Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

doi:10.3934/xx.xx.xx.xx

pp. **X–XX**

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(Communicated by the associate editor name)

FINITE ELEMENT APPROXIMATION OF SPARSE PARABOLIC CONTROL PROBLEMS

Abstract. We study the finite element approximation of an optimal control problem governed by a semilinear partial differential equation and whose objective function includes a term promoting space sparsity of the solutions. We prove existence of solution in the absence of control bound constraints and provide the adequate second order sufficient conditions to obtain error estimates. Full discretization of the problem is carried out, and the sparsity properties of the discrete solutions, as well as error estimates, are obtained.

1. **Introduction.** Throughout this paper, Ω denotes an open, bounded subset of $\mathbb{R}^n$, $1 \le n \le 3$, with boundary Γ, and $0 < T < +\infty$ is fixed. We set $Q = \Omega \times (0, T)$ and $\Sigma = \Gamma \times (0, T)$. The control problem is defined in the way

$$\text{(P)} \quad \min_{u \in L^\infty(Q)} J(u),$$

where $J(u) = F(u) + \mu j(u)$ with $\mu > 0$,

$$F(u) = \frac{1}{2}\int_Q (y_u - y_d)^2 \, dx \, dt + \frac{\nu}{2}\int_Q u^2 \, dx \, dt,$$

$\nu > 0$, and

$$j(u) = \|u\|_{L^1(\Omega;L^2(0,T))} = \int_\Omega \|u(x)\|_{L^2(0,T)} \, dx = \int_\Omega \Big(\int_0^T u^2(x,t) \, dt\Big)^{1/2} dx.$$

2010 *Mathematics Subject Classification.* Primary: 49K20, 35K58, 65M15; Secondary: 49J52.

Key words and phrases. Optimal control, semilinear parabolic equations, sparse solutions, finite element approximation, error estimates.

The first two authors were partially supported by the Spanish Ministerio de Economía y Competitividad under project MTM2014-57531-P.

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For every $u \in L^\infty(Q)$, we denote y_u the solution of

$$\begin{cases} \partial_t y + Ay + a(x,t,y) = u & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } \Omega. \end{cases} \tag{1}$$

Here, A is the linear elliptic operator

$$Ay = -\sum_{i,j=1}^{n} \partial_{x_j}[a_{ij}(x)\,\partial_{x_i} y].$$

Our objective in this work is to study the finite element discretization of the problem: we describe the sparsity pattern of the discrete solutions, prove convergence and provide error estimates.

The first application of L^1-promoting-sparsity terms to optimal control problems was done in [17] for control problems governed by linear elliptic equations. Finite element discretization and error estimates for such a problem were obtained in [18] also for linear elliptic equations. The semilinear case was treated in [6] for piecewise constant approximations of the control and in [5] for continuous piecewise linear approximations. In [3, 2, 15] the case of measure controls for problems governed by linear elliptic equations is studied.

In [11] directional sparsity is introduced and an application to problems governed by linear parabolic equations is considered. In a similar framework, measure-valued controls are considered in [4, 9, 10, 12] for a problem governed by a linear parabolic equation. The measures used in [12] promote, as in the work at hand, a constant-in-time sparsity pattern; a finite element approximation is studied and error estimates for the approximation of the states are provided.

The control of semilinear parabolic equations with measures is quite complicated due to the possible non-existence of solution of the partial differential equation; see [8] for a discussion of this topic for semilinear elliptic equations. To avoid this difficulty, we will use functions to control the nonlinear equation.

The plan of the paper is as follows. At the end of this section the main assumptions are introduced. In Section 2 we recall results about the existence and uniqueness of solution of the state equation and the differentiability properties of the control-to-state mapping and cost functional. Next, in Section 3, we prove existence of solution of the control problem, write the first order necessary optimality conditions and show the regularity and sparsity properties of the optimal controls. Since we are not imposing any bound constraints on the control, existence of solution of problem (P) cannot be deduced by the direct method of calculus of variations as usual, so we employ a truncation method; see Theorem 3.2.

In Section 4 we investigate second order optimality conditions. First and second order necessary and sufficient optimality conditions for control problems governed by semilinear parabolic equations and with a term promoting sparsity in the objective functional have recently been studied in [7]. Three different cases are described in that work, promoting each of them a particular case of sparsity: global sparsity, spatial sparsity whose pattern changes with time and spatial sparsity whose pattern is constant in time. We are interested in this last case. In [7, Theorem 4.12] the authors prove that under adequate second order conditions, the critical point is a strict local minimum in the $L^\infty(\Omega; L^2(0,T))$ sense. This result is not enough to derive error estimates of the numerical estimation of the control problem. The argument we use in Lemma 5.5 to show the existence of a sequence of local minima of the discretized problems converging strongly in $L^2(Q)$ to a strict local minimum of the continuous problem would be incorrect in $L^\infty(\Omega; L^2(0,T))$. To overcome this difficulty, we prove in Theorem 4.2 that under the same second order sufficient conditions, the critical point is also a strict local minimum in the $L^2(Q)$ sense.

Finally, in Section 5, we fully discretize the problem using, in space, continuous piecewise linear elements for the state and piecewise constant approximations for the control and, in time, piecewise constant functions for both variables. We show that the discrete optimal controls follow a sparsity pattern alike the one obtained for the continuous ones and prove convergence and an error estimate in the $L^2(Q)$ norm of the control variable of order $O(\sqrt{\tau}+h)$, where τ denotes the step size in time and h is the mesh size in space. Finally, two numerical experiments are included in Section 6. In the first one we investigate the experimental order of convergence and compare with our theoretical results and in the second one we expose the directional sparsity properties of the solution of (P).

The study of approximations of the control by means of continuous piecewise linear functions in space will be done in a forthcoming paper.

We make the following assumptions.

Assumption 1.— The boundary Γ is of class $C^{1,1}$ or Ω is convex. The coefficients $a_{ij} \in C^{0,1}(\bar\Omega)$ and

$$\exists \Lambda > 0 \ \text{ such that } \ \sum_{i,j=1}^{n} a_{ij}(x)\,\xi_i\,\xi_j \geq \Lambda\,|\xi|^2 \ \forall x \in \Omega \ \text{ and } \ \forall \xi \in \mathbb{R}^n. \tag{2}$$

Assumption 2.— The initial datum $y_0 \in L^\infty(\Omega) \cap H_0^1(\Omega)$, the target state $y_d \in L^{\hat p}(0,T; L^{\hat q}(\Omega))$ where $\hat p, \hat q \in [2,+\infty]$ are such that $\frac{1}{\hat p} + \frac{n}{2\hat q} < 1$, and $a : Q \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function of class C^2 with respect to the last variable, satisfying the following assumptions

$$\left\{\begin{array}{l} a(\cdot,\cdot,0) \in L^{\hat p}(0,T; L^{\hat q}(\Omega)) \text{ and such that} \\ \dfrac{\partial a}{\partial y}(x,t,y) \geq 0 \ \text{ for a.a. } (x,t) \in Q \text{ and } \forall y \in \mathbb{R}, \end{array}\right. \tag{3}$$

$$\left\{\begin{array}{l} \forall M > 0\ \exists C_M > 0 \text{ such that} \\ \left|\dfrac{\partial^j a}{\partial y^j}(x,t,y)\right| \leq C_M \text{ for a.a. } (x,t) \in Q, \forall |y| \leq M, \text{ with } j = 1,2 \end{array}\right. \tag{4}$$

$$\left\{\begin{array}{l} \forall \rho > 0 \text{ and } \forall M > 0\ \exists \varepsilon_{M,\rho} > 0 \text{ such that for a.a. } (x,t) \in Q \\ \left|\dfrac{\partial^2 a}{\partial y^2}(x,t,y_2) - \dfrac{\partial^2 a}{\partial y^2}(x,t,y_1)\right| \leq \rho, \forall |y_i| \leq M, \text{ and } |y_2 - y_1| \leq \varepsilon_{M,\rho}. \end{array}\right. \tag{5}$$

Remark 1. We can deal with non-monotone nonlinearities satisfying

$$\frac{\partial a}{\partial y}(x,t,y) \geq -\delta$$

for some $\delta > 0$ with the change of variable $\tilde y = e^{-\delta t} y$. Denoting $\tilde y_d = e^{-\delta t} y_d$. In this way, problem (P) is equivalent to

$$\min_{u \in L^\infty(Q)} \frac{1}{2}\|e^{\delta t}(\tilde y - \tilde y_d)\|_{L^2(Q)} + \frac{\nu}{2}\|u\|_{L^2(Q)} + \mu\|u\|_{L^1(\Omega; L^2(0,T))}$$

subject to

$$\left\{\begin{array}{l} \partial_t \tilde y + A\tilde y + \tilde a(x,t,\tilde y) = e^{-\delta t} u \ \text{ in } \ Q, \\ \tilde y = 0 \ \text{ on } \ \Sigma, \\ \tilde y(0) = y_0 \ \text{ in } \ \Omega. \end{array}\right.$$

where $\tilde{a}(x,t,\tilde{y}) = \delta\tilde{y} + e^{-\delta t}a(x,t,e^{\delta t}\tilde{y})$ satisfies assumption 2.

2. Analysis of the state equation and the objective functional.

Next we describe the differentiability properties of the control-to-state mapping and later we analyze the cost functional. The next results are quoted from [7].

Theorem 2.1. *Under the assumptions 1 and 2, for all* $u \in L^{\hat{p}}(0,T;L^{\hat{q}}(\Omega))$ *the equation* (1) *has a unique solution* $y_u \in Y = L^\infty(\bar{Q}) \cap H^{2,1}(Q)$. *Moreover, the mapping* $G : L^{\hat{p}}(0,T;L^{\hat{q}}(\Omega)) \longrightarrow Y$*, defined by* $G(u) = y_u$*, is of class* C^2. *For all elements* u, v, v_1 *and* v_2 *of* $L^{\hat{p}}(0,T;L^{\hat{q}}(\Omega))$*, the functions* $z_v = G'(u)\,v$ *and* $z_{v_1v_2} = G''(u)(v_1,v_2)$ *are the solutions of the problems*

$$\begin{cases} \dfrac{\partial z}{\partial t} + Az + \dfrac{\partial a}{\partial y}(x,t,y_u)z = v & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(0) = 0 & \text{in } \Omega, \end{cases} \tag{6}$$

and

$$\begin{cases} \dfrac{\partial z}{\partial t} + Az + \dfrac{\partial a}{\partial y}(x,t,y_u)z + \dfrac{\partial^2 a}{\partial y^2}(x,t,y_u)z_{v_1}z_{v_2} = 0 & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(0) = 0 & \text{in } \Omega, \end{cases} \tag{7}$$

respectively.

In [7] it is proved that $y_u \in L^\infty(Q) \cap W(0,T)$, where $W(0,T) = L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega))$. From Assumption (A2) and the boundness of y_u, we have that $\partial_t y_u + Ay_u \in L^2(Q)$, and hence $y_u \in H^{2,1}(Q) = L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega))$; see e.g. [13, Theorem III-6.1].

Theorem 2.2. *Under the assumptions 1–2,* $F : L^{\hat{p}}(0,T;L^{\hat{q}}(\Omega)) \longrightarrow \mathbb{R}$ *is of class* C^2. *Moreover, for all* u, v, v_1 *and* v_2 *of* $L^{\hat{p}}(0,T;L^{\hat{q}}(\Omega))$ *we have*

$$F'(u)v = \int_Q \big((y_u - y_d)\,z_v + \nu uv\big)\,dx\,dt = \int_Q (\varphi_u + \nu u)\,v\,dx\,dt, \tag{8}$$

$$F''(u)(v_1,v_2) = \int_Q \left[\left(1 - \varphi_u \frac{\partial^2 a}{\partial y^2}(x,t,y_u)\right)z_{v_1}z_{v_2} + \nu v_1 v_2\right]dx\,dt, \tag{9}$$

where $z_{v_i} = G'(u)\,v_i$*,* $i = 1,2$*, and* $\varphi_u \in Y$ *is the solution of*

$$\begin{cases} -\dfrac{\partial \varphi}{\partial t} + A^*\varphi + \dfrac{\partial a}{\partial y}(x,t,y_u)\varphi = y_u - y_d & \text{in } Q, \\ \varphi = 0 & \text{on } \Sigma, \\ \varphi(T) = 0 & \text{in } \Omega, \end{cases} \tag{10}$$

A^* *being the adjoint operator of* A.

