EXAMPLES OF TAUBERIAN OPERATORS ACTING ON $C[0,1]$

MANUEL GONZÁLEZ AND ANTONIO MARTÍNEZ-ABEJÓN

Abstract. We show that some counterexamples in the theory of tauberian operators can be realized as operators acting on $C[0,1]$. Precisely, we show that the set $\tau(C[0,1])$ of tauberian operators acting on $C[0,1]$ is not open, and that $T \in \tau(C[0,1])$ does not imply T^{**} tauberian.

1. Introduction

Tauberian operators arose almost simultaneously in summability theory [6], factorization of operators [4], and certain generalizations of Fredholm theory [17], but were formally introduced by Kalton and Wilansky in [12]. Besides the factorization of weakly compact operators throughout reflexive Banach spaces discovered by Davis et al. [4], the tauberian operators have been successfully applied in other branches of Banach space theory like preservation of isomorphic properties [14], equivalence between the Radon-Nikodym property and the Krein-Milman property [16], and refinements of James' characterization of reflexive spaces [15]. The class of tauberian operators is a semigroup in the sense of [1] associated to the operator ideal of the weakly compact operators [9, Theorem 2], and contains all isomorphic embeddings. We refer to [8] for additional information on the subject.

Let $\mathcal{L}(X,Y)$ denote the set of all bounded operators acting between the Banach spaces X and Y. Given a class $\mathcal{A}$ of operators, let $\mathcal{A}(X,Y) := \mathcal{A} \cap \mathcal{L}(X,Y)$ denote the component of all bounded operators of $\mathcal{A}$ acting between X and Y, and consider the two following statements:

(a) the component $\mathcal{A}(X,Y)$ is open for all X and all Y;
(b) if $T \in \mathcal{A}(X,Y)$ then $T^{**} \in \mathcal{A}(X^{**},Y^{**})$.

The fact that the class of isomorphic embeddings satisfies (a) and (b) is well-known. Nevertheless, it was proved in [2] that neither (a) nor (b) holds for the class of tauberian operators (see Sections 2.1 and 3.1 in [8]). The corresponding counterexamples were obtained by finding some tauberian operators $T\colon X \longrightarrow X$ acting on certain Banach spaces X constructed ad hoc. Thus, there is still some interest in knowing whether (a) and (b) are satisfied by the tauberian operators $T\colon X \longrightarrow Y$ acting between classical Banach spaces. For instance, it is known that both (a) and (b) hold when X is L_1 (see Proposition 6.2.7 and Theorem 6.2.18 in [8] for (a), and Theorem 4.4.2 in [8] for (b)). In this paper we prove that the class of tauberian operators satisfies neither (a) nor (b) when $X = Y = C[0,1]$. Our tools will be the push-out construction of Kisliakov (see [3, Section 1.3] or [5, Lemma 15.14]), and a technical result on embeddings of quotients of separable spaces into ℓ_∞.

We use standard notation in Banach space theory. Capital letters X, Y denote Banach spaces, and the action of an element of the dual space $x^* \in X^*$ on $x \in X$ is denoted by

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$\langle x^*, x\rangle$. The subset of all norm one elements of X is denoted by S_X, and B_X denotes the subset of all elements $x \in X$ such that $\|x\| \leq 1$. Given a subspace Y of X, the annihilator of Y in X^* is $Y^\perp$. The second dual (or bidual) of X is denoted X^{**}. We identify X with a subspace of X^{**}, and denote by X^{co} the quotient X^{**}/X. Operators are continuous linear maps. The range and the kernel of an operator $T\colon X \longrightarrow Y$ are respectively denoted by $R(T)$ and $N(T)$. Moreover, $T^*\colon Y^* \longrightarrow X^*$ is the conjugate of T, and $T^{**}\colon X^{**} \longrightarrow Y^{**}$ is the second conjugate (or biconjugate) of T. The operator $T^{co}\colon X^{co} \longrightarrow Y^{co}$ that maps $x^{**} + X$ to $T^{**}x^{**} + Y$ is called the *residuum operator* of T [10]. Given a subspace E of X, the quotient operator from X onto X/E is denoted by Q_E. As usual, an operator $T\colon X \longrightarrow Y$ is said to be an *isomorphic embedding* if there is a constant $C > 0$ such that $\|Tx\| \geq C\|x\|$ for all $x \in X$; if $\|Tx\| = \|x\|$ for all $x \in X$ then T is said to be an *isometric embedding*.

