

ON CORRECTORS FOR SPECTRAL PROBLEMS IN
THE HOMOGENIZATION OF ROBIN BOUNDARY
CONDITIONS WITH VERY LARGE PARAMETERS

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Abstract: We obtain estimates for convergence rates of the eigenelements $(\lambda^\varepsilon, u^\varepsilon)$ for the Laplace operator in a domain $\Omega \subset \mathbb{R}^3$ periodically perforated along a plane $\gamma = \Omega \cap \{x_1 = 0\}$. The boundary conditions are of the Dirichlet type on $\partial\Omega$ and of the Robin type, involving a large parameter $O(\varepsilon^{-\kappa})$, on the boundary of the cavities. The small parameter ε denotes the period while the size of each cavity is $O(\varepsilon^\alpha)$. Here we consider the most significant case where $\alpha = \kappa = 2$.

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1. Introduction and Setting of the Problem

Let Ω be a bounded domain in \mathbb{R}^3 , with a smooth boundary $\partial\Omega$. Assume that

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$\gamma = \Omega \cap \{x_1 = 0\} \neq \emptyset$ is a domain on the plane $\{x_1 = 0\}$. We denote by G_0 the ball of radius 1 centered at the origin of coordinates. For a domain B , and $\delta > 0$, we denote by $\delta B = \{x \mid \delta^{-1}x \in B\}$, and we set

$$\tilde{G}_\varepsilon = \bigcup_{z \in \mathbb{Z}'} (a_\varepsilon G_0 + \varepsilon z) \equiv \bigcup_{j \in \mathbb{Z}'} G_\varepsilon^j,$$

where \mathbb{Z}' is the set of points of the form $z = (0, z_2, z_3)$ with integer components z_2, z_3 ; $a_\varepsilon = C_0 \varepsilon^\alpha$, C_0 is a fixed positive number, $\varepsilon > 0$ is a parameter that we make converging towards zero, and $\alpha \geq 1$. We define

$$G_\varepsilon = \bigcup_{j \in \Upsilon_\varepsilon} G_\varepsilon^j,$$

where $\Upsilon_\varepsilon = \{j \in \mathbb{Z}' : G_\varepsilon^j \subset \tilde{G}_\varepsilon, \overline{G_\varepsilon^j} \subset \Omega, \rho(\partial\Omega, \overline{G_\varepsilon^j}) \geq 2\varepsilon\}$. The number of G_ε^j with index $j \in \Upsilon_\varepsilon$ is $|\Upsilon_\varepsilon| = O(\varepsilon^{-2})$.

Let Ω_ε be $\Omega_\varepsilon = \Omega \setminus \overline{G_\varepsilon}$, $S_\varepsilon = \partial G_\varepsilon$, $\partial\Omega_\varepsilon = \partial\Omega \cup S_\varepsilon$. Let $H^1(\Omega_\varepsilon, \partial\Omega)$ be the space completion with respect to the norm of $H^1(\Omega_\varepsilon)$ of the set of functions $u \in C^\infty(\overline{\Omega_\varepsilon})$, u vanishing in a neighborhood of $\partial\Omega$.

Let us consider the eigenvalue problem

$$\begin{cases} -\Delta u^\varepsilon = \lambda^\varepsilon u^\varepsilon & \text{in } \Omega_\varepsilon, \\ u^\varepsilon = 0 & \text{on } \partial\Omega, \\ \partial_\nu u^\varepsilon + \varepsilon^{-\kappa} a u^\varepsilon = 0 & \text{on } S_\varepsilon, \end{cases} \tag{1}$$

where ∂_ν denotes the derivative along the unit outward normal vector ν to $\partial\Omega_\varepsilon$ on S_ε , $a \equiv a(x)$ is a strictly positive continuously differentiable function in $\overline{\Omega}$ and $\kappa \in \mathbb{R}$. Throughout this paper, we set the value of the parameters α and κ at $\kappa = \alpha = 2$.