Remark 2. Observe that for every $u \in L^{\hat{p}}(0,T;L^{\hat{q}}(\Omega))$, $G'(u)$ can be extended to a linear and continuous mapping $G'(u) : L^2(Q) \longrightarrow H^{2,1}(Q)$. We also have that $G''(u)$ admits a continuous bilinear extension $G''(u) : L^2(Q) \times L^2(Q) \longrightarrow Y$ and $F'(u)$ and $F''(u)$ can be extended to linear and bilinear continuous forms $F'(u) : L^2(Q) \longrightarrow \mathbb{R}$ and $F''(u) : L^2(Q) \times L^2(Q) \longrightarrow \mathbb{R}$.

Proposition 1. *Given* $u \in L^1(\Omega;L^2(0,T))$ *the following statements hold.*

1. *$\lambda \in \partial j(u)$ is equivalent to $\lambda \in L^\infty(\Omega; L^2(0,T))$ and*

$$\begin{cases} \|\lambda(x)\|_{L^2(0,T)} \le 1 & \text{for a.a. } x \in \Omega_u^0, \\ \lambda(x,t) = \dfrac{u(x,t)}{\|u(x)\|_{L^2(0,T)}} & \text{for a.a. } x \in \Omega_u \text{ and } t \in (0,T), \end{cases} \tag{11}$$

where

$$\Omega_u = \{x \in \Omega : \|u(x)\|_{L^2(0,T)} \neq 0\} \quad \text{and} \quad \Omega_u^0 = \Omega \setminus \Omega_u.$$

2. *For every $v \in L^1(\Omega; L^2(0,T))$, the directional derivative of j at u in the direction v is*

$$j'(u;v) = \int_{\Omega_u^0} \|v(x)\|_{L^2(0,T)}\, dx + \int_{\Omega_u} \frac{1}{\|u(x)\|_{L^2(0,T)}} \int_0^T u\, v\, dt\, dx. \tag{12}$$

3. **Existence of solution for (P), first order optimality conditions and regularity of the optimal controls.** The absence of control bounds leads to some difficulties regarding the existence of optimal controls for (P). We cannot apply the usual direct approach to prove existence of solution of (P), because we cannot conclude the boundedness in $L^\infty(Q)$ of a minimizing sequence. Alternatively, we could have settled the problem in $L^2(Q)$, but in this case Theorems 2.1 and 2.2 do not fulfill. Instead, we are going to introduce an auxiliary problem with bound control constraints to prove existence of a solution of (P).

For $M > 0$ consider the set

$$U_M = \{u \in L^2(Q) : \ -M \le u(x,t) \le M \ \text{ for a.e. } (x,t) \in Q\}.$$

Associated to this set, we have the problem

$$(\mathrm{P}_M) \begin{cases} \min\limits_{u \in U_M} J(u). \end{cases}$$

Existence of a solution $\bar u_M$ for problem (P_M) is standard, see [7, Theorem 1.4], and the following first order optimality conditions are satisfied.

Theorem 3.1. *If $\bar u_M$ is a local minimum of (P_M), then there exist $\bar y_M, \bar\varphi_M \in Y$ and $\bar\lambda_M \in \partial j(\bar u_M)$ such that*

$$\begin{cases} \partial_t \bar y_M + A\bar y_M + a(x,t,\bar y_M) &=& \bar u_M & \text{in } Q, \\ \bar y_M &=& 0 & \text{on } \Sigma, \\ \bar y_M(0) &=& y_0 & \text{in } \Omega, \end{cases} \tag{13}$$

$$\begin{cases} -\partial_t \bar\varphi_M + A^*\bar\varphi_M + \dfrac{\partial a}{\partial y}(x,t,\bar y_M)\,\bar\varphi_M &=& \bar y_M - y_d & \text{in } Q, \\ \bar\varphi_M &=& 0 & \text{on } \Sigma, \\ \bar\varphi_M(T) &=& 0 & \text{in } \Omega, \end{cases} \tag{14}$$

$$\int_Q (\bar\varphi_M + \nu \bar u_M + \mu\bar\lambda_M)(u - \bar u_M)dxdt \ge 0 \ \forall u \in U_M. \tag{15}$$

The proof is standard and can be found in [7, Theorem 2.1]. The projection formula

$$\bar u_M(x,t) = \mathrm{Proj}_{[-M,M]}\left(-\frac{1}{\nu}\left[\bar\varphi_M(x,t) + \mu\bar\lambda_M(x,t)\right]\right) \tag{16}$$

follows in a standard way from (15). Next, we prove existence of solution for (P).

Theorem 3.2. *There exists* $C_\infty > 0$ *independent of* M *such that* $\|\bar u_M\|_{L^\infty(Q)} \le C_\infty$. *Consequently, for every* $M \ge C_\infty$, *any solution* $\bar u_M$ *of* (P_M) *is also a solution of* (P).

Proof. Using the optimality of $\bar u_M$ we have that $J(\bar u_M) \le J(0)$, hence

$$\|\bar u_M\|_{L^2(Q)} \le \frac{1}{\sqrt{\nu}}\|\tilde y - y_d\|_{L^2(Q)}, \tag{17}$$

where $\tilde y$ is the state associated to the control $u \equiv 0$.

Subtracting $a(x,t,0)$ at both sides of the PDE in (13), multiplying by $\bar y_M$ and integrating from 0 to t we have that

$$\begin{aligned}\frac12\|\bar y_M(t)\|^2_{L^2(\Omega)} - \frac12\|y_0\|^2_{L^2(\Omega)} + \int_0^t\int_\Omega \sum_{i,j=1}^n a_{i,j}\partial_{x_i}\bar y_M\partial_{x_j}\bar y_M\,dxds \\ + \int_0^t\int_\Omega (a(x,s,\bar y_M) - a(x,s,0))\bar y_M\,dxds \\ = \int_0^t\int_\Omega (\bar u_M - a(x,s,0))\bar y_M\,dxds\end{aligned}$$

Using the monotonicity of $a(x,t,\cdot)$, we obtain by means of the Cauchy-Schwarz and Friedrichs' inequalities that there exists $C_\Omega > 0$ such that

$$\begin{aligned}&\frac12\|\bar y_M(t)\|^2_{L^2(\Omega)} + \Lambda\int_0^t\int_\Omega |\nabla\bar y_M|^2\,dxds \\ \le\ &\frac12\|\bar y_M(t)\|^2_{L^2(\Omega)} + \int_0^t\int_\Omega \sum_{i,j=1}^n a_{i,j}\partial_{x_i}\bar y_M\partial_{x_j}\bar y_M\,dxds \\ \le\ &\frac12\|y_0\|^2_{L^2(\Omega)} + \left(\|\bar u_M\|_{L^2(Q)} + \|a(x,t,0)\|_{L^2(Q)}\right)\|\bar y_M\|_{L^2(0,T;L^2(\Omega))} \\ \le\ &\frac12\|y_0\|^2_{L^2(\Omega)} + C_\Omega\left(\|\bar u_M\|_{L^2(Q)} + \|a(x,t,0)\|_{L^2(Q)}\right)\|\nabla\bar y_M\|_{L^2(0,T;L^2(\Omega))} \\ \le\ &\frac12\|y_0\|^2_{L^2(\Omega)} + \frac{C_\Omega^2}{2\Lambda}\left(\|\bar u_M\|_{L^2(Q)} + \|a(x,t,0)\|_{L^2(Q)}\right)^2 + \frac{\Lambda}{2}\|\nabla\bar y_M\|^2_{L^2(0,T;L^2(\Omega))}\end{aligned}$$

where Λ is the coercitivity constant of the operator, described in (2). Reordering, we get

$$\|\bar y_M(t)\|_{L^2(\Omega)} \le \|y_0\|_{L^2(\Omega)} + \frac{C_\Omega}{\sqrt{\Lambda}}\left(\|\bar u_M\|_{L^2(Q)} + \|a(x,t,0)\|_{L^2(Q)}\right)$$

Using (17) we obtain

$$\|\bar y_M\|_{L^\infty(0,T;L^2(\Omega))} \le \left[\|y_0\|_{L^2(\Omega)} + \frac{C_\Omega}{\sqrt{\Lambda}}\left(\frac{1}{\sqrt{\nu}}\|\tilde y - y_d\|_{L^2(Q)} + \|a(x,t,0)\|_{L^2(Q)}\right)\right] \tag{18}$$

Now, using the results in [13, Theorem III-7.1], we have that there exists $C > 0$ such that

$$\|\bar\varphi_M\|_{L^\infty(Q)} \le C(\|\bar y_M\|_{L^\infty(0,T;L^2(\Omega))} + \|y_d\|_{L^{\hat p}(0,T;L^{\hat q}(\Omega))})$$

and from estimate (18), we deduce the existence of $C^* > 0$ independent of M such that

$$\|\bar\varphi_M\|_{L^\infty(Q)} \le C^*. \tag{19}$$

Using the variational inequality (15) and the equality in (11), we have that

$$\int_0^T\int_{\Omega_{\bar u_M}}\left(\bar\varphi_M+\left[\nu+\frac{\mu}{\|\bar u_M(x)\|_{L^2(0,T)}}\right]\bar u_M\right)(u(x,t)-\bar u_M(x,t))dxdt\ge 0\ \forall u\in U_M.$$

It can be easily checked that this implies that

$$\bar u_M(x,t)=\text{proj}_{[-M,M]}\left(\frac{-\bar\varphi_M(x,t)}{\nu+\frac{\mu}{\|\bar u_M(x)\|_{L^2(0,T)}}}\right)\text{ for a.a. }(x,t)\in\Omega_{\bar u_M}\times(0,T).$$

Taking into account that $\bar u_M$ vanishes in $\Omega^0_{\bar u_M}\times(0,T)$, we conclude that

$$|\bar u_M(x,t)|\le\frac{|\bar\varphi_M(x,t)|}{\nu+\frac{\mu}{\|\bar u_M(x)\|_{L^2(0,T)}}}\le\frac{1}{\nu}|\bar\varphi_M(x,t)|\text{ for a.a. }(x,t)\in Q.$$

Hence, using (19), we have that the first claim holds for $C_\infty=C^*/\nu$.

Finally, we prove that for $M\ge C_\infty$, $\bar u_M$ is a solution of (P). Let us take $u\in L^\infty(Q)$ and set $M'=\|u\|_{L^\infty(Q)}$. If $M'\le M$, then $u\in U_M$ and $J(\bar u_M)\le J(u)$. If $M'>M$, consider $\bar u_{M'}$, a solution of (P$_{M'}$). We have that $\|\bar u_{M'}\|_{L^\infty(Q)}\le C_\infty\le M$, and hence $\bar u_{M'}\in U_M$, so $J(\bar u_M)\le J(\bar u_{M'})\le J(u)$, and the proof is complete. □

To end this section, we describe the sparsity properties of optimal controls, as well as their regularity.

Theorem 3.3. *If $\bar u$ is a local solution of* (P)*, then there exist $\bar y,\bar\varphi\in Y$ such that relations* (13) *and* (14) *hold withdrawing the subindex M and there exists $\bar\lambda\in\partial j(\bar u)$ such that*

$$\bar\varphi+\nu\bar u+\mu\bar\lambda=0. \tag{20}$$

Moreover, $\bar\varphi\in C(\bar Q)$ $\bar u,\bar\lambda\in C(\bar Q)\cap H^1(Q)$ and the following relations hold for all $(x,t)\in Q$

$$\|\bar u(x)\|_{L^2(0,T)}=0\Leftrightarrow\|\bar\varphi(x)\|_{L^2(0,T)}\le\mu, \tag{21}$$

$$\bar\lambda(x,t)=\begin{cases}-\dfrac{1}{\mu}\bar\varphi(x,t) & \textit{if } x\in\Omega^0_{\bar u},\\[2mm] \dfrac{\bar u(x,t)}{\|\bar u(x)\|_{L^2(0,T)}} & \textit{if } x\in\Omega_{\bar u}.\end{cases} \tag{22}$$

Furthermore, $\bar\lambda$ is unique for $\bar u$ given.

