On multi-scale asymptotic structure of eigenfunctions in a boundary value problem with concentrated masses near the boundary

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Abstract. We construct two-term asymptotics $\lambda_k^\varepsilon = \varepsilon^{m-2}(M + \varepsilon\mu_k + O(\varepsilon^{3/2}))$ of eigenvalues of a mixed boundary-value problem in $\Omega \subset \mathbb{R}^2$ with many heavy ($m > 2$) concentrated masses near a straight part Γ of the boundary $\partial\Omega$. ε is a small positive parameter related to size and periodicity of the masses; $k \in \mathbb{N}$. The main term $M > 0$ is common for all eigenvalues but the correction terms μ_k, which are eigenvalues of a limit problem with the spectral Steklov boundary conditions on Γ, exhibit the effect of asymptotic splitting in the eigenvalue sequence enabling the detection of asymptotic forms of eigenfunctions. The justification scheme implies isolating and purifying singularities of eigenfunctions and leads to a new spectral problem in weighed spaces with a "strongly" singular weight.

Keywords: Spectral analysis, homogenization problems, concentrated masses, asymptotic splitting of eigenvalues, Steklov problem, corner singularities.

AMS Subject Classification: 35B25, 35B27, 35P05, 35B40, 47A75, 74H45

1 Introduction and setting of the problem

In this paper we introduce unaccustomed splitting asymptotic procedure for eigenvalues, purifying singularities of eigenfunctions. The spectral problem under consideration is a mixed boundary value problem for the Laplace operator in a domain $\Omega \subset \mathbb{R}^2$ with heavy concentrated masses periodically distributed along a straight part Γ of the boundary. The density of the concentrated masses is of order ε^{-m}, with $m > 2$, and the period is $\varepsilon \ll 1$. This problem has remained unsolved for a long time and has led to results that are currently the subject of discussion (cf. Remark 1.1 and Section 1.3). Moreover, numerical computations also fail: we refer to [4] for instability effects when approaching numerically the principal mode in close-range problems and for further references. A primary asymptotic analysis (cf. [37]) shows that all the eigenvalues in the low-frequency range of the spectrum have the same main asymptotic term which does not provide a characterization of the corresponding eigenfunctions. We construct the two-term asymptotics, that is, the main term and the first correction term, which gives a much more precise information on the behavior of the eigenvalues as $\varepsilon \to 0^+$ and allows us to describe the asymptotic structure of the corresponding eigenfunctions which exhibit a strongly oscillatory character.

The setting of the problem and some background are outlined in Sections 1.1-1.3 while Section 1.4 of this introduction summarizes the structure of the paper. We emphasize that the strong oscillations of the eigenfunctions detected in this paper along with the singularities coming from the boundary conditions make it difficult to obtain the convergence results. As a consequence, we claim that all the results and proofs that we present are necessary to show the approach of the eigenvalues, and of the eigenfunctions in the natural space of the setting of the problem: cf. the simple statements of Theorems 6.1 and 6.4.

1.1 Formulation of the eigenvalue problem

Let Ω be a domain in the plane $\mathbb{R}^2$ bounded by three line segments

$$\Gamma = \{x = (x_1, x_2) : |x_1| \le L,\, x_2 = 0\}, \quad L > 0,$$

$$\Gamma_\pm = \{x : x_1 = \pm L,\, x_2 \in [0, L_\pm]\}, \quad L_\pm > 0,$$

and a piecewise smooth curve Γ_0 connecting the points $(\pm L, L_\pm)$ inside the upper half-plane $\mathbb{R}^2_+ = \{x : x_2 > 0\}$ (see figure 1). By rescaling, we set $L = 1$ and make the cartesian coordinates x_j and all geometric parameters dimensionless. Let N and $\varepsilon = 2(1 + 2N)^{-1}$ be a large integer in $\mathbb{N} = \{1, 2, 3, \dots\}$ and a small positive parameter, respectively. We divide the base Γ of Ω into small segments, of length ε,

$$\gamma_n^\varepsilon = \{x : x_2 = 0,\, |x_1 - \varepsilon n| \le \varepsilon/2\}, \quad n \in \mathbb{Z}(N) = \{0, \pm 1, \pm 2, \dots, \pm N\},$$

and introduce the sets

$$\theta_n^\varepsilon = \{x : \varepsilon^{-1}(x_1 - \varepsilon n, x_2) \in \theta\}, \quad \Theta^\varepsilon = \bigcup_{n \in \mathbb{Z}(N)} \theta_n^\varepsilon, \tag{1.1}$$

where θ is a new domain in $\mathbb{R}^2_+ = \{\xi : \xi_2 > 0\}$ of the same type as Ω, namely, it is bounded by three line segments

$$\begin{gathered} \tau = \{\xi = (\xi_1, \xi_2) : |\xi_1| \le l,\, \xi_2 = 0\}, \quad l \in (0, 1/2), \\ \tau_\pm = \{\xi : \xi_1 = \pm l,\, \xi_2 \in [0, l_\pm]\}, \quad l_\pm > 0, \end{gathered} \tag{1.2}$$

and a piecewise smooth curve τ_0 connecting the points $(\pm l, l_\pm)$ inside $\mathbb{R}^2_+$. The lower base of θ_n^ε is denoted by τ_n^ε, T^ε is the union of $\tau_{-N}^\varepsilon, \dots, \tau_N^\varepsilon$ and $\tau_\sqcap = \tau_- \cup \tau_0 \cup \tau_+$. Similarly, $\Gamma_\sqcap^\varepsilon$ denotes a union $\Gamma_\sqcap^\varepsilon = \Gamma_\sqcap \cup T^\varepsilon$ with $\Gamma_\sqcap = \Gamma_- \cup \Gamma_0 \cup \Gamma_+$. Also, if no confussion arises, we set $\theta^\varepsilon = \varepsilon\theta$.

In Ω we consider the eigenvalue problem

$$-\Delta u^\varepsilon = \lambda^\varepsilon(1 + \varepsilon^{-m}\chi^\varepsilon)u^\varepsilon \quad \text{in } \Omega, \tag{1.3}$$

$$u^\varepsilon = 0 \quad \text{on } \Gamma_\sqcap^\varepsilon, \tag{1.4}$$

$$\partial_\nu u^\varepsilon = 0 \quad \text{on } \Gamma \setminus \overline{T^\varepsilon}, \tag{1.5}$$

where ∂_ν is the directional derivative along the outward normal, $\partial_\nu = -\dfrac{\partial}{\partial x_2}$ on Γ, and χ^ε is the characteristic function of the set Θ^ε, see (1.1),

$$\chi^\varepsilon(x) = 1 \text{ for } x \in \Theta^\varepsilon \text{ and } \chi^\varepsilon(x) = 0 \text{ for } x \notin \Theta^\varepsilon. \tag{1.6}$$

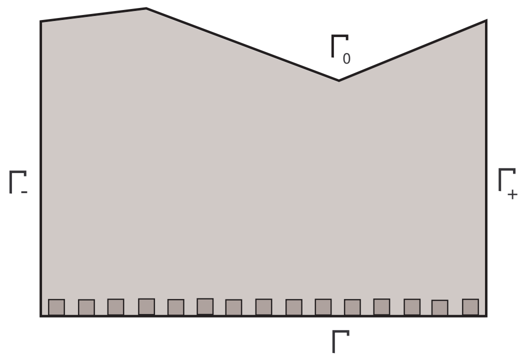

Figure 1: Geometrical configuration of the problem.

Finally, $m \in \mathbb{R}$ is a positive number and in the sequel we assume

$$m > 2. \tag{1.7}$$

The singularly perturbed problem (1.3)–(1.5) can be associated with time dependent harmonic oscillations of a membrane which is fixed over its sides $\Gamma_\sqcap$ and is clamped by a periodic set of small "clips" θ_n^ε, $n \in \mathbb{Z}$. The flexibility of the clips is the same as that of the membrane material but, in view of our assumption (1.7), the weight of each θ_n^ε is much greater than that of the whole membrane. In other words, $\theta_{-N}^\varepsilon, \dots, \theta_N^\varepsilon$ are heavy concentrated masses distributed periodically at the lower flat part Γ of the boundary $\partial\Omega$ and they are fixed over their sides τ_n^ε.

Restriction (1.7) and a special shape of the domains are chosen to reduce the required technicalities to the necessary minimum while preserving all disclosed effects (cf. Remark 3.5).

1.2 The eigenvalue sequence: what is known and what is expected

The variational formulation of problem (1.3)–(1.5) reads: to find a number λ^ε and a function $u^\varepsilon \in H_0^1(\Omega; \Gamma_\sqcap^\varepsilon)$, $u^\varepsilon \neq 0$ such that

$$(\nabla u_\varepsilon, \nabla v_\varepsilon)_\Omega = \varepsilon^{-m} \lambda_\varepsilon (u_\varepsilon, v_\varepsilon)_{\Theta^\varepsilon} + \lambda_\varepsilon (u_\varepsilon, v_\varepsilon)_\Omega, \quad \forall v^\varepsilon \in H_0^1(\Omega; \Gamma_\sqcap^\varepsilon). \tag{1.8}$$

Here, $(\cdot,\cdot)_\Omega$ is the natural inner product in the Lebesgue space $L^2(\Omega)$, and $H_0^1(\Omega; \Gamma_\sqcap^\varepsilon)$ is the Sobolev space of functions satisfying the Dirichlet condition (1.4). We supply this space with the norm

$$\|u^\varepsilon; H_0^1(\Omega)\| = \|\nabla u^\varepsilon; L^2(\Omega)\|$$

and observe that, owing to the restriction $u^\varepsilon = 0$ on $\Gamma_\sqcap$, there holds the inequality

$$\|u^\varepsilon; L^2(\Omega)\| + \|u^\varepsilon; L^2(\Gamma)\| \leq c \|\nabla u^\varepsilon; L^2(\Omega)\|, \quad u^\varepsilon \in H_0^1(\Omega; \Gamma_\sqcap^\varepsilon)$$

with a constant c independent of ε.

The eigenvalues of problem (1.8) form the unbounded monotone positive sequence

$$0 < \lambda_1^\varepsilon < \lambda_2^\varepsilon \leq \lambda_3^\varepsilon \leq \dots \leq \lambda_k^\varepsilon \leq \dots \to +\infty \tag{1.9}$$

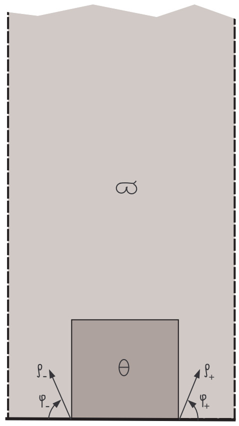

Figure 2: The periodicity cell and the polar coordinates at the collision points

where eigenvalues are repeated according to their multiplicities, and the corresponding eigenfunctions in $H_0^1(\Omega;\Gamma_\sqcap^\varepsilon)$ can be subject to the orthogonality and normalization conditions

$$(\nabla u_j^\varepsilon, \nabla u_k^\varepsilon)_\Omega = \delta_{j,k}, \quad j,k \in \mathbb{N}, \tag{1.10}$$

where $\delta_{j,k}$ is the Kronecker symbol. Due to the strong maximum principle, the first eigenvalue λ_1^ε in (1.9) is simple and the eigenfunction u_1^ε can be fixed to be positive in $\Omega \cup \big(\Gamma \setminus \overline{T^\varepsilon}\big)$.

Rewriting the proof in [37] with minor modifications, we obtain that, as $\varepsilon \to 0$, the asymptotics of spectrum (1.9) has a specific feature, namely all rescaled eigenvalues

$$\Lambda_k^\varepsilon = \varepsilon^{2-m}\lambda_k^\varepsilon \tag{1.11}$$

have the common limit

$$\Lambda_k^\varepsilon \to M \text{ as } \varepsilon \to 0 \text{ for any } k \in \mathbb{N}, \tag{1.12}$$

where M is the principal eigenvalue of an auxiliary problem which is posed in the half-strip $\varpi = (-1/2, 1/2) \times \mathbb{R}_+$, cf. figure 2, with the size of the rescaled clip θ of the reduced mass θ^ε of order 1, and with the periodicity conditions at the lateral sides. Namely, the first eigenvalue of problem

$$\Delta_\xi W = MXW \quad \text{ in } \varpi, \tag{1.13}$$

$$W(\xi_1, 0) = 0, \ |\xi_1| < l, \quad \frac{\partial W}{\partial \xi_2}(\xi_1, 0) = 0, \ |\xi_1| \in \Big(l, \frac{1}{2}\Big), \tag{1.14}$$

$$W\Big(\frac{1}{2}, \xi_2\Big) = W\Big(-\frac{1}{2}, \xi_2\Big), \ \ \frac{\partial W}{\partial \xi_1}\Big(\frac{1}{2}, \xi_2\Big) = \frac{\partial W}{\partial \xi_1}\Big(-\frac{1}{2}, \xi_2\Big), \quad \xi_2 > 0. \tag{1.15}$$

where X is the characteristic function of the set θ, cf. definition (1.6), and ξ is an auxiliary variable (actually the *rapid variable*, cf. (2.1)).

Problem (1.13)-(1.15), the so-called *cell problem*, admits the variational formulation (2.2) (cf. [32] for other geometries of the masses and the formal asymptotic analysis). Having a common limit M, convergence (1.12) cannot help to specify an asymptotic behavior of the eigenfunctions u_k^ε when $\varepsilon \to 0$.

In Section 3, we will construct the two-term asymptotics

$$\Lambda_k^\varepsilon = M + \varepsilon\mu_k + \dots \tag{1.16}$$

of the rescaled eigenvalues (1.11) and demonstrate that the number μ_k in (1.16) and the function u_k appearing in the asymptotic form for the eigenfunction (cf. Section 3.5)

$$u_k^\varepsilon(x) = W(\varepsilon^{-1}x)u_k(x) + \dots \tag{1.17}$$

are an eigenpair of the Steklov spectral problem

$$-\Delta u = 0 \quad \text{in } \Omega, \tag{1.18}$$

$$u = 0 \quad \text{on } \Gamma_\sqcap = \Gamma_- \cup \Gamma_0 \cup \Gamma_+, \tag{1.19}$$

$$-\frac{\partial u}{\partial x_2} = \mu b u \quad \text{on } \Gamma, \tag{1.20}$$

which involves a well determined coefficient $b > 0$ (cf. (2.11) and (3.15)). In (1.17), W stands for the normalized eigenfunction associated with the principal eigenvalue M of (1.13)-(1.15) in the half-strip ϖ that we extend by periodicity in the x_1-direction.

The type of asymptotic expansions (1.16) and (1.17) have been announced without proofs in [32]. This, together with estimates of *the asymptotic remainders*, which for simplicity we denote in this section by ellipsis points, will be proved and evaluated further in Theorems 6.1 and 6.4. In addition, a new focus of the justification scheme provides a powerful and novel contribution in our paper.

The final estimates for the eigenpairs λ_k^ε, u_k^ε of (1.8) can be summarized as follows: for any fixed k, and sufficiently small ε,

$$\left|\lambda_k^\varepsilon - \varepsilon^{m-2}\left(M + \varepsilon\mu_k\right)\right| \le c_k\varepsilon^{m-1/2},$$

and

$$\left\| u_q^\varepsilon - \varepsilon^{1/2} \sum_{j=k}^{k+\varkappa_k-1} a_{qj}^\varepsilon u_j W^\varepsilon; H^1(\Omega)\right\| \le c_k^\sharp \varepsilon^{1/2}, \quad q = k, \dots, k+\varkappa_k - 1,$$

hold, where $W^\varepsilon(x) = W(\varepsilon^{-1}x)$, $\varkappa_k$ is the multiplicity of μ_k (cf. (3.18) and (4.14)), c_k and $c_k^\sharp$ are two constants, and $(a_{qj}^\varepsilon)_{q,j=k,\dots,k+\varkappa_k-1}$, is a well determined ε-dependent constant matrix. See statements of Theorems 6.1 and 6.4 for the precise definitions of the terms in estimates above, and see Section 1.4 for a short summary of intermediate results. Among other things, these intermediate results give the approach of the eigenpairs of the reformulate spectral problem (5.2) to the eigenpairs of the Steklov problem (1.18)-(1.20) (cf. (3.15), (5.55) and Theorem 5.6).

One of the greatest difficulties faced by the authors is in the verification of the uniform boundedness for the $L^2(\Omega)$-norm of the gradient of the fractional function $\mathbf{u}_k^\varepsilon(x) = W(\varepsilon^{-1}x)^{-1}u_k^\varepsilon(x)$,

$$\nabla_x \mathbf{u}_k^\varepsilon(x) = W(\xi)^{-1}\nabla_x u_k^\varepsilon(x) - \varepsilon^{-1}u_k^\varepsilon(x)W(\xi)^{-2}\nabla_\xi W(\xi), \text{ with } \xi = \varepsilon^{-1}x, \tag{1.21}$$

due to the singularities of both eigenfunctions W and u_k^ε at the collision points. Nevertheless, in Section 2.4 we will prove that $\mathbf{u}_k^\varepsilon$ is sufficiently smooth: in particular, it is continuously differentiable at these points.

Remark 1.1. Note that we deal with a singular weight $W(\varepsilon^{-1}x)$ near the boundary, and the equivalence of the original (1.3)–(1.5) and limit (5.1),(5.3),(5.4) problems takes up a voluminous part of the analysis and computations (cf. Sections 2 and 5, and Appendix). It is remarkable that in the gradient formulas (1.21) and (5.20) the last terms get very strong weights as well as the big factor ε^{-1}. The verification of the equivalence is of great importance because there exist many examples (cf., e.g., [33]) when a substitution in a problem with corner singularities leads to incorrect solutions. It is also of great importance when dealing with evolution problems (cf., e.g., [19]). In addition, notice that infinitely many realizations of an elliptic problem as a self-adjoint operator with the discrete spectra occur in domains with corners (cf. Ch.6 in [31]). All together makes it compulsory a thorough analysis to show the equivalence of the above-mentioned problems. Such crucial analysis, with singular weights, is absent in the existing literature on vibrating systems with concentrated masses (see Section 1.3).

1.3 State-of-the-art in the literature and new challenges

The problem under consideration in this paper belongs to a series of problems known with the name of "vibrating systems with concentrated masses" in the literature of applied mathematics (cf. [39, 35, 25, 14] for the first works). Further specifying, we deal with very many concentrated masses near the boundary and strongly alternating boundary conditions. In this framework we refer to [16] and [14] for problems in two and three dimensional domains respectively, the size of the masses being much smaller than the period of the structure. Different relations between sizes of masses and distance between them (cf. terms such as *critical sizes* and *extreme relations*), and different values of the parameter m have been considered in [14, 16] where the authors were concerned with the localization of eigenvalues giving rise to *local vibrations* of the concentrated masses or *global vibrations* of the system; also many questions were formulated in these papers which have been partially solved in further publications such as [15, 17, 18, 36, 38]: see [18, 38] for a long list of references on the subject.

The terms *light* or *heavy concentrated masses* were introduced in the literature to distinguish between the different ranges of the parameter m, namely $m \in (0, 2)$ or $m > 2$ since the asymptotic behavior of the eigenvalues is qualitatively different for m in one of these ranges or $m = 2$. All the cases have been considered in the above mentioned papers but the structure of the eigenfunctions associated with the very *low frequencies* have remained as open questions in the case where $m > 2$, namely, associated with the eigenvalues $\lambda_i^\varepsilon = O(\varepsilon^{m-2})$, for fixed $i = 1, 2, \cdots$ (cf. (1.11) and (1.12)). Even the determination of the structure of the first eigenmode of vibration was an open problem. The same can be said for $m = 2$ and the low frequencies which in this case read $\lambda_i^\varepsilon = O(1)$.

In fact, we note that for the precise distribution of masses (1.1), size of masses $O(\varepsilon)$ and boundary conditions (1.4) and (1.5), the cases where $m \leq 2$ have not been considered in the literature. However, when $m < 2$ (light concentrated masses), the limit problem for the eigenpairs of (1.1)-(1.5) is the Dirichlet eigenvalue problem in Ω. This can be easily proved by standard homogenization techniques: cf. closer results for different problems in [5, 16, 32]. In contrast, when $m = 2$, to determine the asymptotic forms of the eigenfunctions remains an open problem to be considered. Now the limit points of the eigenvalues in the sequence (1.9) can be the eigenvalues the Dirichlet problem in Ω, $\{\beta_j\}_{j=1}^\infty$, and the eigenvalues of the cell problem in ϖ. For fixed k, the convergence of λ_k^ε, as $\varepsilon \to 0$, depends strongly on the position of M, cf. (1.13)-(1.15), in the sequence

$$0 < \beta_1 \leq \beta_2 \leq \beta_3 \leq \cdots \leq \beta_j \leq \cdots \to \infty, \qquad \text{as } j \to \infty\,,$$

and on the possibility for M to coincide with some β_j: we refer to [32] for asymptotic expansions in a close problem. Also the dimension $n = 3$ remains under examination, cf. [14] in the case where the size of the masses is much smaller than the period.

Light and heavy concentrated masses with different boundary conditions, and low and high frequencies, have also been considered in [14, 15, 16, 17, 36, 38] for dimensions 2 and 3 of the space, the boundary of the masses touching $\partial\Omega$. In this respect, we mention [32] for a different geometry of the masses θ^ε which do not touch $\partial\Omega$ and for very different boundary conditions on $\partial\Omega$. In spite of this, there is a big gap in the research on these kinds of problems since the structure of the first eigenmode of vibration has only been glimpsed by means of asymptotic expansions in [32]. However, the existing results in the literature (cf. [18, 36, 37] and references therein) allow us to obtain information on the structures of certain eigenfunctions associated with eigenvalues $\lambda^\varepsilon_{i(\varepsilon)} = \varepsilon^{m-2} M_j + \cdots$, when $j > 1$ and M_j is an eigenvalue of the cell problem (1.13)-(1.15) in the sequence (2.6), but $i(\varepsilon)$ cannot be fixed, (see (1.16) to compare), it converges towards ∞ as $\varepsilon \to 0$. In addition, the structure of the corresponding eigenfunctions is described by *quasimodes* which approach "groups of eigenfunctions" (cf. Lemma 4.1). Obviously, these results deal with the high-frequency range, and they complement those in this paper, which becomes essential for the description of asymptotics of eigenpairs in the low-frequency range.

As outlined in previous works (cf., e.g., [13, 17, 18]) when searching for eigenvalues giving rise to certain kinds of vibrations, the question of the normalization of the corresponding eigenfunctions is crucial. In addition, here we need a reformulation of the problem in weighted Sobolev spaces and a thorough analysis of solutions near the points where the strongly alternating boundary conditions change, the so-called *collision points* (cf. Lemmas 2.1, 2.4 and Proposition A.1). We note that in the asymptotics for eigenfunctions (1.17), fast and slow variables are involved together, and the function u_k (which corresponds to μ_k, the second term of the asymptotic expansion of the rescaled eigenvalues (1.16)) act as an envelope for the fast oscillations of the eigenfunction u_k^ε of our spectral problem (1.8).

It should also be noted that the factorization principle here used has been detected in the literature of homogenization problems: we refer to [40] for the first work, related with perforated media, where weighted Sobolev spaces are also used. However, here we deal with a very singular "weight" $W(\varepsilon^{-1}x)$ near the boundary and we need to obtain smoothness properties for $u^\varepsilon(x)W(\varepsilon^{-1}x)^{-1}$ in the neighborhoods of the points of $\partial\Omega$ where both functions u^ε and W vanish. We use a technique of localization for $u^\varepsilon(x)W(\varepsilon^{-1}x)^{-1}$ near the concentrated masses (cf. figure 3) which allows us to derive its convergence in $H^1(\Omega)$-weak, avoiding approaches with norms in the weighted Sovolev spaces. As a consequence, we obtain sharp bounds for convergence rates of eigenpairs (cf. also Remark 6.2) and, what is very important, this approach is obtained in the norms of the space of the setting of the original problem (cf. (6.1) and (6.5)).

In this connection, it may be worthy mentioning that [22] considers an eigenvalue problem in a planar domain of the dense-comb type which differs from our problem (1.3)–(1.5) both in the geometry and distribution of the heavy masses, and in the boundary conditions, but meets technical difficulties similar to those mentioned in Remark 1.1. However, an analysis of the singularities of the eigenfunctions at the corner points (cf. [10]) and the proof of the equivalence of problems outlined in Remark 1.1 are absent.

Also, it is worthy mentioning that other very different problems in the framework of the vibrating systems with many concentrated masses have been considered in the literature recently. Let us mention, e.g., [28, 30], where asymptotics for eigenvalues are described by means of the spectrum of a certain integral (elliptic pseudo-differential) operator on the torus axis. The geometry and the justification schemes in these papers differ in all aspects from the content of our present paper.

Finally, notice that the Steklov problem (1.18)-(1.20) appears here associated with the second order approach of the eigenvalues. Let us mention references [1, 7, 8, 9, 12, 14] which make it clear how Steklov type boundary conditions can appear associated with the first order approach of eigenvalues of singularly perturbed spectral problems which present a high mass concentration along a part of the boundary or at points along the boundary. See [6] for further recent bibliography on Steklov problems.

1.4 Structure of the paper

The organization of the paper is marked by the asymptotics (1.16) and (1.17) of the eigenpairs of (1.3)-(1.5), and by the tools which we need to justify these asymptotics. We gather the final results in Theorems 6.1 and 6.4 in a simplified way, while other important results appear throughout the paper.

Taking into account that the first term M in the asymptotics (1.16) and that the multiplying function W in (1.17) turn out to be the dominant eigenpair of the cell problem (2.2), Section 2 contains the setting of this problem (*the first limit problem*) along with a detailed study of the dominant eigenmode: properties which allow us to show that $u_k^\varepsilon(x)W(\varepsilon^{-1}x)^{-1}$ belongs to $H^1(\Omega)$ (cf. (2.31)). This seems to be consistent with (1.17) ($u_k \in H^1(\Omega)$) but requires some smoothness results for solutions of boundary value problems near corners (cf. [10] and Ch. 2 in [31] for the general theory), namely, in our case near the points where the boundary condition changes from Dirichlet to Neumann or viceversa. Appendix complements these smoothness results.

Section 3 contains asymptotic expansions for the eigenpairs $(\lambda_k^\varepsilon, u_k^\varepsilon)$ of (1.3)-(1.5), and the Steklov spectral problem (3.17) (*the second limit problem*). The compound asymptotic expansion for the eigenfunctions (3.2) includes terms of the outer expansion and boundary layer functions whose properties prove essential for these justifications. These expansions and functions are in Sections 3.1-3.4 (cf. also Remark 3.2).

Sections 4–6 contain justifications of asymptotic expansions providing precise bounds for convergence rates in terms of the eigenvalue number.

The hard computations in Section 4 rely on results about "near eigenvalues and eigenfunctions" from the spectral perturbation theory (cf. Lemma 4.1). The main result (cf. Theorem 4.2) establishes that for each eigenvalue μ_k of (3.17), μ_k of multiplicity $\varkappa_k$, and for sufficiently small ε, there are at least $\varkappa_k$ eigenvalues of λ_j^ε satisfying

$$\left|\lambda_j^\varepsilon \varepsilon^{2-m} - M - \varepsilon\mu_k\right| \le c_k \varepsilon^{3/2}, \tag{1.22}$$

with a constant c_k independent of ε. The result already improves the convergence (1.12). In addition, in Section 4, a certain approach to the eigenfunctions is stated (cf. (4.29)), which is provided by the asymptotic expansions constructed in Section 3 (cf. (3.2), (3.6)), with the boundary layer functions suitably adapted in such a way that the new function (4.10) belongs to the same space of definition of the eigenfunctions u_k^ε. Nevertheless, these approaches do not provide the convergence expected (cf. (1.11) and (1.16)).

To obtain this convergence, which in some way implies $k = j$ in (1.22), we need to reformulate the original spectral problem (1.8) by introducing a new spectral parameter and a corresponding eigenfunction as follows:

$$\mu^\varepsilon = \lambda^\varepsilon \varepsilon^{2-m} - M, \quad \mathbf{u}^\varepsilon(x) = u^\varepsilon(x) W(\varepsilon^{-1}x)^{-1}. \tag{1.23}$$

These pairs $(\mu^\varepsilon, \mathbf{u}^\varepsilon)$ prove to be eigenpairs of a new spectral problem (5.2) which is formulated in the suitable weighted Sobolev spaces and it turns out to have a dicrete spectrum (cf. Proposition 5.1). Theorem 5.6 states the result of convergence of the *renormalized eigenpairs* $(\varepsilon^{-1}\mu^\varepsilon, \mathbf{u}^\varepsilon)$ towards those of the second limit problem (3.17). The spectral convergence for (5.2) holds with conservation of the multiplicity (see (5.51) and (5.53)), and to derive the convergence in a space independent of ε we use a technique that allows a localization of the new eigenfunctions $\mathbf{u}^\varepsilon$ in *small teeth* near the concentrated masses (see figure 3). This is done in Section 5.2, the main results of this section being summarized in Proposition 5.5.