The variational formulation of (1) is: to find λ^ε , $u^\varepsilon \in H^1(\Omega_\varepsilon, \partial\Omega)$, $u^\varepsilon \neq 0$, such that

$$\int_{\Omega_\varepsilon} \nabla u^\varepsilon \nabla v \, dx + \varepsilon^{-\kappa} \int_{S_\varepsilon} a u^\varepsilon v \, ds = \lambda^\varepsilon \int_{\Omega_\varepsilon} u^\varepsilon v \, dx, \quad \forall v \in H^1(\Omega_\varepsilon, \partial\Omega). \tag{2}$$

For each fixed $\varepsilon > 0$, problem (2) is a standard eigenvalue problem in the couple of spaces $H^1(\Omega_\varepsilon, \partial\Omega) \subset L^2(\Omega_\varepsilon)$, with a discrete spectrum. Let us consider $\{\lambda_k^\varepsilon\}_{k=1}^\infty$ the increasing sequence of eigenvalues, repeated according to their multiplicities, and let $\{u_k^\varepsilon\}_{k=1}^\infty$ denote the associated eigenfunctions which are assumed to form an orthonormal basis in $L^2(\Omega_\varepsilon)$.

As stated in Gómez et al [1], for $\alpha = \kappa = 2$, the homogenized spectral problem of (1) is:

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega^- \cup \Omega^+, \\ u = 0 & \text{on } \partial\Omega, \\ [u] = 0, [\partial_{x_1} u] = 4\pi C_0 h u & \text{on } \gamma, \end{cases} \quad (3)$$

where $h \equiv h(x)$ is the strictly positive continuously differentiable function defined by

$$h(x) = \frac{a(x)C_0}{1 + a(x)C_0}, \quad x \in \bar{\Omega}.$$

Problem (3) has a discrete spectrum; let us consider $\{\lambda_k\}_{k=1}^\infty$ the increasing sequence of its eigenvalues repeated according to their multiplicities, and let $\{u_k\}_{k=1}^\infty$ denote the associated eigenfunctions which are assumed to form an orthonormal basis in $L^2(\Omega)$.

The convergence of the spectrum of (2) towards that of (3) has been proved in Gómez et al [1]. This result in Gómez et al [1] does not provide bounds for convergence rates of eigenvalues and the associated eigenfunctions, since it is obtained from general convergence results for nonlinear stationary problems, and convergence rates for the solutions of these stationary problems rely on the assumption of smoothness of the solution of the limiting problem. Since we are dealing with eigenvalue problems, such an assumption makes no sense.

The aim of this paper is to obtain precise bounds for discrepancies of the eigenvalues of (1) and (3) and for the associated eigenfunctions in terms of the eigenvalue number and the parameter ε . We emphasize that obtaining these bounds proves to be essential in order to determine, e.g., estimates of time in terms of ε . That is, the time in which certain solutions of the associated evolution problems can be approached through time-dependent functions constructed from (3) (see Pérez [5] and Pérez [6] in this connection). Associated evolution problems arise, e.g., in Ecology: see Gómez et al [1] for further references on the model and related works in the literature. For the proofs, we use a strong result from the spectral perturbation theory (cf. Lemma 2) for ε -dependent Hilbert spaces and operators, which provides convergence for the spectrum when a certain convergence for associated stationary problems is known.

In this paper, since we are dealing with a linear problem, we can obtain the required smoothness for the solution of the stationary problem (8) (cf. Lemma 3). Consequently, avoiding the assumptions on smoothness of solutions in Gómez et al [1] we obtain lower order powers of ε in the bounds for the discrepancies but in these bounds we can control the dependence on the data f in

the norm of $L^2(\Omega)$ which is a usual topology for the spectral problems here considered (cf. (13) and (14)). To prove the above mentioned smoothness, we use a variant of results on interior estimates of Sobolev norms for solutions of second order elliptic equations with Dirichlet boundary conditions (cf. Shaposhnikov [7]), and Sobolev embedding theorems which also imply some restriction on the dimension of the space under consideration.