Proof. The continuity $\bar\varphi\in C(\bar Q)$ follows from (14) and [13, Theorem III-10.1]. Relation (20) is a direct consequence of theorems 3.1 and 3.2. From (20) and (11) we get

$$\bar u(x,t)\Big[\nu+\frac{\mu}{\|\bar u(x)\|_{L^2(0,T)}}\Big]=-\bar\varphi(x,t)\ \text{ a.e. in }\Omega_{\bar u}\times(0,T). \tag{23}$$

Taking the $L^2(0,T)$-norm in (23) we infer

$$\|\bar u(x)\|_{L^2(0,T)}=\frac{1}{\nu}\big[\|\bar\varphi(x)\|_{L^2(0,T)}-\mu\big]\text{ a.e. in }\Omega_{\bar u}. \tag{24}$$

Hence, (24) implies that $\|\bar\varphi(x)\|_{L^2(0,T)}>\mu$ if $x\in\Omega_{\bar u}$. On the other hand, if $x\in\Omega^0_{\bar u}$, then (20) implies that $\bar\varphi(x,t)=-\mu\bar\lambda(x,t)$. Then, from (11) we get $\|\bar\varphi(x)\|_{L^2(0,T)}\le\mu$ in $\Omega^0_{\bar u}$. Thus, (21) is proved. The relations (22) are an immediate consequence of (20) and (11) as well. Now combining (21) and (24) we deduce that

$$\|\bar u(x)\|_{L^2(0,T)}=\frac{1}{\nu}\big[\|\bar\varphi(x)\|_{L^2(0,T)}-\mu\big]^+\text{a.e. in }\Omega, \tag{25}$$

where $s^+ = \max(s, 0)$ for every $s \in \mathbb{R}$. Since $\bar\varphi \in C(\bar Q) \cap H^1(Q)$ we obtain from (25) that the function $x \to \|\bar u(x)\|_{L^2(0,T)}$ belongs to $C(\bar\Omega) \cap H^1(\Omega)$. Indeed, it is enough to observe that $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(s) = (s-\mu)^+/\nu$ is Lipschitz continuous, and $g : \Omega \longrightarrow \mathbb{R}$ given by $g(x) = \|\bar\varphi(x)\|_{L^2(0,T)}$ belongs to $C(\bar\Omega) \cap H^1(\Omega)$, hence $(f \circ g) \in H^1(\Omega)$.

Additionally, (23) implies that

$$\bar u(x,t) = -\frac{\|\bar u(x)\|_{L^2(0,T)}}{\nu\|\bar u(x)\|_{L^2(0,T)} + \mu}\bar\varphi(x,t) \quad \forall (x,t) \in Q, \tag{26}$$

The continuity $\bar u \in C(\bar Q)$ follows from (26), $\|\bar u(x)\|_{L^2(0,T)} \in C(\bar\Omega)$ and $\bar\varphi \in C(\bar Q)$. Let us check that $\bar u \in H^1(Q)$. To this end, now we set $f : [0, +\infty) \longrightarrow \mathbb{R}$ with $f(s) = \frac{s}{\nu s+\mu}$, and $g : \Omega \longrightarrow \mathbb{R}$ given by $g(x) = \|\bar u(x)\|_{L^2(0,T)}$. We have that $f \in C^\infty[0,\infty)$, $|f(s)| \le 1/\nu$ and $|f'(s)| \le 1/\mu$ for all $s \ge 0$, $g \in H^1(\Omega)$, and $\bar u(x,t) = -(f \circ g)(x)\bar\varphi(x,t)$. Therefore, we can apply the chain rule to obtain

$$\nabla_x \bar u(x,t) = -f'(g(x))\nabla g(x)\bar\varphi(x,t) - f(g(x))\nabla_x\bar\varphi(x,t),$$

which is in $L^2(\Omega)$ since $(f' \circ g)$, $(f \circ g) \in L^\infty(\Omega)$, $\bar\varphi \in C(\bar Q)$, $\nabla g \in L^2(\Omega)$, and $\nabla_x\bar\varphi(x,t) \in L^2(Q)$.On the other hand, using again relation (26) we deduce

$$\int_Q |\partial_t \bar u(x,t)|^2 dxdt = \int_\Omega \left(\frac{\|\bar u(x)\|_{L^2(0,T)}}{\nu\|\bar u(x)\|_{L^2(0,T)} + \mu}\int_0^T |\partial_t\bar\varphi(x,t)|^2\right) dxdt \le \frac{1}{\nu}\|\partial_t\bar\varphi\|^2_{L^2(Q)}.$$

Hence, the assertion follows from the regularity $\bar\varphi \in H^1(Q)$.

Finally, $\bar\varphi$, $\bar u \in C(\bar Q) \cap H^1(Q)$ and relation (20) imply that $\bar\lambda \in C(\bar Q) \cap H^1(Q)$. □

4. **Second order conditions.** In this section, we provide necessary and sufficient second order optimality conditions. First let us introduce the cone of critical directions

$$C_{\bar u} = \{v \in L^2(Q) : F'(\bar u)v + \mu j'(\bar u; v) = 0\}. \tag{27}$$

Proposition 2. *The set $C_{\bar u}$ is a closed, convex cone in $L^2(Q)$.*

The proof of this proposition can be found in [7, Proposition 3.1] and is based on the observation that

$$F'(\bar u)v + \mu j'(\bar u; v) \ge \int_Q (\bar\varphi + \nu\bar u + \mu\bar\lambda)v\,dx\,dt = 0 \ \forall v \in L^2(Q). \tag{28}$$

We define

$$j''(u;v^2) = \begin{cases} \displaystyle\int_{\Omega_u} \frac{1}{\|u(x)\|_{L^2(0,T)}}\left[\int_0^T v^2(x,t)\,dt - \left(\int_0^T \frac{u(x,t)\,v(x,t)}{\|u(x)\|_{L^2(0,T)}}\,dt\right)^2\right] dx \text{ if } u \ne 0, \\ 0 \text{ if } u = 0. \end{cases} \tag{29}$$

The expression for $j''(u; v^2)$ is just notation, it does not mean that there exists a second derivative in the direction v. In fact, the integral above could be $+\infty$ in some cases. Observe that the integral is well defined because the integrand in Ω_u is nonnegative, which can be proved easily with the Schwarz inequality. In the sequel we will denote $J'(u; v) = F'(u)v + \mu j'(u; v)$ and $J''(u; v^2) = F''(u)v^2 + \mu j''(u; v^2)$.

Necessary conditions are a consequence of [7, Theorem 3.3, Case III].

Theorem 4.1. *Let $\bar u$ be a local minimum of* (P). *Then $J''(\bar u; v^2) \geq 0$ for every $v \in C_{\bar u}$.*

Sufficient conditions are nevertheless different from [7, Theorem 4.12], since in that reference local optimality is proved in $L^\infty(\Omega; L^2(0,T))$, whereas we are able to prove local optimality in $L^2(Q)$. This is essential to prove error estimates for finite dimensional approximations of (P); see Lemma 5.6 below.

Theorem 4.2. *Let $\bar u$ satisfy the first order optimality conditions given by Theorem 3.1 and such that $J''(\bar u; v^2) > 0$ $\forall v \in C_{\bar u} \setminus \{0\}$. Then, there exist $\varepsilon > 0$ and $\delta > 0$ such that*

$$J(\bar u) + \frac{\delta}{2}\|u - \bar u\|^2_{L^2(Q)} \leq J(u) \quad \forall u \in \bar B_\varepsilon(\bar u), \tag{30}$$

where $\bar B_\varepsilon(\bar u) = \{u \in L^\infty(Q) : \|u - \bar u\|_{L^2(Q)} \leq \varepsilon\}$.

Lemma 4.3. *Under the assumptions of Theorem 4.2, if there are no $\delta > 0$ and $\varepsilon > 0$ such that* (30) *holds, then there exist a sequence $\{u_k\}_{k=1}^\infty \subset L^\infty(Q)$ and measurable subsets of Ω, $\{\Omega_k\}_{k=1}^\infty$, such that*

$$|\Omega \setminus \Omega_k| < \frac{1}{k} \quad \forall k \geq 1, \tag{31}$$

$$\|u_k - \bar u\|_{L^\infty(\Omega_k; L^2(0,T))} + \|u_k - \bar u\|_{L^2(Q)} < \frac{1}{k}, \tag{32}$$

$$J(u_k) < J(\bar u) + \frac{1}{2k}\|u_k - \bar u\|^2_{L^2(Q)}. \tag{33}$$

Proof. If (30) does not hold, then for any integer $k \geq 1$ there exists an element $w_k \in L^\infty(Q)$ such that

$$\|w_k - \bar u\|_{L^2(Q)} < \frac{1}{k} \text{ and } J(w_k) < J(\bar u) + \frac{1}{2k}\|w_k - \bar u\|^2_{L^2(Q)}. \tag{34}$$

Since $\|w_k(x) - \bar u(x)\|_{L^2(0,T)} \to 0$ in $L^2(\Omega)$, we can extract a subsequence, denoted in the same way such that $\|w_k(x) - \bar u(x)\|_{L^2(0,T)} \to 0$ for almost all points $x \in \Omega$. Then, from Egorov's theorem we deduce the existence of a subsequence $\{w_{j_k}\}_{k=1}^\infty$ and a sequence $\{\Omega_k\}_{k=1}^\infty$ of measurable subsets of Ω such that (31) holds and

$$\|w_{j_k} - \bar u\|_{L^\infty(\Omega_k; L^2(0,T))} = \text{ess sup}_{x \in \Omega_k}\|w_{j_k}(x) - \bar u(x)\|_{L^2(0,T)} < \frac{1}{2k}.$$

Moreover, j_k can be chosen so that $j_k > 2k$. Then setting $u_k = w_{j_k}$ we get with (34)

$$\begin{aligned}&\|u_k - \bar u\|_{L^\infty(\Omega_k; L^2(0,T))} + \|u_k - \bar u\|_{L^2(Q)}\\ &= \|w_{j_k} - \bar u\|_{L^\infty(\Omega_k; L^2(0,T))} + \|w_{j_k} - \bar u\|_{L^2(Q)} < \frac{1}{2k} + \frac{1}{j_k} < \frac{1}{k}\end{aligned}$$

and

$$J(u_k) = J(w_{j_k}) < J(\bar u) + \frac{1}{j_k}\|w_{j_k} - \bar u\|^2_{L^2(Q)} < J(\bar u) + \frac{1}{2k}\|u_k - \bar u\|^2_{L^2(Q)},$$

which proves (32) and (33). □

Proof of Theorem 4.2. We argue by contradiction. If (30) does not hold, then we get from Lemma 4.3 a sequence $\{u_k\}_{k=1}^\infty$ satisfying (31)-(33). Let us define $\rho_k = \|u_k - \bar u\|_{L^2(Q)} < 1/k$ and $v_k = (u_k - \bar u)/\rho_k$. Since, $\|v_k\|_{L^2(Q)} = 1$ for every k, we can extract a subsequence denoted in the same way so that $v_k \rightharpoonup v$ in $L^2(Q)$. The proof is split into three steps.