An operator $T\colon X \longrightarrow Y$ is said to be *tauberian* if $T^{**}(X^{**}\setminus X) \subset Y^{**}\setminus Y$; equivalently, if T^{co} is injective [8, Proposition 3.1.8]. The class of all tauberian operators will be denoted by $\mathcal{T}$. Therefore, given Banach spaces X and Y, the component $\mathcal{T}(X, Y)$ consists of all tauberian operators in $\mathcal{L}(X, Y)$. In the case $X = Y$ we write $\mathcal{T}(X)$ instead of $\mathcal{T}(X, X)$.

Isomorphic embeddings belong to $\mathcal{T}$, and given operators $S\colon X \longrightarrow Y$ and $T\colon Y \longrightarrow Z$, the following assertions are satisfied (see [8, Section 2.1]):

(i) if $T \in \mathcal{T}$ and $S \in \mathcal{T}$ then $TS \in \mathcal{T}$;
(ii) if $TS \in \mathcal{T}$ then $S \in \mathcal{T}$.

2. The push-out of a pair of operators

Given a pair of operators $A\colon X \longrightarrow Y$ and $B\colon X \longrightarrow Z$, let Δ be the closure of the subspace $D := \{(Bx, -Ax)\colon x \in X\}$. The *push-out space* Σ *of* (B, A) is the range of the quotient operator

$$Q : Z \oplus_1 Y \longrightarrow \Sigma := \frac{Z \oplus_1 Y}{\Delta}$$

and the operators $j_A\colon Z \longrightarrow \Sigma$ and $j_B\colon Y \longrightarrow \Sigma$, defined by $j_A(z) := Q(z, 0)$ and $j_B(y) := Q(0, y)$, produce the *push-out diagram of* (B, A)*:*

$$\begin{array}{ccc} X & \xrightarrow{\ A\ } & Y \\ {\scriptstyle B}\downarrow & & \downarrow{\scriptstyle j_B} \\ Z & \xrightarrow[\ j_A\ ]{} & \Sigma \end{array} \tag{1}$$

It is immediate that $j_A \circ B = j_B \circ A$, $\|j_B\| \leq 1$ and $\|j_A\| \leq 1$. Moreover, if A (or B) is an isomorphic embedding then the subspace D is closed.

The push-out diagram is characterized up to isomorphisms by the following result.

Proposition 2.1. ***(Universal property)*** *For any space U and any pair of operators $u\colon Y \longrightarrow U$ and $v\colon Z \longrightarrow U$ such that $u \circ A = v \circ B$ there exists a unique operator $j\colon \Sigma \longrightarrow U$ such that $u = j \circ j_B$ and $v = j \circ j_A$.*

The operator j is given by $j\big((z, y) + \Delta\big) = v(z) + u(y)$.

Remark 2.2. *The quotient map $Q\colon Z \oplus_1 Y \longrightarrow \Sigma$ satisfies $Q(z, y) := j_A(z) + j_B(y)$.*

Additional properties of the operators A or B produce additional properties of their push-out. Note that the role played by A is symmetric with respect to that one played by B in their push-out diagram of (A, B).

Proposition 2.3. *Let $A\colon X \longrightarrow Y$ and $B\colon X \longrightarrow Z$ be a pair of operators and suppose that B is an isomorphic embedding. Then the following assertions are satisfied:*

(i) *j_B is an isomorphic embedding.*
(ii) *A is injective if and only if j_A is injective.*

Proof. Since B is an isomorphic embedding, Δ equals $\{(Bx, -Ax)\colon x \in X\}$.

(i) Suppose $\|Bx\| \geq C\|x\|$ for each $x \in X$. Given $y \in Y$ with $\|y\| = 1$,

$$\|j_B(y)\| = \inf_{x\in X} \|(0, y) + (Bx, -Ax)\| = \inf_{x\in X} \|Bx\| + \|y - Ax\|.$$

In the case $2\|x\|\|A\| \leq 1$, we get $\|y - Ax\| \geq 1/2$. Otherwise $\|Bx\| \geq C(2\|A\|)^{-1}$. Thus $\|j_B(y)\| \geq \min\{1/2, C(2\|A\|)^{-1}\}\|y\|$ for each $y \in Y$.