Finally, we need to combine the partial results in Sections 4 and 5 to derive the desired approach for eigenvalues and eigenfunctions of the original problem along with precise bounds for convergence rates in terms of the eigenvalue number k and the perturbation parameter ε (cf. (6.1) and (6.5)): j becomes $k, k+1, \cdots \varkappa_k - 1$ in (1.22).

2 The first limit problem in the cell

This section is devoted to the cell problem, namely, the limit problem involved with the first term M of the asymptotics for the eigenvalues (1.16) and the properties of the corresponding eigenfunction W (cf. Sections 2.2-2.3). Some of these properties deal with the required smoothness for (1.23) (cf. Section 2.4) that we need in Sections 4-6.

2.1 The eigenvalue problem in the half-strip ϖ

Considering the rapid variables

$$\xi = (\xi_1, \xi_2) = \varepsilon^{-1} x = (\varepsilon^{-1} x_1, \varepsilon^{-1} x_2), \tag{2.1}$$

we take into account formulas (1.11), (1.7) and recognize $\Delta_\xi + X\Lambda^\varepsilon$ as the main asymptotic part of the differential operator

$$\Delta_x + \lambda^\varepsilon(1 + \varepsilon^{-m}\chi^\varepsilon) = \varepsilon^{-2}(\Delta_\xi + X\Lambda^\varepsilon) + \varepsilon^{m-2}\Lambda^\varepsilon.$$

Considering also the ε-periodicity in x_1, we formulate the first limit problem in the half-strip

$$\varpi = \{\xi : \xi_1 \in (-1/2, 1/2), \xi_2 > 0\}.$$

This limit problem is (1.13)–(1.15), where we note that we have used M to denote the spectral parameter, and X the characteristic function of the set θ.

The variational formulation of problem (1.13)–(1.15) reads: to find a number M and non-trivial function $W \in \mathcal{H}$ such that

$$(\nabla_\xi W, \nabla_\xi V)_\varpi = M(W, V)_\theta, \quad \forall V \in \mathcal{H}. \tag{2.2}$$

The Hilbert space $\mathcal{H}$ in the integral identity (2.2) is determined by completing the linear space $C^\infty_{c,per}(\overline{\varpi}, \tau)$ of infinitely differentiable functions, vanishing on τ, with compact support and 1-periodics in ξ_1, with respect to the norm $\|\nabla_\xi V; L^2(\varpi)\|$. It consists of functions in $H^1_{loc}(\overline{\varpi})$ which have a finite gradient norm and satisfy the stable boundary conditions, namely the first relations in (1.14) and (1.15). Since the Dirichlet condition on τ is included in the space, the classical one-dimensional Hardy inequality proves that the norm introduced in $\mathcal{H}$ is equivalent to the following one:

$$\left(\|\nabla_\xi V; L^2(\varpi)\|^2 + \|(1+\xi_2)^{-1} V; L^2(\theta)\|^2\right)^{1/2}. \tag{2.3}$$

Indeed, the above-mentioned Hardy inequality

$$\int_0^\infty t^{-2}|\mathcal{V}(t)|^2 dt \leq 4 \int_0^\infty \left| \frac{d\mathcal{V}}{dt}(t) \right|^2 dt, \tag{2.4}$$

in particular, requires $\mathcal{V}(0) = 0$ (to verify (2.4) make the change $y \mapsto t = 1/y$ in (5.10)). To fulfil this condition, we multiply V with the cut-off function $\xi \mapsto \mathcal{X}(\xi_2)$,

$$\mathcal{X} \in C^\infty(\mathbb{R}),\ 0 \leq \mathcal{X} \leq 1,\ \mathcal{X}(t) = 0 \text{ for } t \leq 1/2,\ \mathcal{X}(t) = 1 \text{ for } t \geq 1,$$

and apply the Friedrichs inequality

$$\|V; L^2((-l,l) \times (0,1))\| \leq c_l \|\nabla_\xi V; L^2((-l,l) \times (0,1))\| \tag{2.5}$$

which is valid due to the Dirichlet condition $V = 0$ on τ. Setting $\mathcal{V}(t) = \mathcal{X}(t)V(\xi_1, t)$ in (2.4), we integrate the obtained inequality in $\xi_1 \in (-l,l)$, take into account (2.5) to estimate the L^2-norm of the last term in the formula $\mathcal{X}\nabla_\xi V = \nabla_\xi \mathcal{V} - V\partial\mathcal{X}/\partial\xi_2$, and finally observe that $\xi_2^{-2} > (1+\xi_2)^{-2}$ in ϖ. As a result, the weighted Lebesgue norm in (2.3) is bounded by the gradient norm, and this shows the mentioned equivalence of norms.

Owing to the compact embedding $\mathcal{H} \subset L^2(\theta)$, problem (2.2) possesses the unbounded positive increasing sequence of eigenvalues

$$0 < M_1 < M_2 \leq M_3 \leq \cdots \leq M_k \leq \cdots \to +\infty, \tag{2.6}$$

where eigenvalues are repeated according to their multiplicities. We also choose the corresponding eigenfunctions $W_k \in \mathcal{H}$ satisfying the orthogonality and normalization conditions

$$(\nabla_\xi W_j, \nabla_\xi W_k)_\varpi = M(W_j, W_k)_\theta = \delta_{j,k}, \quad j,k \in \mathbb{N}. \tag{2.7}$$

In what follows we address only the principal eigenpair $\{M_1, W_1\}$ of the problem and omit the subscript 1 in the notation. Due to the strong maximum principle, the eigenvalue $M = M_1$ is simple and the eigenfunction $W = W_1$ can be chosen positive in $\overline{\varpi} \setminus \overline{\tau}$.

2.2 Properties of the principal eigenfunction

The 1-periodic function W is a harmonic function in $\varpi \setminus \overline{\theta}$ and becomes infinitely differentiable outside the set $\overline{\theta}$ as well as inside θ where $X = 1$. Hence, the Fourier series

$$W(\xi) = B + \sum_{p=1}^\infty \left(B_{p1}\cos(2\pi p\xi_1) + B_{p2}\sin(2\pi p\xi_1)\right) e^{-2\pi p\xi_2} \tag{2.8}$$

converges for $\xi_2 > h$ with any fixed h,

$$h > h_0 = \max\left\{\xi_2 :\ (\xi_1, \xi_2) \in \overline{\theta}\right\}. \tag{2.9}$$

Since

$$|B|^2 + \sum_{p=1}^\infty e^{-4\pi p h}\left(|B_{p1}|^2 + |B_{p2}|^2\right) < \infty,$$

there are constants $c_{q,h}$ such that there hold the estimates

$$\left|\nabla_\xi^q (W(\xi) - B)\right| \leq c_{q,h} e^{-2\pi\xi_2}, \quad \xi_2 > h,\ q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}. \tag{2.10}$$

To compute the constant B, we insert the functions W and ξ_2 into the Green formula on the rectangle $\varpi(R) = (-1/2, 1/2) \times (0, R)$ (cf. (1.13)), and we take limits as $R \to +\infty$, to obtain

$$M \int\limits_{\theta} \xi_2 W(\xi)\, d\xi = - \lim_{R\to+\infty} \int\limits_{\varpi(R)} \xi_2 \Delta_\xi W(\xi)\, d\xi = \int\limits_{-1/2}^{1/2} \Big(\xi_2 \frac{\partial W}{\partial \xi_2}(\xi) - W(\xi) \Big)\Big|_{\xi_2=0} d\xi_1$$

$$- \lim_{R\to+\infty} \int\limits_{-1/2}^{1/2} \Big(\xi_2 \frac{\partial W}{\partial \xi_2}(\xi) - W(\xi) \Big)\Big|_{\xi_2=R} d\xi_1$$

$$= - \int\limits_{-1/2}^{1/2} W(\xi_1, 0)\, d\xi_1 + B.$$

Thus,

$$B = M \int\limits_{\theta} \xi_2 W(\xi)\, d\xi + \int\limits_{-1/2}^{1/2} W(\xi_1, 0)\, d\xi_1 > 0, \tag{2.11}$$

where the strict inequality is inherited from the positivity of W and the last integral can be reduced to $(-1/2, -l) \cup (l, 1/2)$.

2.3 Asymptotics near collision points

As was mentioned in Section 2.2, the eigenfunction W is smooth everywhere, except at the curve $\tau_\sqcap$, in particular, at the points $P^\pm = (\pm l, 0)$ where the Dirichlet and Neumann conditions (1.14) meet each other. Here, we show that the latter brings the worst singularity $O\big(\mathrm{dist}\,(\xi, P^\pm)^{1/2}\big)$ into $W(\xi)$ and leads W out from the space $H^2_{loc}(\overline{\varpi})$. At the same time, jumps of the second-order derivatives at $\partial\theta \setminus \overline{\tau}$ keep W in $H^2_{loc}(\overline{\varpi} \setminus (P^- \cup P^+))$. Our justification scheme relies upon asymptotic formulas for W near the collision points $P^\pm$ that are obtained below by means of the Kondratiev theory [10]; also, the necessary information about behavior of solutions to the Poisson equation near corner and collision points can be found, e.g., in Ch. 2 in [31].

We need the polar coordinate systems $(\rho_\pm, \varphi_\pm) \in \mathbb{R} \times [0, 2\pi]$ centered at $P^\pm$, see figure 2, and a cut-off function $\varsigma \in C^\infty[0, +\infty)$ such that

$$0 \le \varsigma \le 1, \quad \varsigma(\rho) = 1 \text{ for } \rho \le \frac{1}{2} \min\Big\{l, \frac{1}{2} - l\Big\}, \quad \varsigma(\rho) = 0 \text{ for } \rho \ge \min\Big\{l, \frac{1}{2} - l\Big\}. \tag{2.12}$$

Lemma 2.1. *Let W be the principal mode of (2.2). There holds the decomposition*

$$W(\xi) = \sum_{\pm} \varsigma(\rho_\pm) \Big(K_\pm \rho_\pm^{1/2} \cos\frac{\varphi_\pm}{2} + K^1_\pm \rho_\pm^{3/2} \cos\frac{3\varphi_\pm}{2} \Big) + \widetilde{W}(\xi), \tag{2.13}$$

where $K_\pm$ and $K^1_\pm$ are some coefficients, the remainder $\widetilde{W}$ satisfies the estimates

$$\big|\nabla^p_\xi \widetilde{W}(\xi)\big| \le c^h_p \rho(\xi)^{-p+5/2} \left(1 + |\ln \rho(\xi)|\right), \quad p = 0, 1, 2,\ \xi \in \overline{\varpi(h)}, \tag{2.14}$$

with $\rho(\xi) = \min\{\rho_\pm\}$, and $h > 0$ can be taken to be arbitrary but the constants c^h_p depend on h.

Proof. The detached terms in (2.13) are obtained by the Fourier method which, owing to the separation of variables in the Laplace equation, provides the problem (see figure 2 for the orientation of the angular variables $\varphi_\pm$)

$$-\frac{d^2\mathcal{W}_\pm^\kappa}{d\varphi_\pm^2} - \kappa^2\,\mathcal{W}_\pm^\kappa = 0,\ \varphi_\pm \in (0,\pi), \quad \frac{d\mathcal{W}_\pm^\kappa}{d\varphi_\pm}(0) = 0, \quad \mathcal{W}_\pm^\kappa(\pi) = 0 \tag{2.15}$$

for the exponent κ and the angular part $\mathcal{W}_\pm^\kappa$ in the harmonics

$$\rho_\pm^\kappa \mathcal{W}_\pm^\kappa(\varphi_\pm). \tag{2.16}$$

Solving (2.15), we get

$$\kappa = \pm\Big(j + \frac{1}{2}\Big), \quad \mathcal{W}_\pm^\kappa(\varphi_\pm) = K_\pm^j \cos\bigg(\Big(j + \frac{1}{2}\Big)\varphi_\pm\bigg), \quad j \in \mathbb{N}_0, \tag{2.17}$$

and include in the asymptotic form (2.13) only functions (2.16) with $j = 0$ and $j = 1$. However, these functions bring a discrepancy $O(\rho_\pm^{1/2})$ into equation (2.13) because its right-hand side $MX(\xi)W(\xi)$ has been ignored in the above consideration as a lower-order term. According to the general procedure in [10], see also § 3.3 in [31] and § 3.5 in [31], the main part of the discrepancy is compensated by the next term in the decomposition of W, which must be searched in the form

$$\rho_\pm^{5/2}\bigg(K_\pm^2 \cos\frac{5\varphi_\pm}{2} + C_\pm \ln\rho_\pm \cos\frac{5\varphi_\pm}{2} + \mathcal{W}_\pm^{5/2}(\varphi_\pm)\bigg). \tag{2.18}$$

Inserting (2.18) into problem (1.13), (1.14) and collecting expressions of order $\rho_\pm^{1/2}$ yield the problem

$$\begin{gathered}
-\frac{d^2\mathcal{W}_\pm^{5/2}}{d\varphi_\pm^2}(\varphi_\pm) - \frac{25}{4}\,\mathcal{W}_\pm^{5/2}(\varphi_\pm) = 5C_\pm \cos\frac{5\varphi_\pm}{2} + \mathcal{F}_\pm^{5/2}(\varphi_\pm),\ \varphi_\pm \in (0,\pi),\\
\frac{d\mathcal{W}_\pm^{5/2}}{d\varphi_\pm}(0) = 0, \quad \mathcal{W}_\pm^{5/2}(\pi) = 0,
\end{gathered} \tag{2.19}$$

where

$$\mathcal{F}_\pm^{5/2}(\varphi_\pm) = \begin{cases} 0, & \varphi_\pm \in (0,\pi/2), \\ MK_\pm \cos(\varphi_\pm/2), & \varphi_\pm \in (\pi/2,\pi). \end{cases}$$

Since the homogeneous problem (2.19) has the unique eigenfunction $\cos(5\varphi/2)$ (up to a multiplicative constant), the Fredholm alternative gives the compatibility condition

$$\frac{1}{2}C_\pm \int_0^\pi \Big(\cos\frac{5\varphi}{2}\Big)^{1/2} d\varphi + MK_\pm \int_{\pi/2}^\pi \cos\frac{5\varphi}{2}\cos\frac{\varphi}{2}\,d\varphi = 0$$

which defines the coefficient in (2.18)

$$C_\pm = -\frac{2}{15\pi}\,MK_\pm. \tag{2.20}$$

The formal procedure performed above was worked out and justified in [10] where estimates of remainders are derived in weighted Sobolev norms. Estimates in weighted Hölder norms are obtained in [20] (see also § 3.6 in [31]). We apply these estimates and then join the detected terms (2.18) to the remainder $\widetilde{W}(\xi)$. Since we will prove below that $K_\pm \neq 0$ (cf. (2.21)), the coefficients (2.20) do not vanish and, therefore, the bound in (2.14) is optimal, and this ends the proof of the lemma. □

It should be noted that the results in Lemma 2.1 hold for any eigenfunction of (2.2), and the coefficients $K_\pm$, $K^1_\pm$ in (2.13) and $K^2_\pm$ in (2.18) depend on the whole data in problem (1.13), (1.14). However, for the principal eigenfunction, a method in [21] (see also Ch. 2 in [31]) establishes an integral representation of $K_\pm$ which guarantees the above-mentioned inequality (2.21).

Lemma 2.2. *Under the hypotheses of Lemma* 2.1*, the coefficients in* (2.13) *satisfy*

$$K_\pm \neq 0. \tag{2.21}$$

Proof. We insert the harmonics $Z_\pm(\xi) = \left(\rho_\pm^{-1/2} - \rho_0^{-1}\rho_\pm^{1/2}\right)\cos\dfrac{\varphi_\pm}{2}$ and the eigenfunction W into the Green formula on the semi-annulus

$$\Upsilon^\delta_\pm = \{\xi \in \varpi : \delta < \rho_\pm < \rho_0\},$$

where $\rho_0 = \dfrac{1}{2}\min\left\{l, \dfrac{1}{2} - l\right\}$ and $\delta \geq 0$ is small. We have

$$\begin{aligned}
&- M \int\limits_{\Upsilon^0_\pm} X(\xi)W(\xi)Z_\pm(\xi)\, d\xi = -M \lim_{\delta\to 0^+} \int\limits_{\Upsilon^\delta_\pm} X(\xi)W(\xi)Z_\pm(\xi)\, d\xi \\
&= - \lim_{\delta\to 0^+} \int\limits_{\Upsilon^\delta_\pm} Z_\pm(\xi)\Delta W(\xi)\, d\xi = \rho_0 \int\limits_0^\pi \left(Z_\pm(\xi)\partial_{\rho_\pm} W(\xi) - W(\xi)\partial_{\rho_\pm} Z_\pm(\xi)\right)\big|_{\rho_\pm=\rho_0} d\varphi \\
&- \lim_{\delta\to 0^+} \delta \int\limits_0^\pi \left(Z_\pm(\xi)\partial_{\rho_\pm} W(\xi) - W(\xi)\partial_{\rho_\pm} Z_\pm(\xi)\right)\big|_{\rho_\pm=\delta} d\varphi \\
&= \rho_0^{-1/2} \int\limits_0^\pi W(\xi)\big|_{\rho_\pm\rho_0} \cos\frac{\varphi}{2}\, d\varphi - \frac{1}{2}K_\pm.
\end{aligned}$$

In these calculations, we have used the exact formula for $Z_\pm$ and the asymptotic decomposition (2.13) of W while computing the last limit as $\delta \to 0^+$. Thus, we can write

$$K_\pm = 2\rho_0^{-1/2} \int\limits_0^\pi W(\xi)\big|_{\rho_\pm=\rho_0} \cos\frac{\varphi}{2}\, d\varphi + 2M \int\limits_{\Upsilon^0_\pm} X(\xi)W(\xi)Z_\pm(\xi)\, d\xi.$$

Then, inequality (2.21) follows from the relations

$$W(\xi) > 0 \text{ for } \xi \in \varpi, \quad \cos\frac{\varphi}{2} > 0 \text{ for } \varphi \in (0,\pi), \quad Z_\pm(\xi) > 0 \text{ for } \xi \in \Upsilon^0_\pm,$$

and the lemma is proved. □

Formulas (2.21), (2.11) together with the consequence of the strong maximum principle,

$$\frac{\partial W}{\partial \xi_2}(\xi_1, 0) > 0, \quad \xi_1 \in (-l, l), \tag{2.22}$$

help us to study the behavior of the eigenfunction W in the whole domain ϖ, obtaining the following result.

Corollary 2.3. *For W the principal mode of* (2.2)*, the inequalities*

$$C_1^0 \geq W(\xi) \geq C_1 \quad \text{for} \quad \xi_1 \in \Big[-\frac{1}{2}, \frac{1}{2}\Big],\, \xi_2 \geq 1, \tag{2.23}$$

$$C_2^0 \geq W(\xi) \geq C_2 \quad \text{for} \quad |\xi_1| \in \Big[\frac{1+2l}{4}, \frac{1}{2}\Big],\, \xi_2 \in [0,1], \tag{2.24}$$

$$C_3^0 \xi_2 \geq W(\xi) \geq C_3 \xi_2 \quad \text{for} \quad \xi_1 \in \Big[-\frac{l}{2}, \frac{l}{2}\Big],\, \xi_2 \in [0,1], \tag{2.25}$$

$$C_4^0 \rho_\pm^{1/2}(\pi - \varphi_\pm) \geq W(\xi) \geq C_4 \rho_\pm^{1/2}(\pi - \varphi_\pm) \quad \text{for} \quad \pm\,\xi_1 \in \Big[\frac{l}{2}, \frac{1+2l}{4}\Big],\, \xi_2 \in [0,1], \tag{2.26}$$

are valid with some positive constants C_i and C_i^0, $i = 1,2,3,4$.

Proof. Since the function W is periodic, positive in $\overline{\varpi} \setminus \overline{\tau}$ and continuous in $\overline{\varpi} \setminus P^\pm$, relation (2.24) is evident. Moreover, decomposition (2.8) with $B > 0$ asserts the validity of (2.23). Furthermore, (2.25) follows from (2.22). Finally, formulas (2.26) are based on representation (2.13) and estimates (2.14) with $p = 0, 1$, on account of

$$\cos\frac{\varphi}{2} \geq \frac{\sqrt{2}}{2} \text{ for } \varphi \in \Big[0, \frac{\pi}{2}\Big], \quad \sin\frac{\varphi}{2} \geq \frac{\sqrt{2}}{2} \text{ for } \varphi \in \Big[\frac{\pi}{2}, \pi\Big],$$

$$\text{and} \quad \frac{\partial}{\partial \xi_2}\Big(\rho_\pm^{1/2}\cos\frac{\varphi_\pm}{2}\Big) = \frac{1}{2}\rho_\pm^{-1/2}\sin\frac{\varphi_\pm}{2}.$$

Indeed, when $\xi_1 \in [l/2, l]$ since both functions $\rho^{1/2}(\pi - \varphi_+)$ and $\rho_\pm^{1/2}\cos(\varphi_\pm/2)$ vanish, we can compare the derivatives with respect to ξ_2 and consider the Taylor expansion of $\cos(\varphi_+/2)$ in a neighborhood of $\varphi_+ = \pi$ to obtain (2.26). In the case where $\xi_1 \in (l, (1+2l)/4]$, we deal with the comparison of two strictly positive smooth functions and (2.26) also holds for certain constants. Thus, the estimates (2.23)-(2.26) hold true. □

Note that, obviously, inequalities (2.24) and (2.25) hold for ξ_1 in other larger intervals which do not contain the collisions points; the constants arising in the inequalities depend on the endpoints of these intervals.

2.4 Analysis of eigenfunctions in the original problem

The considerations in Section 2.3 can be applied to problem (1.3)–(1.5) for an examination of its eigenfunctions in the vicinity of the collision points $P^\pm_{(n\varepsilon)} = (\varepsilon n \pm \varepsilon l, 0)$, $n \in \mathbb{Z}(N)$. The polar coordinates systems centered at these points are denoted by $(r_{n\pm}, \varphi_{n\pm})$ and we can state the following result.

Lemma 2.4. *Any eigenfunction u_k^ε of problem* (1.8) *admits the decomposition*

$$u_k^\varepsilon(x) = \sum_{\pm}\sum_{n\in\mathbb{Z}(N)} \varsigma\Big(\frac{r_{n\pm}}{\varepsilon}\Big)\Big(K^\varepsilon_{kn\pm}\rho_{n\pm}^{1/2}\cos\frac{\varphi_{n\pm}}{2} + K^{1\varepsilon}_{kn\pm}\rho_{n\pm}^{3/2}\cos\frac{3\varphi_{n\pm}}{2}\Big) + \widetilde{u}_k^\varepsilon(x), \tag{2.27}$$

where ς is the cut-off function (2.12)*, $K^\varepsilon_{kn\pm}$ and $K^{1\varepsilon}_{kn\pm}$ are some coefficients and the remainder $\widetilde{u}_k^\varepsilon$ satisfies the estimate*

$$\big|\widetilde{u}_k^\varepsilon(x)\big| \leq c_k^\varepsilon r(\varepsilon, x)^{-p+5/2}(1 + |\ln r(\varepsilon,x)|), \quad p = 0,1,2, \tag{2.28}$$

with $r(\varepsilon, x) = \min\{r_{n\pm} : n = 0, \pm 1, \ldots, \pm N\}$.

For further use, in Section 5, we provide with some results on the fractional function

$$\mathbf{u}_k^\varepsilon(x) = \frac{u_k^\varepsilon(x)}{W^\varepsilon(x)}, \tag{2.29}$$

where the weight multiplier $W^\varepsilon(x) = W(\varepsilon^{-1}x)$ is the principal eigenfunction of problem (1.13)–(1.15) written in the rapid variables (2.1) and extended periodically over the half-plane $\mathbb{R}^2_+$. Function (2.29) still belongs to $H^1(\Omega)$ because in the vicinity of each collision point, the numerator and denominator in (2.29) have very similar asymptotic forms (2.27) and (2.13), respectively. Indeed, for a small $r_{n\pm}$, we obtain

$$\begin{aligned}\frac{u_k^\varepsilon(x)}{W(\varepsilon^{-1}x)} =& \frac{K_{kn\pm}^\varepsilon r_{n\pm}^{1/2}\cos(\varphi_{n\pm}/2) + K_{kn\pm}^{1\varepsilon} r_{n\pm}^{3/2}\cos(3\varphi_{n\pm}/2) + \widetilde{u}_k^\varepsilon(x)}{\varepsilon^{1/2}K_\pm r_{n\pm}^{1/2}\cos(\varphi_{n\pm}/2) + \varepsilon^{3/2}K_\pm^1 r_{n\pm}^{3/2}\cos(3\varphi_{n\pm}/2) + \widetilde{W}(\varepsilon^{-1}x)} \\ =& \frac{1}{\sqrt{\varepsilon}}\,\frac{K_{kn\pm}^\varepsilon + K_{kn\pm}^{1\varepsilon} r_{n\pm}\mathbf{C}(\varphi_{n\pm}) + r_{n\pm}^{-1/2}(\cos(\varphi_{n\pm}/2))^{-1}\varepsilon^{-1/2}\widetilde{u}_k^\varepsilon(x)}{K_\pm + \varepsilon K_\pm^1 r_{n\pm}\mathbf{C}(\varphi_{n\pm}) + r_{n\pm}^{-1/2}(\cos(\varphi_{n\pm}/2))^{-1}\varepsilon^{-1/2}\widetilde{W}(\varepsilon^{-1}x)} \\ =& \frac{1}{\sqrt{\varepsilon}}\Big(\frac{K_{kn\pm}^\varepsilon}{K_\pm} + \Big(\frac{K_{kn\pm}^{1\varepsilon}}{K_\pm} - \varepsilon\frac{K_\pm^1 K_{kn\pm}^\varepsilon}{(K_\pm)^2}\Big) r_{n\pm}\mathbf{C}(\varphi_{n\pm})\Big) + \widetilde{\mathbf{u}}_k^\varepsilon(x)\end{aligned} \tag{2.30}$$

with the smooth trigonometric function

$$\mathbf{C}(\varphi) = \frac{\cos(3\varphi/2)}{\cos(\varphi/2)} = 4\Big(\cos\frac{\varphi}{2}\Big)^2 - 3$$

and the remainder $\widetilde{\mathbf{u}}_k^\varepsilon(x)$ having a faster decay as $r_{n\pm} \to 0^+$. Thus, function (2.29) as well as its first-order derivatives, are bounded at the collision points.

Inside smooth, actually flat, parts of the base Γ with either Dirichlet (1.4), or Neumann (1.5) conditions, both u_k^ε and W are smooth while $u_k^\varepsilon(x_1, 0) = 0$ for $|x_1 - \varepsilon n| < l$ and W enjoys properties (2.24)–(2.26). These properties, together with (2.30), demonstrate that $\mathbf{u}_k^\varepsilon$ falls into the Hölder class $C^{0,\alpha}$ near the base with any $\alpha \in (0, 1)$. At the same time, according to square-root singularities, cf. (2.27), the function u_k^ε belongs to the class $C^{0,\alpha}$ under the restriction $\alpha < 1/2$ only. In other words, fraction (2.29) achieves much better differential properties than the eigenfunction u_k^ε itself.

Consequently, from Lemmas 2.1, 2.2 and 2.4 and Corollary 2.3, we have proved the following result.

Proposition 2.5. *The function $\mathbf{u}_k^\varepsilon$ defined by* (2.29) *belongs to $C(\overline{\Omega})$, and*

$$\mathbf{u}_k^\varepsilon \in H^1(\Omega). \tag{2.31}$$

It proves useful to comment several points of the above considerations. First, the function W^ε is positive and differentiable outside a neighborhood of $\partial\theta^\varepsilon$ so that a "bad" behavior of u_k^ε at corner points of the arc $\Gamma_\sqcap$ is not able to disturb the confirmed inclusion (2.31).

Second, the factors $r_{n\pm}^{-1/2}$ and $(\cos(\varphi_{n\pm}/2))^{-1}$ of $\widetilde{u}_k^\varepsilon(x)$ and $\widetilde{W}(\varepsilon^{-1}x)$ in (2.30) bring into the calculation singularities at $r_{n\pm} = 0$ and $\varphi_{n\pm} = \pi$, respectively. The radial singularity is readily compensated by infinitesimal bounds in estimates (2.28) and (2.14) but the angular singularity $O(|\varphi_{n\pm} - \pi|^{-1})$ requires further discussion, see Appendix.

Finally, we emphasize that the coefficients $K_{kn\pm}^\varepsilon$ and $K_{kn\pm}^{1\varepsilon}$ in (2.27), the bound in (2.28) and other characteristics of u_k^ε and $\mathbf{u}_k^\varepsilon$ depend on the small parameter ε, and the derivation in Section 5.2 of the estimate for the norm $\|\nabla\mathbf{u}_k^\varepsilon; L^2(\Omega)\|$, uniformly in $\varepsilon \in (0, \varepsilon_0]$, turns out to be the most intriguing issue in the paper.