Step I. $v \in C_{\bar u}$. Using that $v \to j'(\bar u; v)$ is convex and continuous, we have that

$$j'(\bar u; v) \le \liminf_{k\to\infty} j'(\bar u; v_k) \le \liminf_{k\to\infty} \frac{j(\bar u + \rho_k v_k) - j(\bar u)}{\rho_k} = \liminf_{k\to\infty} \frac{j(u_k) - j(\bar u)}{\rho_k}.$$

The last equality is an immediate consequence of the definition of v_k. From this inequality, (32) and (33) we get

$$\begin{aligned} &F'(\bar u)\, v + \mu\, j'(\bar u; v)\\ &\quad\le \liminf_{k\to\infty} \frac{1}{\rho_k}\{[F(\bar u + \rho_k\, v_k) - F(\bar u)] + \mu[j(\bar u + \rho_k v_k) - j(\bar u)]\}\\ &\quad= \liminf_{k\to\infty} \frac{1}{\rho_k}[J(u_k) - J(\bar u)]\\ &\quad\le \liminf_{k\to\infty} \frac{1}{2k\rho_k}\, \|u_k - \bar u\|^2_{L^2(Q)} = \liminf_{k\to\infty} \frac{\rho_k}{2k} = 0. \end{aligned}$$

This inequality and (28) imply that $F'(\bar u)\, v + \mu\, j'(\bar u; v) = 0$, hence $v \in C_{\bar u}$.

Step II. $v = 0$. For $\beta > 0$ small we define

$$\Omega_\beta = \{x \in \Omega : \|\bar u(x)\|_{L^2(0,T)} \ge \beta\} \ \text{ and } \ j_\beta(u) = \int_{\Omega_\beta} \|u(x)\|_{L^2(0,T)}\, dx$$

and with Lemma 4.3

$$\Omega_{\beta,k} = \Omega_\beta \cap \Omega_k \ \text{ and } \ j_{\beta,k}(u) = \int_{\Omega_{\beta,k}} \|u(x)\|_{L^2(0,T)}\, dx.$$

Since $\|\bar u(x)\|_{L^2(0,T)} \ge \beta > 0$ for every $x \in \Omega_{\beta,k}$, we have that $j_{\beta,k}$ is infinitely differentiable. Making a Taylor expansion we get

$$\begin{aligned} &j_{\beta,k}(\bar u + \rho_k\, v_k) - j_{\beta,k}(\bar u) = \rho_k j'_{\beta,k}(\bar u; v_k) + \frac{\rho_k^2}{2} j''_{\beta,k}(\bar u; v_k^2) + \frac{\rho_k^3}{6} j'''_{\beta,k}(u_{\vartheta_k}; v_k^3)\\ &= \rho_k \int_{\Omega_{\beta,k}} \frac{1}{\|\bar u(x)\|_{L^2(0,T)}} \int_0^T \bar u(x,t)\, v_k(x,t)\, dt\, dx\\ &\quad + \frac{\rho_k^2}{2} \int_{\Omega_{\beta,k}} \frac{1}{\|\bar u(x)\|_{L^2(0,T)}} \left\{ \int_0^T v_k^2(x,t)\, dt - \left(\int_0^T \frac{\bar u(x,t)}{\|\bar u(x)\|_{L^2(0,T)}} v_k(x,t)\, dt \right)^2 \right\} dx\\ &\quad + \frac{\rho_k^3}{6} \int_{\Omega_{\beta,k}} \frac{3}{\|u_{\vartheta_k}(x)\|^3_{L^2(0,T)}} \left\{ \frac{1}{\|u_{\vartheta_k}(x)\|^2_{L^2(0,T)}} \left(\int_0^T u_{\vartheta_k}(x,t)\, v_k(x,t)\, dt \right)^3 \right.\\ &\quad \left. - \left(\int_0^T v_k(x,t)^2\, dt \right) \left(\int_0^T u_{\vartheta_k}(x,t)\, v_k(x,t)\, dt \right) \right\} dx, \end{aligned}$$

where $u_{\vartheta_k} = \bar u + \vartheta_k \rho_k\, v_k$ with $0 \le \vartheta_k(x) \le 1$. Observe that relation (32) and the definition of v_k lead to

$$\begin{aligned} \|u_{\vartheta_k}(x)\|_{L^2(0,T)} &\ge \beta - \vartheta_k \rho_k \|v_k(x)\|_{L^2(0,T)}\\ &\ge \beta - \vartheta_k \|u_k - \bar u\|_{L^\infty(\Omega_k; L^2(0,T))} \ge \beta - \frac{1}{k} \ge \frac{\beta}{2} > 0 \end{aligned}$$

for all $k \ge \frac{2}{\beta}$. Hence, the above integrals are finite for every $k \ge \frac{2}{\beta}$.

Now, using the convexity of the mapping $f \to \|f\|_{L^2(0,T)}$, we get

$$\begin{aligned} j(\bar u + \rho_k\, v_k) - j(\bar u) = \rho_k \int_{\Omega^0_{\bar u}} \|v_k(x)\|_{L^2(0,T)} \\ + \int_{\Omega_{\bar u}\setminus\Omega_{\beta,k}} \left\{\|(\bar u + \rho_k\, v_k)(x)\|_{L^2(0,T)} - \|\bar u(x)\|_{L^2(0,T)}\right\} dx \\ + [j_{\beta,k}(\bar u + \rho_k\, v_k) - j_{\beta,k}(\bar u)] \\ \geq \rho_k\, j'(\bar u; v_k) + \frac{\rho_k^2}{2} j''_{\beta,k}(\bar u; v_k^2) + \frac{\rho_k^3}{6} j'''_{\beta,k}(u_{\vartheta_k}; v_k^3). \end{aligned}$$

From (33) we get

$$\begin{aligned} \frac{\rho_k^2}{2k} > J(\bar u + \rho_k\, v_k) - J(\bar u) \geq \rho_k\{F'(\bar u)\, v_k + \mu\, j'(\bar u; v_k)\} \\ + \frac{\rho_k^2}{2}\{F''(\bar u)\, v_k^2 + \mu\, j''_{\beta,k}(\bar u; v_k^2)\} \\ + \frac{\rho_k^2}{2}[F''(u_{\theta_k}) - F''(\bar u)]v_k^2 + \mu\frac{\rho_k^3}{6} j'''_{\beta,k}(u_{\vartheta_k}; v_k^3), \end{aligned}$$

where $u_{\theta_k} = \bar u + \theta_k\rho_k(u_k - \bar u)$ with $0 \leq \theta_k \leq 1$. We deduce from (28)

$$\frac{\rho_k^2}{2k} > \frac{\rho_k^2}{2}\{F''(\bar u)\, v_k^2 + \mu\, j''_{\beta,k}(\bar u; v_k^2)\} + \frac{\rho_k^2}{2}[F''(u_{\theta_k}) - F''(\bar u)]v_k^2 + \mu\frac{\rho_k^3}{6} j'''_{\beta,k}(u_{\vartheta_k}; v_k^3).$$

Dividing this expression by $\rho_k^2/2$ we obtain

$$F''(\bar u)\, v_k^2 + \mu\, j''_{\beta,k}(\bar u; v_k^2) < |[F''(u_{\theta_k}) - F''(\bar u)]v_k^2| + \mu\frac{\rho_k}{3}|j'''_{\beta,k}(u_{\vartheta_k}; v_k^3)| + \frac{1}{k}. \tag{35}$$

From [7, Lemma 4.2] and the identity $\|v_k\|_{L^2(Q)} = 1$ we deduce

$$\lim_{k\to\infty} |[F''(u_{\theta_k}) - F''(\bar u)]v_k^2| = 0. \tag{36}$$

Let us estimate the second term of (35). By using Hölder's inequality, the expression of $j'''_{\beta,k}(u_{\vartheta_k}; v_k^3)$, that $\|u_{\vartheta_k}(x)\|_{L^2(0,T)} \geq \frac{\beta}{2}$ for every k large enough, (32), and $\|v_k\|_{L^2(Q)} = 1$, we obtain

$$\begin{aligned} |j'''_{\beta,k}(u_{\vartheta_k}; v_k^3)| &\leq 6\int_{\Omega_{\beta,k}} \frac{\|v_k(x)\|^3_{L^2(0,T)}}{\|u_{\vartheta_k}(x)\|^2_{L^2(0,T)}}\, dx. \\ &\leq \frac{24}{\beta^2}\int_{\Omega_{\beta,k}} \|v_k(x)\|^3_{L^2(0,T)} dx \\ &\leq \frac{24}{\beta^2}\|v_k\|_{L^\infty(\Omega_{\beta,k});L^2(0,T)} \int_{\Omega_{\beta,k}} \|v_k(x)\|^2_{L^2(0,T)} dx \leq \frac{24}{\beta^2\rho_k k}. \end{aligned}$$

So we get

$$\mu\frac{\rho_k}{3}|j'''_{\beta,k}(u_{\vartheta_k}; v_k^3)| \leq \frac{8\mu}{k\beta^2} \to 0 \ \text{ as } k \to \infty. \tag{37}$$

Now, from (35), (36) and (37) the following inequality follows

$$F''(\bar u)\, v^2 + \mu\, j''_\beta(\bar u; v^2) \leq \liminf_{k\to\infty}\{F''(\bar u)\, v_k^2 + \mu\, j''_{\beta,k}(\bar u; v_k^2)\} \leq 0 \ \ \forall \beta > 0. \tag{38}$$

Hence, taking the limit as $\beta \to 0$ we conclude that $J''(\bar u; v^2) = F''(\bar u)v^2 + \mu\, j''(\bar u; v^2) \leq 0$. According to the assumption of the theorem, this is possible only if $v = 0$.

Step III. Contradiction. Since $v = 0$, then $z_{v_k} \to 0$ strongly in $L^2(Q)$. Hence, from the expression of F'' given by (9), and using the identity $\|v_k\|_{L^2(Q)} = 1$, we have that

$$\lim_{k\to\infty} F''(\bar{u})v_k^2 = \nu$$

Using now that $j''_{\beta,k}(\bar{u}; v_k^2) \geq 0$, and (38), we deduce

$$\nu \leq \liminf_{k\to\infty}\{F''(\bar{u})\, v_k^2 + \mu\, j''_{\beta,k}(\bar{u}; v_k^2)\} \leq 0,$$

which contradicts the assumption $\nu > 0$.

5. **Numerical approximation.** Next, we will study the approximation of (P) using finite elements. The goal of this section is to show not only convergence of the solutions of the discrete problems to solutions of (P), but also how the sparsity structure of an optimal control (cf. (21)) is inherited by the discrete optimal controls. Both the state and the control will be discretized. In both cases, we will use piecewise constant functions in time, but in space we will use continuous piecewise linear functions for the state and piecewise constant functions for the control. Finally, error estimates are derived. The study of approximations of the control by means of continuous piecewise linear functions will be done in a forthcoming paper. Along this section we will assume that Ω is a convex set.

We consider, cf. [1, definition (4.4.13)], a quasi-uniform family of triangulations $\{\mathcal{K}_h\}_{h>0}$ of $\bar{\Omega}$ and a quasi-uniform family of partitions of size τ of $[0, T]$, $0 = t_0 < t_1 < \cdots < t_{N_\tau} = T$. We will denote $\Omega_h = \text{int} \cup_{K\in\mathcal{K}_h} K$, N_h and $N_{I,h}$ the number of nodes and interior nodes of $\mathcal{K}_h$, $I_j = (t_{j-1}, t_j)$, $\tau_j = t_j - t_{j-1}$, $\tau = \max\{\tau_j\}$ and $\sigma = (h, \tau)$. We assume that every boundary node of Ω_h is a point of Γ. Additionally we suppose that the distance $D(x, \Gamma) \leq C_\Gamma h^2$ for every $x \in \Gamma_h = \partial\Omega_h$, which is always satisfied if $n = 2$ and Γ is of class C^2; see, for instance, [16, Section 5.2]. Under this assumption we have that

$$|\Omega \setminus \Omega_h| \leq Ch^2, \tag{39}$$

where $|\cdot|$ denotes the Lebesgue measure. In the sequel we denote $Q_h = \Omega_h \times (0, T)$.