It should be noted that the critical case here considered where $k = \alpha = 2$ provides the most singular limiting problem among all the possible relations between the parameters α and κ in Gómez et al [1]: it appears a nonlinear dependence on the data $a(x)$ on the transmission condition on γ (see (3)). Also, it should be emphasized that the technique here developed extends, for a three dimensional domain, to all the cases where $\alpha \geq 1$ and $\kappa \in \mathbb{R}$. The cases different from $\kappa = \alpha = 2$ are considered in Gómez et al [2]. In these problems, different test functions are used and different homogenized spectral problems are obtained: either the average on γ depends linearly on $a(x)$ or it does not contain any dependence on $a(x)$.

We note that the spectral problem here considered differs from others in the literature: in this respect, we refer to Gómez et al [2] for comparison. Finally, we mention that Section 2 contains some notations and preliminary results used to prove the convergence in Section 3. Theorem 4 contains the convergence results for the stationary problems and Theorem 5 contains the spectral convergence.

2. Preliminary Results

For the sake of the completeness, we introduce some lemmas bellow; in these lemmas, and in what follows, C and C_k denote constants independent of ε . Lemma 1 provides sharp general estimates for thin domains; Lemma 2 provides results from the spectral perturbation theory. Lemma 3 provides bounds in $W^{1,4}$ and L^∞ norms for solutions.

Lemma 1. *Let Π_ε be $\Pi_\varepsilon = \Omega \cap \{-\varepsilon/2 < x_1 < \varepsilon/2\}$. Then, for all $w \in H_0^1(\Omega)$, we have*

$$\|w\|_{L^2(\Pi_\varepsilon)} \leq C\varepsilon^{1/2} \|\nabla w\|_{L^2(\Omega)} \quad \text{and} \quad \|w\|_{L^4(\Pi_\varepsilon)} \leq C\varepsilon^{1/8} \|\nabla w\|_{L^2(\Omega)}. \quad (4)$$

Sketch of the proof. See, e.g., Lemma 2.6 in Gómez et al [1] for the first inequality in (4). The second one can be obtained from the Hölder inequality $\|w\|_{L^4(\Pi_\varepsilon)} \leq \|w\|_{L^2(\Pi_\varepsilon)}^{1/4} \|w\|_{L^6(\Pi_\varepsilon)}^{3/4}$, the first inequality in (4), and the embedding of $H_0^1(\Omega)$ into $L^6(\Omega)$. We refer to Gómez et al [2] for further explanation. \square

Lemma 2. *Let H_ε and H_0 be two separable Hilbert spaces with the scalar products $(\cdot, \cdot)_\varepsilon$ and $(\cdot, \cdot)_0$ respectively. Let $A^\varepsilon \in \mathcal{L}(H_\varepsilon)$ and $A^0 \in \mathcal{L}(H_0)$. Let \mathcal{W} be a subspace of H_0 such that $\text{Im } A^0 = \{v \mid v = A^0 u : u \in H_0\} \subset \mathcal{W}$. We assume that the following properties are satisfied:*

- (C1) *There exists an operator $\mathcal{R}^\varepsilon \in \mathcal{L}(H_0, H_\varepsilon)$ such that, for any $f \in \mathcal{W}$, $\|\mathcal{R}^\varepsilon f\|_\varepsilon \rightarrow \|f\|_0$ as $\varepsilon \rightarrow 0$.*
- (C2) *A^ε and A^0 are positive, compact and self-adjoint operators on H_ε and H_0 respectively. Besides, the norms $\|A^\varepsilon\|_{\mathcal{L}(H_\varepsilon)}$ are bounded by a constant independent of ε .*
- (C3) *For any $f \in \mathcal{W}$, $\|A^\varepsilon \mathcal{R}^\varepsilon f - \mathcal{R}^\varepsilon A^0 f\|_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$.*
- (C4) *The family of operators A^ε is uniformly compact, i.e., for any sequence f^ε in H_ε such that $\sup_\varepsilon \|f^\varepsilon\|_\varepsilon$ is bounded by a constant independent of ε , we can extract a subsequence $f^{\varepsilon'}$ verifying $\|A^{\varepsilon'} f^{\varepsilon'} - \mathcal{R}^{\varepsilon'} w^0\|_{\varepsilon'} \rightarrow 0$, as $\varepsilon' \rightarrow 0$, for certain $w^0 \in \mathcal{W}$.*

