Algorithm 939: Computation of the Marcum Q-function

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Methods and an algorithm for computing the generalized Marcum $Q-$function ($Q_\mu(x, y)$) and the complementary function ($P_\mu(x, y)$) are described. These functions appear in problems of different technical and scientific areas such as, for example, radar detection and communications, statistics and probability theory, where they are called the non-central chi-square or the non central gamma cumulative distribution functions.

The algorithm for computing the Marcum functions combines different methods of evaluation in different regions: series expansions, integral representations, asymptotic expansions, and use of three-term homogeneous recurrence relations. A relative accuracy close to 10^{-12} can be obtained in the parameter region $(x, y, \mu) \in [0, A] \times [0, A] \times [1, A]$, $A = 200$, while for larger parameters the accuracy decreases (close to 10^{-11} for $A = 1000$ and close to 5×10^{-11} for $A = 10000$).

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1. INTRODUCTION

We define the generalized Marcum $Q-$function by using the integral representation

$$Q_\mu(x, y) = x^{\frac{1}{2}(1-\mu)} \int_y^{+\infty} t^{\frac{1}{2}(\mu-1)} e^{-t-x} I_{\mu-1}\left(2\sqrt{xt}\right) dt, \tag{1}$$

where $x \geq 0$, $y \geq 0$, $\mu > 0$, and $I_\mu(z)$ is the modified Bessel function. We also use the complementary function

$$P_\mu(x,y) = x^{\frac{1}{2}(1-\mu)} \int_0^y t^{\frac{1}{2}(\mu-1)} e^{-t-x} I_{\mu-1}\left(2\sqrt{xt}\right) dt, \qquad (2)$$

and the complementary relation reads

$$P_\mu(x,y) + Q_\mu(x,y) = 1. \qquad (3)$$

There are other notations for the generalized Marcum function in the literature. Among them, probably the most popular one is the following

$$\widetilde{Q}_\mu(\alpha,\beta) = \alpha^{1-\mu} \int_\beta^{+\infty} t^\mu e^{-(t^2+\alpha^2)/2} I_{\nu-1}(\alpha t) dt, \qquad (4)$$

where we have added a tilde in the definition to distinguish it from the definition we are using (1). For $\mu = 1$ this coincides with the original definition of the Marcum $Q-$function [Marcum 1960]. This is the notation used, for instance, in the MATLAB built-in function **marcumq**. The relation with the notation we use is simple:

$$Q_\mu(x,y) = \widetilde{Q}_\mu(\sqrt{2x}, \sqrt{2y}), \qquad (5)$$

and similarly for the $P-$function.

The generalized Marcum $Q-$function is an important function used in radar detection and communications, see [Marcum 1960; Rice 1968; Robertson 1976]. In this field, μ is the number of independent samples of the output of a square-law detector. In our analysis μ is not necessarily a positive integer number and we will consider real values $\mu \geq 1$.

These functions also occur in statistics and probability theory, where they are called non-central chi-square (semi-integer μ) or non-central gamma cumulative distributions [Ashour and Abdel-Samad 1990; Cohen 1988; Dyrting 2004; Knüsel and Bablok 1996; Robertson 1976; Ross 1999]. The central gamma cumulative distribution is in fact the incomplete gamma function, and the relation of the Marcum functions to the incomplete gamma functions is given in §2.1.

In this paper we describe methods and an algorithm for computing the functions $P_\mu(x,y)$ and $Q_\mu(x,y)$ for a large range of the parameters μ, x, y. We consider series expansions in terms of the incomplete gamma functions, recurrence relations, asymptotic expansions, and numerical quadrature. A Fortran 90 module implementing the algorithm is tested, and we conclude it provides a relative accuracy close to $\sim 10^{-12}$ in the parameter region $(x, y, \mu) \in [0, A] \times [0, A] \times [1, A]$, $A = 200$, while for larger parameters the accuracy decreases (close to 10^{-11} for $A = 1000$ and better than 10^{-10} for $A = 10000$).

We include comparisons with the MATLAB built-in function **marcumq**; this function is restricted to integer positive values of the parameter μ, differently from our Fortran 90 module which allows the computation of the Marcum functions for real values (greater or equal to 1) of that parameter. Also, the MATLAB function computes the Marcum $Q-$function but not the complementary $P-$function: computing this function simply as $1 - Q$ when Q is close to 1 can lead to serious cancellation problems. Our tests reveal some bugs in certain parameter regions when computing the Marcum $Q-$function using MATLAB.

2. PROPERTIES OF THE MARCUM FUNCTIONS

2.1. Series expansions

Substituting the Maclaurin series for the modified Bessel function

$$I_\mu(z) = \left(\tfrac{1}{2}z\right)^\mu \sum_{n=0}^\infty \frac{\left(\tfrac{1}{4}z^2\right)^n}{n!\,\Gamma(\mu+n+1)} \tag{6}$$

into the integral representations (1) and (2), we readily obtain the series expansions

$$
\begin{aligned}
P_\mu(x,y) &= e^{-x} \sum_{n=0}^\infty \frac{x^n}{n!} P_{\mu+n}(y), \\
Q_\mu(x,y) &= e^{-x} \sum_{n=0}^\infty \frac{x^n}{n!} Q_{\mu+n}(y).
\end{aligned}
\tag{7}
$$

These expansions are in terms of the incomplete gamma function ratios defined by

$$P_\mu(x) = \frac{\gamma(\mu,x)}{\Gamma(\mu)}, \quad Q_\mu(x) = \frac{\Gamma(\mu,x)}{\Gamma(\mu)}, \tag{8}$$

where the standard incomplete gamma functions are defined by

$$\gamma(\mu,x) = \int_0^x t^{\mu-1} e^{-t}\,dt, \quad \Gamma(\mu,x) = \int_x^{+\infty} t^{\mu-1} e^{-t}\,dt, \quad \Re\mu > 0. \tag{9}$$

Again, we have the complementary relation $P_\mu(x) + Q_\mu(x) = 1$. In [Gil et al. 2012], algorithms are given for the computation of the incomplete gamma function ratios $P_\mu(y)$ and $Q_\mu(y)$ appearing in (7).

From these relations we obtain the particular values

$$
\begin{aligned}
Q_\mu(x,0) &= 1, & Q_\mu(x,+\infty) &= 0, \\
Q_\mu(0,y) &= Q_\mu(y), & Q_\mu(+\infty,y) &= 1, \\
Q_{+\infty}(x,y) &= 1,
\end{aligned}
\tag{10}
$$

and similar complementary relations for $P_\mu(x,y)$.

In the numerical algorithm we compute both functions $P_\mu(x,y)$ and $Q_\mu(x,y)$. The *primary function* in the algorithm is the smallest one of $P_\mu(x,y)$ and $Q_\mu(x,y)$, and the primary function will be computed first. The other one follows from the complementary relation in (3).

As will follow from the relations given later (and in less detail from the relations in (10)), the transition in the (x,y) quarter plane from small values of $Q_\mu(x,y)$ to values close to unity occurs for large values of μ, x, y across the line $y = x + \mu$, and above this line in the (x,y) quarter plane $Q_\mu(x,y)$ is taken as the primary function. Below this line the complementary function $P_\mu(x,y)$ is taken as the primary function.