Now we consider the finite dimensional spaces

$$Y_h = \{z_h \in C(\bar{\Omega}) : \ z_{h|K} \in P_1(K)\ \forall K \in \mathcal{K}_h,\ z_h \equiv 0 \text{ in } \bar{\Omega} \setminus \Omega_h\},$$

$$\mathcal{Y}_\sigma = \{y_\sigma \in L^2(0, T; Y_h) : y_{\sigma|I_j} \in Y_h\ \forall j = 1, \ldots, N_\tau\}.$$

The elements of $\mathcal{Y}_\sigma$ can be written as

$$y_\sigma = \sum_{j=1}^{N_\tau} y_{h,j}\chi_j = \sum_{j=1}^{N_\tau}\sum_{i=1}^{N_{I,h}} y_{i,j} e_i \chi_j$$

where $y_{h,j} \in Y_h$ for $j = 1, \ldots, N_\tau$, $y_{i,j} \in \mathbb{R}$ for $i = 1, \ldots, N_{I,h}$ and $j = 1, \ldots, N_\tau$, $\{e_i\}_{i=1}^{N_{I,h}}$ is the nodal basis associated to the interior nodes $\{x_i\}_{i=1}^{N_{I,h}}$ of the triangulation and χ_j denotes the characteristic function of the interval $I_j = (t_{j-1}, t_j)$. For every $u \in L^\infty(Q_h)$, we define its associated discrete state as the unique element

$y_\sigma(u) \in \mathcal{Y}_\sigma$ such that

$$\int_{\Omega_h}(y_{h,j}-y_{h,j-1})z_h dx+\tau_j a_h(y_{h,j},z_h)+\int_{I_j}\int_{\Omega_h}a(x,t,y_{h,j})z_h dxdt$$
$$=\int_{I_j}\int_{\Omega_h}uz_h dxdt \quad \forall z_h\in Y_h \text{ and all } j=1,\dots,N_\tau,$$
$$\int_{\Omega_h}y_{h,0}z_h dx=\int_{\Omega_h}y_0 z_h dx\ \forall z_h\in Y_h, \tag{40}$$

where, for all $y,z\in H^1(\Omega_h)$,

$$a_h(y,z)=\int_{\Omega_h}\sum_{i,j=1}^n a_{ij}\partial_{x_i}y\partial_{x_j}z dx.$$

From a computational point of view, this scheme can be interpreted as an implicit Euler discretization of the system of ordinary differential equations obtained after spatial finite element discretization.

By using the monotonicity of the nonlinear term $a(x,t,y)$, the proof of the existence and uniqueness of a solution for (40) is standard.

Assuming that $\Omega\subset\mathbb{R}^2$, it is proved in the work by I. Neitzel and B. Vexler [14] that there exist $h_0>0$ and τ_0 such that

$$\|y_\sigma(u)-y_u\|_{L^2(Q_h)}\le C(\tau+h^2)\|y_u\|_{H^{2,1}(Q)}\ \forall\, h<h_0,\ \tau<\tau_0. \tag{41}$$

Remark 3. In the afore-mentioned reference, the estimate is obtained for $n=2$, a polygonal domain and quadrilateral elements. The adaptation of the proofs to convex domains and triangular elements or $n=1$ is straightforward. An extension to $n=3$ is also possible and is currently being written by D. Meidner and B. Vexler.

To discretize the controls, we will use piecewise constant functions. Consider

$$U_h=\{v_h\in L^2(\Omega_h):\ u_{h|K}\in P_0(K)\ \forall K\in\mathcal{K}_h\}$$

and

$$\mathcal{U}_\sigma=\{u_\sigma\in L^2(0,T;U_h):\ u_{\sigma|I_j}\in U_h\ \forall j=1,\dots,N_\tau\}.$$

The elements of $\mathcal{U}_\sigma$ can be written as

$$u_\sigma=\sum_{j=1}^{N_\tau}u_{h,j}\chi_j=\sum_{j=1}^{N_\tau}\sum_{K\in\mathcal{K}_h}u_{K,j}\chi_K\chi_j=\sum_{K\in\mathcal{K}_h}u_K\chi_K.$$

We formulate the discrete problem as

$$(\mathrm{P}_\sigma)\quad \min_{u_\sigma\in\mathcal{U}_\sigma}J_\sigma(u_\sigma),$$

where $J_\sigma(u_\sigma)=F_\sigma(u_\sigma)+\mu j_\sigma(u_\sigma)$,

$$F_\sigma(u)=\frac{1}{2}\int_{Q_h}|y_\sigma(u)-y_d|^2\,dx\,dt+\frac{\nu}{2}\|u\|^2_{L^2(Q_h)}$$

and we define $j_\sigma:\mathcal{U}_\sigma\longrightarrow\mathbb{R}$ by

$$j_\sigma(u_\sigma)=\|u_\sigma\|_{L^1(\Omega_h;L^2(0,T))}=\sum_{K\in\mathcal{K}_h}|K|\|u_K\|_{L^2(0,T)}=\sum_{K\in\mathcal{K}_h}|K|\left(\sum_{j=1}^{N_\tau}\tau_j u^2_{K,j}\right)^{1/2}.$$

The existence of a solution of problem (P_σ) is an obvious consequence of the continuity and the coercivity of J_σ in the finite dimensional space $\mathcal{U}_\sigma$.

Under the assumptions 1-2, $F_\sigma : L^{\hat p}(0,T;L^{\hat q}(\Omega_h)) \to \mathbb{R}$ is of class C^2. Moreover, for every $u,v \in L^{\hat p}(0,T;L^{\hat q}(\Omega_h))$, we have that

$$F'_\sigma(u)v = \int_{Q_h} \varphi_\sigma(u)v\,dx\,dt + \nu \int_{Q_h} uv\,dx\,dt$$

where, for every $u \in L^{\hat p}(0,T;L^{\hat q}(\Omega_h))$, $\varphi_\sigma(u) \in \mathcal{Y}_\sigma$ is its associate discrete adjoint state, which can be written as

$$\varphi_\sigma(u)= \sum_{j=1}^{N_\tau} \varphi_{h,j}\chi_j = \sum_{j=1}^{N_\tau}\sum_{i=1}^{N_{I,h}} \varphi_{i,j}e_i\chi_j$$

and satisfies the equations

$$\varphi_{N_\tau+1,h} = 0$$

$$\int_{\Omega_h} (\varphi_{h,j} - \varphi_{j+1,h})z_h dx + \tau_j a_h(z_h,\varphi_{h,j}) + \int_{I_j}\int_{\Omega_h} \frac{\partial}{\partial y} a(x,t,y_{h,j})\varphi_{h,j} z_h dxdt$$
$$= \int_{I_j}\int_{\Omega_h} (y_{h,j} - y_d)z_h dxdt \quad \forall z_h \in Y_h \text{ for all } j = N_\tau,\dots,1.$$

For every $u_\sigma \in \mathcal{U}_\sigma$, the sets $\mathcal{K}_\sigma$ and $\mathcal{K}^0_\sigma$ are defined as

$$\mathcal{K}_\sigma(u_\sigma) = \{K \in \mathcal{K}_h : \sum_{j=1}^{N_\tau} \tau_j u^2_{K,j} > 0\}, \quad \mathcal{K}^0_\sigma(u_\sigma) = \mathcal{K}_h \setminus \mathcal{K}_\sigma(u_\sigma).$$

Notice that if we define Ω_{h,u_σ} and Ω^0_{h,u_σ} as we did in Proposition 1 using the set Ω_h instead of the set Ω, we have that that $\Omega_{h,u_\sigma} = \operatorname{int}\bigcup_{K\in\mathcal{K}_\sigma(u_\sigma)} K$ and $\Omega^0_{h,u_\sigma} = \bigcup_{K\in\mathcal{K}^0_\sigma(u_\sigma)} K$.

We have that $\lambda_\sigma \in \partial j_\sigma(u_\sigma) \subset \mathcal{U}_\sigma$ if and only if

$$\begin{cases} \Big(\displaystyle\sum_{j=1}^{N_\tau} \tau_j \lambda^2_{K,j}\Big)^{1/2} \le 1 \quad \forall K \in \mathcal{K}^0_\sigma(u_\sigma) \\ \lambda_{K,j} = \dfrac{u_{K,j}}{\|u_K\|_{L^2(0,T)}} \quad \forall K \in \mathcal{K}_\sigma(u_\sigma) \text{ and } \forall 1 \le j \le N_\tau. \end{cases} \tag{42}$$

The directional derivative of j_σ at a point $u_\sigma \in \mathcal{U}_\sigma$ in the direction $v_\sigma \in \mathcal{U}_\sigma$ can be written as

$$j'_\sigma(u_\sigma;v_\sigma) = \sum_{K\in\mathcal{K}^0_\sigma} |K|\|v_K\|_{L^2(0,T)} + \sum_{K\in\mathcal{K}_\sigma} |K| \frac{\displaystyle\sum_{j=1}^{N_\tau} \tau_j u_{K,j}v_{K,j}}{\|u_K\|_{L^2(0,T)}}.$$

In the sequel we denote $J'_\sigma(u_\sigma;v_\sigma) = F'_\sigma(u_\sigma)v_\sigma + \mu j'_\sigma(u_\sigma;v_\sigma)$. We also define $\pi_h : L^1(\Omega) \longrightarrow U_h$ by

$$\pi_h z = \sum_{K\in\mathcal{K}_h} \frac{1}{|K|}\int_K z(x)dx\chi_K.$$

With $\mathcal{P}_\tau$ we denote the space of piecewise constant functions associated with the temporal grid $\{t_0,t_1,\dots t_{N_\tau}\}$. Then, the projection operator $\pi_\tau : L^2(0,T) \longrightarrow \mathcal{P}_\tau$

is given by

$$\pi_\tau u = \sum_{j=1}^{N_\tau} \frac{1}{\tau_j} \int_{t_{j-1}}^{t_j} u(t) dt \chi_j.$$

Then we have $\pi_\tau \pi_h u = \pi_h \pi_\tau u \in \mathcal{U}_\sigma$ for all $u \in L^1(\Omega; L^2(0,T))$. We also have that $\pi_\tau \circ \pi_h : L^2(Q) \longrightarrow \mathcal{U}_\sigma$ is the projection operator.

Theorem 5.1. *If $\bar u_\sigma$ is a local solution of* (P_σ)*, then there exist $\bar y_\sigma = y_\sigma(\bar u_\sigma)$, $\bar\varphi_\sigma = \varphi_\sigma(\bar u_\sigma) \in \mathcal{Y}_\sigma$ and $\bar\lambda_\sigma \in \partial j_\sigma(\bar u_\sigma)$ such that*

$$\pi_h \bar\varphi_\sigma + \nu \bar u_\sigma + \mu \bar\lambda_\sigma = 0. \tag{43}$$

Moreover the inequality $J'_\sigma(\bar u_\sigma; v_\sigma) \ge 0$ holds $\forall v_\sigma \in \mathcal{U}_\sigma$.

Proof. First order optimality conditions follow in a standard way from the convexity of j_σ, the definition of subdifferential and the expression for the derivative of F_σ, taking into account that

$$\int_{Q_h} \pi_h \bar\varphi_\sigma v_\sigma dxdt = \int_{Q_h} \bar\varphi_\sigma v_\sigma dxdt$$

for all $v_\sigma \in \mathcal{U}_\sigma$. □

5.1. **Sparsity properties.** Before proving error estimates, we will show that the discrete optimal controls show a sparsity pattern alike the solutions of Problem (P). Let us introduce the following notation

$$\bar\phi_{K,j} = \frac{1}{|K|} \int_K \bar\varphi_{h,j} dx.$$

Observe that

$$\pi_h \bar\varphi_\sigma = \sum_{j=1}^{N_\tau} \sum_{K \in \mathcal{K}_j} \bar\phi_{K,j} \chi_K \chi_j = \sum_{K \in \mathcal{K}_j} \bar\phi_K \chi_K.$$

Theorem 5.2. *If $\bar u_\sigma$ is a local solution of* (P_σ)*, then*

$$\bar\lambda_{K,j} = \begin{cases} -\dfrac{1}{\mu} \bar\phi_{K,j} & \text{if } K \in \mathcal{K}^0_\sigma(\bar u_\sigma) \\[2ex] \dfrac{\bar u_{K,j}}{\|\bar u_K\|_{L^2(0,T)}} & \text{if } K \in \mathcal{K}_\sigma(\bar u_\sigma) \end{cases} \tag{44}$$

$$K \in K^0_\sigma(\bar u_\sigma) \Leftrightarrow \|\bar\phi_K\|_{L^2(0,T)} \le \mu \tag{45}$$

and $\bar\lambda_\sigma$ is unique for $\bar u_\sigma$ given.