Let $\{\mu_i^\varepsilon\}_{i=1}^\infty$ ($\{\mu_i^0\}_{i=1}^\infty$, respectively) be the sequence of the eigenvalues of A^ε (A^0 , respectively) with the usual convention of repeated eigenvalues. Let $\{w_i^\varepsilon\}_{i=1}^\infty$ and $\{w_i^0\}_{i=1}^\infty$, respectively) be the corresponding eigenfunctions which are assumed to be an orthonormal basis in H_ε (H_0 , respectively). Then, for each fixed k there exist a constant C_k and a $\varepsilon_k > 0$ such that, for $\varepsilon \leq \varepsilon_k$,

$$|\mu_k^\varepsilon - \mu_k^0| \leq C_k \sup \|A^\varepsilon \mathcal{R}^\varepsilon u - \mathcal{R}^\varepsilon A^0 u\|_\varepsilon, \tag{5}$$

where the sup is taken over all the functions u in the eigenspace associated with μ_k^0 , u such that $\|u\|_0 = 1$.

In addition, for any eigenvalue μ_k^0 of A^0 with multiplicity s ($\mu_k^0 = \mu_{k+1}^0 = \dots = \mu_{k+s-1}^0$), and for any w eigenfunction corresponding to μ_k^0 , with $\|w\|_0 = 1$, there exists w^ε , w^ε being a linear combination of eigenfunctions $\{w_j^\varepsilon\}_{j=k}^{j=k+s-1}$ of A^ε corresponding to $\{\mu_j^\varepsilon\}_{j=k}^{j=k+s-1}$, such that

$$\|w^\varepsilon - \mathcal{R}^\varepsilon w\|_\varepsilon \leq C_k \|A^\varepsilon \mathcal{R}^\varepsilon w - \mathcal{R}^\varepsilon A^0 w\|_\varepsilon. \tag{6}$$

We refer to Theorems 1.4 and 1.7 in Chapter III of Oleinik [4] for the proof of Lemma 2.

Let us define $\mathcal{H}^\varepsilon = L^2(\Omega_\varepsilon)$ and $\mathcal{H}^0 = L^2(\Omega)$ with the usual scalar products. Let us introduce the operators $\mathcal{A}^\varepsilon : \mathcal{H}^\varepsilon \rightarrow \mathcal{H}^\varepsilon$ and $\mathcal{A}^0 : \mathcal{H}^0 \rightarrow \mathcal{H}^0$. For $f^\varepsilon \in \mathcal{H}^\varepsilon$,

we set $\mathcal{A}^\varepsilon f^\varepsilon = u_\varepsilon$ where $u_\varepsilon \in H^1(\Omega_\varepsilon, \partial\Omega)$ is the unique solution of

$$\int_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla v \, dx + \varepsilon^{-\kappa} \int_{S_\varepsilon} a u_\varepsilon v \, ds = \int_{\Omega_\varepsilon} f^\varepsilon v \, dx, \quad \forall v \in H^1(\Omega_\varepsilon, \partial\Omega). \quad (7)$$

