Cotauberian operators on $L_1(0,1)$ obtained by lifting

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Abstract. We show that the set $\mathcal{T}^d(L_1(0,1))$ of cotauberian operators acting on $L_1(0,1)$ is not open, and $T \in \mathcal{T}^d(L_1(0,1))$ does not imply T^{**} cotauberian. As a consequence, the derive that set $\mathcal{T}(L_\infty(0,1))$ of tauberian operators acting on $L_\infty(0,1)$ is not open, and that $T \in \mathcal{T}(L_\infty(0,1))$ does not imply T^{**} tauberian.

1. Introduction

Tauberian operators were introduced in [13] as those operators $T\colon X \to Y$ such that the second conjugate satisfies $T^{**-1}(Y) = X$. They have found many applications in Banach space theory like factorization of operators [5], preservation of isomorphic properties [16], equivalence between the Radon-Nikodym property and the Krein-Milman property [18], and refinements of James' characterization of reflexive spaces [17]. The cotauberian operators were introduced by Tacon [19] as those operators T such that T^* is tauberian, and they have found applications in factorization of operators and preservation of isomorphic properties of Banach spaces (see [8]). The classes $\mathcal{T}$ of tauberian operators and $\mathcal{T}^d$ of cotauberian operators are semigroups in the sense of [1] associated to the weakly compact operators [10, Theorem 2]. We refer to [8] for additional information on the subject.

Let $\mathcal{L}(X,Y)$ denote the set of all (bounded) operators acting between X and Y. Given a class $\mathcal{A}$ of operators, $\mathcal{A}(X,Y) := \mathcal{A} \cap \mathcal{L}(X,Y)$ is the component of $\mathcal{A}$ in $\mathcal{L}(X,Y)$. It was proved in [2] that, in general, $\mathcal{T}(X,Y)$ and $\mathcal{T}^d(X,Y)$ are not open subsets of $\mathcal{L}(X,Y)$, and that $T \in \mathcal{T} \not\Rightarrow T^{**} \in \mathcal{T}$ and $T \in \mathcal{T}^d \not\Rightarrow T^{**} \in \mathcal{T}^d$ (see Sections 2.1 and 3.1 in [8]). The corresponding counterexamples were obtained as operators $T\colon X \to X$ acting on certain Banach spaces X constructed ad hoc. So it was interesting to know if there

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are counterexamples among the operators acting between classical Banach spaces. Applying some properties of the push-out construction for a pair of operators and a technical result on the gap between subspaces, it was proved in [9] that the counterexamples for $\mathcal{T}$ can be obtained among the operators acting on $C[0,1]$.

In this paper we obtain some results on the pull-back construction for a pair of operators, and applying a technical result on the gap between subspaces we show that the counterexamples for $\mathcal{T}^d$ can be obtained among the operators acting on $L_1(0,1)$, ℓ_1, or any Banach space Z admitting a quotient isomorphic to ℓ_1. From these results we derive that set $\mathcal{T}(Z^*)$ of tauberian operators acting on the dual Z^* of the mentioned space Z is not open, and that $T \in \mathcal{T}(Z^*)$ does not imply T^{**} tauberian. We observe that the results on tauberian operators acting on Z^* cannot be derived from the construction given in [9], because we cannot guarantee that a quotient of Z^* is isomorphic to a subspace of Z^*. Indeed, taking as Z the space ℓ_1, it follows from the main result in [3] that the quotient ℓ_∞/c_0 is not isomorphic to a subspace of ℓ_∞.

Our notation is standard. Capital letters X, Y, Z denote Banach spaces, and given a subspace Y of X, the annihilator of Y in X^* is $Y^\perp$. The second dual of X is denoted X^{**}, we identify X with a subspace of X^{**}, and we denote by X^{co} the quotient X^{**}/X. We refer to [4, Section 1.3] for a description of the push-out construction for a pair of operators.

Given two closed subspaces M and N of Z, we consider the quantity

$$\delta(M,N) := \sup_{y\in S_M} \operatorname{dist}(y,N),$$

where $S_M := \{y \in M\colon \|y\| = 1\}$ is the unit sphere of M. The *gap between M and N* is defined by $\hat{\delta}(M,N) := \max\{\delta(M,N), \delta(N,M)\}$. Basic results on the gap between subspaces can be found in [15, Section 10], and for Banach space theory we refer to [14].