2.2. Recurrence relations

Considering integration by parts in the integrals in (1) and (2), together with the relation $z^\mu I_{\mu-1}(z) = \frac{d}{dz}\left(z^\mu I_\mu(z)\right)$, it is easy to see that the Marcum functions satisfy the following first order difference equations (see, for instance, [Temme 1993] [1]):

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[1] We note that a factor e^{-y} is missing in [Temme 1993, Eq. (1.4)].

$$Q_{\mu+1}(x,y) = Q_\mu(x,y) + \left(\frac{y}{x}\right)^{\mu/2} e^{-x-y} I_\mu(2\sqrt{xy}),$$

$$P_{\mu+1}(x,y) = P_\mu(x,y) - \left(\frac{y}{x}\right)^{\mu/2} e^{-x-y} I_\mu(2\sqrt{xy}). \tag{11}$$

It is possible to eliminate the Bessel function appearing in (11) and to obtain a homogeneous third order recurrence relation [Temme 1993]:

$$xQ_{\mu+2}(x,y) = (x-\mu)Q_{\mu+1}(x,y) + (y+\mu)Q_\mu(x,y) - yQ_{\mu-1}(x,y). \tag{12}$$

The function $P_\mu(x,y)$ and every constant (with respect to μ) satisfy the same relation. These recurrence relations can be useful for testing.

This first order inhomogeneous equation in (11) can be used for computing the Marcum Q−function in the direction of increasing μ. Observe that the right-hand side in the relation for $Q_\mu(x,y)$ has only positive terms. Also, considering Perron's theorem for this difference equation ([Gil et al. 2007, Thm. 4.17]), it is easy to check that the recurrence relation admits a minimal (or recessive) solution and that the dominant solutions y_μ are such that

$$\lim_{\mu\to+\infty} \frac{y_{\mu+1}}{y_\mu} = 1, \tag{13}$$

as corresponds to the Marcum Q−function, which is therefore dominant and can be computed by forward recursion.

When we consider Perron's theorem, which is also known as the Perron-Kreuser theorem, for the homogeneous recursion in (12), it follows that the Marcum Q−function is neither minimal nor dominant and therefore it cannot be computed neither in the forward or backward direction. A possible way out is to combine two first order recursions of (11) to obtain the three-term homogeneous recurrence relation

$$y_{\mu+1} - (1 + c_\mu)y_\mu + c_\mu y_{\mu-1} = 0, \quad c_\mu = \sqrt{\frac{y}{x}} \frac{I_\mu\left(2\sqrt{xy}\right)}{I_{\mu-1}\left(2\sqrt{xy}\right)}. \tag{14}$$

Both $Q_\mu(x,y)$ and the complementary function $P_\mu(x,y)$ satisfy (14). $Q_\mu(x,y)$ is dominant and can be computed in the forward recursion, while $P_\mu(x,y)$ is minimal and has to be computed in the backward direction.

An advantage of the relation in (14) is that ratios of Bessel functions appear, which are slowly varying with respect to μ. Because dominant factors in the Bessel function representations disappear in the ratios, these are free of overflow/underflow problems. In addition, for the ratios of Bessel functions continued fraction representations can be used.

It is interesting to observe that Pincherle's theorem [Gil et al. 2007, Thm. 2.7] allows us to compute a ratio of $P_\mu(x,y)$ as a continued fraction, the coefficients of which can be represented as continued fractions themselves, namely

$$\frac{P_\mu(x,y)}{P_{\mu-1}(x,y)} = \frac{c_\mu}{1+c_\mu-} \frac{c_{\mu+1}}{1+c_{\mu+1}-} \dots. \tag{15}$$

2.3. Derivatives and monotonicity

Taking the derivative with respect to y in (1) and using (11) we have

$$\frac{\partial Q_\mu(x,y)}{\partial y} = Q_{\mu-1}(x,y) - Q_\mu(x,y). \tag{16}$$

Taking the derivative with respect to x, and using the relation $I'_\nu(z) = I_{\nu+1}(z) + \frac{\nu}{z}I_\nu(z)$, we obtain

$$\frac{\partial Q_\mu(x,y)}{\partial x} = Q_{\mu+1}(x,y) - Q_\mu(x,y). \tag{17}$$

And using (11) we see that $Q_\mu(x,y)$ $(P_\mu(x,y))$ is an increasing (decreasing) function of x and a decreasing (increasing) function of y. With respect to μ, $Q_\mu(x,y)$ is increasing and $P_\mu(x,y)$ is decreasing.

3. USING THE SERIES EXPANSIONS

The series representations in (7) can be computed by using the algorithms for the incomplete gamma ratios described in [Gil et al. 2012]. The recurrence relations

$$Q_{\mu+1}(y) = Q_\mu(y) + \frac{y^\mu e^{-y}}{\Gamma(\mu+1)},$$

$$P_{\mu+1}(y) = P_\mu(y) - \frac{y^\mu e^{-y}}{\Gamma(\mu+1)}, \tag{18}$$

are stable for $Q_\mu(y)$ in the forward direction, and for $P_\mu(y)$ in the backward direction. For the series representation of $P_\mu(x,y)$ we need to find a truncation number n_0 of the infinite series, compute $P_{\mu+n_0}(y)$ and perform the summation and recursion in (7) with $n = n_0, n_0 - 1, \ldots, 0$.

The recursion for $P_{\mu+n}(y)$ is written in the backward form

$$P_{\mu+n}(y) = P_{\mu+n+1}(y) + \frac{y^{\mu+n}e^{-y}}{\Gamma(\mu+n+1)}, \tag{19}$$

and we write the series for $P_\mu(x,y)$ in the form

$$P_\mu(x,y) \simeq e^{-x}P_\mu(y) \sum_{n=0}^{n_0} \frac{x^n}{n!} \frac{P_{\mu+n}(y)}{P_\mu(y)}, \tag{20}$$

where we assume that the first neglected term is smaller than a prescribed precision ε. Observe that the value of the first term in the sum is 1, and then we expect that by considering $S_{n_0} < \varepsilon$, with S_{n_0} the first neglected term, the sum will reach a relative accuracy better than ε.

The first neglected term

$$S_{n_0} = \frac{x^{n_0+1}}{(n_0+1)!} \frac{P_{\mu+n_0+1}(y)}{P_\mu(y)} \tag{21}$$

can be bounded using some monotonicity properties of the incomplete gamma function. Using (8) we have

$$S_{n_0} = \frac{x^{n_0+1}}{(n_0+1)!(\mu)_{n_0+1}} \frac{\gamma(\mu+1,y)}{\gamma(\mu,y)} \frac{\gamma(\mu+2,y)}{\gamma(\mu+1,y)} \cdots \frac{\gamma(\mu+n_0+1,y)}{\gamma(\mu+n_0,y)}. \tag{22}$$

Now, it is easy to check that $h_\nu(y) = \frac{\gamma(\nu,y)}{\gamma(\nu-1,y)} < y$. Indeed, it is known that $h_\nu(y)$ is monotonically increasing as a function on ν [Qi 2002] and, on the other hand, it is easy

to check that $\lim_{\nu \to +\infty} h_\nu(y) = y$ (for instance, by using [Paris 2010, 8.11.4]); these two facts imply that $h_\nu(y) < y$. With this information, we conclude that

$$S_{n_0} < \frac{x^{n_0+1} y^{n_0+1}}{(n_0+1)!(\mu)_{n_0+1}}, \tag{23}$$

and taking

$$\frac{(xy)^{n_0+1}(\mu-1)!}{(n_0+1)!(\mu+n_0)!} = \varepsilon, \tag{24}$$

it is guaranteed that the first neglected term is smaller than ε. Now we use the Stirling approximation for the factorials in the denominators, which is a lower bound for the factorials. This translates into finding the solution of $f(n) = 0$ for

$$f(n) = (n+\mu+1/2)\ln(n+\mu)+(n-1/2)\ln n-2n-n\ln(xy)-C, \; C = \ln\left(\frac{\Gamma(\mu)}{2\pi\varepsilon}\right)+\mu. \tag{25}$$

If $f(n^*) = 0$ with n^* the larger solution then it is guaranteed that $n_0 = n^* - 1$ is such that $S_{n_0} < \varepsilon$.