Consequently, the eigenelements of \mathcal{A}^ε are $\{((\lambda_k^\varepsilon)^{-1}, u_k^\varepsilon)\}_{k=1}^\infty$ with $\{(\lambda_k^\varepsilon, u_k^\varepsilon)\}_{k=1}^\infty$ the eigenelements of (2). In the same way, for $f \in \mathcal{H}^0$, we set $\mathcal{A}^0 f = u$ where $u \in H_0^1(\Omega)$ is the unique solution of

$$\int_{\Omega} \nabla u \nabla v \, dx + 4\pi C_0 \int_{\gamma} h u v \, d\hat{x} = \int_{\Omega} f v \, dx, \quad \forall v \in H_0^1(\Omega), \quad (8)$$

and, the eigenelements of \mathcal{A}^0 are $\{((\lambda_k)^{-1}, u_k)\}_{k=1}^\infty$ with $\{(\lambda_k, u_k)\}_{k=1}^\infty$ the eigenelements of (3). Above \hat{x} denotes $\hat{x} = (x_2, x_3)$. We also set $\mathcal{W} = H_0^1(\Omega)$.

Finally, we define $\mathcal{R}^\varepsilon : L^2(\Omega) \rightarrow L^2(\Omega_\varepsilon)$ the restriction operator; namely, $(\mathcal{R}^\varepsilon f)(x) = f(x)$ if $x \in \Omega_\varepsilon$.

Let us introduce the extension operator \mathcal{P}_ε from $H^1(\Omega_\varepsilon, \partial\Omega)$ into $H_0^1(\Omega)$, such that for $w \in H^1(\Omega_\varepsilon, \partial\Omega)$ we set $\mathcal{P}_\varepsilon w = \tilde{w}$ the function which satisfies: $\tilde{w}(x) = w(x)$ for $x \in \Omega_\varepsilon$, and

$$\|\tilde{w}\|_{H^1(\Omega)} \leq C \|w\|_{H^1(\Omega_\varepsilon)} \quad \text{and} \quad \|\nabla \tilde{w}\|_{L^2(\Omega)} \leq C \|\nabla w\|_{L^2(\Omega_\varepsilon)}. \quad (9)$$

This allows us to prove the estimates for $u_\varepsilon = \mathcal{A}^\varepsilon(\mathcal{R}^\varepsilon f)$ (see Lemma 2.7 and Theorem 2.1 in Gómez et al [1]):

$$\begin{aligned} \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \varepsilon^{-\kappa/2} \|u_\varepsilon\|_{L^2(S_\varepsilon)} &\leq C \|f\|_{L^2(\Omega_\varepsilon)}, \\ \|\tilde{u}_\varepsilon\|_{H^1(\Omega)} &\leq C \|f\|_{L^2(\Omega_\varepsilon)}. \end{aligned} \quad (10)$$

Lemma 3. *Let u be the solution of (8) with $f \in L^2(\Omega)$. Then,*

$$\begin{aligned} \|u\|_{H^1(\Omega)} &\leq C \|f\|_{L^2(\Omega)}, \\ \|u\|_{W^{1,4}(\Omega)} &\leq C \|f\|_{L^2(\Omega)}, \\ \|u\|_{L^\infty(\Omega)} &\leq C \|f\|_{L^2(\Omega)}. \end{aligned} \quad (11)$$

Sketch of the proof. From the Poincaré inequality, taking $v = u$ in (8), we get the first estimate in (11). For the second estimate, we consider the function $\psi(x) = u(x) \exp(g(x))$ where u is the solution of (8) and g is defined by $g(x) = -4\pi C_0 h(0, \hat{x}) x_1$ if $x_1 > 0$ and $g(x) = 0$ otherwise. Then, we show

that ψ satisfies the equation $-\partial_{x_i}(c \partial_{x_i} \psi + b_i \psi) = f$ in Ω , for certain functions $c \in C^{0,1}(\Omega)$ and $b_i \in L^\infty(\Omega)$, $i = 1, 2, 3$, and the Dirichlet condition $\psi = 0$ on $\partial\Omega$. Then, using an adaptation of the proof in Theorem 1 in Shaposhnikov [7] for $n = 3$, $q = 4$ and $p = 12/7$ (see also Section V.5 in Morrey [3]) we have $\|\psi\|_{W^{1,4}(\Omega)} \leq C(\|\psi\|_{L^1(\Omega)} + \|f\|_{L^{12/7}(\Omega)})$. We refer to Gómez et al [2] for further explanation.