Operators are always continuous linear maps. The range and the kernel of an operator $T\colon X \to Y$ are denoted by $\operatorname{ran}(T)$ and $\ker(T)$ respectively, $T^*\colon Y^* \to X^*$ is the conjugate of T, $T^{**}\colon X^{**} \to Y^{**}$ is the second conjugate of T, and the *residuum operator of T* (studied in [11]) is the operator $T^{co}\colon X^{co} \to Y^{co}$ that maps $x^{**} + X$ to $T^{**}x^{**} + Y$.

An operator $T\colon X \to Y$ is said to be *tauberian* if $T^{**}(X^{**} \setminus X) \subset Y^{**} \setminus Y$; equivalently, if T^{co} is injective [8, Proposition 3.1.8], and T is said to be cotauberian if T^* is tauberian; equivalently, if T^{co} has dense range [8, Corollary 3.1.12]. The class of all tauberian operators will be denoted by $\mathcal{T}$. Therefore, given Banach spaces X and Y, the component $\mathcal{T}(X,Y)$ consists of all tauberian operators in $\mathcal{L}(X,Y)$. In the case $X = Y$ we write $\mathcal{T}(X)$ instead of $\mathcal{T}(X,X)$. The class of all cotauberian operators will be denoted by $\mathcal{T}^d$.

Surjective operators belong to $\mathcal{T}^d$, and given operators $S\colon X \to Y$ and $T\colon Y \longrightarrow Z$, the following assertions are satisfied (see [8, Section 3.1]):

(i) if $T \in \mathcal{T}^d$ and $S \in \mathcal{T}^d$ then $TS \in \mathcal{T}^d$;

(ii) if $TS \in \mathcal{T}^d$ then $T \in \mathcal{T}^d$.

2. The pull-back of a pair of operators

Let X, Y and Z be Banach spaces. Given a pair of operators $A\colon Y \to X$ and $B\colon Z \to X$, we consider the operator

$$A - B : (y,z) \in Y \oplus_\infty Z \longrightarrow Ay - Bz \in X,$$

and denote by $PB(A,B)$ (or simply PB if the operators are clear) the space

$$PB(A,B) := \ker(A-B) = \{(y,z) \in Y \oplus_\infty Z \,:\, Ay = Bz\}.$$

Since $A - B$ is continuous, $PB(A,B)$ is a Banach space. It is called the *pull-back space of the pair* (A,B).

The operators $\pi_A\colon PB \to Z$ and $\pi_B\colon PB \to Y$ that take $(y,z) \in PB$ to z and y respectively have norm less or equal than 1, and satisfy the identity $A\pi_B = B\pi_A$. So we have a commutative diagram

$$\begin{tikzcd} PB \arrow[r, "\pi_A"] \arrow[d, "\pi_B"'] & Z \arrow[d, "B"] \\ Y \arrow[r, "A"'] & X \end{tikzcd} \tag{1}$$

which is called the *pull-back diagram of the pair* (A,B). Note that the roles played by A and B in the construction are symmetric.

The following universal property of the pull-back diagram guarantees that it is unique up to isomorphisms. Its proof is straightforward.

Proposition 2.1 (Universal property). *Let $A\colon Y \to X$ and $B\colon Z \to X$ be a pair of operators. Then for any Banach space U and any pair of operators $p_B\colon U \to Y$ and $p_A\colon U \to Z$ such that $Ap_B = Bp_A$ there exists a unique operator $\pi\colon U \to PB$ such that $p_A = \pi_A\pi$ and $p_B = \pi_B\pi$.*

The operator π is given by $\pi(u) = \big(p_B(u), p_A(u)\big)$.

We will need the following properties of the pull-back diagram.

Proposition 2.2. *Let $A\colon Y \to X$ and $B\colon Z \to X$ be a pair of operators and suppose that B is surjective. Then the following assertions are satisfied:*

(i) *π_B is surjective.*
(ii) *A has dense range if and only if so does π_A.*

Proof. (i) Let $y \in Y$. Since B is surjective, there exists $z \in Z$ such that $Ay = Bz$. Thus $(y,z) \in PB$, and $\pi_B(y,z) = y$.