(ii) Clearly j_A injective $\Rightarrow$ $j_AB = j_BA$ injective $\Rightarrow$ A injective. Conversely, suppose that A is injective. Then $j_A(z) = 0$ implies $(z, 0) \in \Delta$. Thus $z = Bx$ for some $x \in X$ such that $Ax = 0$; hence $z = 0$. □

Remark 2.4. When B is an isomorphic embedding, Proposition 2.3 tells us that j_B is an isomorphic embedding of Y into Σ. Thus the operator j_A can be seen as an extension of A. To emphasize this fact, sometimes we will write $\widehat{A}$ instead of j_A in that case.

From now on, when we say that *we can identify operators or diagrams,* we mean that we can identify them up to bijective isomorphisms.

The following result shows that the action of taking biconjugates and that of forming push-outs commute in some cases.

Proposition 2.5. *Let $A\colon X \longrightarrow Y$ be an operator and let $B\colon X \longrightarrow Z$ be an isomorphic embedding. Then the second conjugate of the push-out diagram of (B, A) can be identified with the push-out diagram of (B^{**}, A^{**}).*

Proof. The diagrams to be identified are

$$\begin{array}{ccc} X^{**} & \xrightarrow{A^{**}} & Y^{**} \\ {\scriptstyle B^{**}}\downarrow & & \downarrow{\scriptstyle j_B^{**}} \\ Z^{**} & \xrightarrow{j_A^{**}} & \left(\dfrac{Z \oplus_1 Y}{\Delta}\right)^{**} \end{array} \qquad\qquad \begin{array}{ccc} X^{**} & \xrightarrow{A^{**}} & Y^{**} \\ {\scriptstyle B^{**}}\downarrow & & \downarrow{\scriptstyle j_{B^{**}}} \\ Z^{**} & \xrightarrow{j_{A^{**}}} & \dfrac{Z^{**} \oplus_1 Y^{**}}{\Gamma} \end{array}$$

where $\Delta = \{(Bx, -Ax)\colon x \in X\}$ and $\Gamma = \{(B^{**}x^{**}, -A^{**}x^{**})\colon x^{**} \in X^{**}\}$.

The universal property of the push-out (Proposition 2.1) provides an operator

$$U\colon \frac{Z^{**} \oplus_1 Y^{**}}{\Gamma} \longrightarrow \left(\frac{Z \oplus_1 Y}{\Delta}\right)^{**},$$

given by $U\big((z^{**}, y^{**}) + \Gamma\big) = j_A^{**}(z^{**}) + j_B^{**}(y^{**})$, so that $j_B^{**} = Uj_{B^{**}}$ and $j_A^{**} = Uj_{A^{**}}$.

By Remark 2.2, $R(U) = R(Q^{**})$, where $Q\colon Z \oplus_1 Y \longrightarrow \Sigma$ is the quotient. Hence U is surjective. Moreover, $N(Q^{**}) = \Gamma$; hence U is injective. Thus U is a bijective isomorphism, so the result is proved. □

The following result is known. We give a proof for completeness.

Proposition 2.6. *Given a subspace Y of X, the following statements hold:*

(i) *the quotient* X/Y *embeds isometrically in* $X^{**}/Y^{\perp\perp}$ *via* $\varphi(x+Y) := x+Y^{\perp\perp}$ *and* $X + Y^{\perp\perp}$ *is a closed subspace of* X^{**}*;*

(ii) *the operator*

$$\frac{X^{**}/Y^{\perp\perp}}{\varphi(X/Y)} \longrightarrow \frac{X^{**}}{X+Y^{\perp\perp}}$$

that maps $(x^{**}+Y^{\perp\perp})+\varphi(X/Y)$ *to* $x^{**}+(X+Y^{\perp\perp})$ *is a surjective isometry.*

Proof. (i) Take $x \in X$ such that $\operatorname{dist}(x,Y) = 1$. Since there exists $x^* \in Y^\perp$ with $\|x^*\| = 1$ so that $\langle x^*, x\rangle = 1$, we obtain $\operatorname{dist}(x, Y^{\perp\perp}) = 1$, which proves that φ is an isometry. In particular, $R(\varphi) = (X+Y^{\perp\perp})/Y^{\perp\perp}$ is closed. Moreover, the quotient map $q : X^{**} \longrightarrow X^{**}/Y^{\perp\perp}$ satisfies $X + Y^{\perp\perp} = q^{-1}\big(R(\varphi)\big)$, hence $X + Y^{\perp\perp}$ is closed.