The analysis becomes more simple considering some tiny modifications and taking

$$f(n) = (n+\mu)\ln(n+\mu)+n\ln n-2n-n\ln(xy)-C. \tag{26}$$

For this function the derivative with respect to n is given by

$$f'(n) = \ln\frac{n(n+\mu)}{xy}, \tag{27}$$

and this first derivative vanishes if $n = n_e = (-\mu+\sqrt{\mu^2+4xy})/2$, which is a minimum because $f^{(2)}(n) > 0$. Starting at the right of this minimum we can use Newton's method to solve $f(n) = 0$; convergence will be certain because $f'(n) > 0$ at the right of the minimum and $f^{(2)}(n) > 0$.

We observe that for small values of $P_\mu(x, y)$ (close to the underflow limit), it may happen that the last terms in the series can not be computed because the incomplete gamma ratios may underflow. Similarly, in the the series for the $Q_\mu(x, y)$ it may happen that the first values of $Q_{\mu+n}(y)$ in (7) may underflow, but not the total sum. This means that some error degradation can be expected for values close to underflow. We will discuss this issue later in §6.

4. ASYMPTOTIC REPRESENTATIONS

In [Temme 1993] two types of asymptotic expansions have been derived: one with small values of μ and one with large values. In both cases the expansion is given in terms of the complementary error function

$$\text{erfc } z = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2} \, dt; \tag{28}$$

in this definition z may assume any finite (complex) value.

4.1. Asymptotic representations for large xy

We summarize the results given in [Temme 1993]. We use the function

$$F_\mu(\xi, \sigma) := \int_\xi^\infty e^{-(\sigma+1)t} I_\mu(t) \, dt, \quad \sigma > 0, \tag{29}$$

and we have

$$Q_\mu(x,y) = \tfrac{1}{2}\rho^\mu \left(F_{\mu-1}(\xi,\sigma) - \tfrac{1}{\rho}F_\mu(\xi,\sigma) \right), \quad y > x,$$
$$P_\mu(x,y) = \tfrac{1}{2}\rho^\mu \left(\tfrac{1}{\rho}F_\mu(\xi,\sigma) - F_{\mu-1}(\xi,\sigma) \right), \quad y < x, \tag{30}$$

where the parameters are defined by

$$\xi = 2\sqrt{xy}, \quad \sigma = \frac{(\sqrt{y}-\sqrt{x})^2}{\xi}, \quad \rho = \sqrt{\frac{y}{x}}. \tag{31}$$

We assume that ξ is large and substitute the well-known expansion[2]

$$e^{-t}I_\mu(t) \sim \frac{1}{\sqrt{2\pi t}} \sum_{n=0}^{\infty} (-1)^n \frac{A_n(\mu)}{t^n}, \quad A_n(\mu) = \frac{2^{-n}\Gamma(\tfrac{1}{2}+\mu+n)}{n!\Gamma(\tfrac{1}{2}+\mu-n)}, \quad n = 0,1,2,\ldots, \tag{32}$$

into (29). This gives the expansion

$$F_\mu(\xi,\sigma) \sim \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} (-1)^n A_n(\mu)\Phi_n, \tag{33}$$

where Φ_n is an incomplete gamma function (see (9))

$$\Phi_n = \int_\xi^\infty e^{-\sigma t} t^{-n-\frac{1}{2}} \, dt = \sigma^{n-1/2}\Gamma\left(\tfrac{1}{2}-n, \sigma\xi\right). \tag{34}$$

The function Φ_0 can be written in terms of the complementary error function (see (28))

$$\Phi_0 = \sqrt{\pi/\sigma}\,\mathrm{erfc}\sqrt{\sigma\xi} = \sqrt{\pi/\sigma}\,\mathrm{erfc}(\sqrt{y}-\sqrt{x}). \tag{35}$$

Further terms can be obtained from the recursion

$$\left(n - \tfrac{1}{2}\right)\Phi_n = -\sigma\Phi_{n-1} + e^{-\sigma\xi}\xi^{-n+\frac{1}{2}}, \quad n = 1,2,3,\ldots. \tag{36}$$

Using (30) and (33) we obtain

$$Q_\mu(x,y) \sim \sum_{n=0}^{\infty} \Psi_n, \tag{37}$$

where

$$\Psi_n = \frac{\rho^\mu}{2\sqrt{2\pi}}(-1)^n \left(A_n(\mu-1) - \frac{1}{\rho}A_n(\mu) \right)\Phi_n. \tag{38}$$

For $P_\mu(x,y)$ the expansion reads

$$P_\mu(x,y) \sim \sum_{n=0}^{\infty} \widetilde{\Psi}_n, \tag{39}$$

where $\widetilde{\Psi}_n = -\Psi_n$, $n \geq 1$, and

$$\widetilde{\Psi}_0 = \tfrac{1}{2}\rho^{\mu-\frac{1}{2}}\mathrm{erfc}(\sqrt{x}-\sqrt{y}). \tag{40}$$

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[2]http://dlmf.nist.gov/10.40.E1

Information on the asymptotic nature and error bounds of expansion (33) can be found in [Temme 1986], where numerical aspects of recursion (36) are discussed as well. Recursion of the functions Φ_n should begin at some point n_0 near $\sigma\xi$, and from n_0 backward or forward recursion has to be used. These results are based on [Gautschi 1961], where exponential integrals are considered, which are related to the incomplete gamma functions.

The expansions in (37) and (39) hold for large values of ξ, uniformly with respect to $\sigma \in [0, \infty)$.

Note that the integral defining $F_\mu(\xi, \sigma)$ in (29) becomes undetermined when $\sigma = 0$. However, since we use a combination of two $F-$functions in (30), and ρ tends to unity as $\sigma \to 0$, that is, as $x \to y$, the left-hand sides in (30) are well defined when $x = y$.

4.2. Asymptotic representations for large μ

For numerical calculations the large μ expansion given in [Temme 1993, Eq. (4.3)] is not suitable, and in this section we derive an alternative expansion with coefficients that are easier to evaluate than those derived earlier.

We consider both $Q_\mu(x, y)$ and $P_\mu(x, y)$ and use the integral representations given in (1) and (2).

For using the asymptotic properties of the Bessel function it is convenient to consider this function with order μ. Furthermore, we scale x and y and write

$$Q_{\mu+1}(\mu x, \mu y) = \int_{\mu y}^\infty \left(\frac{t}{\mu x}\right)^{\frac{1}{2}\mu} e^{-t-\mu x} I_\mu(2\sqrt{\mu x t})\, dt. \tag{41}$$

A change of variable $2\sqrt{\mu x t} = \mu z$ gives

$$Q_{\mu+1}(\mu x, \mu y) = \frac{\mu e^{-\mu x}}{(2x)^{\mu+1}} \int_\xi^\infty z^{\mu+1} e^{-\frac{\mu}{4x} z^2} I_\mu(\mu z)\, dz, \tag{42}$$

$$P_{\mu+1}(\mu x, \mu y) = \frac{\mu e^{-\mu x}}{(2x)^{\mu+1}} \int_0^\xi z^{\mu+1} e^{-\frac{\mu}{4x} z^2} I_\mu(\mu z)\, dz, \tag{43}$$

where

$$\xi = 2\sqrt{xy}. \tag{44}$$

4.2.1. Asymptotic expansion of the modified Bessel function. For deriving an asymptotic expansion of the functions $Q_\mu(x, y)$ and $P_\mu(x, y)$ we use the expansion[3]

$$I_\mu(\mu z) \sim \frac{1}{\sqrt{2\pi\mu}} \frac{e^{\mu\eta(z)}}{(1+z^2)^{1/4}} \sum_{k=0}^\infty \frac{u_k(t)}{\mu^k}, \quad \mu \to \infty, \quad z \geq 0, \tag{45}$$

where

$$t = \frac{1}{\sqrt{1+z^2}}, \qquad \eta(z) = \sqrt{1+z^2} + \log\frac{z}{1+\sqrt{1+z^2}}. \tag{46}$$