Proof. From (43) and the definition of $\mathcal{K}^0_\sigma(\bar u_\sigma)$ we have that $\bar\lambda_{K,j} = -\bar\phi_{K,j}/\mu$ if $K \in \mathcal{K}^0_\sigma(\bar u_\sigma)$. The expression for $K \in \mathcal{K}_\sigma(\bar u_\sigma)$ follows from (42) and the fact that $\bar\lambda_\sigma \in \partial j_\sigma(\bar u_\sigma)$.

Using this expression for $\bar\lambda_\sigma$ and (43) and we have that for all $j = 1, \ldots, N_\tau$,

$$\bar u_{K,j} \left[\nu + \frac{\mu}{\|\bar u_K\|_{L^2(0,T)}} \right] = -\bar\phi_{K,j} \text{ if } K \in \mathcal{K}_\sigma(\bar u_\sigma). \tag{46}$$

Multiplying by $\tau_j \bar u_{K,j}$ and making the sum for all j, we get

$$\|\bar u_K\|_{L^2(0,T)} = \frac{1}{\nu} \left[\|\bar\phi_K\|_{L^2(0,T)} - \mu \right] \text{ if } K \in \mathcal{K}_\sigma(\bar u_\sigma). \tag{47}$$

From (47) we deduce that $K \in \mathcal{K}_\sigma(\bar u_\sigma)$ implies $\|\bar\phi_K\|_{L^2(0,T)} > \mu$.

On the other hand, if $K \in \mathcal{K}^0_\sigma(\bar u_\sigma)$, we obtain from (44), (43) and (42) that $\|\bar\phi_K\|_{L^2(0,T)} \le \mu$. □

5.2. **Convergence and error estimates.** We will show that the solutions of the discretized problems converge strongly to solutions of Problem (P) in $L^2(Q)$. Next, we show a kind of reciprocal of this result: strict local solutions of (P) can be approximated by solutions of the discretized problems. Finally, we are able to show an order of convergence for this approximations. Through this section we will assume $n \le 2$, since we use several results from [14]. Nevertheless, B. Vexler has proved recently that the stability results and the error estimates also hold for $\Omega \subset \mathbb{R}^3$. A paper with the details of the proof is in preparation. Using his results we can extend the analysis of this section to the three-dimensional case.

First of all, we need to show boundness of the discrete optimal controls in the adequate norm.

Lemma 5.3. *Let $\bar u_\sigma$ be a local solution of (P_σ). Then there exists $C_\infty > 0$ independent of σ such that*

$$\|\bar u_\sigma\|_{L^\infty(0,T;L^2(\Omega_h))} \le C_\infty$$

Proof. The result follows from a bootstrapping argument using the stability results in [14]. First, we have that

$$\frac{\nu}{2}\|\bar u_\sigma\|^2_{L^2(Q_h)} \le J_\sigma(\bar u_\sigma) \le J_\sigma(0) = \frac{1}{2}\|y_\sigma(0) - y_d\|^2_{L^2(Q_h)} =: C_1$$

where $y_\sigma(0)$ is the discrete state related to the control $u_\sigma \equiv 0$. Now, from the classical stability estimate (see, for instance, the second part of [14, Theorem 4.1]) we have that there exists $C_2 > 0$ independent of σ such that

$$\|\bar y_\sigma\|_{L^\infty(0,T;L^2(\Omega_h))} \le C_2.$$

Analogously, from the discrete adjoint state equation we deduce the existence of a constant $C_3 > 0$ independent of σ such that

$$\|\bar\varphi_\sigma\|_{L^\infty(0,T;L^2(\Omega_h))} \le C_3, \tag{48}$$

and hence, taking into account that π_h is a projection in $L^2(\Omega_h)$ and (46), we get

$$\|\bar u_\sigma\|_{L^\infty(0,T;L^2(\Omega_h))} \le \frac{1}{\nu}\|\pi_h\bar\varphi_\sigma\|_{L^\infty(0,T;L^2(\Omega_h))} \le \frac{1}{\nu}\|\bar\varphi_\sigma\|_{L^\infty(0,T;L^2(\Omega_h))}$$

and the result follows for $C_\infty = C_3/\nu$. □

Remark 4. If we further suppose that $y_d \in L^p(Q)$ for some $p > n$, a slight modification of the proof of the previous Lemma allows us to conclude using [14, Th 3.1 and Th 4.1] that there exists some $\mu_c > 0$ independent of h such that $\|\bar\varphi_\sigma\|_{L^\infty(Q_h)} \le \mu_c$. Using this, (45), and the the fact that $\|\pi_h\bar\varphi_\sigma\|_{L^\infty(Q_h)} \le \|\bar\varphi_\sigma\|_{L^\infty(Q_h)}$, we can deduce the existence of a critical value μ_c such that $\bar u_\sigma \equiv 0$ for all $\mu > \mu_c$. For the analogous property for the continuous solution, see [7, Remark 2.10].

Lemma 5.4. *Let $(\bar u_\sigma)_\sigma$ be a sequence of solutions of (P_σ) with $\sigma \to (0,0)$. Then there exist subsequences of $\{\bar u_\sigma\}_\sigma$, still denoted in the same way, converging weakly* in $L^\infty(0,T;L^2(\Omega))$. If $\bar u_\sigma \rightharpoonup \bar u$ weakly* in $L^\infty(0,T;L^2(\Omega))$, then $\bar u$ is a solution of* (P)*,*

$$\lim_{\sigma\to(0,0)} J_\sigma(\bar u_\sigma) = J(\bar u) = \inf\,(\mathrm{P}) \text{ and } \lim_{\sigma\to(0,0)} \|\bar u_\sigma - \bar u\|_{L^2(Q)} = 0. \tag{49}$$

Since u_σ is not defined on all Q, we have to specify what we mean when we say that u_σ converges weakly* to u in $L^\infty(0,T;L^2(\Omega))$. It means that

$$\int_{Q_h}\psi u_\sigma dxdt \to \int_Q \psi u dxdt \ \forall\psi\in L^1(0,T;L^2(\Omega))$$

Notice that since we suppose that $|\Omega\setminus\Omega_h|\to 0$ this is the same as saying that the extension to $Q\setminus Q_h$ of u_σ by a function in $L^\infty(Q)$, converges weakly* to u. In the following proof, we will consider that the elements of U_σ are extended, for instance, by zero to $(0,T)\times(\Omega\setminus\Omega_h)$.

Proof. From Lemma 5.3 we know that $\{\bar u_\sigma\}_\sigma$ is bounded in $L^\infty(0,T;L^2(\Omega_h))$ We can extract a subsequence, still denoted in the same way, such that $\bar u_\sigma \rightharpoonup \bar u$ weakly* in $L^\infty(0,T;L^2(\Omega))$. We are going to prove that $\bar u$ is a solution of (P). Let $\tilde u$ be a solution of (P) and let u_σ be its projection onto $\mathcal{U}_\sigma$ in the $L^2(Q)$ sense. Denoting $\bar y = y_{\bar u}$, we have that $\bar u_\sigma \rightharpoonup \bar u$ weak* in $L^\infty(0,T;L^2(\Omega))$ implies $\bar u_\sigma \rightharpoonup \bar u$ weakly in $L^2(Q)$ and $y_{\bar u_\sigma}\to\bar y$ in $L^2(Q)$; see Theorem 2.1. On the other hand, (41) implies that $y_\sigma(\bar u_\sigma)-y_{\bar u_\sigma}\to 0$ in $L^2(Q)$, so we have that $y_\sigma(\bar u_\sigma)\to y_{\bar u}$ in $L^2(Q)$. This leads to

$$J(\bar u)\le \liminf_{\sigma\to(0,0)} J_\sigma(\bar u_\sigma)\le \limsup_{\sigma\to(0,0)} J_\sigma(\bar u_\sigma)\le \limsup_{\sigma\to(0,0)} J_\sigma(u_\sigma)=J(\tilde u),$$

where we have used the weak lower semicontinuity of the control cost terms in J_σ. Let us proof now the strong convergence of the optimal controls in $L^2(Q)$. We have just proved that $J_\sigma(\bar u_\sigma)\to J(\bar u)$. This, together with the strong convergence $\bar y_\sigma\to\bar y$, implies that

$$\lim_{\sigma\to(0,0)}\Big(\frac{\nu}{2}\|\bar u_\sigma\|^2_{L^2(Q)}+\mu j(\bar u_\sigma)\Big)=\frac{\nu}{2}\|\bar u\|^2_{L^2(Q)}+\mu j(\bar u). \tag{50}$$

On the other hand, using the convexity of $j(u)$ and the weak convergence $\bar u_\sigma\rightharpoonup\bar u$, we have that

$$j(\bar u)\le\liminf_{\sigma\to(0,0)} j(\bar u_\sigma). \tag{51}$$

Using (50) and (51) we have

$$\begin{aligned}
\frac{\nu}{2}\|\bar u\|^2_{L^2(Q)} &\le \liminf_{\sigma\to(0,0)}\frac{\nu}{2}\|\bar u_\sigma\|^2_{L^2(Q)}\le\limsup_{\sigma\to(0,0)}\frac{\nu}{2}\|\bar u_\sigma\|^2_{L^2(Q)}\\
&\le \limsup_{\sigma\to(0,0)}\Big(\frac{\nu}{2}\|\bar u_\sigma\|^2_{L^2(Q)}+\mu j(\bar u_\sigma)\Big)-\liminf_{\sigma\to(0,0)}\mu j(\bar u_\sigma)\\
&\le \frac{\nu}{2}\|\bar u\|^2_{L^2(Q)}+\mu j(\bar u)-\mu j(\bar u)=\frac{\nu}{2}\|\bar u\|^2_{L^2(Q)}
\end{aligned}$$

from where we readily deduce the strong convergence in $L^2(Q)$. □

In the following we will extend the elements of $\mathcal{U}_\sigma$ by $\bar u$ in $Q\setminus Q_h$, where $\bar u$ is a fixed local solution of (P). Notice that using the sparsity property of the control (21) and the zero boundary condition of the adjoint state equation, we have that for $h>0$ small enough, $\bar u=0$ in $Q\setminus Q_h$.