(ii) Given $x^{**} \in X^{**}$, part (i) yields

$$\|(x^{**}+Y^{\perp\perp})+\varphi(X/Y)\| = \inf_{x\in X} \|(x^{**}+Y^{\perp\perp}) - \varphi(x+Y)\| =$$

$$= \inf_{x\in X} \|x^{**} - x + Y^{\perp\perp}\| = \inf_{x\in X,\, y^{**}\in Y^{\perp\perp}} \|x^{**} - (x+y^{**})\| = \|x^{**} + (X+Y^{\perp\perp})\|$$

which clearly shows the result. □

Corollary 2.7. *Given a quotient map* $Q_M : X \longrightarrow X/M$*, the operator* Q_M^{co} *is surjective.*

The following result shows that the action of passing to residuum operators and that of forming push-outs commute in some cases.

Proposition 2.8. *Let* $A\colon X \longrightarrow Y$ *be an operator and let* $B\colon X \longrightarrow Z$ *be an isomorphic embedding. Then the residuum of the push-out diagram of* (B, A) *can be identified with the push-out diagram of* (B^{co}, A^{co}).

Proof. The proof is formally similar to that of Proposition 2.5. We have to show that we can identify the following diagrams

where $\Delta := \{(Bx, -Ax)\colon x \in X\}$ and $\Upsilon := \{(B^{co}x^{co}, -A^{co}x^{co})\colon x^{co} \in X^{co}\}$.

The universal property of the push-out (Proposition 2.1) provides an operator

$$V\colon \frac{Z^{co}\oplus_1 Y^{co}}{\Upsilon} \longrightarrow \left(\frac{Z\oplus_1 Y}{\Delta}\right)^{co},$$

given by $V\big((z^{co}, y^{co}) + \Upsilon\big) = j_A^{co}(z^{co}) + j_B^{co}(y^{co})$, so that $j_B^{co} = Vj_{B^{co}}$ and $j_A^{co} = Vj_{A^{co}}$.

By Remark 2.2, $R(V) = R(Q^{co})$, where $Q\colon Z\oplus_1 Y \longrightarrow \Sigma$ is the quotient map. By Corollary 2.7 Q^{co} is surjective; hence so is V. Moreover $N(Q^{**}) = \Upsilon$ implies V injective. Thus V is a bijective isomorphism, and the result is proved. □

Proposition 2.9. *Consider the push-out diagram of* (B, A) *given in (1) and assume* B *is an isomorphic embedding. Then the following assertions are satisfied:*

(i) A *is tauberian if and only if so is* j_A.

(ii) A^{**} *is tauberian if and only if so is* j_A^{**}.

Proof. (i) Since B is an isomorphism, B^{co} is injective with closed range [8, Proposition 3.1.15]; hence it is an isomorphic embedding.

Assume A is tauberian; equivalently, assume A^{co} is injective. Let Σ be the push-out of (B, A). Following Proposition 2.8, Σ^{co} is the push-out of (B^{co}, A^{co}), and as B^{co} is an isomorphism and A^{co} is injective, part (ii) in Proposition 2.3 yields that j_A^{co} is also injective, hence j_A is tauberian.

For the reverse, assume that j_A is tauberian. As B is an isomorphism, then $j_A B = j_B A$ is tauberian, hence A is tauberian too.

(ii) Since Proposition 2.5 identifies j_A^{**} with $j_{A^{**}}$, the result follows from (i). □

3. Embedding separable quotients into ℓ_∞

The gap between two subspaces E and F of a given Banach space X measures the closeness of the positions of E and F inside X.

Definition 3.1. *Let E and F be subspaces of a Banach space X. The* gap between E and F *is defined as the real number*

$$\delta(E, F) := \max\{\sup_{x\in S_E} \operatorname{dist}(x, F)\,,\ \sup_{y\in S_F} \operatorname{dist}(y, E)\} \geq 0.$$

Observe that $\delta(E, F) = 0$ if and only if $E = F$. We refer to [13, Section IV.2] for an account of the properties of the gap between subspaces.