3 Two scales asymptotic expansion and related issues

In this section, we obtain the first two terms of the asymptotic expansion for the eigenvalues of problem (1.3)–(1.5) (with variational formulation (1.8)) and the composite asymptotic expansion for the corresponding eigenfunctions. We determine the terms arising in these expansions from the eigenpairs of two spectral problems posed either in Ω (cf. problem (1.18)–(1.20) and Section 3.4) or in the half-strip ϖ (cf. problem (2.2) and Sections 3.2-3.3). We show that both slow and rapid variables are essential to define the first term of the asymptotic expansions for eigenfunctions (cf. Section 3.5 for dominant terms). Section 3.6 contains a two-scale convergence result.

3.1 The second limit problem: a problem in Ω

We introduce the following asymptotic ansätze for an eigenpair of the singularly perturbed problem (1.3)–(1.5)

$$\lambda_k^\varepsilon = \varepsilon^{m-2}(M + \varepsilon\mu_k + \dots), \tag{3.1}$$

$$u_k^\varepsilon(x) = u_k(x) + \varepsilon u_k'(x) + \varsigma_0(x)\left(w_k(x_1, \varepsilon^{-1}x) + \varepsilon w_k'(x_1, \varepsilon^{-1}x)\right) + \dots \tag{3.2}$$

where the dots stand for lower-order terms of the approximations , $M = M_1$ is the principal eigenvalue of the first limit problem (1.13), (1.14), (1.15), and u_k, u_k' are terms of of the regular asymptotic expansion (see Section 3.4 for regularity results). Moreover, ς_0 is a smooth cut-off function such that

$$\varsigma_0(x) = 1 \text{ for } x_2 \le d, \quad \varsigma_0(x) = 0 \text{ for } x_2 \ge 2d, \text{ where } d := \frac{1}{2}\min\{x_2 : x \in \Gamma_0\}, \tag{3.3}$$

and w_k, w_k' are boundary layer terms, namely periodic functions in the half-strip ϖ with an exponential decay at infinity.

Here, and in Sections 3.2-3.3, we successively determine the asymptotic terms in (3.2) and derive the second limit problem which reads (1.18)–(1.20), involving a coefficient $b > 0$ and the correction term $\mu = \mu_k$ in (3.1).

Firstly, we note that the Laplace equation (1.18) asymptotically follows from the differential equation (1.3) because the parameter (3.1) is infinitesimal and the support of the function with the big coefficient $\lambda^\varepsilon\varepsilon^{-m}\chi^\varepsilon$ is located in the $c\varepsilon$-neighborhood of Γ, and hence, it does not appear disappear in Ω when $\varepsilon \to 0$. The Dirichlet condition (1.19) is directly inherited from the boundary condition (1.4) on $\Gamma_\sqcap$. The Steklov spectral condition (1.20) will be found by examining the natural decay property of the boundary layer terms.

3.2 The first term of the boundary layer

In this and the next section we omit the subscript k in the notation, cf. (1.18)–(1.20).

We insert ansätze (3.2) and (3.1) into problem (1.3)–(1.5), consider the rapid variables (2.1) and apply the obvious formulas

$$\frac{\partial^2 w}{\partial x_1^2}\Big(x_1, \frac{x}{\varepsilon}\Big) = \frac{1}{\varepsilon^2}\frac{\partial^2 w}{\partial \xi_1^2}(x_1, \xi) + \frac{2}{\varepsilon}\frac{\partial^2 w}{\partial x_1 \partial \xi_1}(x_1, \xi) + \frac{\partial^2 w}{\partial x_1^2}(x_1, \xi), \quad \xi = \varepsilon^{-1}x,$$

$$\frac{\partial w}{\partial \nu}(x_1, \xi_1, 0) = -\frac{1}{\varepsilon}\frac{\partial w}{\partial \xi_2}(x_1, \xi_1, 0), \tag{3.4}$$

$$u(x) = u(x_1, 0) + x_2\frac{\partial u}{\partial x_2}(x_1, 0) + O(x_2^2) = u(x_1, 0) + \varepsilon\xi_2\frac{\partial u}{\partial x_2}(x_1, 0) + O(\varepsilon^2\xi_2^2),$$

and we gather the coefficients of the same powers of the small parameter ε, then, we arrive at the problem

$$
\begin{gathered}
-\Delta_\xi w(x_1,\xi) = MX(\xi)(w(x_1,\xi)+u(x_1,0)),\ \xi\in\varpi,\\
w(x_1,\xi_1,0) = -u(x_1,0),\ |\xi_1|<l,\quad -\frac{\partial w}{\partial\xi_2}(x_1,\xi_1,0)=0,\ |\xi_1|\in\Big(l,\frac{1}{2}\Big),\\
w\Big(\frac{1}{2},\xi_2\Big)=w\Big(-\frac{1}{2},\xi_2\Big),\ \ \frac{\partial w}{\partial\xi_1}\Big(\frac{1}{2},\xi_2\Big)=\frac{\partial w}{\partial\xi_1}\Big(-\frac{1}{2},\xi_2\Big),\quad \xi_2>0.
\end{gathered}
\tag{3.5}
$$

Notice that the variable $x_1\in(-1,1)$ remains as a parameter in this problem.

Evidently, a solution of (3.5), with the exponential decay as $\xi_2\to+\infty$ takes the form

$$
w(x_1,\xi)=B^{-1}u(x_1,0)\widehat{W}(\xi),\quad \widehat{W}(\xi)=W(\xi)-B, \tag{3.6}
$$

where $W=W_1$ is the principal eigenfunction of the first limit problem (1.13)–(1.15) (cf. norm (2.7)) and B, the first coefficient of the Fourier series (2.8), is defined by (2.11) in its representation (2.8).

3.3 The second term of the boundary layer and the Steklov condition on Γ

Taking into account formulas (3.4) for w, w' and u, u', we collect terms of orders ε^{-1} in (1.3), ε^1 in (1.4) and ε^0 in (1.5). As a result, we obtain the problem

$$
\begin{gathered}
-\Delta_\xi w'(x_1,\xi)-MX(\xi)w'(x_1,\xi)=f'(x_1,\xi),\ \xi\in\varpi,\\
w'(x_1,\xi_1,0)=-u'(x_1,0),\ |\xi_1|<l,\\
-\frac{\partial w'}{\partial\xi_2}(x_1,\xi_1,0)=\frac{\partial u}{\partial x_2}(x_1,0),\ |\xi_1|\in\Big(l,\frac{1}{2}\Big),
\end{gathered}
\tag{3.7}
$$

with the periodicity conditions (1.15) and the right-hand side

$$
\begin{aligned}
f'(x_1,\xi)=&2\frac{\partial^2 w}{\partial x_1\partial\xi_1}(x_1,\xi)+MX(\xi)\Big(u'(x_1,0)+\xi_2\frac{\partial u}{\partial x_2}(x_1,0)\Big)\\
&+\mu X(\xi)\left(w(x_1,\xi)+u(x_1,0)\right)
\end{aligned}
\tag{3.8}
$$

Below we solve problem (3.7) with both conditions $u'(x_1,0)=0$ and $u'(x_1,0)\neq 0$. However, without any restriction, we can assume throughout the paper $u'=0$: see Remark 3.2 when $u'\neq 0$.

Hence, let us assume that $u'(x_1,0)=0$ both in (3.7) and (3.8), and therefore we consider the Dirichlet condition in (3.7) homogeneous. Let also g' denote the right-hand side of the Neumann condition in (3.7) imposed on $\tau^\sharp=(-1/2,-l)\cup(l,1/2)$; namely,

$$
g'(x_1,\xi_1,0)=\frac{\partial u}{\partial x_2}(x_1,0).
$$

Owing to (3.6) and (2.8), the function

$$
\frac{\partial^2 w}{\partial x_1\partial\xi_1}(x_1,\xi)=\frac{1}{B}\frac{\partial u}{\partial x_1}(x_1,0)\frac{\partial W}{\partial\xi_1}(\xi) \tag{3.9}
$$

decays exponentially as $\xi_2 \to +\infty$, and f' also does. Consequently, the variational formulation of problem (3.7), (1.15), which reads

$$(\nabla_\xi w', \nabla_\xi v)_\varpi - M(w', v)_\theta = (f', v)_\varpi + (g', v)_{\tau^\sharp}, \quad v \in \mathcal{H}, \tag{3.10}$$

has on the right-hand side a linear continuous functional in the Hilbert space $\mathcal{H}$ (cf. (2.2)). Since M is a simple eigenvalue, the Fredholm alternative brings the only compatibility condition in problem (3.10)

$$(f', W)_\varpi + (g', W)_{\tau^\sharp} = 0 \tag{3.11}$$

Assuming that (3.11) is satisfied, one solution $w' \in \mathcal{H}$ is defined up to an additive function $C'(x_1)W(\xi)$. According to the above-mentioned relation $f'(x_1, \xi) = O(e^{-2\pi\xi_2})$, we deduce that a particular solution w'_0 of (3.10) admits the representation

$$w'_0(x_1, \xi) = B'_0(x_1) + \widetilde{w}'_0(x_1, \xi)$$

with an exponentially decaying remainder $\widetilde{w}'_0(x_1, \xi)$ and with $B'_0(x_1)$ the constant function in the ξ variable describing the behaviour of $w'_0(x_1, \xi)$ when $\xi_2 \to \infty$. Setting $C'(x_1) = -B^{-1}B'_0(x_1)$ yields the unique solution of (3.7) with the exponential decay

$$w'(x_1, \xi) = w'_0(x_1, \xi) + C'(x_1)\, W(\xi) \in H^1(\varpi) \cap \mathcal{H}. \tag{3.12}$$

Consequently, it suffices to guarantee condition (3.11) for the above defined data f' and g'. Let us examine this condition in further detail. First of all, according to the 1-periodicity of $W(\xi)$ in ξ_1 and formula (3.9), we have

$$2\int_\varpi W(\xi)\frac{\partial^2 w}{\partial x_1 \partial \xi_1}(x_1, \xi)\, d\xi = \frac{1}{B}\frac{\partial u}{\partial x_1}(x_1, 0)\int_\varpi \frac{\partial}{\partial \xi_1}\left(W(\xi)^2\right)\, d\xi = 0. \tag{3.13}$$

Then, recalling (2.11), we obtain

$$\begin{aligned} &M\int_\varpi W(\xi)X(\xi)\xi_2\frac{\partial u}{\partial x_2}(x_1, 0)\, d\xi + \int_{-1/2}^{1/2} W(\xi_1, 0)\frac{\partial u}{\partial x_2}(x_1, 0)\, d\xi_1 \\ =&\frac{\partial u}{\partial x_2}(x_1, 0)\left(M\int_\theta \xi_2 W(\xi)\, d\xi + \int_{-1/2}^{1/2} W(\xi_1, 0)\, d\xi_1\right) = B\frac{\partial u}{\partial x_2}(x_1, 0). \end{aligned}$$

Finally, the relation (3.6) together with the normalization condition (2.7) yield

$$\mu\int_\varpi W(\xi)X(\xi)(w(x_1, \xi) + u(x_1, 0))\, d\xi = \frac{\mu}{B}\int_\theta |W(\xi)|^2\, d\xi\ u(x_1, 0) = B\,\mu b u\,(x_1, 0), \tag{3.14}$$

where

$$b = M^{-1}B^{-2} > 0. \tag{3.15}$$

Thus, formulas (3.13)–(3.14) and (3.8) convert the compatibility condition (3.11) into the Steklov condition (1.20) with coefficient (3.15) and the spectral parameter μ. Considering u the solution of (1.18)-(1.20) gives the solution (3.12) of problem (3.7) and (1.15).

Remark 3.1. As a matter of fact, when the compatibility condition (3.11) is fulfilled, problem (3.7) with $u'(x_1, 0) = 0$ has a unique decaying solution in the form

$$w'(x_1, \xi) = u(x_1, 0)w'_0(\xi) + \sum_{p=1,2} \frac{\partial u}{\partial x_p}(x_1, 0)w'_p(\xi) \tag{3.16}$$

where the functions w'_q are certain 1-periodic functions in ξ_1 which are smooth everywhere in $\overline{\varpi}$, except at the arc $\tau_\sqcap$ and the collision points $P^\pm$ where, respectively, jumps of second derivatives and singularities $O(\rho_\pm^{1/2})$ occur. The assertion on the formula (3.16) is due to the form of the nonhomogeneous term f' (see (3.8) and (3.9)); here u reads u_k in the case where $\mu = \mu_k$ in the sequence (3.18).

Remark 3.2. In the general case where $u'(x_1, 0) \neq 0$ in (3.7), changing $w'(x_1, \xi) + u'(x_1, 0)$ gives the solution of (3.7) that we have obtained above (cf. (3.12)) and this completely solves the problem (3.7). Also, note that we have defined the second term $w'(x_1, \xi)$ of the boundary layer type without imposing any condition on the second term $u'(x)$ of the regular type; hence, we could put $u' = 0$ in ansatz (3.2). Nevertheless, we note that the term u' together with the replacement $w'(x_1, \xi) \mapsto w'(x_1, \xi) + B^{-1}u'(x_1, 0)\widehat{W}(\xi)$ in the boundary layer term, are needed to determine lower-order terms which are, however, omitted in our present study.

3.4 Eigenpairs of the second limit problem: the Steklov problem

The variational formulation of problem (1.18)–(1.20) reads: to find a number μ and a non-trivial function $u \in H^1_0(\Omega; \Gamma_\sqcap)$ such that

$$(\nabla u, \nabla v)_\Omega = \mu b(u, v)_\Gamma, \quad v \in H^1_0(\Omega; \Gamma_\sqcap). \tag{3.17}$$

Here, $H^1_0(\Omega; \Gamma_\sqcap)$ is the Sobolev space of functions vanishing at the arc $\Gamma_\sqcap$. Since the trace operator: $H^1(\Omega) \to L^2(\Gamma)$ is compact, the following assertion becomes evident.

Proposition 3.3. *The variational problem* (3.17) *(equivalently,* (1.18)–(1.20) *in the differential form) has the unbounded monotone positive sequence of eigenvalues*

$$0 < \mu_1 < \mu_2 \leq \mu_3 \leq \cdots \leq \mu_k \leq \cdots \to +\infty \tag{3.18}$$

which repeat according to their multiplicities. The corresponding eigenfunctions $u_k \in H^1_0(\Omega; \Gamma_\sqcap)$ can be subject to the orthogonality and normalization conditions

$$(u_k, u_j)_\Gamma = \delta_{j,k}, \quad j, k \in \mathbb{N}. \tag{3.19}$$

The eigenfunction $u_k \in H^1_0(\Omega; \Gamma_\sqcap)$ has additional smoothness near the base Γ in spite of the corner points $Q^\pm = (\pm L, 0)$ where the Dirichlet and Steklov conditions meet each other. Indeed, these corners have the angle $\pi/2$, and applying the Kondratiev theory [10] again and performing a simple calculation (cf. Ch. 2 in [31]), one may verify the representation

$$\begin{aligned} u_k(x) =& (x_1 \pm L)\frac{\partial u_k}{\partial x_1}(\pm L, 0) + \frac{1}{2}\left((x_1 \mp L)^2 - x_2^2\right)\frac{\partial^2 u_k}{\partial x_1^2}(\pm L, 0) \\ &- (x_1 \mp L)x_2 b\mu_k \frac{\partial u_k}{\partial x_1}(\pm L, 0) - \frac{1}{2}(x_1 \mp L)^2 x_2 b\mu_k \frac{\partial^2 u_k}{\partial^2 x_1}(\pm L, 0) \\ &+ O\left(|x - Q^\pm|^4(1 + |\ln|x - Q^\pm||)\right). \end{aligned}$$

In this way, in a neighborhood of Γ the function u_k falls into the classes H^4 and $C^{3,\alpha}$ with any $\alpha \in (0,1)$. At the same time, we have that $u_k \in H^4(\Gamma)$.

3.5 Transforming the asymptotic expansions of eigenfunctions

Ansatz (3.2) with boundary layer terms proves to be convenient in Section 4 for an estimation of asymptotic remainders in (3.1) and (3.2). However, in order to highlight our approach in Section 6.2, we rewrite the main asymptotic term of the eigenfunction u_k^ε in a different form.

Using formula (3.6) for the boundary layer term $w_k(x_1,\xi)$ in (3.2), we have

$$\begin{aligned} u_k(x) + \varsigma_0(x) w_k(x_1,\xi) =& u_k(x) + \varsigma_0(x) B^{-1} u_k(x_1,0)(W(\varepsilon^{-1}x) - B) \\ =& B^{-1} u_k(x) W(\varepsilon^{-1}x) - (1-\varsigma_0(x)) B^{-1} u_k(x)(W(\varepsilon^{-1}x) - B) \\ & -\varsigma_0(x) B^{-1}(u_k(x) - u_k(x_1,0))(W(\varepsilon^{-1}x) - B). \end{aligned} \quad (3.20)$$

We note that $B^{-1}u_k(x)W(\varepsilon^{-1}x)$ amounts to the dominant term in the asymptotic expansion (3.20) as we can show easily in what follows: The difference $W(\varepsilon^{-1}x) - B$ decays as $O(e^{-2\pi x_2/\varepsilon})$ and $1 - \varsigma_0(x) = 0$ for $x_2 < d$, $d > 0$, see (3.3). Hence, the next to last term in (3.20) is exponentially small. Moreover, $u_k(x) - u_k(x_1,0) = O(x_2)$ and, therefore (see (2.10)), the last product in (3.20) can be bounded by the infinitesimal value $c\varepsilon$ everywhere in Ω.

It should be emphasized that representation (3.20) along with the above-estimates was the main reason to introduce the asymptotic ansatz (1.17) in Section 1.2 and to consider the quotient function (2.29) in Sections 2.4 and 5.1.

3.6 A two scale convergence result

For convenience, we introduce the following result which provides bounds for convergence rates of 1-periodic functions when they satisfy a certain exponential decay in the ξ_2 direction.

Proposition 3.4. *Assume that $z \in H^1(\Omega)$, and $Z \in L^2(\varpi)$ is a function which is extended 1-periodically in ξ_1 over the half-plane $\mathbb{R}^2_+$ and has the exponential decay as $\xi_2 \to +\infty$, namely*

$$\|e^{\beta\xi_2} Z; L^2(\varpi)\| < \infty \text{ with some } \beta > 0.$$

Then

$$\left| \int_\Omega z(x) Z\Big(\frac{x}{\varepsilon}\Big)\, dx - \varepsilon \int_\varpi Z(\xi)\, d\xi \int_{-1}^{1} z(x_1,0)\, dx_1 \right| \le c\varepsilon^{3/2} \|z; H^1(\Omega)\| \, \|e^{\beta\xi_2} Z; L^2(\varpi)\|. \quad (3.21)$$

Proof. Due to the exponential decay of Z, we can restrict the first integral in (3.21) on the rectangle $\Omega_0 = (-1,1)\times(0,l_0) \subset \Omega$ with some fixed $l_0 \in (0, \min\{l_\pm, d\}]$, the committed error being exponentially small in ε. We have

$$\int\limits_{\Omega_0} z(x)Z\Big(\frac{x}{\varepsilon}\Big)\,dx = \sum_{n\in\mathbb{Z}(N)} \int\limits_{\varepsilon(n-1/2)}^{\varepsilon(n+1/2)} \int\limits_0^{l_0} z(x_1,x_2)\,Z\Big(\frac{x_1}{\varepsilon},\frac{x_2}{\varepsilon}\Big)\,dx_2dx_1$$

$$= \sum_{n\in\mathbb{Z}(N)} \int\limits_{\varepsilon(n-1/2)}^{\varepsilon(n+1/2)} z(x_1,0) \int\limits_0^{l_0} Z\Big(\frac{x_1}{\varepsilon},\frac{x_2}{\varepsilon}\Big)\,dx_2dx_1$$

$$+ \int\limits_{-1}^{1} \int\limits_0^{l_0} (z(x_1,x_2)-z(x_1,0))\,Z\Big(\frac{x_1}{\varepsilon},\frac{x_2}{\varepsilon}\Big)\,dx_2dx_1 =: I_0^\varepsilon + I_1^\varepsilon$$

and, furthermore,

$$I_0^\varepsilon = \sum_{n\in\mathbb{Z}(N)} \frac{1}{\varepsilon} \int\limits_{\varepsilon(n-1/2)}^{\varepsilon(n+1/2)} z(\zeta,0)\,d\zeta \int\limits_{\varepsilon(n-1/2)}^{\varepsilon(n+1/2)} \int\limits_0^{l_0} Z\Big(\frac{x_1}{\varepsilon},\frac{x_2}{\varepsilon}\Big)\,dx_2dx_1$$

$$+ \sum_{n\in\mathbb{Z}(N)} \frac{1}{\varepsilon} \int\limits_{\varepsilon(n-1/2)}^{\varepsilon(n+1/2)} \int\limits_{\varepsilon(n-1/2)}^{\varepsilon(n+1/2)} (z(x_1,0)-z(\zeta,0))\,d\zeta \int\limits_0^{l_0} Z\Big(\frac{x_1}{\varepsilon},\frac{x_2}{\varepsilon}\Big)\,dx_2dx_1$$

$$=: I_2^\varepsilon + I_3^\varepsilon.$$

Making the coordinate change $x \mapsto \xi$, it can be easily seen that the expression I_2^ε satisfies

$$\Big| I_2^\varepsilon - \varepsilon \int\limits_{\varpi} Z(\xi)\,d\xi \int\limits_{-1}^{1} z(x_1,0)\,dx_1 \Big| \le c\varepsilon e^{-\beta l_0/(2\varepsilon)} \int\limits_{-1/2}^{1/2} \int\limits_{l_0/\varepsilon}^{\infty} e^{\beta\xi_2/2}\,|Z(\xi)|\,d\xi \int\limits_{-1}^{1} |z(\zeta,0)|\,d\zeta$$

$$\le c\varepsilon e^{-\beta l_0/(4\varepsilon)} \|e^{-\beta\xi_2/2}; L^2(\varpi)\|\,\|e^{\beta\xi_2} Z; L^2(\varpi)\|\,\|z; L^2(\Gamma)\|$$

$$\le c\varepsilon^{3/2} \|z; H^1(\Omega)\|\,\|e^{\beta\xi_2} Z; L^2(\varpi)\|.$$

Another expression will be estimated by means of the Cauchy inequality

$$\sum_{n\in\mathbb{Z}(N)} a_n b_n \le \Big(\sum_{n\in\mathbb{Z}(N)} a_n^2\Big)^{1/2} \Big(\sum_{n\in\mathbb{Z}(N)} b_n^2\Big)^{1/2}.$$

Indeed, recalling the Slobodetskii norm

$$\|z; H^{1/2}(\Gamma)\| = \Big(\|z; L^2(\Gamma)\|^2 + \int\limits_{-1}^{1} \int\limits_{-1}^{1} \frac{|z(x_1,0)-z(\zeta,0)|^2}{|x_1-\zeta|^2}\,dx_1 d\zeta \Big)^{1/2} \tag{3.22}$$

in the trace space $H^{1/2}(\Gamma)$ for $H^1(\Omega)$, and applying the Hölder inequality, we obtain

$$\begin{aligned}
|I_3^\varepsilon| \le & 2 \sum_{n\in\mathbb{Z}(N)} \int\limits_{\varepsilon(n-1/2)}^{\varepsilon(n+1/2)} \int\limits_{\varepsilon(n-1/2)}^{\varepsilon(n+1/2)} \frac{|z(x_1,0)-z(\zeta,0)|}{|x_1-\zeta|+\varepsilon}\, d\zeta \int\limits_0^{l_0} \Big| Z\Big(\frac{x_1}{\varepsilon},\frac{x_2}{\varepsilon}\Big)\Big|\, dx_2 dx_1 \\
\le & 2\Big(\varepsilon \int\limits_{-1}^{1}\int\limits_{-1}^{1} \frac{|z(x_1,0)-z(\zeta,0)|^2}{(|x_1-\zeta|+\varepsilon)^2}\, dx_1\, d\zeta\Big)^{1/2} \Big(\varepsilon(2N+1)\varepsilon^2 \|e^{\beta\xi_2} Z; L^2(\varpi)\|^2\Big)^{1/2} \\
\le & c\varepsilon^{3/2}\|z; H^1(\Omega)\|\, \|e^{\beta\xi_2} Z; L^2(\varpi)\|,
\end{aligned} \tag{3.23}$$

where $2N+1=O(\varepsilon^{-1})$ is the number of cells (see Section 1.1). We complete the proof with the following estimate using the Newton–Leibnitz formula and the integral Hölder inequality:

$$\begin{aligned}
|I_1^\varepsilon| \le & \int\limits_0^{l_0}\int\limits_{-1}^{1} \Big| \int\limits_0^{x_2} \frac{\partial z}{\partial t}(x_1,t)\, dt\Big| \Big| Z\Big(\frac{x_1}{\varepsilon},\frac{x_2}{\varepsilon}\Big)\Big|\, dx_1\, dx_2 \\
\le & \int\limits_0^{l_0}\int\limits_{-1}^{1} \Big(\int\limits_0^{x_2} \Big|\frac{\partial z}{\partial t}(x_1,t)\Big|^2 dt\Big)^{1/2} x_2^{1/2} \Big| Z\Big(\frac{x_1}{\varepsilon},\frac{x_2}{\varepsilon}\Big)\Big|\, dx_1\, dx_2 \\
\le & c\Big\|\frac{\partial z}{\partial x_2}; L^2(\Omega_0)\Big\| \Big(\int\limits_{-1}^{1}\Big(\int\limits_0^{l_0} x_2 \Big| Z\Big(\frac{x_1}{\varepsilon},\frac{x_2}{\varepsilon}\Big)\Big|\, dx_2\Big)^2 dx_1\Big)^{1/2} \\
\le & c\|\nabla z; L^2(\Omega_0)\| \Big(\int\limits_0^{l_0} e^{-2\beta x_2/\varepsilon} x_2\, dx_2 \int\limits_{-1}^{1}\int\limits_0^{l_0} e^{2\beta x_2/\varepsilon}\Big| Z\Big(\frac{x_1}{\varepsilon},\frac{x_2}{\varepsilon}\Big)\Big|^2 dx\Big)^{1/2} \\
\le & c\|z; H^1(\Omega)\| \Big(\varepsilon^2(2N+1)\varepsilon^2 \int\limits_{\varpi} e^{2\beta\xi_2}|Z(\xi)|^2\, d\xi\Big)^{1/2} \\
\le & c\varepsilon^{3/2}\|z; H^1(\Omega)\|\, \|e^{\beta\xi_2} Z; L^2(\varpi)\|.
\end{aligned}$$

Thus, gathering the estimates for I_i^ε, $i=1,2,3$, the result of the proposition holds. □

Remark 3.5. Note that the geometry of the domain chosen (cf. figure 1) provides certain properties for solutions useful mainly to simplify technical proofs, and avoid introducing more cut-off functions which bring more cumbersome computations. However, formal computations hold for a more general geometry of Ω and the proofs can be extended.

4 Estimation of asymptotic remainders

Throughout this section, we justify up to a certain degree asymptotics (3.1) and (3.2) (cf. (3.20)) for eigenvalues and eigenfunctions.We obtain bounds for discrepancies between the eigenvalues of the original problem and the first two terms of the asymptotic expansions, and similarly for the corresponding eigenfunctions. However, this still does not imply the approach of the k-th eigenvalue in the sequence (1.9) through the k-th eigenvalue in the sequence (3.18) with the same k. We set some preliminaires in Section 4.1, while we gather the main results in Section 4.4 (cf. Theorem 4.2 and estimate (4.29)).

4.1 Abstract formulation of the ε-dependent eigenvalue problem

In the Hilbert space $\mathfrak{H}^\varepsilon = H_0^1(\Omega; \Gamma_\sqcap^\varepsilon)$ we introduce the scalar product

$$\langle u^\varepsilon, v^\varepsilon \rangle_\varepsilon = (\nabla u^\varepsilon, \nabla v^\varepsilon)_\Omega + \varepsilon^{-2}(u^\varepsilon, v^\varepsilon)_{\Theta^\varepsilon} + \varepsilon^{m-2}(u^\varepsilon, v^\varepsilon)_\Omega \tag{4.1}$$

and an operator $\mathfrak{K}^\varepsilon$ by the identity

$$\langle \mathfrak{K}^\varepsilon u^\varepsilon, v^\varepsilon \rangle_\varepsilon = \varepsilon^{-2}(u^\varepsilon, v^\varepsilon)_{\Theta^\varepsilon} + \varepsilon^{m-2}(u^\varepsilon, v^\varepsilon)_\Omega, \quad \forall u^\varepsilon, v^\varepsilon \in \mathfrak{H}^\varepsilon. \tag{4.2}$$

Operator $\mathfrak{K}^\varepsilon$ is compact, positive, continuous and symmetric. Therefore, it has a positive monotone infinitesimal sequence of eigenvalues counted according to their multiplicity

$$\kappa_1^\varepsilon \geq \kappa_2^\varepsilon \geq \cdots \geq \kappa_k^\varepsilon \geq \cdots \to 0^+, \tag{4.3}$$

while $\kappa^\varepsilon = 0$ is the only point of the essential spectrum of $\mathfrak{K}^\varepsilon$.