(ii) Suppose that A has dense range; equivalently that the conjugate A^* is injective. Since B is surjective, so is the operator $A - B$. In particular its range $\operatorname{ran}(A-B)$ is closed. Thus

$$PB^* = \frac{Y^* \oplus_1 Z^*}{\ker(A-B)^\perp} = \frac{Y^* \oplus_1 Z^*}{\operatorname{ran}\big((A-B)^*\big)},$$

where $(A - B)^*\colon X^* \to Y^* \oplus_1 Z^*$ is given by $(A - B)^* x^* = (A^* x^*, -B^* x^*)$. Note also that $\pi_A^* : Z^* \to PB^*$ is given by $\pi_A^* z^* = (0, z^*) + \operatorname{ran}\big((A - B)^*\big)$. So suppose that $\pi_A^* z^* = 0$. Then $(0, z^*) = (A^* x^*, -B^* x^*)$ for some x^*. Since A^* is injective we get $x^* = 0$, hence $z^* = -B^* x^* = 0$. Thus π_A^* injective, hence π_A has dense range.

Conversely, if the range of π_A is dense, then $A\pi_B = B\pi_A$ has dense range; hence so does A. □

Remark 2.3. By Proposition 2.2, if B is surjective then π_B is also surjective. In this case we can see the operator π_A as a lifting of A. This is the interpretation in which we are interested in this paper.

From now on, when we say that *we can identify two operators or two diagrams,* we mean that we identify them up to bijective isomorphisms.

The following result shows that the action of taking biconjugates and that of forming pull-backs commute in some cases.

Proposition 2.4. *Let $A\colon Y \to X$ and $B\colon Z \to X$ be a pair of operators with B surjective. Then the second conjugate of the pull-back diagram of (A, B) can be identified with the pull-back diagram of (A^{**}, B^{**}).*

Proof. We have to identify the following diagrams:

$$\begin{tikzcd} PB(A,B)^{**} \arrow[r, "\pi_A^{**}"] \arrow[d, "\pi_B^{**}"'] & Z^{**} \arrow[d, "B^{**}"] \\ Y^{**} \arrow[r, "A^{**}"] & X^{**} \end{tikzcd} \qquad \begin{tikzcd} PB(A^{**},B^{**}) \arrow[r, "\pi_{A^{**}}"] \arrow[d, "\pi_{B^{**}}"'] & Z^{**} \arrow[d, "B^{**}"] \\ Y^{**} \arrow[r, "A^{**}"] & X^{**} \end{tikzcd}$$

where, since $A - B$ has closed range,

$$PB(A,B)^{**} = \ker\big((A{-}B)^{**}\big) = \{(y^{**}, z^{**}) \in Y^{**} \oplus_\infty Z^{**} \;:\; A^{**}y^{**} = B^{**}z^{**}\},$$

and $\pi_A^{**}\colon PB(A,B)^{**} \to Z^{**}$ and $\pi_B^{**}\colon PB(A,B)^{**} \to Y^{**}$ take $(y^{**}, z^{**}) \in PB(A,B)^{**}$ to z^{**} and y^{**} respectively.

The universal property of the pull-back diagram (Proposition 2.1) provides an operator

$$\pi : PB(A,B)^{**} \longrightarrow PB(A^{**}, B^{**})$$

given by

$$\pi(y^{**}, z^{**}) = \big(\pi_B^{**}(y^{**}, z^{**}), \pi_A^{**}(y^{**}, z^{**})\big) = (y^{**}, z^{**}).$$

Since obviously π is a bijective isomorphism, the result is proved. □

The following result shows that the action of passing to residuum operators and that of forming pull-backs commute in some cases.

Proposition 2.5. *Let $A\colon Y \to X$ and $B\colon Z \to X$ be a pair of operators with B surjective. Then the residuum of the pull-back diagram of (A, B) can be identified with the pull-back diagram of (A^{co}, B^{co}).*

Proof. The proof is formally similar to that of Proposition 2.4. We have to show that we can identify the following diagrams:

$$\begin{tikzcd} PB(A,B)^{co} \arrow[r, "\pi_A^{co}"] \arrow[d, "\pi_B^{co}"'] & Z^{co} \arrow[d, "B^{co}"] \\ Y^{co} \arrow[r, "A^{co}"] & X^{co} \end{tikzcd} \qquad \begin{tikzcd} PB(A^{co},B^{co}) \arrow[r, "\pi_{A^{co}}"] \arrow[d, "\pi_{B^{co}}"'] & Z^{co} \arrow[d, "B^{co}"] \\ Y^{co} \arrow[r, "A^{co}"] & X^{co} \end{tikzcd}$$

where, since $A-B$ has closed range, we have (see [8, Proposition 3.1.13])

$$PB(A,B)^{co} = \ker\big((A-B)^{co}\big) = \{(y^{co},z^{co}) \in Y^{co}\oplus_\infty Z^{co} \;:\; A^{co}y^{co} = B^{co}z^{co}\},$$

and $\pi_A^{co}\colon PB(A,B)^{co} \to Z^{co}$ and $\pi_B^{co}\colon PB(A,B)^{co} \to Y^{co}$ take $(y^{co},z^{co}) \in PB(A,B)^{co}$ to z^{co} and y^{co} respectively.