If X is separable and $\delta(E, F)$ is small then E and F can be isometrically embedded in $C[0, 1]$ in such a way that their gap as subspaces of $C[0, 1]$ is also small. Of course, this is a straightforward consequence of the fact that any separable Banach space can be isometrically embedded into $C[0, 1]$. Less evident is the fact that the quotients X/E and X/F can be isometrically identified with a pair of subspaces of $C[0, 1]$ whose gap is also small. In order to show that result, we give a proof of the following technical lemma for the convenience of the reader.

Lemma 3.2. *Let E and F be subspaces of a Banach space X and let $x \in S_E$. Then* $\operatorname{dist}(x, S_F) \leq 2 \operatorname{dist}(x, F)$.

Proof. Given $\varepsilon > 0$ we can find $y \in F$, $y \neq 0$, such that $\|x - y\| < \operatorname{dist}(x, F) + \varepsilon$. Then $y_0 = y/\|y\| \in S_F$ satisfies

$$\|y - y_0\| = |\|y\| - 1| = |\|y\| - \|x\|| \leq \|y - x\|.$$

Hence $\operatorname{dist}(x, S_F) \leq \|x - y\| + \|y - y_0\| < 2 \operatorname{dist}(x, F) + 2\varepsilon$. □

Theorem 3.3. *Let E be a subspace of a separable Banach space X. Then there exists a canonical isometric embedding $G\colon X/E \longrightarrow \ell_\infty$ such that for every subspace F of X with $\delta(E, F) < 1/8$ we can find an isomorphic embedding $G_F\colon X/F \longrightarrow \ell_\infty$ satisfying* $\|GQ_E - G_F Q_F\| \leq 2\delta(E, F)$.

Proof. We denote $\delta := \delta(E, F)$. Since $\delta = 0$ implies $E = F$, we can assume $0 < \delta < 1/8$. Let $\{u_i + E\}_{i=1}^\infty$ be a countable dense subset of $S_{X/E}$. For each i, we choose $x_i^* \in S_{E^\perp}$ such that $\langle x_i^*, u_i\rangle = 1$. It is easy to check that the operator $G\colon X/E \longrightarrow \ell_\infty$ defined by $G(x + E) := (\langle x_i^*, x\rangle)$ is an isometric embedding.

Now, fix a subspace F of X such that $\delta := \delta(E, F) < 1/8$. Since $\delta(E^\perp, F^\perp) = \delta(E, F)$ [13, Theorem IV.2.9], for each i we can find $y_i^* \in F^\perp$ such that $\|x_i^* - y_i^*\| < 2\delta$. We define

$G_F\colon X/F \longrightarrow \ell_\infty$ by $G_F(x+F) := (\langle y_i^*, x\rangle)$. Note that

$$\|GQ_Ex - G_FQ_Fx\| = \sup_{i\in\mathbb{N}} |\langle x_i^* - y_i^*, x\rangle| \leq 2\delta\|x\|.$$

In order to prove that G_F is an isomorphic embedding, we claim that for each $x \in X$,

$$\|Q_Fx\| \leq (1+\delta)\|Q_Ex\| + \delta\|x\|. \tag{2}$$

Indeed, given $\varepsilon > 0$ we can find $u \in E$ such that $\|x-u\| \leq \operatorname{dist}(x,E)+\varepsilon$. We choose $v \in F$ such that $\|u-v\| \leq \operatorname{dist}(u,F)+\varepsilon$. Then

$$\operatorname{dist}(x,F) \leq \|x-v\| \leq \operatorname{dist}(x,E)+\operatorname{dist}(u,F)+2\varepsilon \leq \operatorname{dist}(x,E)+\|u\|\delta(E,F)+2\varepsilon.$$

Since $\|u\| \leq \|x\|+\|x-u\| \leq \|x\|+\operatorname{dist}(x,E)+\varepsilon$, we get

$$\operatorname{dist}(x,F) \leq \operatorname{dist}(x,E)+\|x\|\delta+\operatorname{dist}(x,E)\delta+(2+\delta)\varepsilon,$$

and Formula (2) is proved.