In view of (4.2) and (4.1), the integral identity (1.8) is equivalent to the abstract equation

$$\mathfrak{K}^\varepsilon u^\varepsilon = \kappa^\varepsilon u^\varepsilon \quad \text{in} \quad \mathfrak{H}^\varepsilon,$$

where, in addition, a simple calculation shows that sequences (4.3) and (1.9) satisfy the relationship

$$\kappa_k^\varepsilon = \frac{1}{1+\Lambda_k^\varepsilon} = \frac{\varepsilon^{m-2}}{\lambda_k^\varepsilon + \varepsilon^{m-2}}. \tag{4.4}$$

In what follows, to show the above mentioned proximity of the asymptotic formulas (3.1) and (3.2), we use the following simple consequence of the spectral decomposition of the resolvent, also known as result on "near eigenvalues and eigenvectors", see, e.g. [42] and Ch. 6 in [3].

Lemma 4.1. *Let $\mathfrak{U}^\varepsilon \in \mathfrak{H}^\varepsilon$ and $\mathfrak{k}^\varepsilon \in \mathbb{R}$ possess the properties*

$$\|\mathfrak{U}^\varepsilon; \mathfrak{H}^\varepsilon\| = \langle \mathfrak{U}^\varepsilon, \mathfrak{U}^\varepsilon \rangle_\varepsilon^{1/2} = 1, \quad \|\mathfrak{K}^\varepsilon \mathfrak{U}^\varepsilon - \mathfrak{k}^\varepsilon \mathfrak{U}^\varepsilon; \mathfrak{H}^\varepsilon\| =: \delta \in (0, \mathfrak{k}^\varepsilon). \tag{4.5}$$

Then, there exists an eigenvalue κ_q^ε of the operator $\mathfrak{K}^\varepsilon$ such that

$$\left|\kappa_q^\varepsilon - \mathfrak{k}^\varepsilon\right| \leq \delta.$$

Moreover, for any $\delta_\bullet \in (\delta, \mathfrak{k}^\varepsilon)$, one finds some coefficients $\mathfrak{a}_{Q^\varepsilon}^\varepsilon, \ldots, \mathfrak{a}_{Q^\varepsilon+X^\varepsilon-1}^\varepsilon$ satisfying

$$\left\|\mathfrak{U}^\varepsilon - \sum_{q=Q^\varepsilon}^{Q^\varepsilon+X^\varepsilon-1} \mathfrak{a}_q^\varepsilon \mathfrak{V}_q^\varepsilon; \mathfrak{H}^\varepsilon\right\| \leq 2\frac{\delta}{\delta_\bullet}, \qquad \sum_{q=Q^\varepsilon}^{Q^\varepsilon+X^\varepsilon-1} \left|\mathfrak{a}_q^\varepsilon\right|^2 = 1, \tag{4.6}$$

where $\kappa_{Q^\varepsilon}^\varepsilon, \ldots, \kappa_{Q^\varepsilon+X^\varepsilon-1}^\varepsilon$ stand for all the eigenvalues of $\mathfrak{K}^\varepsilon$ in the segment $[\mathfrak{k}^\varepsilon - \delta_\bullet, \mathfrak{k}^\varepsilon + \delta_\bullet]$ and $\mathfrak{V}_{Q^\varepsilon}^\varepsilon, \ldots, \mathfrak{V}_{Q^\varepsilon+X^\varepsilon-1}^\varepsilon$ are the corresponding eigenvectors subject to the orthogonality and normalization conditions

$$\langle \mathfrak{V}_q^\varepsilon, \mathfrak{V}_p^\varepsilon \rangle_\varepsilon = \delta_{p,q}, \quad p, q \in \mathbb{N}. \tag{4.7}$$

Comparing (4.7) and (1.10), we recall formulas (4.2) and (1.8) to arrive at the relation

$$\mathfrak{V}_k^\varepsilon = \left(1 + \varepsilon^{m-2}(\lambda_k^\varepsilon)^{-1}\right)^{-1/2} u_k^\varepsilon. \tag{4.8}$$

$\mathfrak{k}^\varepsilon$ and $\mathfrak{U}^\varepsilon$ arising in (4.5) are the so-called near eigenvalue and eigenvector respectively for the operator $\mathfrak{K}^\varepsilon$; we perform a suitable choice in Section 4.2, and we compute δ in Section 4.3.

4.2 Choosing near eigenvalues and eigenvectors

According to formulas (3.1), (3.2), (3.6), Proposition 3.3 and Remark 3.2, we set

$$\mathfrak{k}_k^\varepsilon = (1 + M + \varepsilon\mu_k)^{-1}, \quad \mathfrak{U}_k^\varepsilon = \|\mathfrak{u}_k^\varepsilon; \mathfrak{H}^\varepsilon\|^{-1}\mathfrak{u}_k^\varepsilon, \tag{4.9}$$

with the function $\mathfrak{u}_k^\varepsilon$ being

$$\mathfrak{u}_k^\varepsilon(x) = u_k(x) + \varsigma_0(x)\left(B^{-1}u_k(x_1, 0)\widehat{W}(\varepsilon^{-1}x) + \varepsilon\varsigma_\varepsilon'(x_1)w'(x_1, \varepsilon^{-1}x)\right), \tag{4.10}$$

where w' is the decaying solution (3.12) of problem (3.7) with $u' = 0$. The cut-off functions ς_0 from (3.3) and ς_ε',

$$\begin{gathered}\varsigma_\varepsilon^1 \in C^\infty(\mathbb{R}), \quad 1 \le \varsigma_\varepsilon^1 \le 1, \quad \left|\frac{\partial^p \varsigma_\varepsilon^1}{\partial x_1^p}(x_1)\right| \le c_p\varepsilon^{-p}, \; p \in \mathbb{N}, \\ \varsigma_\varepsilon^1(x_1) = 1 \text{ for } |x_1| \le 1 - \frac{2\varepsilon}{3}\Big(\frac{1}{2} - l\Big), \quad \varsigma_\varepsilon^1(x_1) = 0 \text{ for } |x_1| \ge 1 - \frac{\varepsilon}{3}\Big(\frac{1}{2} - l\Big),\end{gathered} \tag{4.11}$$

are introduced into (4.10) in order to fulfil the boundary condition (1.4) on $\Gamma_\sqcap^\varepsilon$. Notice that $\varsigma_0 = 0$ on Γ_0, $\varsigma_\varepsilon' = 0$ on $\Gamma_\pm$ while, owing to the Dirichlet condition in problems (1.13)–(1.15) and (3.5), (3.7), the function $\mathfrak{u}_k^\varepsilon$ also vanishes at T^ε and therefore belongs to $\mathfrak{H}^\varepsilon = H_0^1(\Omega; \Gamma_\sqcap^\varepsilon)$ due to the properties of the functions u_k and W mentioned in Sections 3.4 and 2.2, 2.3, respectively.

Let us consider the discrepancy $\delta^\varepsilon = \delta_k^\varepsilon$ in (4.5), namely

$$\begin{aligned}\delta_k^\varepsilon =& \|\mathfrak{K}^\varepsilon\mathfrak{U}_k^\varepsilon - \mathfrak{k}_k^\varepsilon\mathfrak{U}_k^\varepsilon; \mathfrak{H}^\varepsilon\| = \sup\left|\langle\mathfrak{K}^\varepsilon\mathfrak{U}_k^\varepsilon - \mathfrak{k}_k^\varepsilon\mathfrak{U}_k^\varepsilon, v\rangle_\varepsilon\right| \\ =& \|\mathfrak{u}_k^\varepsilon; \mathfrak{H}^\varepsilon\|^{-1}\mathfrak{k}_k^\varepsilon \sup\Big|(1 + M + \varepsilon\mu_k)\left(\varepsilon^{-2}(\mathfrak{u}_k^\varepsilon, v)_{\Theta^\varepsilon} + \varepsilon^{m-2}(\mathfrak{u}_k^\varepsilon, v)_\Omega\right) \\ &\qquad - (\nabla\mathfrak{u}_k^\varepsilon, \nabla v)_\Omega - \varepsilon^{-2}(\mathfrak{u}_k^\varepsilon, v)_{\Theta^\varepsilon} - \varepsilon^{m-2}(\mathfrak{u}_k^\varepsilon, v)_\Omega\Big| \\ =& \|\mathfrak{u}_k^\varepsilon; \mathfrak{H}^\varepsilon\|^{-1}\mathfrak{k}_k^\varepsilon \sup\left|(\nabla\mathfrak{u}_k^\varepsilon, \nabla v)_\Omega - \varepsilon^{-2}(M + \varepsilon\mu_k)(\mathfrak{u}_k^\varepsilon, v)_{\Theta^\varepsilon} - \varepsilon^{m-2}(M + \varepsilon\mu_k)(\mathfrak{u}_k^\varepsilon, v)_\Omega\right| \\ =& \|\mathfrak{u}_k^\varepsilon; \mathfrak{H}^\varepsilon\|^{-1}\mathfrak{k}_k^\varepsilon \sup\Big|(\Delta\mathfrak{u}_k^\varepsilon, v)_\Omega - \Big(\frac{\partial\mathfrak{u}_k^\varepsilon}{\partial x_2}, v\Big)_{\Gamma\setminus T^\varepsilon} + \varepsilon^{-2}(M + \varepsilon\mu_k)(\mathfrak{u}_k^\varepsilon, v)_{\Theta^\varepsilon} \\ &\qquad + \varepsilon^{m-2}(M + \varepsilon\mu_k)(\mathfrak{u}_k^\varepsilon, v)_\Omega\Big|,\end{aligned} \tag{4.12}$$

where the supremum is taken over all $v \in \mathfrak{H}^\varepsilon$ such that $\|v; \mathfrak{H}^\varepsilon\| = 1$. We postpone computing δ_k^ε to Section 4.3, although some of the bounds below will be used in this computation.

We proceed by calculating the scalar products

$$\langle\mathfrak{u}_k^\varepsilon, \mathfrak{u}_j^\varepsilon\rangle_\varepsilon = (\nabla\mathfrak{u}_k^\varepsilon, \nabla\mathfrak{u}_j^\varepsilon)_\Omega + \varepsilon^{-2}(\mathfrak{u}_k^\varepsilon, \mathfrak{u}_j^\varepsilon)_{\Theta^\varepsilon} + \varepsilon^{m-2}(\mathfrak{u}_k^\varepsilon, \mathfrak{u}_j^\varepsilon)_\Omega, \tag{4.13}$$

see (4.1), for the functions $\mathfrak{u}_k^\varepsilon$ and $\mathfrak{u}_j^\varepsilon$ in (4.10) corresponding to the same eigenvalue μ_k of multiplicity $\varkappa_k$,

$$\mu_{k-1} < \mu_k = \cdots = \mu_{k+\varkappa_k-1} < \mu_{k+\varkappa_k}. \tag{4.14}$$

For $\varkappa_k = 1$, we have $j = k$ in (4.13) but in Section 4.4 we will also need to deal with the case $j \ne k$. In fact, our aim in the rest of the section is to show the relationships

$$\left|\langle\mathfrak{u}_k^\varepsilon, \mathfrak{u}_j^\varepsilon\rangle_\varepsilon - \varepsilon^{-1}(M + 1)b\delta_{j,k}\right| \le c_{j,k}\varepsilon^{-1/2}, \tag{4.15}$$

for certain constants $c_{j,k}$.

First, we observe that the last term in (4.13) meets the estimate

$$\begin{aligned}\varepsilon^{m-2}\left|(\mathfrak{u}_k^\varepsilon,\mathfrak{u}_j^\varepsilon)_\Omega\right| \leq & c\varepsilon^{m-2}\left(\|u_k^\varepsilon;L^2(\Omega)\|+\|u_k^\varepsilon;L^2(\Gamma)\|+O(\varepsilon)\right)\times \\ & \left(\|u_j^\varepsilon;L^2(\Omega)\|+\|u_j^\varepsilon;L^2(\Gamma)\|+O(\varepsilon)\right)\leq c_{jk}\varepsilon^{m-2}.\end{aligned} \tag{4.16}$$

Second, considering the first two terms on the right-hand side of (4.13), we use Proposition 3.4 and we proceed by evaluating scalar products for the part $\mathfrak{u}_k^{\varepsilon 0}=u_k+\varsigma_0 B^{-1}u_k^\Gamma\widehat{W}^\varepsilon$ of the expression (4.10) where u_k^Γ stands for the trace of u_k on Γ. Since

$$\begin{aligned}\varepsilon^{-2}(\mathfrak{u}_k^{\varepsilon 0},\mathfrak{u}_j^{\varepsilon 0})_{\Theta^\varepsilon}= & \varepsilon^{-2}B^{-2}(u_k^\Gamma\widehat{W}^\varepsilon,u_j^\Gamma\widehat{W}^\varepsilon)_{\Theta^\varepsilon}+\varepsilon^{-2}B^{-1}(u_k,u_j^\Gamma\widehat{W}^\varepsilon)_{\Theta^\varepsilon} \\ & +\varepsilon^{-2}B^{-1}(u_k^\Gamma\widehat{W}^\varepsilon,u_j)_{\Theta^\varepsilon}+\varepsilon^{-2}B^{-2}(u_k,u_j)_{\Theta^\varepsilon},\end{aligned}$$

for the first term on the right-hand side, we introduce the functions z and Z in Proposition 3.4 as follows: $z(x)=u_k(x)u_j(x)$ either for $x\in\Omega$ or $x\in\Gamma$, and $Z(\xi)=(W(\xi)-B)^2$ when $\xi\in\Theta$, $Z(\xi)=0$ outside. We proceed similarly for the rest of the terms, and we derive that

$$\begin{aligned}\varepsilon^{-2}(\mathfrak{u}_k^{\varepsilon 0},\mathfrak{u}_j^{\varepsilon 0})_{\Theta^\varepsilon} &= \varepsilon^{-1}\int\limits_\Gamma u_k(x_1,0)u_j(x_1,0)\,dx_1 \\ &\times\left(B^{-2}\int\limits_\theta (W(\xi))-B)^2\,d\xi+2B^{-1}\int\limits_\theta (W(\xi))-B)\,d\xi+\int\limits_\theta d\xi\right)+O(\varepsilon^{-1/2}) \\ &=\varepsilon^{-1}\int\limits_\Gamma u_k(x_1,0)u_j(x_1,0)\,dx_1\,B^{-2}\|W;L^2(\theta)\|^2+O(\varepsilon^{-1/2}) \\ &=\varepsilon^{-1}(u_k,u_j)_\Gamma\,B^{-2}M^{-1}+O(\varepsilon^{-1/2})\varepsilon^{-1}b\delta_{j,k}+O(\varepsilon^{-1/2}).\end{aligned} \tag{4.17}$$

Here, we have used formulas (3.19), (2.7) and (3.15). Taking into account the exponential decay of the difference $\widehat{W}=W-B$ and definition (3.3) of the cut-off function ς_0, and its derivative, we obtain

$$\begin{aligned}(\nabla\mathfrak{u}_k^{\varepsilon 0},\nabla\mathfrak{u}_j^{\varepsilon 0})_\Omega= & \varepsilon^{-2}B^{-2}(u_k^\Gamma\varsigma_0\nabla_\xi W^\varepsilon,u_j^\Gamma\varsigma_0\nabla_\xi W^\varepsilon)_\Omega+\varepsilon^{-1}B^{-1}(\nabla u_k,u_j^\Gamma\varsigma_0\nabla_\xi W^\varepsilon)_\Omega \\ & +\varepsilon^{-1}B^{-1}(u_k^\Gamma\varsigma_0\nabla_\xi W^\varepsilon,\nabla u_j)_\Omega+(\nabla u_k,\nabla u_j)_\Omega+O(1) \\ = & \varepsilon^{-1}B^{-2}(u_k,u_j)_\Gamma\|\nabla_\xi W;L^2(\varpi)\|^2+O(\varepsilon^{-1/2}) \\ = & \varepsilon^{-1}Mb\delta_{j,k}+O(\varepsilon^{-1/2}).\end{aligned} \tag{4.18}$$

Finally, in view of (4.16), to conclude with the sought-for relationship (4.15), it suffices to mention that (4.17) and (4.18) lead to the formulas

$$\|\nabla\mathfrak{u}_k^{\varepsilon 0};L^2(\Omega)\|\leq c_k\varepsilon^{-1/2}\quad\text{and}\quad\varepsilon^{-1}\|\mathfrak{u}_k^{\varepsilon 0};L^2(\Theta^\varepsilon)\|\leq c_k\varepsilon^{-1/2}$$

as can be easily checked, while a similar calculation shows that the rest $\mathfrak{u}_k^{\varepsilon\prime}=\mathfrak{u}_k^\varepsilon-\mathfrak{u}_k^{\varepsilon 0}=\varepsilon\zeta_0\zeta_\varepsilon' w'$ admits the estimates

$$\|\nabla\mathfrak{u}_k^{\varepsilon\prime};L^2(\Omega)\|\leq c_k\varepsilon^{1/2}\quad\text{and}\quad\varepsilon^{-1}\|\mathfrak{u}_k^{\varepsilon\prime};L^2(\Theta^\varepsilon)\|\leq c_k\varepsilon^{1/2}.$$

4.3 Estimating discrepancies

Here, we obtain an estimate for δ_k^ε given by (4.12), cf. (4.24).

Inequality (4.15), in particular, means that the first factor on the right-hand side of (4.12) is less than $c\varepsilon^{1/2}$ for a certain constant c. Let us evaluate the scalar products under the sign sup. The last one evidently meets the estimate

$$\varepsilon^{m-2}(M_k + \varepsilon\mu_k)\big|(\mathfrak{u}_k^\varepsilon, v)_\Omega\big| \le c_k\varepsilon^{m-2}\|v; L^2(\Omega)\|.$$

By definition (4.10), the entry of the first scalar product becomes

$$\begin{aligned}\Delta_x \mathfrak{u}_k^\varepsilon =&\Delta_x u_k + \varsigma_0 B^{-1}\Big(\frac{\partial^2 u_k^\Gamma}{\partial x_1^2}\widehat{W} + \frac{2}{\varepsilon}\frac{\partial u_k^\Gamma}{\partial x_1}\frac{\partial W}{\partial \xi_1} + \frac{1}{\varepsilon^2}u_k^\Gamma \Delta_\xi W\Big)\\ &+ \varepsilon\varsigma_0\Big(\frac{\partial^2 w_k'}{\partial x_1^2} + \frac{2}{\varepsilon}\frac{\partial^2 w_k'}{\partial x_1 \partial \xi_1} + \frac{1}{\varepsilon^2}\Delta_\xi w_k'\Big)\\ &+ [\Delta, \varsigma_0]\left(B^{-1}u_k^\Gamma w_k + \varepsilon\varsigma_\varepsilon' w_k'\right) - \varepsilon\varsigma_0[\Delta, 1-\varsigma_\varepsilon']w_k'.\end{aligned} \tag{4.19}$$

Above, $[\Delta, V]U$ denotes the commutator operator $[\Delta, V]U = \Delta(VU) - V\Delta U = 2\nabla V.\nabla U + U\Delta V$, and the derivatives involve partial derivatives with respect to x and ξ; namely, $\partial_{x_i} = \frac{\partial}{\partial x_i} + \varepsilon^{-1}\frac{\partial}{\partial \xi_i}$. Clearly, $\Delta_x u_k = 0$ by virtue of (1.18). According to definitions (3.3) and (4.11), the commutators $[\Delta, \varsigma_0]$ and $[\Delta, 1-\varsigma_\varepsilon'] = -[\Delta, \varsigma_\varepsilon']$ vanish outside the subdomains $\Omega(d) = \{x \in \Omega : d < x_2 < 2d\}$ and

$$\Omega_\varepsilon' = \Big\{x \in \Omega : |x_1| \in \Big(1 - \frac{2\varepsilon}{3}\Big(\frac{1}{2} - l\Big), 1 - \frac{\varepsilon}{3}\Big(\frac{1}{2} - l\Big)\Big)\Big\},$$

respectively. In the set $\Omega(d)$ the functions $\widehat{W}$ and w' are exponentially small so that

$$\Big|\Big([\Delta, \varsigma_0](B^{-1}u_k^\Gamma\widehat{W} + \varepsilon\varsigma_\varepsilon' w_k', v\Big)_\Omega\Big| \le ce^{-\tau/\varepsilon}, \quad \tau \in (0, 2\pi d)$$

(see (2.10) and (3.3)). The two thin vertical strips touching the sides $\Gamma_\pm$ and composing the set Ω_ε', do not contain the collision points and therefore w' is twice differentiable in Ω_ε'. Hence, on account of the bounds for the derivatives of ς_ε' and the Fourier expansion of w' similar to (2.8) with $B = 0$, we have

$$\begin{aligned}|\varepsilon(\varsigma_0[\Delta, \varsigma_\varepsilon']w_k', v)_\Omega| \le& c\varepsilon\frac{1}{\varepsilon^2}\int\limits_{\Omega_\varepsilon'} e^{-2\pi x_2/\varepsilon}|v(x)|\,dx\\ \le& c\frac{1}{\varepsilon}\Big(\int\limits_{\Omega_\varepsilon'}(|x_1| - 1)^2 e^{-4\pi x_2/\varepsilon}\,dx\Big)^{1/2}\Big(\int\limits_{\Omega_\varepsilon'}(|x_1| - 1)^{-2}|v(x)|^2\,dx\Big)^{1/2}\\ \le& c\varepsilon^{-1}\left(\varepsilon\varepsilon^3\right)^{1/2}\|(|x_1| - 1)^{-1}v; L^2(\Omega_0)\| \le c\varepsilon\|v; \mathfrak{H}^\varepsilon\| \le c\varepsilon.\end{aligned}$$

Here, we have used the normalization condition $\|v; \mathfrak{H}^\varepsilon\| = 1$ in (4.12) as well as the consequence

$$\|(|x_1| - 1)^{-1}v; L^2(\Omega_0)\| \le 4\|\nabla v; L^2(\Omega_0)\| \tag{4.20}$$

of the Hardy inequality (2.4) in the variable $t = 1 \mp x_1$ together with the Dirichlet condition $v = 0$ on $\Gamma_\pm$. Also the integral in Ω_ε' of the function $e^{-4\pi x_2/\varepsilon}$ has been computed, and Ω_0 denotes the set $\{x \in \Omega : 0 < x_2 < d\}$.

Some of the terms in (4.19) can be treated as follows:

$$\left|(\varsigma_0 B^{-1}\widehat{W}\partial^2 u_k^\Gamma/\partial x_1^2, v)_\Omega\right| \le c\varepsilon\|v;H^1(\Omega)\|,$$

$$\left|\varepsilon(\varsigma_0\partial^2 w'/\partial x_1^2, v)_\Omega\right| \le c\varepsilon\|v;H^1(\Omega)\|,$$

$$\left|(\varsigma_0\partial^2 w'/\partial x_1\partial\xi_1, v)_\Omega\right| \le c\varepsilon^{3/2}\|v;H^1(\Omega)\|.$$

These estimates are obtained by Proposition 3.4 as a consequence of the following: in the first two estimates, the subtrahend on the left-hand side of inequality (3.21) has been considered for the corresponding bound. In the third estimate, similarly to (3.13), the formula $\displaystyle\int_\varpi \frac{\partial w'}{\partial\xi_1}(\xi)\,dx = 0$ for the 1-periodic function has been taken into account. We emphasize that the function Z in Proposition 3.4 is a product of the test function v with the trace on Γ of $\partial^j u_k/\partial x_1^j$, with $j = 0,\dots,3$ (cf. Remark 3.1), which belongs to $H_0^1(\Omega;\Gamma_\sqcap)$ according to the differential properties of the eigenfunction u_k described in Section 3.4.

With the help of equations (3.5) and (3.7), (3.8), we rewrite the scalar products with the remaining terms in (4.19) in the form:

$$\begin{aligned}
\varepsilon^{-2}B^{-1}\left(\varsigma_0 u_k^\Gamma\Delta_\xi W, v\right)_\Omega &= -\,\varepsilon^{-2}MB^{-1}\left(u_k^\Gamma W, v\right)_{\Theta^\varepsilon},\\
\varepsilon^{-1}\Big(\varsigma_0\Delta_\xi w_k' + 2B^{-1}\varsigma_0\frac{\partial u_k^\Gamma}{\partial x_1}\frac{\partial W}{\partial\xi_1}, v\Big)_\Omega &= -\,\varepsilon^{-1}M(w_k', v)_{\Theta^\varepsilon} - \varepsilon^{-1}M\Big(\xi_2\Big(\frac{\partial u_k}{\partial x_2}\Big)^\Gamma, v\Big)_{\Theta^\varepsilon}\\
&\quad -\,\varepsilon^{-1}\mu_k B^{-1}\left(u_k^\Gamma W, v\right)_{\Theta^\varepsilon}.
\end{aligned} \tag{4.21}$$

These scalar products will be considered together with the other term in (4.12)

$$\begin{aligned}
&\varepsilon^{-2}(M+\varepsilon\mu_k)(\mathfrak{u}_k^\varepsilon, v)_{\Theta^\varepsilon} = \varepsilon^{-2}M\left(u_k + u_k^\Gamma B^{-1}(W-B), v\right)_{\Theta^\varepsilon}\\
&+\varepsilon^{-1}\left(\mu_k\left(u_k + u_k^\Gamma B^{-1}(W-B), v\right)_{\Theta^\varepsilon} + M(w_k', v)_{\Theta^\varepsilon}\right) + \mu_k(w_k', v)_{\Theta^\varepsilon}.
\end{aligned} \tag{4.22}$$

In (4.21) and (4.22), we have taken into account that, by definitions (3.3) and (4.11), both the cut-off functions ς_0 and ς_ε' equal 1 on the union Θ^ε of the concentrated masses.

The first term on the right-hand side of (4.22) becomes

$$\begin{aligned}
&\varepsilon^{-2}MB^{-1}\left(u_k^\Gamma W, v\right)_{\Theta^\varepsilon} + \varepsilon^{-2}M\left(u_k - u_k^\Gamma, v\right)_{\Theta^\varepsilon} = \varepsilon^{-2}MB^{-1}\left(u_k^\Gamma W, v\right)_{\Theta^\varepsilon}\\
&+\varepsilon^{-1}M\Big(\xi_1\Big(\frac{\partial u_k}{\partial x_2}\Big)^\Gamma, v\Big)_{\Theta^\varepsilon} + \varepsilon^{-2}M\Big(u_k - u_k^\Gamma - x_1\Big(\frac{\partial u_k}{\partial x_2}\Big)^\Gamma, v\Big)_{\Theta^\varepsilon}.
\end{aligned}$$

The first and second scalar products can be readily found in (4.21) so that all of them cancel each other under the last sign sup in (4.12). By definition (4.1) and the Taylor formula, the modulo of the third scalar product does not exceed the expression

$$c\varepsilon^{-2}\varepsilon^2\int_{\Theta^\varepsilon}|v(x)|\,dx \le c\varepsilon^{1/2}\|v;L^2(\Theta^\varepsilon)\| \le c\varepsilon^{3/2}. \tag{4.23}$$

Furthermore, the coefficient of ε^{-1} in (4.22) coincides with

$$\varepsilon^{-1}\mu_k B^{-1}\left(u_k^\Gamma W, v\right)_{\Theta^\varepsilon} + \varepsilon^{-1}\mu_k\left(u_k - u_k^\Gamma, v\right)_{\Theta^\varepsilon} + \varepsilon^{-1}M(w_k', v)_{\Theta^\varepsilon}$$

while the first and third scalar products have also appeared in (4.21) and thus are canceled in (4.12). Finally, we obtain the estimate

$$\varepsilon^{-1}\mu_k\left|\big(u_k-u_k^\Gamma,v\big)_{\Theta^\varepsilon}\right|+\mu_k\big|(w_k',v)_{\Theta^\varepsilon}\big|\le c\int\limits_{\Theta^\varepsilon}|v(x)\,dx\le c\varepsilon$$

which is quite similar to (4.23).