Consequently, by definition of ψ , the smoothness of h , the embedding of the spaces L^r with $1 \leq r \leq \infty$, and the first estimate in (11), we obtain the second estimate in (11). The last estimate in (11) can be obtained directly from the second estimate, namely the estimate for the $W^{1,4}(\Omega)$ -norm, and the embedding of $W^{1,4}(\Omega)$ into $L^\infty(\Omega)$. \square

Finally, in order to prove the convergence, we introduce the test function W_ε . Let P_ε^j be the center of the ball G_ε^j and we denote by T_ε^j the ball of radius $\varepsilon/4$ with center P_ε^j . Let us consider the functions w_ε^j ($j \in \Upsilon_\varepsilon$) as the solutions of the following problems

$$\Delta w_\varepsilon^j = 0 \text{ in } T_\varepsilon^j \setminus \overline{G_\varepsilon^j}, \quad w_\varepsilon^j = 1 \text{ on } \partial G_\varepsilon^j, \quad w_\varepsilon^j = 0 \text{ on } \partial T_\varepsilon^j. \tag{12}$$

We define the function $W_\varepsilon \in H^1(\mathbb{R}^n)$ by extending by 1 for $x \in \overline{G_\varepsilon}$ and by 0 for $x \in \mathbb{R}^n \setminus \bigcup_{j \in \Upsilon_\varepsilon} T_\varepsilon^j$. As is well-known (see, e.g., Gómez et al [1]), the solution of (12) can be constructed explicitly, $0 \leq W_\varepsilon \leq 1$, and the weak convergence $W_\varepsilon \rightharpoonup 0$ in $H_0^1(\Omega)$, as $\varepsilon \rightarrow 0$, holds.

3. Convergence Results when $\alpha = \kappa = 2$

Theorem 4. *Let W_ε and h be the functions defined by (12) and (3) respectively. For $f \in L^2(\Omega)$, let u be the solution of (8) and let u_ε be the solution of (7) for f^ε equal to the restriction of f to Ω_ε . Then, we have*

$$\|u_\varepsilon - u + W_\varepsilon h u\|_{H^1(\Omega_\varepsilon)}^2 + \varepsilon^{-2} \|u_\varepsilon - u + h u\|_{L^2(S_\varepsilon)}^2 \leq C \varepsilon^{1/8} \|f\|_{L^2(\Omega)}^2 \tag{13}$$

and

$$\|u_\varepsilon - u\|_{L^2(\Omega_\varepsilon)}^2 \leq C \varepsilon^{1/8} \|f\|_{L^2(\Omega)}^2. \tag{14}$$

Proof. Let us consider (7) and (8) with $v = u_\varepsilon - u + W_\varepsilon h u \in H^1(\Omega_\varepsilon, \partial\Omega)$ and $v = \tilde{u}_\varepsilon - u + W_\varepsilon h u \in H_0^1(\Omega)$ as test functions, respectively. Subtracting both equalities, we obtain

$$\|\nabla(u_\varepsilon - u + W_\varepsilon h u)\|_{L^2(\Omega_\varepsilon)}^2 + \varepsilon^{-2} \int_{S_\varepsilon} a (u_\varepsilon - u + h u)^2 ds = S_1 + S_2 + S_3,$$

where

$$S_1 = \int_{G_\varepsilon} \nabla u \nabla (\tilde{u}_\varepsilon - u + W_\varepsilon h u) \, dx, \quad S_2 = - \int_{G_\varepsilon} f (\tilde{u}_\varepsilon - u + W_\varepsilon h u) \, dx,$$

$$\begin{aligned} S_3 &= \int_{\Omega_\varepsilon} \nabla (W_\varepsilon h u) \nabla (u_\varepsilon - u + W_\varepsilon h u) \, dx \\ &\quad + 4\pi C_0 \int_{\gamma} h u (\tilde{u}_\varepsilon - u + W_\varepsilon h u) \, d\hat{x} \\ &\quad - \varepsilon^{-2} \int_{S_\varepsilon} a (1-h) u (u_\varepsilon - u + h u) \, ds. \end{aligned}$$