The first coefficients $u_k(t)$ are

$$u_0(t) = 1, \qquad u_1(t) = \frac{3t - 5t^3}{24}, \qquad u_2(t) = \frac{81t^2 - 462t^4 + 385t^6}{1152}, \tag{47}$$

and other coefficients can be obtained by applying the formula

$$u_{k+1}(t) = \tfrac{1}{2}t^2(1-t^2)u_k'(t) + \tfrac{1}{8}\int_0^t (1-5s^2)u_k(s)\, ds, \quad k = 0, 1, 2, \ldots. \tag{48}$$

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[3]http://dlmf.nist.gov/10.41.E3

4.3. Asymptotic expansion of $Q_{\mu+1}(\mu x, \mu y)$

We write (42) in the form

$$Q_{\mu+1}(\mu x, \mu y) = \frac{\mu e^{-\mu x}}{(2x)^{\mu+1}} \int_\xi^\infty z e^{-\mu\phi(z)} e^{-\mu\eta(z)} I_\mu(\mu z)\, dz, \tag{49}$$

where

$$\phi(z) = -\ln z + \frac{1}{4x} z^2 - \eta(z). \tag{50}$$

The saddle point follows from the equation $\phi'(z) = 0$. We have $\eta'(z) = \sqrt{1+z^2}/z$, and the saddle point is obtained by solving the equation

$$\phi'(z) = -\frac{1}{z} + \frac{z}{2x} - \frac{\sqrt{1+z^2}}{z} = \frac{-2x + z^2 - 2x\sqrt{1+z^2}}{2xz} = 0. \tag{51}$$

It follows that the positive saddle point z_0 is given by

$$z_0 = 2\sqrt{x(1+x)}. \tag{52}$$

The saddle point is located outside the interval of integration if $\xi > z_0$, that is, when $y > x + 1$, in which case $Q_{\mu+1}(\mu x, \mu y) \lesssim \frac{1}{2}$ (when the parameters are large). As discussed earlier, see also [Temme 1993, §4], the relation $y = x + 1$ (in the scaled variables) indicates a transition from small values of $Q_{\mu+1}(\mu x, \mu y)$ (when $y > x + 1$) to values close to 1 when ($y < x + 1$) (see also the first line of (10)).

So, for the $Q-$function the case $\xi > z_0$ is of special interest, because in that case this function is smaller than the $P-$function. We will derive an expansion in which the cases $y < x + 1$, $y = x + 1$, and $y > x + 1$ are included; however, for $y < x + 1$ we use an expansion for the $P-$function, see §4.4.

We use a method introduced in [Bleistein 1966], see also [Olver 1997, pp. 344–351]. As a standard form in asymptotic analysis with the above described features (a saddle point coalescing with an end point of integration) we can consider

$$\int_0^\infty e^{-\mu(\frac{1}{2}w^2 - \zeta w)} f(w)\, dw, \tag{53}$$

with saddle point ζ, which may be any complex number. We transform the integral in (49) into this form by writing

$$\phi(z) - \phi(\xi) = \tfrac{1}{2} w^2 - \zeta w, \tag{54}$$

where ζ has to be determined. The point $z = \xi$ corresponds to $w = 0$, and we assume that the $z-$saddle point z_0 corresponds to the $w-$saddle point at $w = \zeta$. Also, we assume that $\mathrm{sign}(z - z_0) = \mathrm{sign}(w - \zeta)$. This gives

$$\tfrac{1}{2}\zeta^2 = \phi(\xi) - \phi(z_0). \tag{55}$$

Because z_0 is the saddle point, the right-hand side is always nonnegative.

The sign of ζ is chosen as follows. As mentioned after (52), the saddle point z_0 is inside the domain of integration if $y < x + 1$. The same condition is used for the sign of ζ:

$$\zeta = \mathrm{sign}(x + 1 - y)\sqrt{2\left(\phi(\xi) - \phi(z_0)\right)}. \tag{56}$$

For the numerical evaluation of ζ from this relation, we refer to §4.6.1.

The transformation (54) applied to (49) gives

$$Q_{\mu+1}(\mu x, \mu y) = \sqrt{\frac{\mu}{2\pi}} e^{-\frac{1}{2}\mu\zeta^2} \int_0^\infty e^{-\mu(\frac{1}{2}w^2 - \zeta w)} f(w)\, dw, \tag{57}$$

where

$$f(w) = \sqrt{2\pi\mu}\frac{z}{2x}e^{-\mu\eta(z)}I_\mu(\mu z)\frac{dz}{dw}. \tag{58}$$

In the representation in (57) we have used the relation

$$(2x)^{-\mu}e^{-\mu x}e^{-\mu\phi(\xi)} = e^{-\frac{1}{2}\mu\zeta^2}, \tag{59}$$

which follows from observing that

$$\phi(z_0) = -x - \ln(2x). \tag{60}$$

In Bleistein's method an asymptotic expansion is obtained by expanding (interpolating, actually) this function at the points $w = 0$ and $w = \zeta$. We modify this method, as in [Olver 1997, pp. 346–348], by expanding at $w = \zeta$ only. Also, we replace the Bessel function with its asymptotic expansion in (45). In this way, we obtain

$$Q_{\mu+1}(\mu x, \mu y) \sim \sqrt{\frac{\mu}{2\pi}}\sum_{k=0}^{\infty}\frac{\Phi_k}{\mu^k}, \tag{61}$$

where

$$\Phi_k = e^{-\frac{1}{2}\mu\zeta^2}\int_0^\infty e^{-\mu(\frac{1}{2}w^2 - \zeta w)}f_k(w)\,dw, \tag{62}$$

with (see (45))

$$f_k(w) = \frac{z}{2x}\frac{u_k(t)}{(1+z^2)^{\frac{1}{4}}}\frac{dz}{dw}, \tag{63}$$

and t is as in (46). The relation between z and w follows from the transformation given in (54).

We expand

$$f_k(w) = \sum_{j=0}^{\infty}f_{jk}(w-\zeta)^j, \tag{64}$$

and obtain

$$\Phi_k \sim \sum_{j=0}^{\infty}f_{jk}\Psi_j(\zeta), \tag{65}$$

where

$$\Psi_j(\zeta) = e^{-\frac{1}{2}\mu\zeta^2}\int_0^\infty e^{-\mu(\frac{1}{2}w^2 - \zeta w)}(w-\zeta)^j\,dw. \tag{66}$$

These functions can be expressed in terms of the complementary error function defined in (28). The first ones are

$$\Psi_0(\zeta) = \sqrt{\frac{\pi}{2\mu}}\mathrm{erfc}\left(-\zeta\sqrt{\mu/2}\right), \quad \Psi_1(\zeta) = \frac{1}{\mu}e^{-\frac{1}{2}\mu\zeta^2}, \tag{67}$$

and the other ones follow from integrating by parts in (66), giving

$$\Psi_j(\zeta) = \frac{j-1}{\mu}\Psi_{j-2}(\zeta) + \frac{(-\zeta)^{j-1}}{\mu}e^{-\frac{1}{2}\mu\zeta^2}, \quad j = 2, 3, 4, \ldots. \tag{68}$$

By substituting in (66) $w = \zeta + \sqrt{2/\mu}s$ we have

$$\Psi_j(\zeta) = \left(\frac{2}{\mu}\right)^{(j+1)/2} \int_{-\zeta\sqrt{\mu/2}}^{\infty} e^{-s^2} s^j \, ds. \tag{69}$$

It follows that $\Psi_j(\zeta)$ can be written in terms of the incomplete gamma functions, see (9).