The universal property of the pull-back diagram (Proposition 2.1) provides an operator

$$\pi : PB(A,B)^{co} \longrightarrow PB(A^{co},B^{co})$$

given by

$$\pi(y^{co},z^{co}) = \big(\pi_B^{co}(y^{co},z^{co}), \pi_A^{co}(y^{co},z^{co})\big) = (y^{co},z^{co}).$$

Since obviously π is a bijective isomorphism, the result is proved. □

The following result is an application of the previous identifications.

Proposition 2.6. *Consider the pull-back diagram of (A,B) given in (1) and assume B is surjective. Then the following assertions are satisfied:*

(i) *A is cotauberian if and only if so is π_A.*
(ii) *A^{**} is cotauberian if and only if so is π_A^{**}.*

Proof. (i) Assume that A is cotauberian; equivalently, that A^{co} has dense range. By Proposition 2.5, the space $PB(A,B)^{co}$ can be identified with $PB(A^{co},B^{co})$. The operator B^{co} is surjective because so is B [8, Proposition 3.1.15]. Thus using Proposition 2.2 we get that π_A^{co} has dense range, hence π_A is cotauberian.

For the converse, assume that π_A is cotauberian. As B is surjective, $B\pi_A = A\pi_B$ is cotauberian, hence A is cotauberian too.

(ii) Since Proposition 2.4 identifies π_A^{**} with $\pi_{A^{**}}$, the result follows from (i). □

3. Applications

Here we apply the results of the previous section to show that some counterexamples in the theory of tauberian operators obtained in [2] can be realized as operators acting on $L_1(0,1)$ or ℓ_1, or in any Banach space Z admitting a quotient isomorphic to ℓ_1. We will give the results for $Z = L_1(0,1)$, but the proofs in the general case are identical.

Theorem 3.1. *There exists a cotauberian operator* $S\colon L_1(0,1) \to L_1(0,1)$ *such that* S^{**} *is not cotauberian.*

Proof. It is proved in [2] (see also Theorem 3.1.18 in [8]) that there exists a separable Banach space Y such that Y^{co} is isomorphic to ℓ_1, and an operator $T\colon Y \to Y$ such that T^{co} can be identified to the operator $A\colon \ell_1 \to \ell_1$ given by $A(x_n) = (x_n/n)$.

The operator A has dense range, hence so does T^{co}, and T is cotauberian. Moreover, since A is compact, the range of A^{**} is separable, so it is not dense. Taking into account that we can identify $(T^{co})^{**}$ and $(T^{**})^{co}$ [8, Proposition 3.1.11], we conclude that T^{**} is not cotauberian.

Since Y is separable, there is a surjective operator $B\colon L_1(0,1) \to Y$, and the space $PB(T,B)$ is separable. So there exists a surjective operator $Q\colon L_1(0,1) \to PB(T,B)$, and we have the following commutative diagram:

$$\begin{tikzcd}
L_1(0,1) \arrow[r, "Q"] & PB(T,B) \arrow[r, "\pi_T"] \arrow[d, "\pi_B"'] & L_1(0,1) \arrow[d, "B"] \\
& Y \arrow[r, "T"'] & Y
\end{tikzcd}$$

The remaining of the proof is a repeated application of the properties of the class $\mathcal{T}^d$ of cotauberian operators given at the end of the introduction.

On the one hand, by Proposition 2.6, π_T is cotauberian. Since Q is surjective, the operator $S : \pi_T Q\colon L_1(0,1) \to L_1(0,1)$ is also cotauberian.

On the other hand, by Propositions 2.4 and 2.6, π_T^{**} is not cotauberian. Therefore $S^{**} = \pi_T^{**} Q^{**}$ is not cotauberian. □

Corollary 3.2. *There exists a tauberian operator* $T\colon L_\infty(0,1) \to L_\infty(0,1)$ *such that* T^{**} *is not tauberian.*

Proof. It is enough to take $T = S^*$, where S is the operator obtained in Theorem 3.1. □

Remark 3.3. *There exist separable Banach spaces* X *such that the set* $\mathcal{T}^d(X)$ *is not open in* $\mathcal{L}(X)$.