Lemma 5.5. *Conversely, let $\bar u$ be a strict local minimum of* (P) *in the $L^2(Q)$ sense. Then there exist $\varepsilon_0>0$, $h_0>0$ and $\tau_0>0$ such that* (P$_\sigma$) *has a local minimum $\bar u_\sigma\in B_{\varepsilon_0}(\bar u)$, where $B_{\varepsilon_0}(\bar u)=\{u\in L^2(Q):\ \|\bar u-u\|_{L^2(Q)}<\varepsilon_0\}$, for every $h<h_0$, $\tau<\tau_0$ and the convergences* (49) *hold.*

Proof. Suppose now $\bar{u}$ is a strict local minimum of (P). This means that there exist $\varepsilon_0 > 0$ such that $\bar{u}$ is the unique solution of

$$(\mathrm{P}^{\varepsilon_0}) \quad \min_{u \in L^\infty(Q) \cap \bar{B}_{\varepsilon_0}(\bar{u})} J(u),$$

where $\bar{B}_{\varepsilon_0}(\bar{u}) = \{u \in L^2(Q) : \ \|\bar{u} - u\|_{L^2(Q)} \leq \varepsilon_0\}$. Associated to this problem, we consider

$$(\mathrm{P}_\sigma^{\varepsilon_0}) \quad \min_{u_\sigma \in \mathcal{U}_\sigma \cap \bar{B}_{\varepsilon_0}(\bar{u})} J_\sigma(u_\sigma).$$

Let $u_\sigma = \pi_\tau \pi_h \bar{u}$ be the projection of $\bar{u}$ onto $\mathcal{U}_\sigma$ in the $L^2(Q_h)$ sense. We extend u_σ to Q by taking $u_\sigma(x,t) = \bar{u}(x,t)$ in $Q \setminus Q_h$. Since $u_\sigma \to \bar{u}$ in $L^2(Q)$, there exist $h_1 > 0$ and $\tau_1 > 0$ such that $u_\sigma \in \mathcal{U}_\sigma \cap \bar{B}_{\varepsilon_0}(\bar{u})$ and hence this set is not empty for every $h < h_1$, $\tau < \tau_1$ and therefore $(\mathrm{P}_\sigma^{\varepsilon_0})$ has a solution $\bar{u}_\sigma$. Moreover, from the definition of the projection we infer that $\|u_\sigma\|_{L^\infty(Q)} \leq \|\bar{u}\|_{L^\infty(Q)}$. Now let us considered a subsequence, still denoted in the same way, converging weakly in $L^2(Q)$ to $\tilde{u}$. Arguing as in the proof of Lemma 5.4, we have that $\tilde{u}$ is a solution of $(\mathrm{P}^{\varepsilon_0})$, and the convergence is strong. Since $\bar{u}$ is the unique solution of this problem, we have that $\tilde{u} = \bar{u}$. Since all the convergent subsequences converge to the same point, the whole sequence converges to $\bar{u}$. Finally, this strong convergence implies that there exist $h_0 > 0$ and $\tau_0 > 0$ such that $\bar{u}_\sigma \in B_{\varepsilon_0}(\bar{u})$ for every $h < h_0$, $\tau < \tau_0$ and therefore $\bar{u}_\sigma$ is also a local solution of (P_σ). □

Lemma 5.6. *Let $\bar{u}$ be a solution of* (P) *such that $J''(\bar{u}; v^2) > 0$ for all $v \in C_{\bar{u}} \setminus \{0\}$ and let $\bar{u}_\sigma$ be the solution of (P_σ) described in Lemma 5.5. Then there exist $\bar{h} > 0$ and $\bar{\tau} > 0$ such that*

$$\frac{\delta}{2}\|\bar{u}_\sigma - \bar{u}\|^2_{L^2(Q)} \leq J(\bar{u}_\sigma) - J(\bar{u}),$$

for every $h < \bar{h}$, $\tau < \bar{\tau}$, where δ is given in Theorem 4.2.

Proof. The strong convergence (49) $\bar{u}_\sigma \to \bar{u}$ in $L^2(Q)$ shown in Lemma 5.5 implies that for the $\varepsilon > 0$ given in Theorem 4.2, there exist $\bar{h} > 0$ and $\bar{\tau} > 0$ such that $\bar{u}_\sigma \in \bar{B}_\varepsilon(\bar{u})$ for all $h < \bar{h}$, $\tau < \bar{\tau}$, and the result follows from (30). □

Theorem 5.7. *Let $\bar{u}$ be a solution of* (P) *such that $J''(\bar{u}; v^2) > 0$ for all $v \in C_{\bar{u}} \setminus \{0\}$ and let $\bar{u}_\sigma$ be the solution of (P_σ) and τ_0 and h_0 be as described in Lemma 5.5. Let us assume that there exists $h_1 > 0$ such that $y_d \in L^\infty(Q \setminus Q_h)$ $\forall h \leq h_1$. Then, for every $h \leq \min\{h_1, h_0\}$ and every $\tau < \tau_0$, we have*

$$\frac{\delta}{2}\|\bar{u}_\sigma - \bar{u}\|^2_{L^2(Q)} \leq c(\tau + h^2).$$

Proof. Using Lemma 5.6, we have to estimate $J(\bar{u}_\sigma) - J(\bar{u})$. We split into the following parts

$$\begin{aligned} J(\bar{u}_\sigma) - J(\bar{u}) \quad = \quad & J(\bar{u}_\sigma) - J_\sigma(\bar{u}_\sigma) && (52)\\ & + J_\sigma(\bar{u}_\sigma) - J_\sigma(u_\sigma) && (53)\\ & + J_\sigma(u_\sigma) - J(u_\sigma) && (54)\\ & + J(u_\sigma) - J(\bar{u}) && (55) \end{aligned}$$

We choose $u_\sigma = \pi_\tau \pi_h u_\sigma$, the $L^2(Q_h)$-projection of $\bar{u}$ to the space of piecewise constant functions. We extend u_σ to Q by taking $u_\sigma(x,t) = \bar{u}(x,t)$ in $Q \setminus Q_h$. We also recall that $\|u_\sigma\|_{L^\infty(Q)} \leq \|\bar{u}\|_{L^\infty(Q)}$. Because of optimality we have for (53)

$$J_\sigma(\bar{u}_\sigma) - J_\sigma(u_\sigma) \leq 0.$$

To obtain the estimates for the terms in (52) and (54) we use the assumption $y_d \in L^\infty(Q \setminus Q_h)$, the existence of $C > 0$ independent of σ such that $\|y_{\bar u_\sigma}\|_{L^\infty(Q\setminus Q_h)} + \|y_{u_\sigma}\|_{L^\infty(Q\setminus Q_h)} \le C$ and assumption (39), together with estimate (41) to obtain

$$J(\bar u_\sigma) - J_\sigma(\bar u_\sigma) + J_\sigma(u_\sigma) - J(u_\sigma) \le c(\tau + h^2).$$

It remains to estimate term (55).

$$\begin{aligned} J(u_\sigma) - J(\bar u) \;\; = \;\; & \frac{1}{2}\|y_{u_\sigma} - y_d\|^2_{L^2(Q)} - \frac{1}{2}\|\bar y - y_d\|^2_{L^2(Q)} \\ & + \frac{\nu}{2}\|u_\sigma\|^2_{L^2(Q)} - \frac{\nu}{2}\|\bar u\|^2_{L^2(Q)} \\ & + \mu\|u_\sigma\|_{L^1(\Omega;L^2(0,T))} - \mu\|\bar u\|_{L^1(\Omega;L^2(0,T))} \end{aligned} \tag{56}$$

First, using that $\bar u \in H^1(Q) \cap L^2(0,T;H^1_0(\Omega))$ and $\bar y,\ y_{u_\sigma} \in L^2(0,T;H^1_0(\Omega))$, we get

$$\begin{aligned} & \frac{1}{2}\|y_{u_\sigma} - y_d\|^2_{L^2(Q)} - \frac{1}{2}\|\bar y - y_d\|^2_{L^2(Q)} \le c\|u_\sigma - \bar u\|_{L^2(0,T;H^{-1}(\Omega))} \\ & \le c\Big(\|\pi_\tau(\pi_h \bar u - \bar u)\|_{L^2(0,T;H^{-1}(\Omega))} + \|\pi_\tau \bar u - \bar u\|_{L^2(0,T;H^{-1}(\Omega))}\Big) \\ & \le c\Big(\|\pi_h \bar u - \bar u\|_{L^2(0,T;H^{-1}(\Omega))} + \|\pi_\tau \bar u - \bar u\|_{L^2(0,T;H^{-1}(\Omega))}\Big) \\ & \le C(h^2 + \tau)\Big(\|\bar u\|_{L^2(0,T;H^1(\Omega))} + \|\bar u\|_{H^1(0,T;H^{-1}(\Omega))}\Big), \end{aligned}$$

where we have used the well known approximation property

$$\|\pi_\tau u - u\|_{L^2(0,T)} \le C\tau\|u\|_{H^1(0,T)} \ \forall u \in H^1(0,T)$$

and

$$\|\pi_h u - u\|_{H^{-1}(\Omega)} \le Ch^2\|u\|_{H^1_0(\Omega)} \ \forall u \in H^1_0(\Omega),$$

which follows using a classical duality argument.

Now, recalling that $u_\sigma = \bar u$ in $Q \setminus Q_h$ and that u_σ is the projection in the $L^2(Q_h)$-sense of $\bar u$, we infer

$$\frac{\nu}{2}\|u_\sigma\|^2_{L^2(Q)} - \frac{\nu}{2}\|\bar u\|^2_{L^2(Q)} \le 0.$$

Let us estimate the last part of (56):

$$\begin{aligned} & \|u_\sigma\|_{L^1(\Omega_h;L^2(0,T))} = \int_{\Omega_h} \|u_\sigma\|_{L^2(0,T)}\,dx = \int_{\Omega_h} \|\pi_\tau\pi_h \bar u\|_{L^2(0,T)}\,dx \\ & \le \int_{\Omega_h} \|\pi_h \bar u\|_{L^2(0,T)}\,dx = \sum_{K\in\mathcal{K}_h} \int_K \left(\int_0^T \left(\frac{1}{|K|}\int_K \bar u(\xi,t)\,d\xi\right)^2 dt\right)^{1/2} dx \\ & = \sum_{K\in\mathcal{K}_h} \left(\int_0^T \left(\int_K \bar u(\xi,t)\,d\xi\right)^2 dt\right)^{1/2} = \sum_{K\in\mathcal{K}_h} \|\int_K \bar u(\xi,.)\,d\xi\|_{L^2(0,T)} \\ & \le \sum_{K\in\mathcal{K}_h} \int_K \|\bar u(\xi,.)\|_{L^2(0,T)}\,d\xi = \|\bar u\|_{L^1(\Omega_h;L^2(0,T))}. \end{aligned}$$

Since u_σ was extended by $\bar u$ in $Q \setminus Q_h$, we get that

$$\|u_\sigma\|_{L^1(\Omega;L^2(0,T))} \le \|\bar u\|_{L^1(\Omega;L^2(0,T))}.$$

Hence, we finally find with Lemma 5.6

$$\frac{\delta}{2}\|\bar{u}_\sigma - \bar{u}\|^2_{L^2(Q)} \leq J(\bar{u}_\sigma) - J(\bar{u}) \leq c(\tau + h^2).$$

□

Remark 5. It remains an open question whether our error estimate $\mathcal{O}(\sqrt{\tau} + h)$ is sharp. There are several facts that suggest that the order of convergence for the error should be $\mathcal{O}(\tau + h)$: the finite element error for the state equation is $\mathcal{O}(\tau + h^2)$; the $H^1(Q)$-regularity of the optimal controls implies that they can be approximated by elements of $\mathcal{U}_\sigma$ with an approximation error $\mathcal{O}(\tau + h)$ (using $L^2(Q)$-projections, for instance); the experimental order of convergence found in our numerical experiment also supports this idea; finally, the available error estimate in [14] for a problem governed by a semilinear parabolic equation and quadratic differentiable functional is also $\mathcal{O}(\tau + h)$.

Nevertheless, we have not been able to prove such an estimate for our problem. Sharp estimates for problems involving differentiable functionals make use of the second derivative and the mean value theorem, which are not applicable in our setting, since we deal with a non-differentiable functional.

6. **Numerical experiments.** We report on two numerical experiments. In the first one, we describe an example with known solution and show error estimates (cf. Theorem 5.7). In the second one, we show how the sparsity properties of the solution change as μ changes; cf. Remark 4 and [7, Remark 2.10].

6.1. **Experiment 1. Error estimates for an example with known solution.** Let $\Omega = (0, 1) \subset \mathbb{R}$ and let $T = 1$. We are going to describe all the parameters, data and solution, of a model example for (P) when $a(x, t, y) \equiv 0$ and $y_0 \equiv 0$.

Consider two real numbers $0 < a_1 < a_2 < 1$ and a continuous function $U(x)$ supported in $[a_1, a_2]$. For instance

$$U(x) = \chi_{(a_1,a_2)}(x - a_1)(a_2 - x)$$

Consider also a continuous function $V(t)$ such that $V(T) = 0$. For simplicity, we will choose one such that $\|V\|_{L^2(0,T)} = 1$. In our example $V(t) = \sqrt{2}\sin(2\pi t)$. The optimal control is

$$\bar{u}(x, t) = U(x)V(t).$$

With an expression for $\bar{u}$, we can compute (an approximation of) $\bar{y}$.