It remains to consider the scalar product in (4.12) with the derivative $\partial_{x_2}\mathfrak{u}_k^\varepsilon$, namely

$$\left|\Big(\frac{\partial\mathfrak{u}_k^\varepsilon}{\partial x_2},v\Big)_{\Gamma\setminus T^\varepsilon}\right|=\left|\Big(\frac{\partial u_k}{\partial x_2}+\varsigma_\varepsilon'\frac{\partial w_k'}{\partial\xi_2},v\Big)_{\Gamma\setminus T^\varepsilon}+\frac{1}{\varepsilon}B^{-1}\Big(u_k^\Gamma\frac{\partial W}{\partial\xi_2},v\Big)_{\Gamma\setminus T^\varepsilon}\right|$$

$$=\left|\Big((1-\varsigma_\varepsilon')\frac{\partial u_k}{\partial x_2},v\Big)_{\Gamma\setminus T^\varepsilon}\right|\le c\int\limits_{1-\varepsilon(1-2l)/3}^{1}(|v(x_1,0)|+|v(-x_1,0)|)\;dx_1\le c\varepsilon^{3/2}.$$

Here, we took into account that $\dfrac{\partial W}{\partial\xi_2}\Big(\dfrac{x_1}{\varepsilon},0\Big)v(x_1,0)=0$ on $\Gamma\setminus T^\varepsilon$ due to the Neumann boundary condition (1.14) for W as well as the second formula in (3.7) for w', and formulas (4.11) for ς_ε' and (4.20) for v.

Finally, note that all the constants c arising in the bounds throughout the section depend on k and they are bounded by $c_k\varepsilon$ for some constant c_k.

Thus, gathering our calculations through the section, we observe that all the terms of the expression under the last sign sup in (4.12) are either canceled out, or bounded by $c_k\varepsilon$. This together with formula (4.15), which gives the estimate from below of the norm $\|\mathfrak{u}_k^\varepsilon;\mathfrak{H}^\varepsilon\|$, while $\mathfrak{k}_k^\varepsilon$ from (4.9), lead to the inequality

$$\delta_k^\varepsilon\le c_k\varepsilon^{3/2}. \tag{4.24}$$

4.4 The intermediate result on asymptotics

Let us apply Lemma 4.1 with $\mathfrak{k}_k^\varepsilon$ and $\mathfrak{U}_k^\varepsilon$ in (4.9) and $\delta^\varepsilon=c_k^0\varepsilon^{3/2}$ with c_k^0 a positive constant. Lemma 4.1 provides us with an eigenvalue κ_p^ε of the operator $\mathfrak{K}^\varepsilon$ such that

$$\left|\kappa_p^\varepsilon-\mathfrak{k}_k^\varepsilon\right|\le c_k^0\varepsilon^{3/2}. \tag{4.25}$$

Now, using (4.4) and (4.9), we obtain

$$\left|\Lambda_p^\varepsilon-M-\varepsilon\mu_k\right|\le c_k^0\varepsilon^{3/2}(1+\Lambda_p^\varepsilon)(1+M+\varepsilon\mu_k). \tag{4.26}$$

Choosing $\varepsilon_k>0$ to fulfil $c_k^0\varepsilon^{3/2}(1+M+\varepsilon\mu_k)\le\dfrac{1}{2}$ for $\varepsilon\in(0,\varepsilon_k]$, from (4.26) we derive the chain of inequalities

$$\begin{gathered}\Lambda_p^\varepsilon\le M+\varepsilon\mu_k+c_k^0\varepsilon^{3/2}(1+\Lambda_p^\varepsilon)(1+M+\varepsilon\mu_k),\\ \Lambda_p^\varepsilon\le 2\left(M+\varepsilon\mu_k+c_k^0\varepsilon^{3/2}(1+M+\varepsilon\mu_k)\right),\\ \left|\Lambda_p^\varepsilon-M-\varepsilon\mu_k\right|\le c_k^1\varepsilon^{3/2}\text{ for }\varepsilon\in(0,\varepsilon_k],\end{gathered} \tag{4.27}$$

where c_k^1 is expressed through c_k^0, M and $\varepsilon_k\mu_k$. Note that above we have used that Λ_p^ε is bounded by a constant depending on k. Thus, we have found a rescaled eigenvalue (1.11) of problem (1.3)–(1.5) in the $c_k^1\varepsilon^{3/2}$-neighborhood of the point $M+\varepsilon\mu_k$.

Let us verify that in the case (4.14) a neighborhood of $M + \varepsilon\mu_k$ contains at least $\varkappa_k$ eigenvalues. We take

$$\delta_\bullet^\varepsilon = S\varepsilon^{3/2} \tag{4.28}$$

in Lemma 4.1 where $S > 0$ is a big number to be fixed later. Then, considering μ_k of multiplicity $\varkappa_k$, for the same $\mathfrak{k}_k^\varepsilon$ in (4.9) we define $\mathfrak{U}_j^\varepsilon = \|\mathfrak{u}_j^\varepsilon; \mathfrak{H}^\varepsilon\|^{-1}\mathfrak{u}_j^\varepsilon$ for each $j = k, \dots, k+\varkappa_k-1$, and obtain (4.12) for $\delta_k^\varepsilon = c_k^0\varepsilon^{3/2}$. Thus, using Lemma 4.1, we derive that for each $j = k, \dots, k+\varkappa_k-1$ there is the coefficient column $\mathfrak{a}_{(j)}^\varepsilon = \left(a_{jQ^\varepsilon}^\varepsilon, \dots, \mathfrak{a}_{jQ^\varepsilon+X^\varepsilon-1}^\varepsilon\right)$ which, according to (4.6), is normalized in $\mathbb{R}^{X^\varepsilon}$ and satisfies

$$\left\|\mathfrak{U}_j^\varepsilon - \sum_{q=Q^\varepsilon}^{Q^\varepsilon+X^\varepsilon-1} \mathfrak{a}_{jq}^\varepsilon \mathfrak{V}_q^\varepsilon; \mathfrak{H}^\varepsilon\right\| \le 2\frac{c_k^\bullet}{S}, \tag{4.29}$$

where $c_k^\bullet = \max\{c_k^1, \dots, c_{k+\varkappa_k-1}^1\}$. Moreover, according to (4.6) and (4.7) we have

$$\left|\left\langle \mathfrak{U}_j^\varepsilon, \mathfrak{U}_l^\varepsilon \right\rangle_\varepsilon - \left\langle \mathfrak{a}_{(j)}^\varepsilon, \mathfrak{a}_{(l)}^\varepsilon \right\rangle_{\mathbb{R}^{X^\varepsilon}}\right| = \left|\left\langle \mathfrak{U}_j^\varepsilon, \mathfrak{U}_l^\varepsilon \right\rangle_\varepsilon - \left\langle \sum \mathfrak{a}_{jq}^\varepsilon \mathfrak{V}_q^\varepsilon, \sum \mathfrak{a}_{lm}^\varepsilon \mathfrak{V}_m^\varepsilon \right\rangle_\varepsilon\right|$$

$$= \left|\left\langle \mathfrak{U}_j^\varepsilon, \mathfrak{U}_l^\varepsilon - \sum \mathfrak{a}_{lm}^\varepsilon \mathfrak{V}_m^\varepsilon \right\rangle_\varepsilon + \left\langle \mathfrak{U}_j^\varepsilon - \sum \mathfrak{a}_{jq}^\varepsilon \mathfrak{V}_q^\varepsilon, \sum \mathfrak{a}_{lm}^\varepsilon \mathfrak{V}_m^\varepsilon \right\rangle_\varepsilon\right| \le 4\frac{c_k^\bullet}{S}$$

with $j, l = k, \dots, k+\varkappa_k-1$ and the summation over $q, m = Q^\varepsilon, \dots, Q^\varepsilon + X^\varepsilon - 1$. For Q^ε and X^ε we use the same notation as in Lemma 4.1, X^ε being the total number of eigenvalues κ_p^ε of $\mathfrak{K}^\varepsilon$ in the interval $[\mathfrak{k}_k^\varepsilon - \delta_\bullet^\varepsilon, \mathfrak{k}_k^\varepsilon + \delta_\bullet^\varepsilon]$. On the other hand, formulas (4.9) and (4.15) show that, for sufficiently small ε,

$$\begin{aligned}\left|\left\langle \mathfrak{U}_j^\varepsilon, \mathfrak{U}_l^\varepsilon \right\rangle_\varepsilon - \delta_{j,l}\right| =& \|\mathfrak{u}_j^\varepsilon; \mathfrak{H}^\varepsilon\|^{-1}\|\mathfrak{u}_l^\varepsilon; \mathfrak{H}^\varepsilon\|^{-1}\left|\left\langle \mathfrak{u}_j^\varepsilon, \mathfrak{u}_l^\varepsilon \right\rangle_\varepsilon - \delta_{j,l}\|\mathfrak{u}_j^\varepsilon; \mathfrak{H}^\varepsilon\|\,\|\mathfrak{u}_l^\varepsilon; \mathfrak{H}^\varepsilon\|\right| \\ \le& c_k^\natural \varepsilon^{-1/2}\left(\varepsilon^{-1}Mb\right)^{-1} = C_k^\natural \varepsilon^{1/2}.\end{aligned}$$

Thus, we obtain the inequality

$$\left|\left\langle \mathfrak{a}_{(j)}^\varepsilon, \mathfrak{a}_{(l)}^\varepsilon \right\rangle_{\mathbb{R}^{X^\varepsilon}} - \delta_{j,l}\right| \le C_k^\natural \varepsilon^{1/2} + 4c_k^\bullet S^{-1}. \tag{4.30}$$

In other words, the columns $a_{(k)}^\varepsilon, \dots, a_{(k+\varkappa_k-1)}^\varepsilon$ are normalized and "almost orthogonal" in $\mathbb{R}^{X^\varepsilon}$ for a small ε and a big S. This may happen for a sufficiently small ε and a sufficiently large S, only under the restriction $X^\varepsilon \ge \varkappa_k$, as can be shown by contradiction, and, again by Lemma 4.1, we detect at least $\varkappa_k$ eigenvalues of the operator $\mathfrak{K}^\varepsilon$ in the $S\varepsilon^{3/2}$-neighborhood of the point $\mathfrak{k}_k^\varepsilon$ in (4.9). Since we can show that the Λ_p^ε are bounded by a constant depending on k only, the replacement $c_k^0 \mapsto S$ in (4.25) does not affect our conclusion (4.27) and we formulate the result that we have obtained.

Theorem 4.2. *Let μ_k be an eigenvalue of the limit Steklov problem* (1.18)–(1.20) *with multiplicity $\varkappa_k$, cf.* (4.14). *Then there exist positive ε_k, c_k and a rescaled eigenvalue $\Lambda_{p(k)}^\varepsilon$, cf.* (1.11), *of the original problem* (1.3)–(1.5) *such that the estimate*

$$\left|\Lambda_j^\varepsilon - M - \varepsilon\mu_k\right| \le c_k\varepsilon^{3/2} \text{ for } \varepsilon \in (0, \varepsilon_k] \tag{4.31}$$

is valid with $j = p(k), \dots, p(k)+\varkappa_k-1$.

Finally, let us note that the equality $p(k) = k$ will be proved in Section 6.1.

5 The convergence theorem

In this section, we reformulate the original eigenvalue problem (1.8) in terms of new spectral parameters and eigenfunctions (1.23). The new spectral problem reads (5.2) and we show that it has a discrete spectrum (cf. Sections 5.1 and 5.2). Its rescaled eigenvalues $\{\varepsilon^{-1}\mu_k^\varepsilon\}_{k=1}^\infty$ converge towards the eigenvalues $\{\mu_k\}_{k=1}^\infty$ of (3.17) with conservation of multiplicity; also a certain convergence of the corresponding eigenfunctions holds. In order to obtain this convergence, we formulate problem (5.2) in weighted Sobolev spaces with the singular weight W^ε which is obtained from the principal eigenmode of the cell problem (2.2) (cf. Sections 5.2 and 5.3). The main results of the section are stated in Proposition 5.5 and Theorem 5.6. The above-mentioned convergence, with conservation of the multiplicity, is derived at the beginning of Section 6.1.

5.1 Reformulation of the ε-dependent eigenvalue problem

Considering the first eigenvalue M and the corresponding eigenfunction $W = W_1$ of problem (1.13)–(1.15), we recall the weight multiplier $W^\varepsilon(x) = W(\varepsilon^{-1}x)$ in (2.29) is positive in $\mathbb{R}^2_+$ and ε-periodic in x_1 (cf. Section 2). In this way, function (2.29) is properly defined in the domain Ω for any eigenfunction $u^\varepsilon = u_k^\varepsilon$ of the original problem (1.3)–(1.5). Considering the inclusion (2.31), we reformulate this spectral problem to get a new eigenvalue problem for the fractional function $\mathbf{u}^\varepsilon = u^\varepsilon / W^\varepsilon$ in (2.29).

Recalling that $-\Delta W^\varepsilon = \varepsilon^{-2} M \chi^\varepsilon W^\varepsilon$, we have

$$\begin{aligned}\Delta u^\varepsilon = \Delta(W^\varepsilon \mathbf{u}^\varepsilon) &= W^\varepsilon \Delta \mathbf{u}^\varepsilon + 2\nabla W^\varepsilon \cdot \nabla \mathbf{u}^\varepsilon + \mathbf{u}^\varepsilon \Delta W^\varepsilon \\ &= W^\varepsilon \Delta \mathbf{u}^\varepsilon + 2\nabla W^\varepsilon \cdot \nabla \mathbf{u}^\varepsilon - \mathbf{u}^\varepsilon \varepsilon^{-2} M \chi^\varepsilon W^\varepsilon.\end{aligned}$$

Hence, function (2.29) verifies the differential equation

$$-W^\varepsilon \Delta \mathbf{u}^\varepsilon - 2\nabla W^\varepsilon \cdot \nabla \mathbf{u}^\varepsilon - \varepsilon^{m-2} M W^\varepsilon \mathbf{u}^\varepsilon = \varepsilon^{m-2} \mu^\varepsilon (1 + \varepsilon^{-m} \chi^\varepsilon) W^\varepsilon \mathbf{u}^\varepsilon \quad \text{in} \quad \Omega \tag{5.1}$$

with the new spectral parameter

$$\mu^\varepsilon = \varepsilon^{2-m} \lambda^\varepsilon - M = \Lambda^\varepsilon - M.$$

Multiplying (5.1) by the test function $W^\varepsilon \mathbf{v}^\varepsilon$, $\mathbf{v}^\varepsilon$ being any smooth function in $\overline{\Omega}$ vanishing on $\Gamma_\sqcap$, we take into account the relation

$$2(\nabla W^\varepsilon \cdot \nabla \mathbf{u}^\varepsilon, W^\varepsilon \mathbf{v}^\varepsilon)_\Omega = (\nabla (W^\varepsilon)^2 \cdot \nabla \mathbf{u}^\varepsilon, \mathbf{v}^\varepsilon)_\Omega$$

and, integrating by parts, derive the integral identity

$$\begin{aligned}&(W^\varepsilon \nabla \mathbf{u}^\varepsilon, W^\varepsilon \nabla \mathbf{v}^\varepsilon)_\Omega - \varepsilon^{m-2} M (W^\varepsilon \mathbf{u}^\varepsilon, W^\varepsilon \mathbf{v}^\varepsilon)_\Omega \\ &= \varepsilon^{m-2} \mu^\varepsilon (W^\varepsilon \mathbf{u}^\varepsilon, W^\varepsilon \mathbf{v}^\varepsilon)_\Omega + \varepsilon^{-2} \mu^\varepsilon (W^\varepsilon \mathbf{u}^\varepsilon, W^\varepsilon \mathbf{v}^\varepsilon)_{\Theta^\varepsilon}.\end{aligned} \tag{5.2}$$

Let us note that, according to (2.29) and (1.4), (1.5), the function $\mathbf{u}^\varepsilon$ satisfies the boundary conditions

$$\mathbf{u}^\varepsilon = 0 \text{ on } \Gamma_\sqcap, \tag{5.3}$$

$$\partial_\nu (W^\varepsilon \mathbf{u}^\varepsilon) = 0 \text{ on } \Gamma \setminus \overline{T^\varepsilon}. \tag{5.4}$$

This makes null the line integral over $\partial\Omega \setminus T^\varepsilon$ while the integral over T^ε vanishes because $W = 0$ on τ and, therefore, $W^\varepsilon = 0$ on T^ε. In other words, the differential equation (5.1) equipped with the boundary conditions (5.3) and (5.4), does not need any boundary condition on the set T^ε where the weight multiplier W^ε becomes null. This peculiarity of differential equations with degenerating coefficients is a subject which has been investigated in the literature, cf. [41] and [23].

The eigenvalue problem (5.2) must be posed in the space $\mathbf{H}^\varepsilon$ obtained by completing the space $C_0^\infty(\overline{\Omega};\Gamma_\sqcap)$ with the weighted norm

$$\|\mathbf{v}^\varepsilon;\mathbf{H}^\varepsilon\| = \|W^\varepsilon \nabla \mathbf{v}^\varepsilon; L^2(\Omega)\|.$$

$C_0^\infty(\overline{\Omega};\Gamma_\sqcap)$ denotes the linear space of C^∞ functions vanishing in a neighborhood of $\Gamma_\sqcap$. We emphasize that, in view of Corollary 2.3, Proposition 2.5 ensures that for the eigenfunctions u^ε of the original problem (1.8), the fractional function $\mathbf{u}^\varepsilon = u^\varepsilon/W^\varepsilon$ satisfies $\mathbf{u}^\varepsilon \in C(\overline{\Omega})$ and $\mathbf{u}^\varepsilon \in \mathbf{H}^\varepsilon$.

As a consequence, the new spectral problem reads: find μ^ε, $\mathbf{u}^\varepsilon \in \mathbf{H}^\varepsilon$, $\mathbf{u}^\varepsilon \neq 0$, satisfying (5.2) for any $\mathbf{v}^\varepsilon \in \mathbf{H}^\varepsilon$. In addition, we have the following result.

Proposition 5.1. *The spectral problem* (5.2) *in* $\mathbf{H}^\varepsilon$ *has the monotone unbounded positive sequence of eigenvalues*

$$0 < \mu_1^\varepsilon < \mu_2^\varepsilon \le \mu_3^\varepsilon \le \cdots \le \mu_k^\varepsilon \le \cdots \to +\infty \tag{5.5}$$

and the corresponding eigenfunctions $\mathbf{u}_k^\varepsilon \in \mathbf{H}^\varepsilon$ *can be subject to the orthogonality and normalization conditions*

$$(W^\varepsilon \nabla \mathbf{u}_k^\varepsilon, W^\varepsilon \nabla \mathbf{u}_j^\varepsilon)_\Omega - \varepsilon^{m-2} M (W^\varepsilon \mathbf{u}_k^\varepsilon, W^\varepsilon \mathbf{u}_j^\varepsilon)_\Omega = \delta_{k,j}, \quad k, j \in \mathbb{N}. \tag{5.6}$$

Proof. To prove the conclusion of the statement, we need the inequality

$$\|\mathbf{u}^\varepsilon; L^2(\Omega)\|^2 \le c_\Omega \|W^\varepsilon \nabla \mathbf{u}^\varepsilon; L^2(\Omega)\|^2, \quad \mathbf{u}^\varepsilon \in \mathbf{H}^\varepsilon, \tag{5.7}$$

which follows from the classical one-dimensional Hardy inequality, namely, from (5.10).

Indeed, to derive (5.7) from (5.10), we extend u^ε as null over the half-strip $(-1,1)\times\mathbb{R}_+$, set $\mathbf{U}(y) = \mathbf{u}^\varepsilon(x_1, y)$, integrate in $x_1 \in (-1,1)$ and observe that, in view of Corollary 2.3,

$$W^\varepsilon(x) \ge C \min\{1, \varepsilon^{-1} x_2\} \ge C_\Omega x_2, \quad C_\Omega > 0.$$

Notice that the constant C_Ω and, therefore, c_Ω in (5.7) are independent of $\varepsilon \in (0, \varepsilon_0]$.

In addition, introducing $\mathbf{L}^\varepsilon$ the weighted Lebesgue space with the norm

$$\|\mathbf{u}^\varepsilon; \mathbf{L}^\varepsilon\| = \|W^\varepsilon \mathbf{u}^\varepsilon; L^2(\Omega)\|, \tag{5.8}$$

we show that the embedding $\mathbf{H}^\varepsilon \subset \mathbf{L}^\varepsilon$ is compact. To do this, we consider any sufficiently small $\delta > 0$ and we represent the embedding operator $\mathbf{I}^\varepsilon$ as the sum $\mathbf{I}^\varepsilon(\delta) + \mathbf{I}_\delta^\varepsilon$ of the restriction operators onto the sets $\Omega \setminus \mathbf{T}_\delta^\varepsilon$ and $\mathbf{T}_\delta^\varepsilon$, respectively. Here, $\mathbf{T}_\delta^\varepsilon$ is the union of the small rectangles

$$\mathbf{t}_{n\delta}^\varepsilon = \{x : |x_1 - \varepsilon n| \le \varepsilon(l+\delta), x_2 \in (0, \varepsilon(h_0+\delta))\}, \quad n \in \mathbb{Z}(N), \tag{5.9}$$

where $W^\varepsilon(x) \le C\delta^{1/2}$ according to (2.26) and (2.25). Hence, the operator $\mathbf{I}_\delta^\varepsilon$ has a small norm $O(\delta^{1/2})$ due to the weight multiplier W^ε in (5.8) while $\mathbf{I}^\varepsilon(\delta)$ stays compact because $W^\varepsilon(x) \ge c(\delta,\varepsilon) > 0$ on $\Omega \setminus \mathbf{T}_\delta^\varepsilon$ and therefore $\mathbf{I}^\varepsilon(\delta)\mathbf{H}^\varepsilon = H^1(\Omega \setminus \mathbf{T}_\delta^\varepsilon; \Gamma_\sqcap)$. As a matter of fact, since the function $W^\varepsilon(x)$ is bounded, it suffices to consider that the embedding $\mathbf{H}^\varepsilon \subset H^1(\Omega \setminus \mathbf{T}_\delta^\varepsilon; \Gamma_\sqcap)$ holds continuously. This amounts to say that $\mathbf{I}^\varepsilon$ can be approximated by compact operators in the operator norm and thus is compact too, and this shows the compactness of the embedding $\mathbf{H}^\varepsilon \subset \mathbf{L}^\varepsilon$.

Furthermore, inequality (5.7) ensures that

$$\varepsilon^{m-2} M \|W^\varepsilon \mathbf{u}^\varepsilon; L^2(\Omega)\|^2 \leq c\varepsilon^{m-2} \|\mathbf{u}^\varepsilon; L^2(\Omega)\|^2 \leq C\varepsilon^{m-2} \|\mathbf{u}^\varepsilon; \mathbf{H}^\varepsilon\|^2$$

and, owing to assumption (1.7), on the left hand side of (5.2), the second term is a small perturbation of the first one. Then, we note that the right-hand side of (5.2) can be written as

$$\varepsilon^{m-2} \mu^\varepsilon (W^\varepsilon \mathbf{u}^\varepsilon (1 + \varepsilon^m \chi^\varepsilon), W^\varepsilon \mathbf{v}^\varepsilon)_\Omega,$$

for χ^ε in (1.6), and the scalar product here defines a norm equivalent to that of $\mathbf{L}^\varepsilon$, cf.(5.8). Consequently, problem (5.2) is a standard eigenvalue problem in the framework of sesquilinear, continuous and coercive forms on the couple of Hilbert spaces $\mathbf{H}^\varepsilon \subset \mathbf{L}^\varepsilon$ and the result of the proposition holds. $\square$

In the next section, we establish the equivalence between both spectral problems, (1.8) and (5.2). In order to do it, we show Proposition 5.3 below which provides properties for eigenfunctions of (5.2) complementing those in Proposition 2.5 for (1.8). Its proof uses the estimate in the next lemma, which readily follows from the classical Hardy inequality

$$\int_0^\infty |\mathbf{U}(y)|^2 dy \leq 4 \int_0^\infty y^2 \left| \frac{d\mathbf{U}}{dy}(y) \right|^2 dy, \quad \mathbf{U} \in C_c^\infty[0, +\infty). \tag{5.10}$$

Lemma 5.2. *For fixed T_1 and T_2, $0 < T_1 < T_2$, and for any $U \in C^\infty[0, +\infty)$, the inequality*

$$\int_0^{T_1} |U(\tau)|^2 d\tau \leq C \left(\int_0^{T_2} \tau^2 \left| \frac{dU}{d\tau}(\tau) \right|^2 d\tau + \int_{T_1}^{T_2} |U(\tau)|^2 d\tau \right) \tag{5.11}$$

holds, with C a constant depending on T_1 and T_2 but independent of U.

Proof. Introducing the cut-off function

$$\mathcal{X}_T \in C^\infty(\mathbb{R}),\ 0 \leq \mathcal{X}_T \leq 1,\ \mathcal{X}(\tau) = 1 \text{ for } \tau \leq T_1,\ \mathcal{X}_T(\tau) = 0 \text{ for } t \geq T_2,$$

we take $U(\tau) = \mathbf{U}(\tau)\mathcal{X}_T(\tau)$ and apply (5.10) to obtain (5.11). $\square$

Proposition 5.3. *Let $\mathbf{u}^\varepsilon = \mathbf{u}_k^\varepsilon$ be any eigenfunction of (5.2) in $\mathbf{H}^\varepsilon$. Then, $W^\varepsilon \mathbf{u}^\varepsilon$ belongs to $H^1(\Omega)$.*

Proof. On account of the boundedness of W^ε it suffices to show that $\mathbf{u}^\varepsilon \nabla W^\varepsilon \in L^2(\Omega)$, which holds due to the inequality

$$\|\mathbf{u}^\varepsilon \nabla W^\varepsilon; L^2(\Omega)\|^2 \leq c_\varepsilon \|W^\varepsilon \nabla \mathbf{u}^\varepsilon; L^2(\Omega)\|^2 \tag{5.12}$$

with some ε-dependent constant c_ε. To prove (5.12), we decompose the domain Ω into subdomains which (after changing to variables (2.1)) are contained in regions of the plane where bounds in Corollary 2.3 hold.

Let Ω^ε denote $\Omega \cap \{x_2 > \varepsilon\}$. Let $0 < \delta_1 < \min\{l, \frac{1}{2} - l\}$. For each $n \in \mathbb{Z}(N)$, we consider the intervals

$$N_n^\varepsilon = \{x_1 \,:\, |x_1 - \varepsilon(n + \frac{1}{2})| < (\frac{1}{2} - l - \delta_1)\varepsilon\} \quad \text{and} \quad D_n^\varepsilon = \{x_1 \,:\, |x_1 - \varepsilon n| < (l - \delta_1)\varepsilon\}$$

which overlap with the regions of Γ where the Neumann and Dirichlet conditions for the eigenfunction of (1.3)-(1.5) are imposed, repectively. Also, considering polar coordinates in a neighborhood of the collision point $P=\varepsilon(n\pm l)$ (cf. Lemma 2.1 and figure 2), we denote by $\mathcal{E}_n^\varepsilon$ the half-disk

$$\mathcal{E}_n^\varepsilon=\{(\rho_\pm,\varphi_\pm)\,:\,\rho_\pm\in(0,\varepsilon r_1),\,\varphi_\pm\in(0,\pi)\}.$$

Above, we take δ_1 and r_1 in such a way that $\Omega=\widetilde{\Omega}^\varepsilon\cup\bigcup\limits_{n\in\mathbb{Z}(N)}\mathcal{E}_n^\varepsilon\cup(D_n^\varepsilon\times(0,\epsilon])$, where $\widetilde{\Omega}^\varepsilon$ denotes the subdomain $\widetilde{\Omega}^\varepsilon=\Omega^\varepsilon\cup\bigcup\limits_{n\in\mathbb{Z}(N)}N_n^\varepsilon\times(0,\varepsilon]$.