Combining these expansions, we find

$$Q_{\mu+1}(\mu x, \mu y) \sim \sqrt{\frac{\mu}{2\pi}} \sum_{j=0}^{\infty} A_j \Psi_j(\zeta), \quad A_j \sim \sum_{k=0}^{\infty} \frac{f_{jk}}{\mu^k}. \tag{70}$$

We can rearrange this expansion in the form

$$Q_{\mu+1}(\mu x, \mu y) \sim \sqrt{\frac{\mu}{2\pi}} \sum_{k=0}^{\infty} B_k, \quad B_k = \sum_{j=0}^{k} \frac{f_{j,k-j}\Psi_j(\zeta)}{\mu^{k-j}}. \tag{71}$$

The first coefficients f_{jk} are given in (90).

By separating the first term in this expansion, and using (67), we obtain

$$Q_{\mu+1}(\mu x, \mu y) \sim \tfrac{1}{2}\mathrm{erfc}\left(-\zeta\sqrt{\mu/2}\right) + \sqrt{\frac{\mu}{2\pi}} \sum_{k=1}^{\infty} B_k. \tag{72}$$

If we wish we can find the asymptotic representation for $Q_\mu(\mu x, \mu y)$ by using (11), which with the present notation can be written as

$$Q_\mu(\mu x, \mu y) = Q_{\mu+1}(\mu x, \mu y) - e^{-\frac{1}{2}\mu\zeta^2} e^{-\mu\eta(\xi)} I_\mu(\mu\xi). \tag{73}$$

This follows from

$$\tfrac{1}{2}\ln(y/x) - x - y + \eta(\xi) = \phi(z_0) - \phi(\xi) = -\tfrac{1}{2}\zeta^2. \tag{74}$$

The result is:

$$Q_\mu(\mu x, \mu y) \sim \tfrac{1}{2}\mathrm{erfc}\left(-\zeta\sqrt{\mu/2}\right) + \sqrt{\frac{\mu}{2\pi}} \sum_{k=1}^{\infty} B_k - e^{-\frac{1}{2}\mu\zeta^2} e^{-\mu\eta(\xi)} I_\mu(\mu\xi). \tag{75}$$

4.4. Asymptotic expansion of $P_{\mu+1}(\mu x, \mu y)$

The computation of the complementary function $P_\mu(x, y)$ is required when $y < x + \mu$. To avoid using the relation $P_\mu(x, y) = 1 - Q_\mu(x, y)$, which may cause a large relative error, we derive an expansion that can be used for the P−function, and that has a similar form and is in fact complementary relation with the expansion in (72).

Starting from (43) we have (cf. (49))

$$P_{\mu+1}(\mu x, \mu y) = \frac{\mu e^{-\mu x}}{(2x)^{\mu+1}} \int_0^\xi z e^{-\mu\phi(z)} e^{-\mu\eta(z)} I_\mu(\mu z) \, dz, \tag{76}$$

where ξ is defined in (44) and $\phi(z)$ is the same as in (49). This function has a saddle point z_0 defined in (52). When $y < x + 1$ (in the scaled variables used in (76)), we have $z_0 > \xi$.

We use the same mapping as in (54), now with the conditions that $z = 0$ corresponds to $w = +\infty$ and (again) $z = \xi$ to $w = 0$. This gives (cf. (57))

$$P_{\mu+1}(\mu x, \mu y) = -\sqrt{\frac{\mu}{2\pi}} e^{-\frac{1}{2}\mu\zeta^2} \int_0^\infty e^{-\mu(\frac{1}{2}w^2 - \zeta w)} f(w) \, dw, \tag{77}$$

where $f(w)$ has the same form as in (58), although the relation between z and w is different; for example $dz/dw < 0$. The quantity ζ is defined in (56) and it is positive in this case, because $y < x + 1$.

We can repeat the procedure that we used for the $Q-$function, and the main change is the choice of the square root in a_1 that occurs in the expansion in (87), which has its effect on signs of the coefficients $f_{j,k}$ given in (90). In this way we obtain the expansion (cf. (72))

$$P_{\mu+1}(\mu x, \mu y) \sim \tfrac{1}{2}\mathrm{erfc}\left(\zeta\sqrt{\mu/2}\right) + \sqrt{\frac{\mu}{2\pi}}\sum_{k=1}^{\infty}B_k^*, \tag{78}$$

where

$$B_k^* = \sum_{j=0}^{k}(-1)^j\frac{f_{j,k-j}\Psi_j(-\zeta)}{\mu^{k-j}}, \tag{79}$$

and $\Psi_j(-\zeta)$ can be obtained in the same way as $\Psi_j(\zeta)$ in (67)–(69). In this representation, the coefficients $f_{j,k}$ are the same as those used for the $Q-$function, and the first few are given in (90).

From (73) we obtain

$$P_\mu(\mu x, \mu y) = P_{\mu+1}(\mu x, \mu y) + e^{-\frac{1}{2}\mu\zeta^2}e^{-\mu\eta(\xi)}I_\mu(\mu\xi). \tag{80}$$

4.5. Where to use the expansions

From (69) we see that $\{\Psi_j(\zeta)\}$ is an asymptotic sequence for large μ and bounded values of $\zeta\sqrt{\mu/2}$. By assuming $-b \le \zeta\sqrt{\mu/2} \le b$, for some $b > 0$, we try to find a domain in the $(x, y)-$plane where we can use the expansions in (72) and (79).

For large values of μ the inequalities can only be satisfied when $|\zeta|$ is small. In that case we consider the expansion given in (85), and we use for small values of ζ the approximation $\zeta \sim -(y - x - 1)/\sqrt{(2x + 1}$. This gives for y the inequalities

$$x + 1 - b\sqrt{2/\mu}\sqrt{2x + 1} < y < x + 1 + b\sqrt{2/\mu}\sqrt{2x + 1}. \tag{81}$$

This is in the scaled variables for $Q_\mu(\mu x, \mu y)$. For the unscaled variables we have

$$x + \mu - b\sqrt{4x + 2\mu} < y < x + \mu + b\sqrt{4x + 2\mu}. \tag{82}$$

From numerical tests we will conclude which values μ and b can be used for a given set of coefficients in the expansions.

4.6. Computational aspects of the expansion

We consider some numerical aspects of the expansion of $Q_\mu(\mu x, \mu y)$. First we observe that verifying numerically the parameter domain for applying the asymptotic expansion for $Q_\mu(\mu x, \mu y)$ we can use the inhomogeneous recursion given in (73).

4.6.1. Expanding ζ. The value of ζ, see (55) and (56), vanishes when $z_0 = \xi$, that is, when $y = x + 1$. To avoid numerical cancellation we expand as follows. We have

$$\phi(z_0) = -x - \ln(2x), \quad \phi(\xi) = y - \sqrt{1 + 4xy} - \ln\left(\sqrt{1 + 4xy} - 1\right). \tag{83}$$

This gives

$$\tfrac{1}{2}\zeta^2 = \phi(\xi) - \phi(z_0) = x + y - \sqrt{1 + 4xy} + \ln\frac{1 + \sqrt{1 + 4xy}}{2y}. \tag{84}$$

Expanding the right-hand side of (84) in powers at $y = x + 1$, we obtain an expansion that can be written in the form

$$\zeta = -\frac{y - x - 1}{\sqrt{2x + 1}} \sum_{k=0}^{\infty} c_k z^k, \quad z = \frac{y - x - 1}{(2x + 1)^2}. \tag{85}$$

The first few coefficients are

$$c_0 = 1,$$
$$c_1 = -\tfrac{1}{3}(3x + 1),$$
$$c_2 = \tfrac{1}{36}(72x^2 + 42x + 7), \tag{86}$$
$$c_3 = -\tfrac{1}{540}(2700x^3 + 2142x^2 + 657x + 73),$$
$$c_4 = \tfrac{1}{12960}(15972x + 76356x^2 + 177552x^3 + 181440x^4 + 1331).$$

4.6.2. Computing the coefficients f_{jk}. The computation of the coefficients f_{jk} is rather straightforward, although we need a computer algebra package for obtaining enough coefficients for performing numerical calculations.