Indeed, let Z be a non-reflexive separable Banach space. We consider the space

$$\ell_2(Z) := \{(x_n) \subset Z \,:\, \sum_{n=1}^{\infty} \|x_n\|^2 < \infty\}$$

which endowed with the ℓ_2-norm is a Banach space.

We consider the operator $T\colon \ell_2(Z) \to \ell_2(Z)$ that maps (x_n) to (x_n/n), and for every positive integer n, we consider the operator $T_n\colon \ell_2(Z) \to \ell_2(Z)$ that sends each (x_n) to

$$(x_1, \frac{x_2}{2}, \ldots\ldots, \frac{x_n}{n}, 0, 0, \ldots\ldots\ldots).$$

It is not difficult to show (see [8, Example 2.1.7]) that $\ell_2(Z)^{co}$ can be identified with $\ell_2(Z^{co})$, that the operators T_n are not cotauberian, that $T_n \underset{n}{\longrightarrow} T$, and that T is cotauberian. It turns out that $\mathcal{T}^d\big(\ell_2(Z)\big)$ is not open in $\mathcal{L}\big(\ell_2(Z)\big)$.

Let X be a Banach space. We denote by B_X and S_X the closed unit ball and the unit sphere of X respectively. Moreover, given a nonempty subset S of X and $\delta > 0$, a subset C of S is called a *δ-net* in S if for every $x \in S$ there exists $y \in C$ such that $\|x - y\| \leq \delta$.

Later we will need the following technical result. It is known, but we give a proof for the convenience of the reader.

Lemma 3.4. *Let (x_n) be a $(1/2)$-net in the unit sphere S_X of a Banach space X, and let $T \in \mathcal{L}(\ell_1, X)$ the operator defined by $Te_n := x_n$ $(n \in \mathbb{N})$, where (e_n) is the unit vector basis of ℓ_1. Then $T(B_{\ell_1})$ contains $(1/2)B_X$. In particular T is surjective.*

Proof. Note that for every $r > 0$, (rx_n) is a $(r/2)$-net in rS_X.

Let $x \in X$ with $\|x\| = 1/2$, and set $t_1 = 1/2$. We select x_{n_1} in the $(1/2)$-net such that $\|x - t_1 x_{n_1}\| \leq 1/4$, and set $t_2 = \|x - x_{n_1}\|$. Then we select x_{n_2} such that $\|x - t_1 x_{n_1} - t_2 x_{n_2}\| \leq 1/8$, and set $t_3 = \|x - x_{n_1} - x_{n_2}\|$.

Proceeding in this way we obtain $0 \leq t_k \leq 1/2^k$ $(k \in \mathbb{N})$ and a subsequence (x_{n_k}) of the $(1/2)$-net. Taking $a = (a_n)$ with $a_{n_k} = t_k$ and $a_n = 0$ otherwise, we obtain $a \in B_{\ell_1}$ such that $Ta = x$, and the result is proved. $\square$

Let us show that $\mathcal{T}^d\big(L_1(0,1)\big)$ is not open in $\mathcal{L}\big(L_1(0,1)\big)$.

Theorem 3.5. *There exists a cotauberian operator in the boundary of $\mathcal{T}^d\big(L_1(0,1)\big)$.*

Proof. We write L_1 instead of $L_1(0,1)$. Applying the arguments in the proof of Theorem 3.1 to the operators T_n and T given in Remark 3.3, we obtain the following push-out diagrams:

$$\begin{array}{ccccc} L_1 & \xrightarrow{Q} & PB(T,B) & \xrightarrow{\pi_T} & L_1 \\ & & \downarrow{\scriptstyle \pi_B} & & \downarrow{\scriptstyle B} \\ & & \ell_2(Z) & \xrightarrow{T} & \ell_2(Z) \end{array} \qquad \begin{array}{ccccc} L_1 & \xrightarrow{Q_n} & PB(T_n,B_n) & \xrightarrow{\pi_{T_n}} & L_1 \\ & & \downarrow{\scriptstyle \pi_{B_n}} & & \downarrow{\scriptstyle B_n} \\ & & \ell_2(Z) & \xrightarrow{T_n} & \ell_2(Z) \end{array}$$

The operators $S_n := \pi_{T_n} Q_n$ are not cotauberian, $S := \pi_T Q$ is cotauberian, and S_n, S belong to $\mathcal{L}\big(L_1(0,1)\big)$. It remains to show that we can arrange the constructions so that $\|S_n - S\| \underset{n}{\longrightarrow} 0$.