Now, given $x+F = Q_Fx \in S_{X/F}$, we can assume $\|x\| < 4/3$, and Formula (2) gives $1 \leq (9/8)\|Q_Ex\| + 1/6$; hence $\|Q_Ex\| \geq 2/3$. Thus

$$\|G_F(x+F)\| = \|G_FQ_Fx\| \geq \|GQ_Ex\| - \|GQ_E - G_FQ_F\|\,\|x\| \geq 2/3 - 1/3 = 1/3,$$

and the proof is finished. □

4. Applications

Here we show that some counterexamples in the theory of tauberian operators obtained in [2] can be realized as operators in $\mathcal{L}(C[0,1])$.

Theorem 4.1. *There exists a tauberian operator $S\colon C[0,1] \longrightarrow C[0,1]$ such that S^{**} is not tauberian.*

Proof. By Theorem 3.1.18 in [8], there exists a separable Banach space Y and an operator $T\colon Y \longrightarrow Y$ which is tauberian but T^{**} is not.

Since Y is separable, there is an isometric embedding $i\colon Y \longrightarrow C[0,1]$, and as the push-out Σ of (i,T) is also separable, there is another isometric embedding $J\colon \Sigma \longrightarrow C[0,1]$.

$$\begin{array}{ccccc} Y & \xrightarrow{\;T\;} & Y & & \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j_i} & & \\ C[0,1] & \xrightarrow[\widehat{T}]{} & \Sigma & \xrightarrow[J]{} & C[0,1]. \end{array}$$

Moreover, by Proposition 2.3, j_i is an isomorphic embedding.

What follows is a repeated application of the properties of the class $\mathcal{T}$ of tauberian operators mentioned at the end of the introduction. By Proposition 2.9, $T \in \mathcal{T}$ implies $\widehat{T} \in \mathcal{T}$. Therefore $S := J\widehat{T}$ is tauberian.

Since T^{**} is not tauberian, $j_i^{**}T^{**} = \widehat{T}^{**}i^{**} \notin \mathcal{T}$. But i^{**} and J^{**} are isomorphic embeddings, so they belong to $\mathcal{T}$. Therefore $\widehat{T}^{**}$ and $S^{**} := J^{**}\widehat{T}^{**}$ are not tauberian. □

Theorem 4.2. *There exists a tauberian operator in the boundary of $\mathcal{T}(C[0,1])$.*

Proof. Following Example 2.1.7 in [8], given a non-reflexive separable space X, the operator $T\colon \ell_2(X) \longrightarrow \ell_2(X)$ that maps (x_n) to (x_n/n) is tauberian, and for every $k \in \mathbb{N}$, the operator $T_k\colon \ell_2(X) \longrightarrow \ell_2(X)$ that maps each (x_n) to

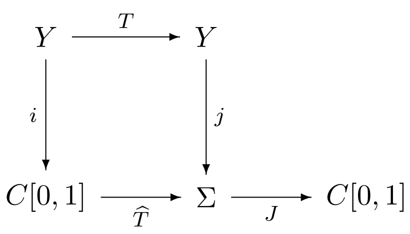
$$\Big(x_1, \frac{x_2}{2}, \dots, \frac{x_k}{k}, 0, 0, 0, \dots\dots\Big)$$

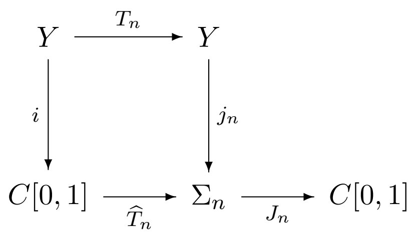
is non-tauberian and $\|T_k - T\| = 1/(k+1)$. So T is in the boundary of $\mathcal{T}(\ell_2(X))$.