First we need the coefficients a_k in the expansion

$$z = \sum_{k=0}^{\infty} a_k (w - \zeta)^k, \quad a_0 = z_0. \tag{87}$$

These follow from the transformation in (54). To simplify the notation we introduce $u \in (0, 1]$ by writing

$$u = \frac{1}{\sqrt{2x + 1}}, \quad x = \frac{1 - u^2}{2u^2}. \tag{88}$$

Then the first few coefficients a_k are

$$a_0 = z_0 = 2\sqrt{x(1 + x)} = \frac{1}{u^2}\sqrt{1 - u^4},$$
$$a_1 = \sqrt{\frac{(2x + 1)x}{x + 1}} = \frac{1}{u}\sqrt{\frac{1 - u^2}{1 + u^2}},$$
$$a_2 = -a_1 \frac{u^3 \left(u^2 - 2\right)}{6 \left(1 + u^2\right)}, \tag{89}$$
$$a_3 = -a_1 \frac{u^4 \left(2u^6 + u^4 - 10u^2 + 9\right)}{36 \left(1 + u^2\right)^2},$$
$$a_4 = -a_1 \frac{u^5 \left(40u^{10} + 30u^8 - 141u^6 - 158u^4 + 432u^2 - 216\right)}{1080 \left(1 + u^2\right)^3}.$$

The coefficients f_{jk} are polynomials in u and the first few are

$$f_{0,0} = 1,$$

$$f_{0,1} = \frac{1}{24}u^2\left(3 - 5u^4\right),$$

$$f_{1,0} = \frac{1}{6}u\left(3 + u^2\right),$$

$$f_{0,2} = \frac{1}{1152}u^4\left(81 - 462u^4 + 385u^8\right),$$

$$f_{1,1} = -\frac{1}{144}u^3\left(9 - 21u^2 - 75u^4 + 95u^6\right),$$

$$f_{2,0} = -\frac{1}{24}u^2\left(3 - 5u^4\right),$$

$$f_{0,3} = \frac{1}{414720}u^6\left(30375 - 369603u^4 + 765765u^8 - 425425u^{12}\right),$$

$$f_{1,2} = -\frac{1}{6912}u^5\left(729 - 1053u^2 - 9702u^4 + 11550u^6 + 12705u^8 - 14245u^{10}\right),$$

$$f_{2,1} = \frac{1}{576}u^4\left(27 - 144u^2 - 402u^4 + 1440u^6 - 925u^8\right),$$

$$f_{3,0} = \frac{1}{2160}u^3\left(135 - 117u^2 - 675u^4 + 625u^6\right).$$

(90)

5. A QUADRATURE METHOD

Numerical quadrature of suitable integral representations can be an important tool for evaluating special functions, as explained in [Gil et al. 2007, Chapter 5]. In the case of the Marcum functions, the integrals in (1) and (2) give stable integral representations, but we prefer a representation in terms of elementary functions. Also, one important point for applying the trapezoidal rule efficiently is the vanishing of the integrand with many (or all) derivatives at the endpoints of integration.

The derivation of the integral representations in terms of elementary functions for the Marcum functions is described in detail in [Gil et al. 2013]. The starting point is the contour integral representation (see [Temme 1993, Eq. (2.3)])

$$Q_\mu(x, y) = \frac{e^{-x-y}}{2\pi i}\int_{\mathcal{L}_Q}\frac{e^{x/s+ys}}{1-s}\frac{ds}{s^\mu},$$

(91)

where $\mathcal{L}_Q$ is a vertical line that cuts the real axis in a point s_0, with $0 < s_0 < 1$.

For the complementary function we have

$$P_\mu(x, y) = \frac{e^{-x-y}}{2\pi i}\int_{\mathcal{L}_P}\frac{e^{x/s+ys}}{s-1}\frac{ds}{s^\mu},$$

(92)

now with a vertical line $\mathcal{L}_P$ that cuts the real axis at a point s_0 with $s_0 > 1$.

Introducing scaled variables x, y, we write the representation in (91) in the form

$$Q_\mu(\mu x, \mu y) = \frac{e^{-\mu(x+y)}}{2\pi i}\int_{\mathcal{L}_Q}\frac{e^{\mu\phi(s)}}{1-s}\,ds,$$

(93)

where

$$\phi(s) = \frac{x}{s} + ys - \ln s.$$

(94)

Using saddle point analysis for this expression, we arrive to a final integral representation in the form:

$$Q_\mu(\mu x, \mu y) = \frac{e^{-\frac{1}{2}\mu\zeta^2}}{2\pi} \int_{-\pi}^{\pi} e^{\mu\psi(\theta)} f(\theta)\, d\theta, \tag{95}$$

where ζ is the same quantity as used in §4.3, see (55), and

$$f(\theta) = \frac{\sin\theta\, r'(\theta) + (\cos\theta - r(\theta))\, r(\theta)}{r^2(\theta) - 2r\cos\theta r(\theta) + 1}, \tag{96}$$

$$\psi(\theta) = \cos\theta\rho(\theta,\xi) - \sqrt{1+\xi^2} - \ln\frac{\dfrac{\theta}{\sin\theta} + \rho(\theta,\xi)}{1 + \sqrt{1+\xi^2}}, \tag{97}$$

with $\xi = 2\sqrt{xy}$, and $r(\theta)$ and $\rho(\theta,\xi)$ defined as follows

$$r(\theta) = \frac{1}{2y}\left(\frac{\theta}{\sin\theta} + \rho(\theta,\xi)\right), \quad \rho(\theta,\xi) = \sqrt{\left(\frac{\theta}{\sin\theta}\right)^2 + \xi^2}. \tag{98}$$

Some extra care is needed when computing the expressions for small values of θ. Details are given in [Gil et al. 2013] on how to proceed in this situation when computing (96), (97) and (98).

Finally, an important point is to determine the domain of applicability of the quadrature method for computing the Marcum-Q function:

In §4.5 we have explained where we can use the asymptotic expansion derived in §4.3. These are valid around the transition line $y = x + \mu$ for large values of y, μ. See in particular the domain given in (82). The quadrature method is not valid inside this domain because the saddle point tends to unity (where the pole is located in the representation (91)) as $y \to x+\mu$. So, in the resulting algorithm the quadrature method will be used outside the domain specified by (82) with a proper value of b. Numerical test leads us to consider $b = 1$. It should be observed that the quadrature method does not necessarily need large parameters, although it also performs quite well in that case.

In [Helstrom 1992] the trapezoidal rule is used by including the pole at $s = 1$ in (93) in the function $\phi(s)$. That is, by writing (compare (94))

$$\widetilde{\phi}(s) = \frac{x}{s} + ys - \ln s - \frac{1}{\mu}\ln(1 - s). \tag{99}$$

In that case the saddle point has to be calculated from a cubic polynomial and the contour follows from $\Im\widetilde{\phi}(s) = 0$ through that saddle point. Helstrom used an approximation of this contour by taking a parabola centered at the saddle point. The results in [Helstrom 1992, Table I] show correct values at the critical value $y = x + 1$, also for large values of the parameters, as we will comment later.