To shorten the arguments, we denote $PB := PB(T,B)$ and $PB_n := PB(T_n, B_n)$, and take $B_n = B$ for all n. So $PB = \ker(T - B)$ and $PB_n = \ker(T_n - B)$ are closed subspaces of $\ell_2(Z) \oplus_\infty L_1$.

We fix a surjective operator $p\colon L_1 \to \ell_1$, select a dense sequence (u_k) in the unit sphere S_{PB}, and define $q\colon \ell_1 \to PB$ by $qe_k := u_k$ $(k \in \mathbb{N})$, where (e_k)

is the unit vector basis of ℓ_1. The operator $Q := qp\colon L_1 \to PB$ is surjective (Lemma 3.4).

Since the operators $T-B$ and T_n-B are surjective and $\|T_n-T\| \underset{n}{\longrightarrow} 0$, it follows from [15, Theorem 10.17] that the gap between the kernels satisfy

$$\hat{\delta}\big(\ker(T_n-B), \ker(T-B)\big) \xrightarrow[n]{\qquad} 0.$$

Let us denote $\delta_n := \hat{\delta}\big(\ker(T_n-B), \ker(T-B)\big)$. Given M and N closed subspaces of a Banach space, for each $x \in S_M$ we have $\operatorname{dist}(x, S_N) \leq 2\operatorname{dist}(x, N)$ [9, Lemma 3.2]. Therefore, for each $n \in \mathbb{N}$, we can select a sequence $(u_{n,k})_{k\in\mathbb{N}}$ in S_{PB_n} such that $\|u_k - u_{n,k}\| \leq 3\delta_n$ for each k.

We define $q_n\colon \ell_1 \to PB_n$ by $q_n e_k := u_{n,k}$ $(k \in \mathbb{N})$, and $Q_n := q_n p\colon L_1 \to PB$.

$$\begin{tikzcd}
& PB \arrow[dr, "\pi_T"] & \\
L_1(0,1) \arrow[ur, "Q=qp"] \arrow[dr, "Q_n=q_np"'] & & L_1(0,1) \\
& PB_n \arrow[ur, "\pi_{T_n}"'] &
\end{tikzcd}$$

We claim that the operator Q_n is surjective for n big enough. Indeed, for each v in S_{PB_n} we can find u in S_{PB} such that $\|v-u\| \leq 3\delta_n$. Since (u_k) is dense in S_{PB}, we can find $k \in \mathbb{N}$ such that $\|v - u_{n,k}\| \leq 7\delta_n$. Thus $(u_{n,k})_{k\in\mathbb{N}}$ is a $7\delta_n$-net in S_{PB_n}, and the claim follows from Lemma 3.4.

Let $f \in L_1(0,1)$. Then

$$\|S_n f - Sf\| = \|\pi_{T_n} Q_n f - \pi_T Qf \leq \|Q_n f - Qf\| \leq \|q_n - q\|\,\|f\|.$$

Hence $\|S_n - S\| \leq \|q_n - q\| = \sup_{k\in\mathbb{N}} \|u_k - u_{n,k}\| \leq 3\delta_n \underset{n}{\longrightarrow} 0$. Thus S is a cotauberian operator in the boundary of $\mathcal{T}^d\big(L_1(0,1)\big)$. $\square$

Corollary 3.6. *There exists a tauberian operator in the boundary of* $\mathcal{T}\big(L_\infty(0,1)\big)$.

Proof. It is enough to take the conjugate operator of the operator obtained in Theorem 3.5. $\square$

Remark 3.7. (a) It was proved in [7] that $\mathcal{T}\big(L_1(0,1)\big)$ is open in $\mathcal{L}\big(L_1(0,1)\big)$, and that $T \in \mathcal{T}\big(L_1(0,1)\big)$ implies T^{**} tauberian.

(b) Since reflexive quotients of $L_\infty(0,1)$ are superreflexive [12], it follows from Proposition 20 and Theorem 22 in [6] that $\mathcal{T}^d\big(L_\infty(0,1)\big)$ is open in $\mathcal{L}\big(L_\infty(0,1)\big)$, and applying [8, Proposition 6.6.5] we get that $T \in \mathcal{T}^d\big(L_1(0,1)\big)$ implies T^{**} cotauberian.

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