Let us denote $Y := \ell_2(X)$, let $i\colon Y \longrightarrow C[0,1]$ be an isometric embedding, and consider the subspaces of $C[0,1] \oplus_1 Y$ given by

$$\begin{aligned} \Delta &:= \big\{\big(i(x), -Tx\big)\colon x \in Y\big\} \\ \Delta_n &:= \big\{\big(i(x), -T_n x\big)\colon x \in Y\big\},\ n \in \mathbb{N}. \end{aligned}$$

By construction, the push-out of (i, T) is $\Sigma := (C[0,1] \oplus_1 Y)/\Delta$, and for every n, the push-out of (i, T_n) is $\Sigma_n := (C[0,1] \oplus_1 Y)/\Delta_n$, producing the following push-out diagrams:

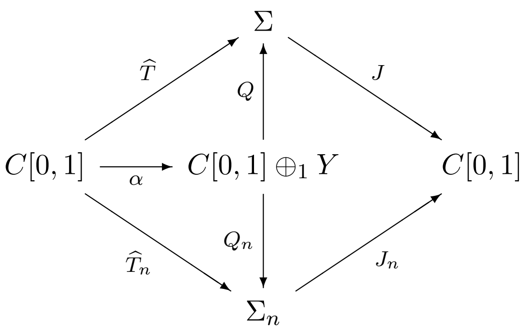




Since i is an isometric embedding and $\|T - T_n\| \underset{n}{\longrightarrow} 0$, it is not difficult to check that $\delta(\Delta_n, \Delta) \underset{n}{\longrightarrow} 0$. Thus, Theorem 3.3 provides us with an isometric embedding $G\colon \Sigma \longrightarrow \ell_\infty$ and a sequence of isomorphic embeddings $G_n\colon \Sigma_n \longrightarrow \ell_\infty$ such that $\|GQ - G_n Q_n\| \underset{n}{\longrightarrow} 0$, where Q and Q_n are the quotient operators from $C[0,1] \oplus_1 Y$ onto Σ and onto Σ_n respectively. Since the ranges $R(G)$ and $R(G_n)$ are separable for all n, the space $Z := \overline{\text{span}}\,\{\cup_{n=1}^{\infty} R(G_n) \cup R(G)\}$ is separable too. Thus there is an isometric embedding H from Z into $C[0,1]$, and the compositions $J := H \circ G$ and $J_n := H \circ G_n$ are isomorphic embeddings.

On the one hand, as T is tauberian, with the same argument of Theorem 4.1 we can prove that $S := J\widehat{T}$ is tauberian too.

On the other hand, let $\alpha\colon C[0,1] \longrightarrow C[0,1] \oplus_1 Y$ denote the operator that maps each f to $(f, 0)$. As the push-out operator extensions of T and T_n are $\widehat{T} = Q \circ \alpha$ and $\widehat{T}_n = Q_n \circ \alpha$, denoting $S_n := J_n \widehat{T}_n$, it follows that $\|S_n - S\| \le \|J_n Q_n - JQ\| \underset{n}{\longrightarrow} 0$. And as i, j_n and J_n are all tauberian and T_n is not, then $\widehat{T}_n \notin \mathcal{T}$, hence $S_n \notin \mathcal{T}$.



That proves that S is a tauberian operator belonging to the topological boundary of $\mathcal{T}(C[0,1])$. □

An operator T is said to be *cotauberian* if T^* is tauberian. The class of all cotauberian operators is denoted by $\mathcal{T}^d$. We refer to [8, Section 3.1] for information about this class of operators.

Remark 4.3. The existence of a space X and an operator $T \in \mathcal{T}^d(X)$ such that T^{**} is not cotauberian is proved in [8, Theorem 3.1.18]. Besides, it is not difficult to adapt the construction of [8, Example 2.1.17] to obtain a space Y such that $\mathcal{T}^d(Y)$ is not open in $\mathcal{L}(Y)$. However, the negative role of X and Y cannot be played by $C[0,1]$ or more generally by a C^*-algebra Z.

Indeed the reflexive quotients of a C^*-algebra are superreflexive [11, Corollary 2]. Therefore $\mathcal{T}^d(Z)$ is open in $\mathcal{L}(Z)$ [7, Proposition 20 and Theorem 22], and the biconjugate of each operator in $\mathcal{T}^d(Z)$ is cotauberian [8, Proposition 6.6.5].

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Departamento de Matemáticas, Facultad de Ciencias, Universidad de Cantabria, E-39071 Santander, Spain
E-mail address: manuel.gonzalez@unican.es

Departamento de Matemáticas, Facultad de Ciencias, Universidad de Oviedo, E-33007 Oviedo, Spain
E-mail address: ama@uniovi.es