6. ALGORITHM FOR THE MARCUM FUNCTIONS, NUMERICAL TESTS AND COMPARISONS

We propose an algorithm for computing the Marcum functions which combines different methods of evaluation in different regions: series expansions, integral representations, asymptotic expansions, and use of the three-term homogeneous recurrence relation given in (14).

The algorithm uses one or another method based on the expected region of validity of each method supported by numerical tests of each method, particularly near the boundaries of the domains where we switch methods.

From the numerical tests that we later discuss in detail, the following scheme of computation has been adopted.

We use unscaled variables. Let $\xi = 2\sqrt{xy}$ and

$$f_1(x,\mu) = x + \mu - \sqrt{4x + 2\mu}, \quad f_2(x,\mu) = x + \mu + \sqrt{4x + 2\mu}. \tag{100}$$

Then the scheme is as follows:

(1) If $x < 30$, then compute the series expansion (§3).
(2) If $\xi > 30$ and $\mu^2 < 2\xi$, then compute the asymptotic expansion (§4.1).
(3) If $f_1(x,\mu) < y < f_2(x,\mu)$ and $\mu < 135$, then compute the Marcum functions using the recurrence relations (14).
(4) If $f_1(x,\mu) < y < f_2(x,\mu)$ and $\mu \geq 135$, then use the asymptotic expansion (§4.2).
(5) In other case: compute the integral representation (§5).

Next we give details on the numerical tests for each method and on the global tests for the complete algorithm.

6.1. Testing

For testing the accuracy of the different methods described in the previous sections, we have used the recurrence relation given in (12). Because, as discussed previously, the Q is neither minimal nor domininant, the recursion should not be tested in the backward of forward direction; instead, we write the recurrence in the form

$$\frac{(x - \mu)Q_{\mu+1}(x,y) + (y + \mu)Q_\mu(x,y)}{xQ_{\mu+2}(x,y) + yQ_{\mu-1}(x,y)} = 1. \tag{101}$$

and the deviations from 1 of the left-hand side of (101) (in absolute value) will measure the accuracy of the tested methods. We use (101) when $y \geq x + \mu$ and the same expression but for $P_\mu(x,y)$ when $y < x + \mu$. For $x > \mu$ all terms in the numerator and the denominator are positive. For the case $x < \mu$ a negative terms appears in the numerator and a more stable test, both for P (when $y < x + \mu$) and Q (when $y > x + \mu$) is provided by writing the recurrence as

$$\frac{(y + \mu)Q_\mu(x,y)}{xQ_{\mu+2}(x,y) + (\mu - x)Q_{\mu+1}(x,y) + yQ_{\mu-1}(x,y)} = 1. \tag{102}$$

We should note that the tests involve four functions and that the errors in the individual evaluations of each function turn out to be slightly smaller than the error of the test of the recurrences.

We have first checked the implementation of the series expansions given in (7), which are used for $x < 30$. For computing the ratios of gamma functions appearing in the expressions, we have used the algorithms given in [Gil et al. 2012]. As discussed before, the series may present some error degradation for values of the functions close to the underflow limit; also, a mild degradation takes place as large parameters are considered. We have conducted a series of tests using 10^8 random points in regions such that $(x,y,\mu) \in [0,30) \times [0,A] \times [1,A]$ with increasing A. For values of P greater than 10^{-280}, our tests show that a relative accuracy better than 10^{-12} when testing (101) and (102) can be obtained for $A = 200$, while the relative accuracies obtained for $A = 1000, 5000, 10000$ are approximately 3×10^{-12}, 3×10^{-11} and 5×10^{-11} respectively. For function values smaller than 10^{-280} and larger than 10^{-290} the relative accuracies are quite uniform for different values of A and around 5×10^{-11}. For function values smaller than 10^{-290} the loss of accuracy becomes more severe. For this reason, we have

x/μ	$Q_\mu(x,y)$	$P_\mu(x,y)$	$Q_\mu(x,y)$ with MATLAB
0.01	1.9845278031193e−4	0.9998015472196881	1.9845278031**062**e−4
0.02	4.138241872117e−3	0.9958617581278824	4.13**4063851051513**e−3
0.03	0.04000364971081	0.9599963502891851	0.0**3999947168921022**
0.04	0.191650654805848	0.8083493451941514	0.1916**464767819643**
0.05	0.498535453743169	0.5014645462568305	0.49853**12757146799**
0.06	0.803520373008492	0.1964796269915073	0.8035203730**33010**
0.07	0.95565734175388	0.04434265824612003	0.9556573417**477112**
0.08	0.9944737609126645	0.005526239087335513	0.9944737609**062473**
0.09	0.9996249723836407	0.00037502761635937467	0.999624972383644**7**
0.1	0.9999861372355183	0.00001386276448162126	0.999986137235518**4**

Table 6.1. Values for $P_\mu(x,y)$ and $Q_\mu(x,y)$ obtained by using the asymptotic expansion of (72). The values of μ and y are fixed to μ=8192 and $y = 1.05\mu$. The last column shows the values obtained for $Q_\mu(x,y)$ with the MATLAB function **marcumq**; bold font indicates the wrong digits in the MATLAB computations.

limited the output of the algorithms to values larger than 10^{-290}; and when a smaller value is obtained, we set it to zero.

The relations (101) and (101) have also been tested for the asymptotic representation for large values of $\xi = 2\sqrt{xy}$ derived in §4.1. Our results show that an accuracy better than 6×10^{-14} can be obtained in the parameter region defined by $\xi > 30$, $\mu^2 < 2\xi$. For this case 10^{11} random points have been considered in the domain $(x, y, \mu) \in [30, 10000] \times [0, 10000] \times [1, 10000]$ and the values corresponding to random points inside the region of interest and with values not underflowing have been tested (close to 8×10^8 points).

The asymptotic expansion for large μ given in (72) has been tested first for some of the values of μ, x and y of Table 5.1 in [Temme 1993] and Table I of [Robertson 1976]. In these tables, the values of μ and y are fixed to $\mu = 8192$ and $y = 1.05\mu$. The values obtained for $P_\mu(x,y)$ and $Q_\mu(x,y)$ are shown in Table 1 and they are in agreement with those of reference [Temme 1993]. For comparison, we include the values given by MATLAB, and we observe that most of them present loss of accuracy; we will come back to this point next.

We have also used the asymptotic expansion of §4 for the large values of μ (up to $\mu = 10^9$) as shown in [Helstrom 1992, Table I], and we confirm the shown values of that table. These results have been tested with Maple. Finally, similarly to the other methods, we have checked the accuracy obtained with the large μ-expansion when testing the recurrence relation in the parameter region defined by the relation $x + \mu - \sqrt{4x + 2\mu} < y < x + \mu + \sqrt{4x + 2\mu}$. We consider 10^{10} random points for values of the parameters such that $(\mu, x, y) \in [136, A] \times [0, A] \times [0, A]$ and we analyze the accuracy when the asymptotic expansion is evaluated. As expected, our results show that the highest errors occur for the smaller values of μ, and the accuracy tends to decrease as A becomes large, particularly close to the transition line $y = x + \mu$. We get accuracies better than 3×10^{-13}, 2×10^{-12} and 10^{-11} for $A = 200, 1000, 10000$ respectively; the percentage of points in these regions where asymptotics for large μ is used is between 0.01% ($A = 200$) and 0.03% ($A = 10000$). We have performed an additional test for values of μ close to 136 and the accuracies confirm the previous results and the fact that the accuracy is better as μ is higher.

