational orthogonally labelled Hamiltonian cycle in $G_{n,r}$ with set-sequence $H_{n,r}$.*

DEFINITION 1.9: Let n, r be positive integers with $r \leqslant n$, and let $H_{n,r}$ be defined recursively as follows:

1. $H_{n,n} = X_n$.
2. $H_{n,1} = \{1\} \dots \{n\}$.
3. For $1 < r < n$, given that $H_{n-1,r-1} = A_1A_2 \dots A_{\binom{n-1}{r-1}}$, let $H^{rev}_{n-1,r-1} \oplus n$ be the list

$$\left(A_{\binom{n-1}{r-1}} \cup \{n\}\right) \dots \left(A_2 \cup \{n\}\right)\left(A_1 \cup \{n\}\right),$$

that results by adjoining n to each set of $H_{n-1,r-1}$ and then reversing the order of the resulting listing.

4. For $1 < r < n$, let $H_{n,r} = H_{n-1,r}\ (H^{rev}_{n-1,r-1} \oplus n)$ be the list that results from concatenating $H_{n-1,r}$ and $H^{rev}_{n-1,r-1} \oplus n$.

EXAMPLE 1.2.

$$H_{3,2} = H_{2,2}(H^{rev}_{2,1} \oplus 3) = \{12\}\{23\}\{13\},$$
$$H_{4,2} = H_{3,2}(H^{rev}_{3,1} \oplus 4) = \{12\}\{23\}\{13\}\{34\}\{24\}\{14\},$$
$$H_{4,3} = \{123\}\{134\}\{234\}\{124\},$$
$$H_{5,3} = H_{4,3}(H^{rev}_{4,2} \oplus 5) = \{123\}\{134\}\{234\}\{124\}\{145\}\{245\}\{345\}\{135\}\{235\}\{125\}.$$

Notice that the base step of the inductive proof of Proposition 1.3 involves the cycle $H_{5,2}$. The inductive procedure used to $(d\,2)$-label Hamiltonian cycles in Proposition 1.3 leads to set-sequences which are $H_{d+2,2}$ cycles. The construction used in Proposition 1.4 guarantees that if the two given cycles are $H_{n-1,r}$ and $H_{n,r+1}$, then the resulting $\tau \oplus 1$-labelled cycle is $H_{n+1,r+1}$. Thus, we may assume that for $d \geqslant 3$, the $(d\,2\,1^{r-2})$-labelled Hamiltonian cycles used in the proof of Theorem 1.2 are all $H_{n,r}$ cycles.

The Hamiltonian cycle in $G_{5,3}$ in Figure 1 is $H_{5,3}$. We can assume the Hamiltonian cycles of $G_{r+1,r}$ used in the proof in Lemma 1.6 are $H_{r+1,r}$ cycles. Once again, the inductive procedure used in the proof of Lemma 1.6 leads to $H_{r+2,r}$ cycles for $2^2\,1^{r-2}$ cases. Thus we may assume that all the orthogonally labelled cycles in Corollary 1.7 are $H_{n,r}$ cycles. Theorem 1.8 follows.

2. CONCLUSION

In this work the improvement over existing literature involves the "permutational" aspect of our main theorem. Indeed in [3], the present authors and R. B. McFadden prove that *for any* Hamiltonian cycle $\mathcal{A}$ there exists a partition sequence Π such that $(\mathcal{A}, \Pi)$ is an orthogonally labelled *Hamiltonian* cycle. They provide a highly efficient algorithm that on input (n, r) outputs an orthogonally labelled Hamiltonian cycle. However, except for the $(3, 2)$ case, the partition sequence associated with their algorithm is not permutational. In [3] the *Transposition Listing Conjecture* is stated: for $n \geqslant 2r$, the authors conjecture that there exists a permutational orthogonally labelled Hamiltonian cycle such that all permutations involved are transpositions. The authors show that the validity of the Transposition Listing Conjecture is a logical consequence of the celebrated *Middle Levels Conjecture* (for a reference on the Middle Levels Conjecture, see [6]).

The partition type τ is said to be *exceptional* ([2]) if the number of distinct partitions of type τ is less than $\binom{n}{r}$. Clearly if τ is an exceptional partition type, no orthogonally τ-labelled list exists. In [2], the first author and J. Lehel prove existence of orthogonally τ-labelled lists for all non-exceptional partition types τ with classes of size at most two, a result we used in the paper. Moreover they show that for $1 \leqslant s < r$, there exist orthogonally $2^s 1^{r-s}$-labelled Hamiltonian cycles. In [**4**], the authors extend this result and show that even for non-exceptional τ of the form 2^r, there exist orthogonally 2^r-labelled Hamiltonian cycles.

In [**2**] it is conjectured that for every non-exceptional type τ, there exists orthogonally τ-labelled list. The present paper is a part of a series of papers directed towards proving this conjecture.

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