Note that $\widetilde{\Omega}^\varepsilon$ denotes a teeth domain which is nothing but Ω minus small rectangles of height ε and width $2(l+\delta_1)\varepsilon$ containing the collision points (cf. similar domains in (5.9) and figure 3). Let us denote by $\mathbf{t}_n^\varepsilon$ the larger, but still small, rectangles

$$\mathbf{t}_n^\varepsilon=\{x\,:\,|x_1-\varepsilon n|\leq\varepsilon(l+\delta_1),x_2\in(0,2\varepsilon)\},\quad n\in\mathbb{Z}(N).$$

In $\widetilde{\Omega}^\varepsilon$, we take into account that $\mathbf{u}^\varepsilon$ vanishes on Γ_0 and apply the Friedrichs inequality to obtain

$$\|\mathbf{u}^\varepsilon;L^2(\widetilde{\Omega}^\varepsilon)\|^2\leq c\|\nabla\mathbf{u}^\varepsilon;L^2(\widetilde{\Omega}^\varepsilon)\|^2.$$

As a consequence of properties (2.23) and (2.24), we get (5.12) in $\widetilde{\Omega}^\varepsilon$, namely,

$$\|\mathbf{u}^\varepsilon\nabla W^\varepsilon;L^2(\widetilde{\Omega}^\varepsilon)\|^2\leq c_\varepsilon\|W^\varepsilon\nabla\mathbf{u}^\varepsilon;L^2(\widetilde{\Omega}^\varepsilon)\|^2 \tag{5.13}$$

Let us proceed obtaining the desired estimates in the small rectangles $\mathbf{d}_n^\varepsilon\subset\mathbf{t}_n^\varepsilon$, $\mathbf{d}_n^\varepsilon:=D_n^\varepsilon\times(0,\varepsilon)$, as follows. Let $x_1\in D_n^\varepsilon$, we apply (5.11) taking $U=\mathbf{u}^\varepsilon$, $\tau=x_2$, $T_1=\varepsilon$ and $T_2=2\varepsilon$, and we deduce

$$\int_0^\varepsilon|\mathbf{u}^\varepsilon(x_1,x_2)|^2\,dx_2\leq c_\varepsilon\left(\int_0^{2\varepsilon}x_2^2\left|\frac{\partial\mathbf{u}^\varepsilon}{\partial x_2}(x_1,x_2)\right|^2\,dx_2+\int_\varepsilon^{2\varepsilon}|\mathbf{u}^\varepsilon(x_1,x_2)|^2dx_2\right).$$

Then, we take the integral over D_n^ε, consider the sum for $n\in\mathbb{Z}(N)$ and, since u^ε vanishes at $x_1=\pm L$, apply the Friedrichs inequality in the rectangle $(-L,L)\times(\varepsilon,2\varepsilon)$. As a consequence of the boundedness of $|\nabla W^\varepsilon|$ and properties (2.23) and (2.25), we can write

$$\|\mathbf{u}^\varepsilon\nabla W^\varepsilon;L^2(\bigcup_{n\in\mathbb{Z}(N)}\mathbf{d}_n^\varepsilon)\|^2\leq c_\varepsilon\Big(\|W^\varepsilon\nabla\mathbf{u}^\varepsilon;L^2(\bigcup_{n\in\mathbb{Z}(N)}\mathbf{d}_n^\varepsilon)\|^2+\|W^\varepsilon\nabla\mathbf{u}^\varepsilon;L^2(\widetilde{\Omega}^\varepsilon)\|^2\Big). \tag{5.14}$$

In order to obtain estimates near the collision points (namely, in subdomains of $\mathbf{t}_n^\varepsilon$ out of $\mathbf{d}_n^\varepsilon$ and $\widetilde{\Omega}^\varepsilon$), we consider the half-disk $\mathcal{E}_n^\varepsilon$ centered at the collision point $P=(n+l)\varepsilon$, where the boundary conditions change from Dirichlet to Neumann, obtaining suitable bounds, and we proceed in the same way with the other collision point $(n-l)\varepsilon$ in each half-strip of width ε.

Let us fix n. If no confusion arises we skip indexes ε and n, and denote by (r,φ) the polar coordinates. We write the half-disk $\mathcal{E}_n^\varepsilon$ as the union of three sectors which are contained in S_D, S and S_N, where

$$S_D=\{x\,:\,r\in(0,\varepsilon r_1),\,\varphi\in(\frac{\pi}{2},\pi)\},\quad S_N=\{x\,:\,r\in(0,\varepsilon r_1),\,\varphi\in(0,\frac{\pi}{2}-\varphi_0)\}$$

and

$$S=\{x\,:\,r\in(0,\varepsilon r_2),\,|\varphi-\frac{\pi}{2}|<\varphi_0\},$$

and the constants $\varphi_0 \in (0, \frac{\pi}{4})$ and $0 < r_1 < r_2$ are chosen to ensure that $\overline{S} \cap \overline{\mathbf{d}}_n^\varepsilon \neq \emptyset$ and $S \cap \{r \in (\varepsilon r_1, \varepsilon r_2)\}$ is contained in $\gamma_n^\varepsilon \times (\varepsilon, 2\varepsilon)$ (cf. Section 1.1 for the definition of γ_n^ε). These choices restrict those for δ_0 and r_1 performed above and can be done as follows: let us fix $\varphi_0 < \pi/4$ and choose δ_1 such that $\overline{S} \cap \overline{\mathbf{d}}_n^\varepsilon$ is only the point of polar coordinates $(\varepsilon r_1, \varphi_0)$, then take r_2 such that $S \cap \{r \in (\varepsilon r_1, \varepsilon r_2)\} \subset \gamma_n^\varepsilon \times (\varepsilon, 2\varepsilon)$. Obviously, by symmetry $\overline{S} \cap (\overline{N}_n^\varepsilon \times [0, \varepsilon])$ is the point with the coordinates $(\varepsilon r_1, -\varphi_0)$.

For the half-disk $\mathcal{E}_n^\varepsilon$, we recall (cf. Lemma 2.1) that $|\nabla W^\varepsilon| \leq \widetilde{C}_\varepsilon r^{-1/2}$, that $W^\varepsilon(x) \geq \widetilde{c}_\varepsilon r^{1/2}$ with a constant $\widetilde{c}_\varepsilon > 0$ in S because $\cos(\varphi/2) \geq C > 0$ for $\varphi \in (\frac{\pi}{2} - \varphi_0, \frac{\pi}{2} + \varphi_0)$ and $\varphi_0 \in (0, \frac{\pi}{4})$. Similarly, $W^\varepsilon(x) \geq \widetilde{c}_\varepsilon r^{1/2}$ in S_D, while $W^\varepsilon(x) \geq \widetilde{c}_\varepsilon r^{1/2}(\pi - \varphi)$ in S_N (cf. also (2.26)) .

Let us start getting bounds for the integrals of $|\mathbf{u}^\varepsilon \nabla W^\varepsilon|^2$ over the symmetric sector S. Taking $\tau = r$, $T_1 = \varepsilon r_1$ and $T_1 = \varepsilon r_1$ in (5.11), we can write

$$\int_{S \cap \mathcal{E}_n^\varepsilon} |\mathbf{u}^\varepsilon \nabla W^\varepsilon|^2 dx \leq c_\varepsilon \int_{\pi/2-\varphi_0}^{\pi/2+\varphi_0} \int_0^{\varepsilon r_1} r^{-1} |\mathbf{u}^\varepsilon|^2 r \, dr d\varphi$$

$$\leq c_\varepsilon \left(\int_{\pi/2-\varphi_0}^{\pi/2+\varphi_0} \int_0^{\varepsilon r_2} r \left| \frac{\partial \mathbf{u}^\varepsilon}{\partial r} \right|^2 r \, dr d\varphi + \int_{\pi/2-\varphi_0}^{\pi/2+\varphi_0} \int_{\varepsilon r_1}^{\varepsilon r_2} r^{-1} |\mathbf{u}^\varepsilon|^2 r \, dr d\varphi \right),$$

and consequently,

$$\int_{S \cap \mathcal{E}_n^\varepsilon} |\mathbf{u}^\varepsilon \nabla W^\varepsilon|^2 dx \leq c_\varepsilon \left(\int_S |W^\varepsilon \nabla \mathbf{u}^\varepsilon|^2 dx + \int_{\gamma_n^\varepsilon \times (\varepsilon, 2\varepsilon)} |\mathbf{u}^\varepsilon(x)|^2 dx \right),$$

where we note that when considering sums for $n \in \mathbb{Z}(N)$, the second integral on the right-hand side can be estimated in terms of (5.12), as has been done in (5.14).

As regards the sector S_D, we apply (5.11) with $\tau = \pi - \varphi$, $T_1 = \frac{\pi}{2} - \varphi_0$, $T_2 = \frac{\pi}{2}$. We have

$$\int_0^{\varepsilon r_1} \int_{\pi/2+\varphi_0}^{\pi} r^{-1} |\mathbf{u}^\varepsilon|^2 r \, d\varphi dr$$

$$\leq c_\varepsilon \left(\int_0^{\varepsilon r_1} \int_{\pi/2}^{\pi} r^{-1} (\pi - \varphi)^2 \left| \frac{\partial \mathbf{u}^\varepsilon}{\partial \varphi} \right|^2 r \, d\varphi dr + \int_0^{\varepsilon r_1} \int_{\pi/2}^{\pi/2+\varphi_0} r^{-1} |\mathbf{u}^\varepsilon|^2 r \, d\varphi dr \right).$$

The second integral on the right-hand side has been estimated in the previous step (cf. estimates in the sector S), while the first integral can be rewritten and bounded as follows

$$\int_0^{\varepsilon r_1} \int_{\pi/2}^{\pi} r (\pi - \varphi)^2 \left| \frac{1}{r} \frac{\partial \mathbf{u}^\varepsilon}{\partial \varphi} \right|^2 r \, d\varphi dr \leq c_\varepsilon \int_{S_D} |W^\varepsilon \nabla \mathbf{u}^\varepsilon|^2 dx.$$

Finally, in the sector S_N, we apply the inequality

$$\int_0^{\pi/2-\varphi_0} |\mathbf{u}^\varepsilon|^2 \, d\varphi \leq c_\varepsilon \left(\int_0^{\pi/2} \left| \frac{\partial \mathbf{u}^\varepsilon}{\partial \varphi} \right|^2 d\varphi + \int_{\pi/2-\varphi_0}^{\pi/2} |\mathbf{u}^\varepsilon|^2 d\varphi \right)$$

and take integrals over $r \in (0, r_1 \varepsilon)$. We have

$$\int_0^{\varepsilon r_1} \int_0^{\pi/2-\varphi_0} r^{-1} |\mathbf{u}^\varepsilon|^2 r \, dr d\varphi$$

$$\leq c_\varepsilon \left(\int_0^{\varepsilon r_1} \int_0^{\pi/2} r \left| \frac{1}{r} \frac{\partial \mathbf{u}^\varepsilon}{\partial \varphi} \right|^2 r \, dr d\varphi + \int_0^{\varepsilon r_1} \int_{\pi/2-\varphi_0}^{\pi/2} r^{-1} |\mathbf{u}^\varepsilon|^2 r dr d\varphi \right).$$

The second integral on the right-hand side has been estimated above (cf. estimates in the sector S), while the first integral is bounded by $c_\varepsilon\|W^\varepsilon\nabla\mathbf{u}^\varepsilon; L^2(\mathcal{E}_n^\varepsilon)\|^2$.

Gathering the bounds on the three sub-sectors S, S_N and S_D, we can write

$$\|\mathbf{u}^\varepsilon\nabla W^\varepsilon; L^2(\bigcup_{n\in\mathbb{Z}(N)}\mathcal{E}_n^\varepsilon)\|^2 \le c_\varepsilon\Big(\|W^\varepsilon\nabla\mathbf{u}^\varepsilon; L^2(\bigcup_{n\in\mathbb{Z}(N)}\mathbf{t}_n^\varepsilon)\|^2 + \|W^\varepsilon\nabla\mathbf{u}^\varepsilon; L^2(\Omega)\|^2\Big). \tag{5.15}$$

From (5.13)–(5.15) we derive (5.12), and the proposition is proved. □

5.2 The equivalence of the ε-dependent problems and the crucial estimate for eigenfunctions

Clearly, relations (5.2), (5.6), Proposition 5.3, and properties (2.23)–(2.26) of the weight function W^ε show that any eigenpair $\{\mu_k^\varepsilon, \mathbf{u}_k^\varepsilon\} \in \mathbb{R}_+ \times \mathbf{H}^\varepsilon$ of problem (5.2) gives rise to an eigenpair

$$\{\lambda_K^\varepsilon, \widehat{u}_K^\varepsilon\} = \{\varepsilon^{m-2}(M+\mu_k^\varepsilon), W^\varepsilon\mathbf{u}_k^\varepsilon\} \in \mathbb{R}_+ \times H_0^1(\Omega;\Gamma_\sqcap^\varepsilon) \tag{5.16}$$

of problem (1.8), together with the estimate for the eigenfunction

$$\|\widehat{u}_K^\varepsilon; L^2(\Omega)\| = \|W^\varepsilon\mathbf{u}_k^\varepsilon; L^2(\Omega)\| \le c\|\mathbf{u}_k^\varepsilon; L^2(\Omega)\| \le C\|\mathbf{u}_k^\varepsilon; \mathbf{H}^\varepsilon\| \tag{5.17}$$

(cf. (5.7)) where C does not depend on $\varepsilon \in (0, \varepsilon_0]$, and $k \in \mathbb{N}$.

The converse follows from the two following observations. First, (2.31) confirms that function (2.29) lives in $H^1(\Omega;\Gamma_\sqcap)$, therefore, in $\mathbf{H}^\varepsilon$. Second, taking $W^\varepsilon\mathbf{v}^\varepsilon$ as a test function in the integral identity (1.8), the latter turns into (5.2) by a simple algebraic transformation which involves applying the Green formula to ΔW^ε multiplied by $W^\varepsilon\mathbf{v}^\varepsilon\mathbf{u}_k^\varepsilon$ (see arguments in Section 5.1). Hence, any eigenpair $\{\lambda_k^\varepsilon, u_k^\varepsilon\}$ of problem (1.8) generates an eigenpair

$$\{\mu_K^\varepsilon, \widehat{\mathbf{u}}_K^\varepsilon\} = \{\varepsilon^{2-m}\lambda_k^\varepsilon - M, (W^\varepsilon)^{-1}u_k^\varepsilon\} \in \mathbb{R}_+ \times \mathbf{H}^\varepsilon \tag{5.18}$$

of problem (5.2), that together with (5.16) means that in both cases, numbers k and K of eigenvalues coincide with each other. In this way, we can assert that the eigenvalue problems (1.8) and (5.2) are equivalent and

$$\mu_k^\varepsilon = \varepsilon^{2-m}\lambda_k^\varepsilon - M, \quad k \in \mathbb{N}. \tag{5.19}$$

It should be noted that Proposition 5.1 shows that the values defined by (5.19) are positive. However, we observe that this can also be obtained independently by proving that $\varepsilon^{2-m}\lambda_1^\varepsilon - M > 0$, and applying the technique in [36], which uses the minimax principle and comparison results for eigenvalues in different domains (cf. Lemma 5.4). At the same time, we note that the eigenfunctions $\widehat{u}_k^\varepsilon$ in (5.16) and $\widehat{\mathbf{u}}_k^\varepsilon$ in (5.18) are not normalized according to conditions (1.10) and (5.6), respectively. Besides, although the $L^2(\Omega)$-norm of $\widehat{u}_k^\varepsilon$ is uniformly bounded in $\varepsilon \in (0, \varepsilon_0]$, see (5.17), the gradient norm $\|\nabla\widehat{u}_k^\varepsilon; L^2(\Omega)\|$ grows unboundedly as $\varepsilon \to 0$ because of the last term in the formula

$$\nabla(W^\varepsilon(x)\mathbf{u}_k^\varepsilon(x)) = W(\varepsilon^{-1}x)\nabla\mathbf{u}_k^\varepsilon(x) + \varepsilon^{-1}\mathbf{u}_k^\varepsilon(x)\nabla_\xi W(\xi), \text{ with } \xi = \varepsilon^{-1}x. \tag{5.20}$$

Our immediate objective is to show the uniform estimate

$$\|\nabla\mathbf{u}_k^\varepsilon; L^2(\Omega)\| \le c_k \tag{5.21}$$

for the eigenfunctions $\mathbf{u}_k^\varepsilon$ normalized by (5.6) (see statement of Proposition 5.5). This estimate becomes the key point when proving Theorem 5.6 on convergence for eigenpairs of (5.2) in Section 5.3.

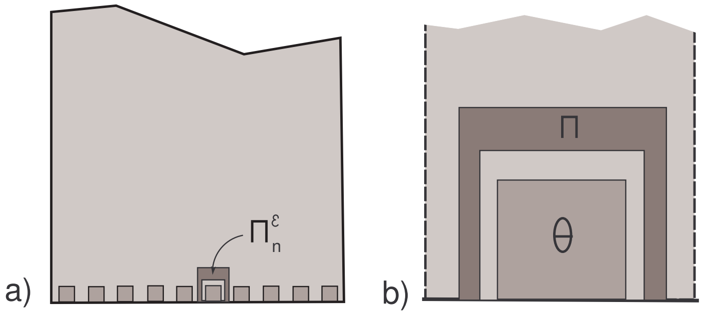

Figure 3: Localizing $\mathbf{u}^{\varepsilon}_{k}$ near the concentrated masses

Since $W^{\varepsilon}(x) \geq c_{\delta} > 0$ for $x \in \Omega \setminus \mathbf{T}^{\varepsilon}_{\delta}$ with any $\delta > 0$, see (5.9) and (2.23)–(2.26), and, by virtue of (5.6) and (5.7), we have

$$\|W^{\varepsilon}\nabla\mathbf{u}^{\varepsilon}_{k}; L^{2}(\Omega))\| \leq c \text{ for } \varepsilon \in (0, \varepsilon_{0}] \text{ and } k \in \mathbb{N}. \tag{5.22}$$

Then, it is sufficient to derive an appropriate estimate of the gradient $\nabla\mathbf{u}^{\varepsilon}_{k}$ on the union of the small rectangles (5.9) with some fixed $\delta > 0$ to be determined. To this end, we localize the solution $\mathbf{u}^{\varepsilon}_{k}$ of problem (5.1), (5.3), (5.4) onto a neighborhood of $\overline{\mathbf{t}^{\varepsilon}_{n\delta}}$ with the help of the cut-off function

$$\varsigma^{\varepsilon}_{n} = \varsigma^{1}(\varepsilon^{-1}(x_{1} - \varepsilon n))\varsigma^{2}(\varepsilon^{-1}x_{2}) \tag{5.23}$$

where

$$\varsigma^{1}(\xi_{1}) = \begin{cases} 0 & \text{for} \quad |\xi_{1}| > l + \frac{2}{3}\Big(\frac{1}{2} - l\Big), \\ 1 & \text{for} \quad |\xi_{1}| < l + \frac{1}{3}\Big(\frac{1}{2} - l\Big), \end{cases} \qquad \varsigma^{2}(\xi_{2}) = \begin{cases} 0 & \text{for} \quad \xi_{2} > h_{0} + 2, \\ 1 & \text{for} \quad \xi_{2} < h_{0} + 1, \end{cases}$$

with $l \in \big(0, \frac{1}{2}\big)$ and $h_{0} > 0$ taken from (1.2) and (2.9), respectively. The function $|\nabla\varsigma^{\varepsilon}_{n}|$ has a support in the set $\Pi^{\varepsilon}_{n} = \overline{\Xi^{2\varepsilon}_{n}} \setminus \Xi^{1\varepsilon}_{n}$, see figure 3, a,

$$\Xi^{p\varepsilon}_{n} = \Big\{x : \ |x_{1} - \varepsilon n| < \varepsilon l + \varepsilon\frac{p}{3}\Big(\frac{1}{2} - l\Big), x_{2} \in (0, \varepsilon h_{0} + \varepsilon p)\Big\}, \quad p = 1, 2, \tag{5.24}$$

and we define

$$\mathbf{c}^{\varepsilon}_{kn} = (\mathrm{mes}_{2}\Pi^{\varepsilon}_{n})^{-1} \int\limits_{\Pi^{\varepsilon}_{n}} \mathbf{u}^{\varepsilon}_{k}(x)\, dx, \tag{5.25}$$

$$\mathbf{U}^{\varepsilon}_{kn}(\xi) = \varsigma^{\varepsilon}_{n}(x)\,(\mathbf{u}^{\varepsilon}_{k}(x) - \mathbf{c}^{\varepsilon}_{kn}) \tag{5.26}$$

where $\xi = \varepsilon^{-1}(x_{1} - \varepsilon n, x_{2})$ are the stretched coordinates (2.1) and $\mathbf{U}^{\varepsilon}_{kn}$ is the function $\mathbf{u}^{\varepsilon}_{k}$ somehow localized.

From (5.1), we derive the differential equation

$$-W(\xi)\Delta_{\xi}\mathbf{U}^{\varepsilon}_{kn}(\xi) - 2\nabla_{\xi}W(\xi)\cdot\nabla_{\xi}\mathbf{U}^{\varepsilon}_{kn}(\xi) = \mathbf{F}^{\varepsilon}_{kn}(\xi), \quad \xi \in \Xi^{2}, \tag{5.27}$$

where

$$\Xi^{p} = \Big\{\xi : \ |\xi_{1}| < l + \frac{p}{3}\Big(\frac{1}{2} - l\Big), \xi_{2} \in (0, h_{0} + p)\Big\}, \quad p = 1, 2, \tag{5.28}$$

cf. (5.24), and

$$\begin{aligned}\mathbf{F}^\varepsilon_{kn}(\xi) =&W(\xi)\left(\varepsilon^m(M+\mu^\varepsilon_k)+\mu^\varepsilon_k X(\xi)\right)\varsigma(\xi)\,\mathbf{c}^\varepsilon_{kn}\\&+W(\xi)\left(\varepsilon^m(M+\mu^\varepsilon_k)+\mu^\varepsilon_k X(\xi)\right)\,\mathbf{U}^\varepsilon_{kn}(\xi)+\widehat{\mathbf{F}}^\varepsilon_{kn}(\xi).\end{aligned}\tag{5.29}$$

We emphasize that the factor ε^2 comes to the right-hand side of (5.27) from the relationship $W^\varepsilon(x)\Delta_x=\varepsilon^{-2}W(\xi)\Delta_\xi$, and the additional term in (5.29), that is,

$$\begin{aligned}\widehat{\mathbf{F}}^\varepsilon_{kn}(\xi) =&W(\xi)\left(2\varepsilon\nabla_x\mathbf{u}^\varepsilon_k\cdot\nabla_\xi\varsigma(\xi)+(\mathbf{u}^\varepsilon_k(x)-\mathbf{c}^\varepsilon_{kn})\Delta_\xi\varsigma(\xi)\right)\\&+2(\mathbf{u}^\varepsilon_k(x)-\mathbf{c}^\varepsilon_{kn})\nabla_\xi W(\xi)\cdot\nabla_\xi\varsigma(\xi)\end{aligned}\tag{5.30}$$

involves the commutator (cf. (4.19) for the definition) of the differential operator from the left hand side of (5.27) with the cut-off function $\varsigma(\xi)=\varsigma^1(\xi_1)\varsigma^2(\xi_2)$ obtained from (5.23). We finally note that function (5.26) satisfies the boundary conditions

$$\mathbf{U}^\varepsilon_{kn}(\xi)=0,\quad \xi\in\partial\Xi^2,\xi_2>0,\tag{5.31}$$

$$\frac{\partial}{\partial\xi_2}\left(W(\xi)\mathbf{U}^\varepsilon_{kn}(\xi)\right)=0,\quad \xi_2=0,l<|\xi_1|<l+\frac{2}{3}\Big(\frac{1}{2}-l\Big).\tag{5.32}$$

The Dirichlet condition (5.31) is due to the definition of the cut-off function (5.23) and the Neumann condition (5.32) is additionally inherited from (1.5) and (1.14). As has been mentioned previously, there is no need to impose any condition on the segment $\tau=(-l,l)\times\{0\}$.

Owing to (5.25), the orthogonality condition

$$\int\limits_{\Pi^\varepsilon_n}(\mathbf{u}^\varepsilon_k(x)-\mathbf{c}^\varepsilon_{kn})\,dx=0$$

is satisfied. Thus, the Poincaré inequality in $\Pi=\overline{\Xi^2}\setminus\Xi^1$ (before the rescaling) ensures that

$$\int\limits_{\Pi^\varepsilon_n}\left|\mathbf{u}^\varepsilon_k(x)-\mathbf{c}^\varepsilon_{kn}\right|^2\,dx\le c_\Pi\varepsilon^2\int\limits_{\Pi^\varepsilon_n}\left|\nabla_x\mathbf{u}^\varepsilon_k(x)\right|^2\,dx\tag{5.33}$$

while the factor ε^2 is due to the small size of the set Π^ε_n in figure 3, a. Function (5.30) has a support in Π, see figure 3, b, and

$$\begin{aligned}\|\widehat{\mathbf{F}}^\varepsilon_{kn};L^2(\Pi)\|^2\le&c\big(\varepsilon^2\|W\nabla_x\mathbf{u}^\varepsilon_k;L^2(\Pi)\|^2+\|\mathbf{u}^\varepsilon_k(x)-\mathbf{c}^\varepsilon_{kn};L^2(\Pi)\|^2\big)\\=&c\big(\|W^\varepsilon\nabla_x\mathbf{u}^\varepsilon_k;L^2(\Pi^\varepsilon_n)\|^2+\varepsilon^{-2}\|\mathbf{u}^\varepsilon_k(x)-\mathbf{c}^\varepsilon_{kn};L^2(\Pi^\varepsilon_n)\|^2\big)\\\le&c\big(\|W^\varepsilon\nabla_x\mathbf{u}^\varepsilon_k;L^2(\Pi^\varepsilon_n)\|^2+\|\nabla_x\mathbf{u}^\varepsilon_k(x);L^2(\Pi^\varepsilon_n)\|^2\big)\\\le&c\|W^\varepsilon\nabla_x\mathbf{u}^\varepsilon_k;L^2(\Pi^\varepsilon_n)\|^2.\end{aligned}\tag{5.34}$$

Here, we have applied inequality (5.33) together with the formulas $d\xi=\varepsilon^{-2}dx$ and $W^\varepsilon(x)\ge c_\Pi>0$, $x\in\Pi^\varepsilon_n$.

In what follows, we employ the notation

$$\mathbf{n}^\varepsilon_{kn}=\|W^\varepsilon\nabla_x\mathbf{u}^\varepsilon_k;L^2(\Omega^\varepsilon_n)\|^2,\quad \Omega^\varepsilon_n=\{x\in\Omega:\,|x_1-\varepsilon n|\le 1/2\}\tag{5.35}$$

so that, according to (5.22),

$$\sum_{n\in\mathbb{Z}(N)}\mathbf{n}^\varepsilon_{kn}\le c\text{ for }\varepsilon\in(0,\varepsilon_0]\text{ and }k\in\mathbb{N}$$

while estimate (5.34) can be simplified as follows:

$$\|\widehat{\mathbf{F}}^\varepsilon_{kn}; L^2(\Pi)\|^2 \le c\mathbf{n}^\varepsilon_{kn}. \tag{5.36}$$

Let us estimate the constant (5.25). Writing the Newton–Leibnitz formula

$$|\mathbf{u}^\varepsilon_k(x_1,x_2)|^2 = \Big(\int\limits_{x_2}^{+\infty} \frac{\partial \mathbf{u}^\varepsilon_k}{\partial y}(x_1,y)\,dy\Big)^2$$

for the function $\mathbf{u}^\varepsilon_k$ extended by zero from Ω onto $(-1,1)\times\mathbb{R}_+$ (recall the Dirichlet condition (5.3) on Γ_0), we integrate in $x \in \Xi^\varepsilon_n \setminus \Xi^{1\varepsilon}_n \supset \Pi^\varepsilon_n$ where

$$\Xi^\varepsilon_n = \{x \in \Omega^\varepsilon_n : x_2 < \varepsilon h_0 + 3\varepsilon\}.$$

As a result, we obtain

$$\begin{aligned}
\|\mathbf{u}^\varepsilon_k; L^2(\Pi^\varepsilon_n)\|^2 \le & \|\mathbf{u}^\varepsilon_k; L^2(\Xi^\varepsilon_n \setminus \Xi^{1\varepsilon}_n)\|^2 = \int\limits_{\Xi^\varepsilon_n \setminus \Xi^{1\varepsilon}_n} |\mathbf{u}^\varepsilon_k(x)|^2\,dx \\
\le & c_\Omega \int\limits_{\Xi^\varepsilon_n \setminus \Xi^{1\varepsilon}_n} \int\limits_{x_2}^{+\infty} \Big|\frac{\partial \mathbf{u}^\varepsilon_k}{\partial y}(x_1,y)\Big|^2\,dy\,dx \\
\le & c_\Omega\,\varepsilon\,(h_0+3) \int\limits_{\Omega^\varepsilon_n \setminus \Xi^{1\varepsilon}_n} \Big|\frac{\partial \mathbf{u}^\varepsilon_k}{\partial y}(x_1,y)\Big|^2\,dy\,dx \le C_\Omega\,\varepsilon\,\|\nabla_x \mathbf{u}^\varepsilon_k; L^2(\Omega^\varepsilon_n \setminus \Xi^{1\varepsilon}_n)\|^2 \\
\le & C_{W,\Omega}\,\varepsilon\,\|W^\varepsilon \nabla_x \mathbf{u}^\varepsilon_k; L^2(\Omega^\varepsilon_n \setminus \Xi^{1\varepsilon}_n)\|^2 \le C_{W,\Omega}\,\varepsilon\,\mathbf{n}^\varepsilon_{kn}.
\end{aligned}$$

Thus, taking into account the area $\mathrm{mes}_2\,(\Pi^\varepsilon_n) = O(\varepsilon^2)$, we conclude that

$$|\mathbf{c}^\varepsilon_{kn}|^2 \le c\varepsilon^{-4}\mathrm{mes}_2\,(\Pi^\varepsilon_n)\,\|\mathbf{u}^\varepsilon_k; L^2(\Pi^\varepsilon_n)\|^2 \le c\varepsilon^{-1}\mathbf{n}^\varepsilon_{kn}. \tag{5.37}$$

Now, we are in position to evaluate all the terms on the right-hand side of (5.29). First of all, we notice that for any $k \in \mathbb{N}$, there exist positive ε_k and C_k supporting the inequality

$$\mu^\varepsilon_k \le C_k\varepsilon \text{ for } \varepsilon \in (0,\varepsilon_k]. \tag{5.38}$$

This estimate will be proved at the beginning of Section 5.3 (cf. (5.51)) based on results in Section 4, that is, independent of the results of this section. Thus, there is no problem assuming for the rest of this section that (5.38) is satisfied, and we observe that the coefficients in the terms

$$\mathbf{f}^{\varepsilon\mathbf{c}}_{kn} := W(\varepsilon^m(M+\mu^\varepsilon_k) + \mu^\varepsilon_k X)\varsigma\mathbf{c}^\varepsilon_{kn} \in H^1(\Xi^2) \tag{5.39}$$

of (5.29) admit the estimates:

$$\begin{gathered}
\varepsilon^m(M+\mu^\varepsilon_k)|\mathbf{c}^\varepsilon_{kn}| \le C_k\varepsilon^{m-1/2}(\mathbf{n}^\varepsilon_{kn})^{1/2} \le C_k\varepsilon^{3/2}(\mathbf{n}^\varepsilon_{kn})^{1/2} \\
\mu^\varepsilon_k|\mathbf{c}^\varepsilon_{kn}| \le C_k\varepsilon^{1/2}(\mathbf{n}^\varepsilon_{kn})^{1/2},
\end{gathered} \tag{5.40}$$

according to (5.37), (5.38) and (1.7). The last two formulas also allow us to consider the term

$$\mathbf{f}^{\varepsilon\mathbf{U}}_{kn} := \mathbf{F}^\varepsilon_{kn} - \widehat{\mathbf{F}}^\varepsilon_{kn} - \mathbf{f}^{\varepsilon\mathbf{c}}_{kn} = W(\varepsilon^m(M+\mu^\varepsilon_k) + \mu^\varepsilon_k X)\mathbf{U}^\varepsilon_{kn} \tag{5.41}$$

as a small perturbation of the differential expression on the left hand side of (5.27).