We have that

$$\Omega_{\bar{u}} = (a_1, a_2)$$

and also, since $U(x) \geq 0$,

$$\|\bar{u}(x)\|_{L^2(0,T)} = |U(x)| = U(x).$$

Therefore, we can define the element of the subdifferential and the adjoint state in $\Omega_{\bar{u}}$ according to Theorem 3.3 as

$$\bar{\lambda}(x, t) = V(t) \text{ if } x \in \Omega_{\bar{u}}$$

$$\bar{\varphi}(x, t) = -\nu\bar{u}(x, t) - \mu V(t) \text{ if } x \in \Omega_{\bar{u}}$$

We have just to define $\bar{\varphi}(x, t)$ for $x \in \Omega^0_{\bar{u}}$. $\bar{\varphi}$ has to satisfy some conditions:

1. $\bar{\varphi} \in C(\bar{Q}) \cap H^1(Q)$.
2. $\bar{\varphi}(x, t) = 0$ if $x = 0$ or $x = 1$ or $t = 1$.
3. $\|\bar{\varphi}(x)\| \leq \mu$ if $x \in \Omega^0_{\bar{u}}$

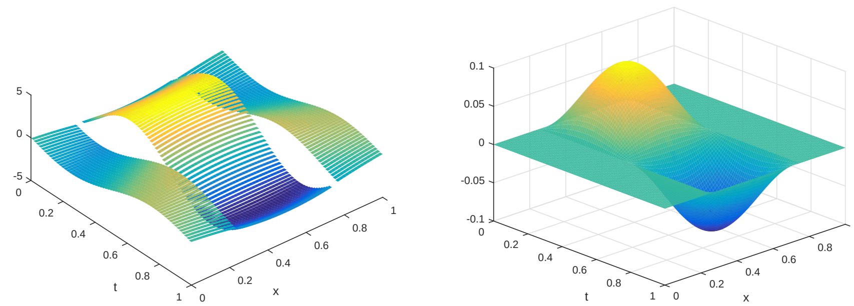

FIGURE 1. Desired state (left) and Optimal control (right)

An easy way to achieve all these requirements is to look for an adjoint state that is also in $C^1(\bar{Q})$. We will build an adjoint state of the form

$$\bar{\varphi}(x,t) = V(t) \cdot \begin{cases} (A_1x^2 + B_1x + C_1) & \text{if } 0 \le x \le a_1 \\ (-\nu U(x) - \mu) & \text{if } a_1 < x < a_2 \\ (A_2x^2 + B_2x + C_2) & \text{if } a_2 \le x \le 1 \end{cases}$$

The parameters $A_i, B_i, C_i,\ i = 1,2$ are univocally determined by the boundary conditions and the condition $\bar{\varphi} \in C^1(\bar{Q})$.

$$\begin{aligned} A_1 &= (\nu a_1^2 - a_2\nu a_1 + \mu)/a_1^2 \\ B_1 &= -(\nu a_1^2 - a_2\nu a_1 + 2\mu)/a_1 \\ C_1 &= 0 \\ A_2 &= (\mu + a_1\nu - a_2\nu + a_2^2\nu - a_1a_2\nu)/(a_2-1)^2 \\ B_2 &= -(2a_2\mu + a_1\nu - a_2\nu + a_2^3\nu - a_1a_2^2\nu)/(a_2-1)^2 \\ C_2 &= -(\mu - 2a_2\mu + a_2^2\nu - a_2^3\nu - a_1a_2\nu + a_1a_2^2\nu)/(a_2-1)^2 \end{aligned}$$

Once this numbers are obtained, the condition $\|\bar{\varphi}(x)\| \le \mu$ if $x \in \Omega_{\bar{u}}^0$ will give us a lower bound for the values of μ that we can select.

$$\begin{aligned} \mu &> \nu(a_2 - a_1)a_1/2 \\ \mu &> \nu(a_2 - a_1)(1 - a_2)/2 \end{aligned}$$

Now that we have the adjoint state and (an approximation of) the state, we can define (an approximation of) the desired target y_d using the adjoint state equation. We get

$$y_d(x,t) = \bar{y} + \partial_t\bar{\varphi}(x,t) + \partial_{xx}^2\bar{\varphi}(x,t)$$

Notice that $\partial_{xx}^2\bar{\varphi}(x,t)$ is not continuous in x and neither is y_d.

We fix the following parameters. The resulting desired state and the optimal control are represented in Figure 1.

$$a_1 = 0.25,\ a_2 = 0.75,\ \nu = 1,\ \mu = 0.1$$

We obtain a value for the objective functional of $J(\bar{u}) = 1.3927$.

Theorem 5.7 gives the estimate

$$\|\bar{u} - \bar{u}_\sigma\|_{L^2(Q)} = O(\sqrt{\tau} + h),$$

but our experiments apparently show

$$\|\bar{u} - \bar{u}_\sigma\|_{L^2(Q)} = O(\tau + h).$$

A similar superconvergence in τ is observed in the experiments performed in [12, §5.1]. In that reference, the authors obtain an experimental order of convergence slightly better than the predicted one, concretely $O(\tau^{0.8})$. This observation is based on an experiment with 512 time steps. Motivated by this, we have performed our experiments using 8192 time steps.

We take two families of uniform partitions in space and time, with $h = 2^{-i}$, $i = i_0 : I$, and $\tau = 2^{-j}$ $j = j_0 : J$ for some values of I and J big enough. We have been able to achieve $I = J = 13$ in a PC with Matlab. To solve the discrete problems, we use a semismooth Newton method as described in [11].

Let us denote $\sigma_{i,j} = (h_i, \tau_j)$. We perform three tests:

1. $\sigma_{i,i}$, $i = i_0 : I$. This is $h = \tau$
2. $\sigma_{i,J}$, $i = i_0 : I^*$. This is fix small τ and refine only in space.
3. $\sigma_{I,j}$, $j = j_0 : J^*$. And this is fix small h and refine only in time.

To measure the error, we compute

$$e_\sigma = \|\bar{u}_\sigma - \tilde{\pi}_\sigma \bar{u}\|_{L^2(Q)}$$

where $\tilde{\pi}_\sigma \bar{u} = \tilde{\pi}_\tau \tilde{\pi}_h \bar{u}$. The operator $\tilde{\pi}_\tau$ is the numerical approximation of the $L^2(0,T)$ projection onto the set of piecewise constant functions given by the midpoint rule: $\tilde{\pi}_\tau f = \sum_{j=1}^{N_\tau} f((t_{j-1} + t_j)/2)\chi_{(t_{j-1},t_j)}$. The operator $\tilde{\pi}_h$ is the usual nodal interpolation in space for the experiment with continuous piecewise linear functions in space and $\tilde{\pi}_h$ is the numerical approximation of the $L^2(\Omega)$ projection onto the set of piecewise constant functions given by the midpoint rule. The experimental order of convergence is measured as

$$EOC_i = \frac{\log(e_{\sigma_{i,i}}) - \log(e_{\sigma_{i-1,i-1}})}{\log(h_i) - \log(h_{i-1})}$$

in the first cases and analogously in the other cases.

For the first test ($h = \tau$), we obtain the results shown in Table 1.

i	e_i	EOC_i
6	4.37E − 3	–
7	2.22E − 3	0.98
8	1.12E − 3	0.99
9	5.60E − 4	0.99
10	2.81E − 4	1.00
11	1.40E − 4	1.00
12	7.03E − 5	1.00
13	3.51E − 5	1.00

Table 1. Results for $h_i = \tau_i = 2^{-i}$.

It looks a lot like

$$\|\bar{u}_\sigma - \bar{u}\|_{L^2(Q)} \leq C(\tau + h) \text{ for } \tau = h$$

For the second test (τ fixed and small, refinements only in the space step), we get the results summarized in Table 2. The error due to $\tau = 2^{-13}$ is small, but not

i	e_i	EOC_i	
6	2.99E − 3	−	
7	1.48E − 3	1.01	
8	7.44E − 4	1.00	
9	3.76E − 4	0.98	$*$
10	1.94E − 4	0.96	
11	1.03E − 4	0.91	
12	5.75E − 5	0.84	
13	3.51E − 5	0.71	

TABLE 2. Results for fixed $\tau = 2^{-13}$ and decreasing $h_i = 2^{-i}$

zero. So the values obtained for the error due to the discretization in space are not of the form Ch_i, but of the form $Ch_i \pm E_{\tau_J}$. So it seems reasonable to discard the results for which the error in time starts to be big enough. For $i \geq 10$ it maybe more than 10% of the error, so we stop at $I = 9^*$. We obtain an order of convergence of $O(h)$, as expected.

In Table 3 we show the results for the third test (h fixed and small, refinements in the time step). Since the spatial error is not zero, we discard the results for which

j	e_j	EOC_j	
6	1.71E − 3	−	
7	8.84E − 4	0.95	
8	4.57E − 4	0.95	$*$
9	2.40E − 4	0.93	
10	1.30E − 4	0.88	
11	7.54E − 5	0.79	
12	4.83E − 5	0.64	
13	3.51E − 5	0.46	

TABLE 3. Results for fixed $h = 2^{-13}$ and $\tau_j = 2^{-j}$.

it is at least the 10% of the global error and stop at $J^* = 8$. We obtain an order of convergence close to $O(\tau)$.

6.2. **Experiment 2. Directional sparsity properties of the control.** Let $\Omega = (0, 1) \subset \mathbb{R}$ and let $T = 1$. We have solved the unconstrained version of the example shown in [7, Remark 2.11]. The data for the example are $\nu = 1e - 4$, $\mu = \mu_0 = 4e - 3$ and

$$y_d(x, t) = \exp(-20[(x - 0.2)^2 + (t - 0.2)^2]) + \exp(-20[(x - 0.7)^2 + (t - 0.9)^2]).$$

We solve the problem in a rough mesh with $h = \tau = 2^{-4}$. In Figure 2, we show the support of the optimal control for the values $\mu = M\mu_0$, $M = 0 : 8$. For $\mu = 0$, we have no sparsity pattern for the control. Then we see how the control is directionally sparse for $\mu > 0$ and how the support of the control is smaller as μ increases. After a few essays, we find that $\bar{u} \equiv 0$ for $\mu \geq 7.4540\mu_0$. As expected, the value of the objective functional increases as μ increases. You may find the obtained numerical values for $J_\sigma(\bar{u}_\sigma)$ in Table 4.

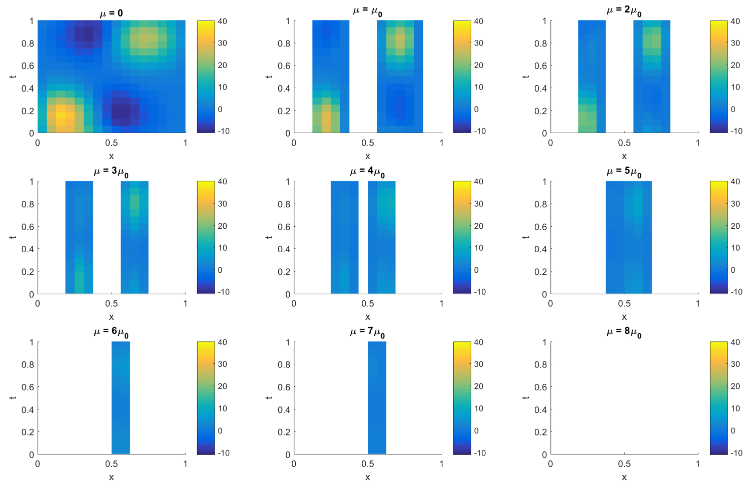

FIGURE 2. Experiment 2. Support of the optimal control for different values of μ

μ	0	μ_0	$2\mu_0$	$3\mu_0$	$4\mu_0$
$J_\sigma(\bar{u}_\sigma)$	0.00935	0.03465	0.04879	0.05738	0.06273
μ	$5\mu_0$	$6\mu_0$	$7\mu_0$	$8\mu_0$	
$J_\sigma(\bar{u}_\sigma)$	0.06705	0.06803	0.06896	0.06906	

TABLE 4. Experiment 2. Value of the objective functional as the parameter μ increases

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Received xxxx 20xx; revised xxxx 20xx.

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