Finally, we have tested (101) and (102) when the quadrature method is used in the regions $[0, A] \times [0, A] \times [1, A]$. We consider 10^8 random points for three different values. We obtain an accuracy better than 6×10^{-13}, 1.1×10^{-12} and 10^{-11} for $A = 200, 1000, 10000$ respectively. It is important to observe that more than 95% of the points in these regions corresponds to quadrature, which means that our algorithm

uses quadrature most of the time. However, the rest of methods are crucial in order to cover regions where the quadrature method looses accuracy.

6.1.1. Global test. The algorithm has been implemented in the Fortran 90 module **MarcumQ**, which includes the Fortran 90 routine **marcum** for the computation of $Q_\mu(x, y)$ and $P_\mu(x, y)$. After the assembly of the different methods, we have tested that in the regions $(x, y, \mu) \in [0, A] \times [0, A] \times [1, A]$ we obtain relative accuracies close to 10^{-12}, 10^{-11} and 5×10^{-11} for $A = 200, 1000, 10000$ respectively when the function values are larger that 10^{-280}. A total of 10^9 random points was considered for each of these tests. For values between 10^{-280} and 10^{-290} the accuracy is close to 5×10^{-11} for the three values of A. We set all values smaller than 10^{-290} to be equal to zero.

6.2. Comparison against other software

As far as we know, the only available software for computing Marcum functions are the functions **marcumq** of MATLAB, which according to MathWorks documentation implements the algorithm given in [Shnidman 1989], and the function **MarcumQ** of Mathematica. The MATLAB function is a fixed precision algorithm while the Mathematica function is a variable precision one. It makes sense to compare our algorithm against MATLAB's, but it is no so clear how to compare against a variable precision algorithm, particularly when Mathematica gives no clue about the methods used.

Mathematica computes the function $\widetilde{Q}_m(\alpha, \beta)$ of (4), in our notation $Q_\mu(x, y)$ with $x = \alpha^2/2$ and $y = \beta^2/2$; it also computes the complementary function P. Real parameters m, α and β are accepted. It appears that the algorithm gives correct results if a sufficiently high accuracy is considered, but that it may fail if the demanded accuracy is not high enough. We give two examples of this. For instance, computing with 16 digits accuracy $\widetilde{P}_1(40, 20)$ with the command **N[MarcumQ[1,40,0,20],16]** gives the wrong result $-7.170326438841136 \times 10^{-42}$ while if no accuracy is declared (with **N[MarcumQ[1,40,0,20]]**) Mathematica gives 1.94499×10^{-89} which is correct; with 64 digits accuracy Mathematica still returns a negative value, but the order of magnitude is correct and with 65 we get three correct digits. A second example is provided by the computation of $\widetilde{Q}_1(31, 20)$. Computing with 16 digits accuracy $\widetilde{Q}_1(31, 20)$ with the command **N[MarcumQ[1,31,20],16]** gives the wrong result $7.607098470200109 \times 10^{20}$ while if no accuracy is declared (with **N[MarcumQ[1,31,20]]**) Mathematica gives 1 which is correct; declaring 37 digits of accuracy is needed to obtain three correct digits. These examples seem to indicate that Mathematica uses some approximations suffering from severe cancellations which need to be computed with high accuracy. It is always possible to get the correct result, but the accuracy needed is uncertain. Apart from this, the Mathematica algorithm appears to be much slower than our algorithm. For instance, in the region $(x, y, \mu) \in [0, 200] \times [0, 200] \times [1, 200]$ the average time per function evaluation is of the order of 1 second for computing $\widetilde{Q}$, and slightly larger for $\widetilde{P}$; in the region $(x, y, \mu) \in [0, 20] \times [0, 20] \times [1, 200]$ the time is of the order of $0.1\,s$ per evaluation. This has to be compared with the $2 \times 10^{-4}\,s$ per evaluation of our algorithm in the region $(x, y, \mu) \in [0, 200] \times [0, 200] \times [1, 200]$. We have used Mathematica 8 in our tests; the numerical values given before can also be checked online at the WolframAlpha site (www.wolframalpha.com).

The comparison with MATLAB is more natural because it computes the Marcum function with fixed precision arithmetic. It is important to observe that our method is more general than the function **marcumq** of MATLAB, which can only be used for integer μ. MATLAB computes $\widetilde{Q}_m(\alpha, \beta)$, but it does not compute the $P-$function; the only way to obtain values of P from MATLAB is by computing them as $1 - Q$, but then small values of P loose accuracy. In terms of speed, our algorithm appears to be

much faster. In average, MATLAB spends around $8 \times 10^{-3} s$ per evaluation for $(x, y, \mu) \in [0, 200] \times [0, 200] \times [1, 200]$, while our algorithm spends $2 \times 10^{-4} s$; when $(x, y, \mu) \in [0, 20] \times [0, 20] \times [1, 200]$ MATLAB spends an average time of $2 \times 10^{-3} s$ per evaluation and our algorithm $3 \times 10^{-5} s$.

Apart from being faster, our algorithm is more accurate and reliable than MATLAB's **marcumq** and we have found ranges of the parameters where our algorithm is accurate while MATLAB gives wrong results.

Some of these accuracy problems occur for large μ, in the region where we consider the asymptotic expansion of (72), as shown in Table 6.1.

Severe problems occur with the MATLAB algorithm near the transition line $y = x + \mu$ for μ large and x small even more, but also for larger values of y. For example, the minimum (maximum) error found when computing $Q_{800}(x, y)$ with $x \in [0.3, 5]$ and $y \in [806, 870]$ (taking a step-size in the intervals of $\Delta x = 0.2$ and $\Delta y = 1$, respectively) is 0.87 (1.96). The error is computed as $2 \|Q_1 - Q_2\| / (Q_1 + Q_2)$, Q_1, Q_2 being the values obtained with MATLAB's **marcumq** and our Fortran 90 module, respectively. In other words, none of the 1650 computed MATLAB values are correct. As a specific numerical example, MATLAB gives $Q_{800}(0.4, 810) \simeq 0.0053$ while the correct value is approximately 0.3632.

This type of errors is so clear that it becomes noticeable by just plotting function values. The plot of the decreasing function $Q_{800}(1, y)$ as a function of y shows different abrupt changes when computed with MATLAB. For small y the results are correct, but plotting the function for $y \in [750, 850]$ we observe a steep jump close to $y = 800$ and for $800 < y < 1100$ all the values are wrong (smaller than the correct values); for $y > 1100$ a different type of error occurs and the algorithm returns approximately machine-epsilon and for for $y = 2348$ the result jumps to approximately 10^{-300} and it jumps again to machine-epsilon when $2375 < y < 2384$. Finally, for $y > 2384$ the algorithm gives 0, which is reasonable because the value of Q is already close to underflow. These anomalies are present for smaller values of m. For $m = 550$ the first jump is still noticeable, and the other phases (machine-epsilon and close to underflow) remain. For smaller values of m the first jump becomes unnoticeable, but the other (wrong) phases remain for large y.

There exist other regions where MATLAB gives no accuracy. For instance, the plot of the increasing function $Q_2(x, 200)$ as a function of x shows that the MATLAB function oscillates very rapidly for $0 < x < 70$, with most values smaller than machine-epsilon; all these values are wrong. Similar errors occur also for large m and even some negative results are obtained. We observe that if for x small enough, Q becomes smaller than machine-epsilon, then the MATLAB values are wrong. As x increases, the error tends to disappear.

None of these anomalies occur for our algorithm, which is able to compute accurately for parameters smaller than 10000. Our main loss of accuracy comes from the use of the series when the functions are close to the underflow limit, but the results are accurate if they are larger than 10^{-290}.

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