From Section 2.3 we know that $\mathbf{U}^\varepsilon_{kn} \in H^1(\Xi^2)$, although the corresponding norm is not estimated yet. We insert into problem (5.27), (5.31), (5.32) the representation formula

$$\mathbf{U}^\varepsilon_{kn}(\xi) = W(\xi)^{-1} U^\varepsilon_{kn}(\xi), \quad U^\varepsilon_{kn}(\xi) = \varsigma^\varepsilon_n(\xi)\,(u^\varepsilon_k(x) - \mathbf{c}^\varepsilon_{kn} W^\varepsilon(x)) \tag{5.42}$$

and after long but simple algebraic computations which involve computing $\nabla(W^{-1})$ and $\Delta(W^{-1})$ we arrive at the differential equation

$$-\Delta_\xi U^\varepsilon_{kn}(\xi) - (M + \mu^\varepsilon_k) X(\xi) U^\varepsilon_{kn}(\xi) - \varepsilon^m (M + \mu^\varepsilon_k) U^\varepsilon_{kn}(\xi) = \mathbf{f}^{\varepsilon\mathbf{c}}_{kn}(\xi) + \widehat{\mathbf{F}}^\varepsilon_{kn}(\xi), \tag{5.43}$$

for $\xi \in \Xi^2$, with the boundary conditions

$$\begin{gathered} U^\varepsilon_{kn}(\xi) = 0, \quad \xi \in \partial\Xi^2, \xi_2 > 0, \\ U^\varepsilon_{kn}(\xi_1, 0) = 0,\ |\xi_1| < l, \quad \frac{\partial U^\varepsilon_{kn}}{\partial \xi_2}(\xi_1, 0) = 0,\ |\xi_1| \in \Big(l, l + \frac{2}{3}\Big(\frac{1}{2} - l\Big)\Big). \end{gathered} \tag{5.44}$$

It should be noted that the above-mentioned term $\mathbf{f}^{\varepsilon\mathbf{U}}_{kn}$ in (5.41) has moved to the left hand side in (5.43) and indeed exhibits a small perturbation of the differential operator $\Delta_\xi + MX$. The right-hand side (5.39) of (5.43) is a smooth function everywhere in Ξ^2, except at the arc $\tau_\sqcap$, and inherits from W the singularities $O(\rho_\pm^{1/2})$ at the endpoints $P^\pm$ of the line segment τ, see (1.2) and (2.13). Moreover, we have bounds (5.40) for coefficients of the operator involved with $\mathbf{f}^{\varepsilon\mathbf{c}}_{kn}$ and the result of the following lemma allows us to show that, for sufficiently small ε, problem (5.43)-(5.44) has a unique solution in $H^1(\Xi^2)$.

Lemma 5.4. *The first eigenvalue $\mathcal{M}$ of the differential equation*

$$-\Delta_\xi \mathcal{U} = \mathcal{M} X \mathcal{U} \qquad \text{in } \Xi^2, \tag{5.45}$$

with the mixed boundary conditions (5.44) *is strictly bigger than the first entry $M = M_1$ in the eigenvalue sequence* (2.6) *for problem* (1.13)–(1.15).

Proof. The variational formulation of problem (5.45), (5.44) and the discreteness of its spectrum are clear. Let us revisit problem (1.13)–(1.15). In the Hilbert space $\mathcal{H}$ we introduce the scalar product

$$\langle W, V \rangle = (\nabla_\xi W, \nabla_\xi V)_\varpi + (W, V)_\theta$$

and a compact positive self-adjoint operator $\mathcal{K}$

$$\langle \mathcal{K} W, V \rangle = (W, V)_\theta, \quad W, V \in \mathcal{H}.$$

Then the variational problem (2.2) reduces to the abstract spectral equation

$$\mathcal{K} W = \kappa W \text{ in } \mathcal{H}$$

with the new spectral parameter $\kappa = (1 + M)^{-1}$. The operator $-\mathcal{K}$ (with minus) is below semi-bounded and hence the minimum principle (see, e.g., Theorem 10.2.1 in [3]) assures that

$$-\frac{1}{1 + M_1} = \inf_{V \in \mathcal{H}} \frac{-\langle \mathcal{K} V, V \rangle}{\langle V, V \rangle} = \inf_{V \in \mathcal{H}} \frac{-\|V; L^2(\theta)\|^2}{\|\nabla_\xi V; L^2(\varpi)\|^2 + \|V; L^2(\theta)\|^2}\,.$$

We insert into the Raileigh quotient the principal eigenfunction $\mathcal{U}_1$ of problem (5.45), (5.44) extended by zero from Ξ^2 onto the half-strip ϖ, cf. the first condition in (5.44). Then

$$-\frac{1}{1+M_1} < \frac{-\|\mathcal{U}_1; L^2(\theta)\|^2}{\|\nabla_\xi \mathcal{U}_1; L^2(\varpi)\|^2 + \|\mathcal{U}_1; L^2(\theta)\|^2} = -\frac{1}{1+\mathcal{M}_1}$$

and the desired relation $\mathcal{M}_1 > M_1$ follows immediately. The strict inequality is due to the fact that the principal eigenfunction W_1 of (1.13)–(1.15) is positive in ϖ and therefore cannot coincide with our test function. $\square$

Let us return to problem (5.43), (5.44) whose differential operator is a small perturbation of $-\Delta_\xi - MX$. Then, Lemma 5.4 shows that, for a small $\varepsilon > 0$, this problem is uniquely solvable and thus, owing to (5.36) and (5.39), (5.40), there holds the estimate

$$\|U^\varepsilon_{kn}; H^1(\Xi^2)\|^2 \le c\left(\|\mathbf{f}^{\varepsilon\mathbf{c}}_{kn}; L^2(\Xi^2)\|^2 + \|\widehat{\mathbf{F}}^\varepsilon_{kn}; L^2(\Xi^2)\|^2\right) \le c\mathbf{n}^\varepsilon_{kn}. \tag{5.46}$$

As we have mentioned above, the function $\mathbf{f}^{\varepsilon\mathbf{c}}_{kn}$ in (5.43) is "good". At the same time, the other function (5.30) is supported in Π, that is $\widehat{\mathbf{F}}^\varepsilon_{kn}$ vanishes inside Ξ^1 and its L^2-norm has been properly bounded in (5.36). In this way, using local estimates [2] of solutions to the Neumann problem (compare the last relation in (5.44)) for the Helmholtz equation, we may restrict our consideration on the smaller rectangle

$$\Xi^{1/2} = \left\{\xi : |\xi_1| < l + \frac{1}{6}\Big(\frac{1}{2} - l\Big), \xi_2 \in \Big(0, h_0 + \frac{1}{2}\Big)\right\}, \tag{5.47}$$

cf. (5.28), namely to deal with rectangles of the type (5.9) for a width $\delta = \dfrac{1}{6}\Big(\dfrac{1}{2} - l\Big) > 0$ after the coordinate compression $\xi \mapsto x$.

Let us review the situation. Inside the bigger rectangle $\Xi^1 \supset \Xi^{1/2}$, the function U^ε_{kn} satisfies the differential equation (5.43) with $\widehat{\mathbf{F}}^\varepsilon_{kn} = 0$. According to (5.38) and (1.7), the coefficient $M + \mu^\varepsilon_k$ in the differential operator is a small perturbation of M and the other coefficient $\varepsilon^m(M + \mu^\varepsilon_k)$ is small itself. The remaining right-hand side $\mathbf{f}^{\varepsilon\mathbf{c}}_{kn}$ takes the convenient form (5.39) with coefficients estimated in (5.40). Finally, U^ε_{kn} meets the homogeneous Dirichlet and Neumann conditions (5.44) at the base $\{\xi \in \partial\Xi^1 : \xi_2 = 0\}$. These facts allow us to apply the Kondratiev theory and, based on the theorem on asymptotics in weighted Sobolev [10] and Hölder [21] classes (see also Section 2.3, Appendix and, e.g., Ch. 3 in [31]), we conclude that in the smaller rectangle $\Xi^{1/2}$ there hold the same asymptotic forms for U^ε_{kn} as we have used in Section 2.3 to examine the fractional function (2.29).

The most profitable inference of the performed localization of the problem is undoubtedly the possibility to estimate all the necessary terms of the function U^ε_{kn} in $\Xi^{1/2}$ by the expression

$$C_k\left(\|U^\varepsilon_{kn}; L^2(\Xi^2)\|^2 + \|\mathbf{f}^{\varepsilon\mathbf{c}}_{kn}; L^2(\Xi^2)\|^2 + \left(\varepsilon^{2m}(M + \mu^\varepsilon_k)^2 + |\mu^\varepsilon_k|^2\right)|\mathbf{c}^\varepsilon_{kn}|^2\right) \tag{5.48}$$

which is nothing but the sum of the "weak norms" of the solution and the right-hand side in a "bigger" domain Ξ^2 and a "strong norm" of the right-hand side in a "intervening" domain Ξ^1. In this context, the weak norm means the $L^2(\Xi^2)$-norm but the strong norm is a complicated weighted norm which will be minutely explained in the Appendix. Aiming to estimate a strong norm of the solution in the "small domain" $\Xi^{1/2}$, we note that $\Xi^{1/2} \subsetneqq \Xi^1 \subsetneqq \Xi^2$, cf. (5.28). Moreover, the strong norm of the right-hand side reduces to the sum of moduli of coefficients in the linear combination (5.39). The constant C_k is independent of $n \in \mathbb{Z}(N)$ and $\varepsilon \in (0, \varepsilon_0]$ because of the cell's identity and the above-mentioned property of coefficients in the differential operator of the problem. As a result, we obtain the desired local estimate which shows that the squared norm $\|\nabla_\xi (W^{-1} U^\varepsilon_{kn}); L^2(\Xi^{1/2}\|^2$ does not exceed expression (5.48).

According to (5.42) and (5.26), for $\xi \in \Xi^{1/2}$ and therefore $x \in \Xi^{1/2\,\varepsilon}_n$ (cf. (5.24) for $p = 1/2$), we have

$$\nabla_\xi \left(W(\xi)^{-1} U^\varepsilon_{kn}(\xi) \right) = \nabla_\xi \mathbf{U}^\varepsilon_{kn}(\xi) = \nabla_\xi \left(\mathbf{u}^\varepsilon_k(x) - \mathbf{c}^\varepsilon_{kn} \right) = \nabla_\xi \mathbf{u}^\varepsilon_k(x) = \varepsilon \nabla_x \mathbf{u}^\varepsilon_k(x).$$

Thus, the estimate derived above reads:

$$\begin{aligned} \|\nabla_x \mathbf{u}^\varepsilon_k; L^2(\Xi^{1/2\,\varepsilon}_n)\|^2 &= \|\nabla_\xi \mathbf{u}^\varepsilon_k; L^2(\Xi^{1/2}_n)\|^2 = \|\nabla_\xi (W^{-1} U^\varepsilon_{kn}); L^2(\Xi^{1/2})\|^2 \\ &\leq C_k \left(\|U^\varepsilon_{kn}; L^2(\Xi^2)\|^2 + \|\mathbf{f}^{\varepsilon\mathbf{c}}_{kn}; L^2(\Xi^2)\|^2 + \left(\varepsilon^{2m} (M + \mu^\varepsilon_k)^2 + |\mu^\varepsilon_k|^2 \right) |\mathbf{c}^\varepsilon_{kn}|^2 \right) \\ &\leq C^1_k \mathbf{n}^\varepsilon_{kn} = C^1_k \|W^\varepsilon \nabla_x \mathbf{u}^\varepsilon_k; L^2(\Omega^\varepsilon_n)\|^2. \end{aligned} \tag{5.49}$$

Here, we have used relations (5.46), (5.36), (5.40) and notation (5.35).

Proposition 5.5. *Let an eigenvalue μ^ε_k in* (5.5) *satisfy inequality* (5.38). *Then the corresponding eigenfunction $\mathbf{u}^\varepsilon_k$ of problem* (5.2) *belongs to $H^1(\Omega; \Gamma_\sqcap)$ and admits the estimate*

$$\|\nabla \mathbf{u}^\varepsilon_k; L^2(\Omega)\|^2 \leq c_k \|W^\varepsilon \nabla \mathbf{u}^\varepsilon_k; L^2(\Omega)\|^2, \tag{5.50}$$

where the constant c_k is independent of $\varepsilon \in (0, \varepsilon_k]$ with some $\varepsilon_k > 0$. Also, under the normalization condition (5.6)*, the estimate* (5.21) *holds uniformly in ε.*

Proof. Summing up inequalities (5.49), $n \in \mathbb{Z}(N)$, and adding the self-evident inequality

$$\|\nabla_x \mathbf{u}^\varepsilon_k; L^2(\Omega \setminus \mathbf{T}^\varepsilon_\delta)\|^2 \leq c_W \|W^\varepsilon \nabla_x \mathbf{u}^\varepsilon_k; L^2(\Omega \setminus \mathbf{T}^\varepsilon_\delta)\|^2$$

yields (5.50). Here, $\delta > 0$ is chosen $\delta = (1 - 2l)/12$, namely, to be as it was indicated below (5.47). In addition, from (5.50), (5.7) and (5.22) it follows (5.21). ☐

5.3 Passing to the limit in the integral identity

According to Theorem 4.2, the $c_k \varepsilon^{3/2}$-neighborhood of the point $M + \varepsilon \mu_k$ with the $\varkappa_k$-multiple eigenvalue μ_k, cf. (4.14), contains at least $\varkappa_k$ rescaled eigenvalues $\Lambda^\varepsilon_{p(k)} = M + \mu^\varepsilon_{p(k)}, \dots, \Lambda^\varepsilon_{p(k)+\varkappa_k-1} = M + \mu^\varepsilon_{p(k)+\varkappa_k-1}$, see (1.11) and (5.19). Hence, if $\varepsilon > 0$ is sufficiently small, we have detected at least $k + \varkappa_k - 1$ different eigenvalues in the segment $\left[0, M + \varepsilon \mu_{k+\varkappa_k-1} + c_k \varepsilon^{3/2}\right]$ and, therefore, we conclude the relations $k \leq p(k)$ and

$$\Lambda^\varepsilon_{k+\varkappa_k-1} = M + \mu^\varepsilon_{k+\varkappa_k-1} \leq \Lambda^\varepsilon_{p(k)+\varkappa_k-1} \leq M + C_k \varepsilon.$$

This proves inequality (5.38) used in the previous section. Moreover, we can choose a positive infinitesimal sequence $\{\varepsilon^\bullet_j\}_{j \in \mathbb{N}}$, along which there holds

$$\varepsilon^{-1} \mu^\varepsilon_k \to \mu^\bullet_k, \tag{5.51}$$

for some $\mu^\bullet_k > 0$. Recalling inequalities (5.7) and (5.50), the eigenfunctions $\left\{ \mathbf{u}^{\varepsilon^\bullet_j}_k \right\}_{j \in \mathbb{N}}$ normalized according to (5.6) satisfy the estimates

$$\|W^\varepsilon \nabla \mathbf{u}^\varepsilon_k; L^2(\Omega)\| + \|\mathbf{u}^\varepsilon_k; L^2(\Omega)\| \leq \mathbf{c}_k \tag{5.52}$$

and

$$\|\mathbf{u}_k^\varepsilon; H^1(\Omega)\| \le c_k \|W^\varepsilon \mathbf{u}_k^\varepsilon; L^2(\Omega)\| \le c_k c_W \mathbf{c}_k.$$

Thus, along a subsequence (the above notation $\{\varepsilon_j^\bullet\}_{j\in\mathbb{N}}$ is kept) we have

$$\mathbf{u}_k^\varepsilon \to u_k^\bullet \text{ weakly in } H^1(\Omega) \text{ and strongly in } L^2(\Omega), \tag{5.53}$$

for a certain function $u_k^\bullet \in H^1(\Omega)$. By the compactness of the trace operator from $H^1(\Omega)$ into $L^2(\Gamma)$, we also detect the strong convergence $\mathbf{u}_k^\varepsilon \to u_k^\bullet$ in $L^2(\Gamma)$.

In order to identify the pairs $(\mu_k^\bullet, u_k^\bullet)$, let the test function $\mathbf{v}$ in the integral identity (5.2) belong to $C_0^\infty(\overline{\Omega};\Gamma_\sqcap) \subset \mathbf{H}^\varepsilon$; we take it independent of ε. Clearly, an eigenpair $\{\mu_k^\varepsilon, \mathbf{u}_k^\varepsilon\}$ satisfies

$$\varepsilon^{m-2}(M+\mu_k^\varepsilon)\,(W^\varepsilon \mathbf{u}_k^\varepsilon, W^\varepsilon \mathbf{v})_\Omega \ \to 0.$$

In view of the exponential decay $W^\varepsilon(x) - B = O(e^{-2\pi x_2/\varepsilon})$, see (2.10), we can write

$$(W^\varepsilon)^2 \nabla \mathbf{v} \ \to\ B^2 \nabla \mathbf{v} \text{ strongly in } L^2(\Omega),$$

which easily follows from the smoothness of $\nabla\mathbf{v}$ and from the fact that the function $W^2(\xi) - B^2 \in L^2(\varpi)$ is periodic in the ξ_1 variable (cf. (2.1)).

Proposition 3.4 with $Z = XW^2 \in L^2(\varpi)$ and $z = \mathbf{v}\mathbf{u}_k^\varepsilon \in H^1(\Omega)$ shows that

$$\begin{aligned}\left|(W^\varepsilon \mathbf{u}_k^\varepsilon, W^\varepsilon \mathbf{v})_{\Theta^\varepsilon} - \varepsilon \int_\Theta W(\xi)^2 d\xi\, (\mathbf{u}_k^\varepsilon, \mathbf{v})_\Gamma\right| &\le c\varepsilon^{3/2}\|\mathbf{v}\mathbf{u}_k^\varepsilon; H^1(\Omega)\| \\ &\le c(\mathbf{v})\varepsilon^{3/2}\|\mathbf{u}_k^\varepsilon; H^1(\Omega)\|.\end{aligned} \tag{5.54}$$

Notice that, by virtue of (2.7), the integral over Θ in (5.54) is equal to M^{-1}.

The above-mentioned facts allow us to perform the limit passage as $\varepsilon_j^\bullet \to 0$ in the integral identity (5.2) with $\mu^\varepsilon = \mu_k^{\varepsilon_j^\bullet}$, $\mathbf{u}^\varepsilon = \mathbf{u}_k^{\varepsilon_j^\bullet}$ and $\mathbf{v}^\varepsilon = \mathbf{v}$. As a result, we get the integral identity

$$B^2(\nabla \mathbf{u}_k^\bullet, \nabla \mathbf{v})_\Omega = \mu_k^\bullet M^{-1}(\mathbf{u}_k^\bullet, \mathbf{v})_\Gamma \tag{5.55}$$

which, in view of (3.15), takes the form (3.17). By a completion argument, we can take any test function $\mathbf{v} \in H_0^1(\Omega;\Gamma_\sqcap)$.

Theorem 5.6. *For any $k \in \mathbb{N}$, the limits $\mu_k^\bullet$ and $\mathbf{u}_k^\bullet \in H^1(\Omega;\Gamma_\sqcap)$ computed by formulas* (5.51) *and* (5.53) *through the eigenpair $\{\mu_k^\varepsilon, \mathbf{u}_k^\varepsilon\}$ of problem* (5.2) *in $\mathbf{H}^\varepsilon$ (or equivalently problem* (5.1)*,* (5.3)*,* (5.4) *in its differential form), are an eigenpair of the limit Steklov problem* (1.18)*–*(1.20)*.*

Proof. From (5.55), it suffices to verify that the limit $\mathbf{u}_k^\bullet$ in (5.53) is not trivial. To this end, cf. (5.55), we will prove the relation

$$\mu_k^\bullet M^{-1}\left(\mathbf{u}_k^\bullet, \mathbf{u}_q^\bullet\right)_\Gamma = \delta_{k,q} \tag{5.56}$$

for any $k, q \in \mathbb{N}$ bearing in mind the case $k = q$. This formula also shows that the limits $\mathbf{u}_k^\bullet$ and $\mathbf{u}_q^\bullet$ with $k \ne q$ are orthogonal in $L^2(\Gamma)$ and therefore differ from each other.

First of all, we write the immediate consequence of formulas (5.2), (5.6)

$$\delta_{k,q} = \varepsilon^{-2}\mu_k^\varepsilon\left(W^\varepsilon \mathbf{u}_k^\varepsilon, W^\varepsilon \mathbf{u}_q^\varepsilon\right)_{\Theta^\varepsilon} + \varepsilon^{m-2}\mu_k^\varepsilon\left(W^\varepsilon \mathbf{u}_k^\varepsilon, W^\varepsilon \mathbf{u}_q^\varepsilon\right)_\Omega \tag{5.57}$$

and we observe that, according to (5.51), (5.52) and (1.7),

$$\varepsilon^{m-2}\mu_k^\varepsilon\left|\left(W^\varepsilon \mathbf{u}_k^\varepsilon, W^\varepsilon \mathbf{u}_q^\varepsilon\right)_\Omega\right| \le c_{kq}\varepsilon^{m-1} \ \to 0.$$

Then, obtaining (5.56) follows the idea in Proposition 3.4. We apply the Hardy inequality (2.4) with $t = x_2$ and $\mathcal{V}(t) = \mathbf{u}_j^\varepsilon(x_1, t) - \mathbf{u}_j^\varepsilon(x_1, 0)$ in order to obtain the relation (cf. (3.3))

$$\begin{aligned}\frac{1}{\varepsilon^2}\int\limits_{\Theta^\varepsilon}\left|\mathbf{u}_j^\varepsilon(x) - \mathbf{u}_j^{\varepsilon\Gamma}(x_1)\right|^2|W^\varepsilon|^2dx &\le c\int\limits_{\Omega} x_2^{-2}\left|\varsigma_0(x_1)\left(\mathbf{u}_j^\varepsilon(x) - \mathbf{u}_j^{\varepsilon\Gamma}(x_1)\right)\right|^2|W^\varepsilon|^2dx\\ &\le c\int\limits_{\Omega}\left|\nabla\left(\varsigma_0(x_1)\left(\mathbf{u}_j^\varepsilon(x) - \mathbf{u}_j^{\varepsilon\Gamma}(x_1)\right)\right)\right|^2dx \le c\|\mathbf{u}_j^\varepsilon; H^1(\Omega)\|^2 \le C_j,\end{aligned}\tag{5.58}$$

for the trace $\mathbf{u}_j^{\varepsilon\Gamma}$ of $\mathbf{u}_j^\varepsilon$ on Γ. In addition, from (5.57)–(5.58) we get

$$\varepsilon^{-2}\mu_k^\varepsilon\left(W^\varepsilon\mathbf{u}_k^{\varepsilon\Gamma}, W^\varepsilon\mathbf{u}_q^{\varepsilon\Gamma}\right)_{\Theta^\varepsilon} = \delta_{k,q} + o(1) \text{ as } \varepsilon \to 0. \tag{5.59}$$

Furthermore, recalling formulas (2.7) and (3.22), we proceed similarly to (3.23) using $Z = W^2$ and that $\int_0^\varepsilon |W(x_1/\varepsilon, x_2/\varepsilon)|^2 dx_2 \le C\varepsilon$, and obtain

$$\begin{aligned}&\mu_k^\varepsilon\left|\frac{1}{\varepsilon^2}\sum_{n\in\mathbb{Z}(N)}\int\limits_{\theta_n^\varepsilon}W\Big(\frac{x}{\varepsilon}\Big)^2\mathbf{u}_k^\varepsilon(x_1,0)\mathbf{u}_q^\varepsilon(x_1,0)dx - \frac{1}{\varepsilon M}\sum_{n\in\mathbb{Z}(N)}\int\limits_{\varepsilon(n-1/2)}^{\varepsilon(n+1/2)}\mathbf{u}_k^\varepsilon(\zeta,0)\mathbf{u}_q^\varepsilon(\zeta,0)d\zeta\right|\\ =&\varepsilon^{-3}\mu_k^\varepsilon\left|\sum_{n\in\mathbb{Z}(N)}\int\limits_{\theta_n^\varepsilon}\int\limits_{\varepsilon(n-1/2)}^{\varepsilon(n+1/2)}\left(\mathbf{u}_k^\varepsilon(x_1,0)\mathbf{u}_q^\varepsilon(x_1,0) - \mathbf{u}_k^\varepsilon(\zeta,0)\mathbf{u}_q^\varepsilon(\zeta,0)\right)W\Big(\frac{x}{\varepsilon}\Big)^2 d\zeta\, dx_1\right|\\ \le&c_{kq}\sum_{n\in\mathbb{Z}(N)}\int\limits_{\varepsilon(n-1/2)}^{\varepsilon(n+1/2)}\int\limits_{\varepsilon(n-1/2)}^{\varepsilon(n+1/2)}\frac{\left|\mathbf{u}_k^\varepsilon(x_1,0)\mathbf{u}_q^\varepsilon(x_1,0) - \mathbf{u}_k^\varepsilon(\zeta,0)\mathbf{u}_q^\varepsilon(\zeta,0)\right|}{|x_1-\zeta|+\varepsilon}d\zeta\, dx_1\\ \le&c_{kq}\sqrt{\varepsilon}\Big(\|\mathbf{u}_k^\varepsilon; H^{1/2}(\Gamma)\| + \|\mathbf{u}_k^\varepsilon; L^2(\Gamma)\|\Big)\Big(\|\mathbf{u}_q^\varepsilon; H^{1/2}(\Gamma)\| + \|\mathbf{u}_q^\varepsilon; L^2(\Gamma)\|\Big) \le C_{kq}\sqrt{\varepsilon}.\end{aligned}$$

Consequently, from the above estimates, after subtracting $\varepsilon^{-1}\mu_k^\varepsilon M^{-1}\left(\mathbf{u}_k^\varepsilon, \mathbf{u}_q^\varepsilon\right)_\Gamma$ on the right and left-hand sides of (5.59), it remains to mention that the limit of this subtrahend equals nothing else but $\mu_k^\bullet M^{-1}\left(\mathbf{u}_k^\bullet, \mathbf{u}_q^\bullet\right)_\Gamma$. This shows (5.56) and the theorem holds. ☐

6 Asymptotics of eigenvalues and eigenfunctions

This section contains the main results related to the asymptotics of eigenpairs of the original problem (1.8). We combine results obtained in Section 4 and 5 to conclude the convergence rates for the discrepancies between the eigenvalues and eigenfunctions and the terms on the right-hand side of (1.16) and (1.17) respectively (cf. (6.1) and (6.5)).

6.1 Asymptotic splitting of eigenvalues

Let us analyze Theorem 4.2. At the beginning of Section 5.3 we have shown that that the number $p(k)$ in relation (4.31) satisfies $p(k) \ge k$. Actually, $p(k) = k$. Indeed, by contradiction, if $p(k) > k$, we find an eigenvalue $\mu_{P(\varepsilon)}^\varepsilon$ of problem (5.1), (5.3), (5.4) such that, for a small $\varepsilon > 0$,

$$\mu_{P(\varepsilon)}^\varepsilon \le \varepsilon\left(\mu_k + c_k\varepsilon^{1/2}\right)$$

and

$$\varepsilon^{-2}\mu^\varepsilon_{P(\varepsilon)}\left(\left(W^\varepsilon \mathbf{u}^\varepsilon_{P(\varepsilon)}, W^\varepsilon \mathbf{u}^\varepsilon_q\right)_{\Theta^\varepsilon} + \varepsilon^m\left(W^\varepsilon \mathbf{u}^\varepsilon_{P(\varepsilon)}, W^\varepsilon \mathbf{u}^\varepsilon_q\right)_\Omega\right) = 0,\ q = 1, \dots, k + \varkappa_k - 1,$$

cf. (4.31), (5.19), (5.6) and (5.2). Theorem 5.6 on convergence, in particular, formulas (5.55) and (5.56) show that the limits $\mu^\bullet_P = \lim \varepsilon^{-1}\mu^\varepsilon_{P(\varepsilon)}$ and $\mathbf{u}^\bullet_P = \lim \mathbf{u}^\varepsilon_{P(\varepsilon)} \in H^1_0(\Omega; \Gamma_\sqcap)$ give an eigenpair of problem (1.18)–(1.20) while

$$\mu^\bullet_P \le \mu_k \quad \text{and} \quad \left(\mathbf{u}^\bullet_P, \mathbf{u}^\bullet_q\right)_\Gamma = 0, \quad q = 1, \dots, k + \varkappa_k - 1.$$

The latter conclusion contradicts the way in which the eigenvalue sequence (3.18) was constructed in Proposition 3.3 (cf. (3.19)): the eigenfunction $\mathbf{u}^\bullet_P$ is orthogonal in $L^2(\Gamma)$ to $k + \varkappa_k - 1$ different eigenfunctions $\mathbf{u}^\bullet_q$ would imply $\mu^\bullet_P > \mu_k$. Thus, $p(k) = k$ in (4.31) and we can formulate our main result on eigenvalues.

Theorem 6.1. *1) For any $k \in \mathbb{N}$, there exist positive ε_k and c_k such that the corresponding entries of the eigenvalue sequences* (1.9) *and* (3.18) *of the original problem* (1.3)–(1.5) *and limit problem* (1.18)–(1.20) *satisfy the relationship*

$$\left|\lambda^\varepsilon_k - \varepsilon^{m-2}\left(M + \varepsilon\mu_k\right)\right| \le c_k \varepsilon^{m-1/2} \text{ for } \varepsilon \in (0, \varepsilon_k]. \tag{6.1}$$

Here M is the first eigenvalue of (1.13)-(1.15).

2) If μ_k is an eigenvalue of (1.18)–(1.20) *of multiplicity $\varkappa_k$, as in* (4.14)*, then there exists $\varepsilon^\bullet_k \in (0, \varepsilon_k]$ such that, for $\varepsilon \in (0, \varepsilon^\bullet_k]$, neither $\lambda^\varepsilon_{k-1}$ nor $\lambda^\varepsilon_{k+\varkappa_k}$ falls into the $c_k\varepsilon^{m-1/2}$-neighborhood of the point $\varepsilon^{m-2}(M + \varepsilon\mu_k)$.*

The proof of the first part of Theorem 6.1 follows from (4.31) for $p(k) = k$ and (1.11) (see the reasoning above the statement).

The second part of Theorem 6.1, of course, follows from the first part applied simultaneously for $k-1, k, k+\varkappa_k$ while clearly the relation $\varepsilon^\bullet_k \le \varepsilon_k$ occurs.

Notice that estimate (6.1) readily implies convergence (1.12) but also describes how the eigenvalues in sequence (1.9) deviate from each other by lower-order asymptotic terms. Such asymptotic splitting of the eigenvalues also helps to describe in the next section asymptotics of eigenfunctions.

Remark 6.2. In connection with Theorem 6.1 and its proof, we note that a procedure of the direct and inverse reduction developed in Ch. 7 in [26] and applied, e.g., in [27, 13], shows that the decay rate of $\varepsilon^\bullet_k$ is much faster than the rate of ε_k. This procedure proves assertions similar to Theorem 6.1 but also provides bounds for ε_k, $\varepsilon^\bullet_k$ and c_k, for which an explicit dependence is exhibited on the values of μ_k, $\varkappa_k$ and the relative distance $d_k = \min\left\{1 - \mu_{k-1}\mu_k^{-1}, 1 - \mu_k\mu^{-1}_{k+\varkappa_k}\right\}$ from μ_k to other points in the spectrum. However, the procedure is rather complicated and cumbersome and we skip it in this paper.

6.2 Asymptotic forms for eigenfunctions

Considering in Section 4.4 the $\varkappa_k$-multiple eigenvalue μ_k from (4.14), we have found the coefficient columns $\mathfrak{a}^\varepsilon_{(k)}, \dots, \mathfrak{a}^\varepsilon_{(k+\varkappa_k-1)} \in \mathbb{R}^{X^\varepsilon}$ such that relations (4.29) are valid for the approximate eigenfunctions $\mathfrak{U}^\varepsilon_k, \dots, \mathfrak{U}^\varepsilon_{k+\varkappa_k-1}$ defined in (4.9), (4.10). Moreover, by Theorem 6.1, we now know that $Q^\varepsilon = k$, $X^\varepsilon = \varkappa_k$ in (4.29) and the magnitude $\delta^\varepsilon_\bullet = S\varepsilon$ can be taken instead of the lower-order magnitude (4.28) where S is chosen such that both the eigenvalues $\kappa^\varepsilon_{k-1} = (1+M+\varepsilon\mu_{k-1})^{-1}+O(\varepsilon^{3/2})$ and $\kappa^\varepsilon_{k+\varkappa_k} = (1+M+\varepsilon\mu_{k+\varkappa_k})^{-1}+O(\varepsilon^{3/2})$

of the operator $\mathfrak{K}^\varepsilon$, cf. (4.4), do not belong to the $C_k\varepsilon$-neighborhood of $(1+M+\varepsilon\mu_k)^{-1}$ containing the eigenvalues $\kappa_j^\varepsilon = (1+M+\varepsilon\mu_k)^{-1} + O(\varepsilon^{3/2})$, $j=k,\dots,k+\varkappa_k-1$. Such a constant S depending on k exists since the distance between two consecutive values $(1+M+\varepsilon\mu_k)^{-1}$, $(1+M+\varepsilon\mu_{k-1})^{-1}$ and $(1+M+\varepsilon\mu_{k+\varkappa_k})^{-1}$ is $O(\varepsilon)$ and, for sufficiently small ε, Theorem 6.1 ensures that the eigenvalues of the operator $\mathfrak{K}^\varepsilon$ above mentioned are in intervals of smaller amplitude. Hence, the above-mentioned columns involved in the specified inequality (4.29)

$$\left\| \mathfrak{U}_j^\varepsilon - \sum_{q=k}^{k+\varkappa_k-1} \mathfrak{a}_{jq}^\varepsilon \mathfrak{V}_q^\varepsilon ; \mathfrak{H}^\varepsilon \right\| \leq 2\frac{c^1}{S}\varepsilon^{1/2}, \tag{6.2}$$

meet the estimate

$$\left| \left\langle \mathfrak{a}_{(j)}^\varepsilon, \mathfrak{a}_{(l)}^\varepsilon \right\rangle_{\mathbb{R}^{\varkappa_k}} - \delta_{j,l} \right| \leq c_k \varepsilon^{1/2} \tag{6.3}$$

which replaces (4.30).

Taking (4.8) into account, we rewrite (6.2) as follows:

$$\left\| \mathfrak{u}_j^\varepsilon - |\langle \mathfrak{u}_j^\varepsilon, \mathfrak{u}_j^\varepsilon \rangle_\varepsilon|^{1/2} \left(1+\varepsilon^{m-2}(\lambda_k^\varepsilon)^{-1}\right)^{-1/2} \sum_{q=k}^{k+\varkappa_k-1} \mathfrak{a}_{jq}^\varepsilon u_q^\varepsilon ; \mathfrak{H}^\varepsilon \right\| \leq c_k,$$

for $j=k,\dots,k+\varkappa_k-1$. Consequently, using (6.1) and (4.15) we obtain:

$$\left\| \mathfrak{u}_j^\varepsilon - \varepsilon^{-1/2} M^{1/2} b^{1/2} \sum_{q=k}^{k+\varkappa_k-1} \mathfrak{a}_{jq}^\varepsilon u_q^\varepsilon ; \mathfrak{H}^\varepsilon \right\| \leq c_k, \quad j=k,\dots,k+\varkappa_k-1. \tag{6.4}$$

Recalling definition (4.1) and formula (3.20), we see that the asymptotic expression (4.10) satisfies (see also (4.15))

$$\|\mathfrak{u}_j^\varepsilon - B^{-1}u_j W^\varepsilon ; \mathfrak{H}^\varepsilon\| \leq C_j, \quad \|u_j W^\varepsilon ; \mathfrak{H}^\varepsilon\| = O(\varepsilon^{-1/2})$$

and therefore the term $\mathfrak{u}_j^\varepsilon$ can be replaced in (6.4) by $B^{-1}u_jW^\varepsilon$, so that

$$\left\| \frac{u_j W^\varepsilon}{\|u_j W^\varepsilon;\, H^1(\Omega)\|} - \sum_{q=k}^{k+\varkappa_k-1} \widetilde{\mathfrak{a}}_{jq}^\varepsilon u_q^\varepsilon ;\, H^1(\Omega) \right\| \leq c_k \varepsilon^{1/2}, \quad j=k,\dots,k+\varkappa_k-1.$$

which already provides information on the structure of the eigenfunctions (cf. (3.2) and (3.20)). Above $\widetilde{\mathfrak{a}}_{jq}^\varepsilon = \varepsilon^{-1/2}\|u_jW^\varepsilon;\, H^1(\Omega)\|^{-1} B M^{1/2} b^{1/2} \mathfrak{a}_{jq}^\varepsilon$ and the norm in $H^1(\Omega)$ is that of the gradient; in particular, $\|u_jW^\varepsilon;\, H^1(\Omega)\| = \|\nabla(u^jW^\varepsilon); L^2(\Omega)\| = O(\varepsilon^{-1/2})$.

Furthermore, we observe that, in view of (6.3), the $\varkappa_k \times \varkappa_k$-matrix $\mathfrak{a}^\varepsilon$ composed from the above-mentioned coefficient columns, $\mathfrak{a}^\varepsilon = \left(\mathfrak{a}_{(k)}^\varepsilon, \dots, \mathfrak{a}_{(k+\varkappa_k-1)}^\varepsilon\right)$, is "almost orthogonal" and this implies an "almost orthogonality" property for the functions

$$\{u_jW^\varepsilon \|u_jW^\varepsilon;\, H^1(\Omega)\|^{-1}\}_{j=k}^{k+\varkappa_k-1}.$$

Then, a simple algebraic lemma (see, e.g., Lemma 7.1.7 in [27]), gives us an orthogonal $\varkappa_k \times \varkappa_k$-matrix a^ε such that $a^\varepsilon\mathfrak{a}^\varepsilon$ becomes the unit matrix of size $\varkappa_k \times \varkappa_k$ and allows us to write the eigenfunctions u_p^ε of the original problem in terms of the products of eigenfunctions of the two limit problems u_jW^ε. For the sake of completeness, we introduce here a variant of this lemma (see Lemma 1.5 in [13]).

Lemma 6.3. *Let* $y^1, \dots, y^n \in \mathrm{H}$ *and* $\mathcal{Y}^1, \dots, \mathcal{Y}^N \in \mathrm{H}$ *fulfill the relations*

$$\langle y^j, y^k \rangle_{\mathrm{H}} = \delta_{j,k}\,,\ ||\mathcal{Y}^q; \mathrm{H}|| = 1,$$

$$|\langle \mathcal{Y}^q\,,\, \mathcal{Y}^p \rangle_{\mathrm{H}} - \delta_{q,p}| \leq \tau, \|\mathcal{Y}^q - \sum_{j=1}^{n} a_j^q y^j;\ \mathrm{H}\| \leq \sigma,$$

for certain constants $\{a_j^q\}_{j=1}^n$ *and* σ *and* τ *positive constants independent of* p *and* q*;* $p, q = 1, 2, \cdots N$. *The conditions* $n = N$ *and* $n(\tau + (2+\sigma)\sigma) < 1$ *ensure the existence of the orthogonal* $n \times n$*-matrix* $\theta = (\theta_q^j)_{q,j=1,2,\cdots,n}$ *such that*

$$\|y^j - \sum_{q=1}^{n} \theta_q^j \mathcal{Y}^q; \mathrm{H}\| \leq n(\tau + (3+\sigma)\sigma).$$

It suffices to consider $y_j = u_j^\varepsilon$ satisfying (1.10), $\mathrm{H} = H_0^1(\Omega; \Gamma_\sqcap^\varepsilon)$ with the gradient norm, and $n = N = \varkappa_k$, $\mathcal{Y}^j = u_j W^\varepsilon \|u_j W^\varepsilon;\, H^1(\Omega)\|^{-1}$, and τ and σ certain k-dependent constants multiplied by $\varepsilon^{1/2}$, to apply Lemma 6.3 and to obtain the result stated below.

Theorem 6.4. *Let* μ_k *be a* $\varkappa_k$*-multiple eigenvalue of* (1.18)–(1.20) *in* (4.14)*. Then, there exist positive numbers* $\varepsilon_k^\sharp$, $c_k^\sharp$ *and an orthogonal matrix* $a^{k\varepsilon}$ *of size* $\varkappa_k \times \varkappa_k$ *such that, for* $\varepsilon \in (0, \varepsilon_k^\sharp]$, *the inequalities*

$$\left\| u_q^\varepsilon - \varepsilon^{1/2} \sum_{j=k}^{k+\varkappa_k-1} a_{qj}^\varepsilon u_j W^\varepsilon; H^1(\Omega) \right\| \leq c_k^\sharp \varepsilon^{1/2}, \quad q = k, \dots, k+\varkappa_k-1, \tag{6.5}$$

are valid, where $u_k^\varepsilon, \dots, u_{k+\varkappa_k-1}^\varepsilon$ *and* $u_k, \dots, u_{k+\varkappa_k-1}$ *are eigenfunctions of the original problem* (1.3)–(1.5) *and the limit problem* (1.18)–(1.20)*, repectively, which are orthonormalized according to* (1.10) *and* (3.19)*, respectively. Above,* $W^\varepsilon(x) = W(\varepsilon^{-1}x)$ *is* ε*-periodic in* x_1*,* W *being the principal eigenmode of problem* (1.13)–(1.15)*, normalized by* (2.7)*.*

We underline that $\|\nabla(u^j W^\varepsilon); L^2(\Omega)\| = O(\varepsilon^{-1/2})$ and, thus, formula (6.5) indeed exhibits an asymptotics of the eigenfunctions.

A Appendix

The material of this appendix complements Section 2 and supports the estimate (5.49) which lead us to Proposition 5.5.

A.1 The homogeneous Kondratiev norms

Let us consider the model mixed boundary-value problem in the half-plane

$$-\Delta v = f \quad \text{in } \mathbb{R}_+^2, \tag{A.1}$$

$$v = 0 \quad \text{on } \{x : x_2 = 0,\, x_1 > 0\}, \quad \frac{\partial v}{\partial x_2} = 0 \quad \text{on } \{x : x_2 = 0,\, x_1 > 0\} \tag{A.2}$$

within the Kondratiev theory [10]. By $V_\beta^l(\mathbb{R}_+^2)$, with the indexes of *smoothness* $l \in \mathbb{N}_0$ and *weight* $\beta \in \mathbb{R}$, we denote the completion of the linear space $C_c^\infty\left(\overline{\mathbb{R}_+^2} \setminus \mathcal{O}\right)$ in the homogeneous weighted norm

$$\|u; V_\beta^l(\mathbb{R}_+^2)\| = \left(\sum_{j=0}^{l} \left\| r^{\beta-l+j} \nabla^j u; L^2(\mathbb{R}_+^2) \right\|^2 \right)^{1/2},$$

where (r, φ) is the polar coordinate system centered at the coordinate origin $\mathcal{O}$, the collision point, and $\nabla^j u$ denotes all partial derivatives of u of order j. It is known, see [10] and, e.g., Ch. 2 in [31], that, for any $l \in \mathbb{N}$, the operator

$$\mathcal{A}^l_\beta : \Big\{ v \in V^{l+1}_\beta(\mathbb{R}^2_+) : v \text{ satisfies (A.2)} \Big\} \quad \to \quad V^{l-1}_\beta(\mathbb{R}^2_+) \tag{A.3}$$

of problem (A.1), (A.2) is Fredholm if and only if $\beta - l \neq \beta_{\pm j} := \pm(j + 1/2)$ with $j \in \mathbb{N}_0$; otherwise, the range of (A.3) is not closed in $V^{l-1}_\beta(\mathbb{R}^2_+)$. The forbidden indexes $\beta_{\pm j}$ are closely connected to exponents (2.17) in harmonics (2.16).

If $v \in V^{l+1}_\beta(\mathbb{R}^2_+)$ is a solution of problem (A.1), (A.2) with the right-hand side $f \in V^{l-1}_{\beta-2}(\mathbb{R}^2_+)$ and

$$|\beta - l| < \frac{1}{2}, \tag{A.4}$$

$$\operatorname{supp} f \subset \left\{ x \in \mathbb{R}^2_+ : r \leq 1 \right\}, \tag{A.5}$$

then

$$v(x) = \varsigma(r)\Big(K r^{1/2} \sin\frac{\varphi}{2} + K^1 r^{3/2} \sin\frac{3\varphi}{2} \Big) + \widetilde{v}(x) \tag{A.6}$$

where the coefficients K, K^1 and the remainder $\widetilde{u} \in V^{l+1}_{\beta-2}(\mathbb{R}^2_+)$ satisfy the estimate

$$|K| + |K^1| + \|\widetilde{v}; V^{l+1}_{\beta-2}(\mathbb{R}^2_+)\| \leq c\|f; V^{l-1}_{\beta-2}(\mathbb{R}^2_+)\|. \tag{A.7}$$

We emphasize that mapping (A.3) becomes an isomorphism under restriction (A.4) and the inclusion $f \in V^{l-1}_{\beta-2}(\mathbb{R}^2_+)$ implies a faster decay rate as $r \to 0^+$ than the decay rate of $\Delta v \in V^{l-1}_\beta(\mathbb{R}^2_+)$ prescribed by the original inclusion $v \in V^{l+1}_\beta(\mathbb{R}^2_+)$. In the same way, formula (A.6) gives the asymptotics of the solution v in the radial variable r.

A.2 The multi-scaled weighted norms

Considering the Rayleigh principle for the spectral problem

$$-\Phi''(t) = \Lambda\Phi(t) \ \text{ for } \ t \in (0, \pi), \quad \Phi(0) = 0, \Phi'(\pi) = 0,$$

and the angular variable $t = \varphi$, a function in $\{v \in V^1_\beta(\mathbb{R}^2_+) : v(x_1, 0) = 0,\ x_1 > 0\}$ satisfies

$$\int\limits_{\mathbb{R}^2_+} r^{2\beta-2} |v(x)|^2 \, dx = \int\limits_{\mathbb{R}_+} \int\limits_0^\pi r^{2\beta-2} |v(x)|^2 \, r dr d\varphi$$

$$\leq 4 \int\limits_{\mathbb{R}_+} \int\limits_0^\pi r^{2\beta} \Big| \frac{1}{r} \frac{\partial v}{\partial \varphi}(x) \Big|^2 r dr d\varphi \leq 4 \int\limits_{\mathbb{R}^2_+} r^{2\beta} |\nabla v(x)|^2 \, dx.$$

This apparent observation was suggested in [24] to introduce multi-scaled weighted space $\mathcal{V}^{l,0}_{\beta,\gamma}(\mathbb{R}^2_+)$ in two-dimensional angular domains with the norm

$$\|u; \mathcal{V}^{l,0}_{\beta,\gamma}(\mathbb{R}^2_+)\| = \Big(\sum_{j=0}^{l} \big\| r^{\beta-l+j} \varphi^{\gamma-l+j} \nabla^j u; L^2(\mathbb{R}^2_+) \big\|^2 \Big)^{1/2} \tag{A.8}$$

involving two weights, radial and angular, with different weight exponents β and γ. Such function spaces are convenient in the investigation of different perturbations of the boundary of the angular domain, cf. [24, 29].

If restrictions (A.4) and

$$\gamma - l \in \Big(-\frac{1}{2}, \frac{1}{2} \Big) \tag{A.9}$$

are satisfied, the operator

$$\mathcal{A}^l_{\beta,\gamma} : \Big\{ v \in \mathcal{V}^{l+1,0}_{\beta,\gamma}(\mathbb{R}^2_+) : \frac{\partial v}{\partial x_2}(x_1, 0) = 0, \ x_1 < 0 \Big\} \quad \to \quad \mathcal{V}^{l-1}_{\beta,\gamma}(\mathbb{R}^2_+) \tag{A.10}$$

of problem (A.1), (A.2) is an isomorphism and in the case of the right-hand side $f \in V^{l-1}_{\beta-2,\gamma^1}(\mathbb{R}^2_+)$ with the compact support (A.5) and the second weight index $\gamma^1 \in (0, \gamma]$ the asymptotic representation (A.6) is valid where the remainder $\widetilde{v} \in \mathcal{V}^{l+1,0}_{\beta,\gamma^1}(\mathbb{R}^2_+)$ and the coefficients K, K^1 satisfy appropriately modified estimate (A.7).

We emphasize that the Dirichlet condition at the semi-axis $\mathbb{R}_+ \ni x_1$ does not appear explicitly in the domain of the operator (A.10) because, by virtue of the restriction $\gamma < 1/2$, the assumption $v(r, 0) \neq 0$ leads to the divergent integral

$$\int_0^\pi \varphi^{2(\gamma-l-1)} |v(r, \varphi)|^2 d\varphi. \tag{A.11}$$

A.3 Weighted spaces with detached asymptotics

Under condition (A.9) operator (A.10) stays Fredholm for any $\beta \in \mathbb{R}$ with the exception of the forbidden indexes $\beta_{\pm j}$ indicated in Section A.1. However, denying (A.9) deprives the operator of the Fredholm property. For example, in the case $\gamma - l < -1/2$ a function $v \in \mathcal{V}^{l+1}_{\beta,\gamma}(\mathbb{R}^2_+)$ has a finite norm only under the two conditions $v(x_1, 0) = 0$ and $\dfrac{\partial v}{\partial x_2}(x_1, 0) = 0$ on the semi-axis $\mathbb{R}_+$. The latter is not possible for a non-trivial harmonics due to the theorem on unique extension. To vary the second weight index γ requires detaching asymptotics, cf. Ch. 12 in [31], namely to deal with functions in the form

$$v(x) = \mathcal{K}(r) r^{1/2} \sin \frac{\varphi}{2} + \mathcal{K}^1(r) r^{3/2} \sin \frac{3\varphi}{2} + \widetilde{v}(x). \tag{A.12}$$

The remainder $\widetilde{v}$ must belong to the space $\mathcal{V}^{l+1}_{\beta,\gamma}(\mathbb{R}^2_+)$ with the weighted norm

$$\|\widetilde{v}; \mathcal{V}^{l+1}_{\beta,\gamma}(\mathbb{R}^2_+)\| = \Big(\sum_{j=0}^{l+1} \big\| r^{\beta-l-1+j} \varphi^{\gamma-l-1+j} (\pi - \varphi)^{\gamma-l-1+j} \nabla^j \widetilde{v}; L^2(\mathbb{R}^2_+) \big\|^2 \Big)^{1/2} \tag{A.13}$$

and the weight indexes

$$\beta - l \in \Big(-\frac{1}{2}, \frac{1}{2} \Big), \quad \gamma - l \in \Big(-\frac{3}{2}, -\frac{1}{2} \Big). \tag{A.14}$$

We emphasize that now, in contrast to (A.8), weights are introduced in (A.13) at both endpoints of the arc $(0, \pi)$ while the restriction on γ in (A.14) demands that the remainder $\widetilde{v} \in \mathcal{V}^{l+1}_{\beta,\gamma}(\mathbb{R}^2_+)$ satisfies formulas

$$\widetilde{v}(x_1, 0) = 0, \ \frac{\partial \widetilde{v}}{\partial x_2}(x_1, 0) = 0 \quad \text{for } x_1 \in \mathbb{R} \setminus \{0\};$$

otherwise, norm (A.13) cannot be finite because of the divergence of the integral (similar to (A.11))

$$\int_0^\pi \varphi^{2(\gamma-l-1)}(\pi-\varphi)^{2(\gamma-l-1)}|\widetilde{v}(r,\varphi)|^2 d\varphi.$$

Hence, according to (A.12), we have

$$\begin{aligned} &v(r,0)=0,\ \frac{\partial v}{\partial x_2}(r,0)=\frac{1}{2}\,r^{-1/2}\mathcal{K}(r)+\frac{3}{2}\,r^{1/2}\mathcal{K}^1(r) \qquad \text{for } x_1=r>0, \\ &v(r,0)=r^{1/2}\mathcal{K}(r)-r^{3/2}\mathcal{K}^1(r),\ \frac{\partial v}{\partial x_2}(r,0)=0 \qquad \text{for } x_1=-r<0. \end{aligned} \tag{A.15}$$

Roughly speaking, to compose from functions (A.12) a weighted space with detached asymptotics by means of a procedure in Ch. 12 of [31] requires setting the coefficient functions $\mathcal{K}(r)$, $\mathcal{K}^1(r)$ in a certain weighted Kondratiev space and incorporating their norms together with norm (A.13) into the norm of the whole function v. Additional difficulties originate in insufficient smoothness properties of the coefficients: according to (A.15) none of the traces $v\big|_{\mathbb{R}_\pm}$ and $\dfrac{\partial v}{\partial x_2}\Big|_{\mathbb{R}_\pm}$ and, therefore, none of $\mathcal{K}$ and $\mathcal{K}^1$ belongs to the proper space $H^{l+1}(\mathbb{R}_\pm)$. The latter requires the introduction of special extension operators into the asymptotic forms of type (A.12) (cf. [11, 34] and Ch. 12 in [31]). To avoid unnecessary complications, we consider a particular case with an infinitely differentiable right-hand side f vanishing near the coordinate origin, we deal with the model differential equation corresponding to the original problem at collision points

$$-\Delta v-\mu H v=f \quad \text{in} \quad \mathbb{B}_R^+, \tag{A.16}$$

and we write only an asymptotic formula for a solution of (A.16), (A.2) near the point $\mathcal{O}$. In (A.16), $\mu\in\mathbb{R}_+$, $H=H(x_1)$ is the Heaviside unit step function and $\mathbb{B}_R^+=\{x: |x|<R, x_2>0\}$ is the upper half-disk of radius $R>0$. We have the following result.

Proposition A.1. *Let $v\in H^1(\mathbb{B}_R^+)\cap H^2_{loc}(\overline{\mathbb{B}_R^+}\setminus\mathcal{O})$ satisfy equation* (A.16) *with*

$$f\in L^2(\mathbb{B}_R^+) \ \text{ and } f(x)=0 \text{ for } r<R/2,$$

and the boundary conditions (A.2) *for $r\in(0,R)$. Then v falls into $\mathcal{V}^2_{1,\gamma}(\mathbb{B}^+_{2R/3})$ with any $\gamma\in(1/2,3/2)$ and admits the asymptotic form*

$$v(x)=\varsigma(r)\left(\mathcal{K}(r)r^{1/2}\sin\frac{\varphi}{2}+\mathcal{K}^1(r)r^{3/2}\sin\frac{3\varphi}{2}\right)+\widetilde{v}(x) \ \text{ for } r\le\frac{1}{3}R, \tag{A.17}$$

where the remainder $\widetilde{v}$ and the coefficients

$$\mathcal{K}(r)=K+\widetilde{\mathcal{K}}(r), \quad \mathcal{K}^1(r)=K^1+\widetilde{\mathcal{K}}^1(r)$$

fulfill the estimate

$$\begin{aligned} \sum_{j=0,1}&\Big(r^{-2+j}\big|\partial_r^j\widetilde{\mathcal{K}}(r)\big|+r^{-1+j}\big|\partial_r^j\widetilde{\mathcal{K}}^1(r)\big|+r^{-2+j}\varphi^{-2+j}(\pi-\varphi)^{-2+j}\big|\nabla^j\widetilde{v}(x)\big|\Big) \\ &+|K|+|K^1|\le c\Big(\|f;L^2(\mathbb{B}_R^+\setminus\mathbb{B}^+_{R/2})\|+\|v;H^1(\mathbb{B}_R^+)\|\Big) \quad \text{for } r\le\frac{1}{3}R. \end{aligned} \tag{A.18}$$

Formulas (A.17) and (A.18) suffice to support all the calculations and estimations for the quotient function (2.29) in Sections 2.4 and 5.2.

Acknowledgments. This research work has been partially supported by Spanish MINECO, MTM2013-44883-P. Also, the research work of the first author has been partially supported by Russian Foundation of Basic research (project 15–01–02175).

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