Now, considering the volume of G_ε , the definition and the boundedness of W_ε in $H^1(\Omega)$, the smoothness of h , (11), (10) and (4), we obtain

$$|S_1| \leq \|\nabla u\|_{L^4(G_\varepsilon)} |G_\varepsilon|^{1/4} \|\nabla(\tilde{u}_\varepsilon - u + W_\varepsilon h u)\|_{L^2(\Omega)} \leq C\varepsilon \|f\|_{L^2(\Omega)}^2 \quad \text{and}$$

$$|S_2| \leq \|f\|_{L^2(G_\varepsilon)} \varepsilon^{1/2} \|\nabla(\tilde{u}_\varepsilon - u + W_\varepsilon h u)\|_{L^2(\Omega)} \leq C\varepsilon^{1/2} \|f\|_{L^2(\Omega)}^2.$$

Let us estimate S_3 . Using

$$\begin{aligned} \int_{\Omega_\varepsilon} \nabla (W_\varepsilon h u) \nabla w \, dx &= \int_{\Omega_\varepsilon} \nabla W_\varepsilon \nabla (h u w) \, dx - \int_{\Omega_\varepsilon} \nabla W_\varepsilon \nabla (h u) w \, dx \\ &\quad + \int_{\Omega_\varepsilon} W_\varepsilon \nabla (h u) \nabla w \, dx \end{aligned}$$

for $w = u_\varepsilon - u + W_\varepsilon h u$, the Green formula in the first integral on the right hand side above, and the definition of W_ε we have that $S_3 = S_{3a} + S_{3b} + S_{3c}$ where

$$\begin{aligned} S_{3a} &= \sum_{j \in \Upsilon_\varepsilon} \int_{\partial T_\varepsilon^j} \partial_\nu w_\varepsilon^j h u (u_\varepsilon - u + W_\varepsilon h u) \, ds \\ &\quad + 4\pi C_0 \int_{\gamma} h u (\tilde{u}_\varepsilon - u + W_\varepsilon h u) \, d\hat{x}, \end{aligned}$$

$$S_{3b} = \sum_{j \in \Upsilon_\varepsilon} \int_{\partial G_\varepsilon^j} \partial_\nu w_\varepsilon^j h u (u_\varepsilon - u + W_\varepsilon h u) ds - \varepsilon^{-2} \int_{S_\varepsilon} a (1-h) u (u_\varepsilon - u + h u) ds,$$

$$S_{3c} = - \int_{\Omega_\varepsilon} \nabla W_\varepsilon \nabla (h u) (u_\varepsilon - u + W_\varepsilon h u) dx + \int_{\Omega_\varepsilon} W_\varepsilon \nabla (h u) \nabla (u_\varepsilon - u + W_\varepsilon h u) dx.$$

Taking into account the explicit computation of the normal derivatives of w_ε^j , the estimate

$$\left| \sum_{j \in \Upsilon_\varepsilon} \int_{\partial T_\varepsilon^j} w ds - \frac{\pi}{4} \int_\gamma w d\hat{x} \right| \leq C\varepsilon^{1/2} \|\nabla w\|_{L^2(\Omega)}, \quad \forall w \in H_0^1(\Omega)$$

(see Lemma 2.5 in Gómez et al [1] for the proof), and the trace theorem in $H^1(\Omega)$, we have

$$|S_{3a}| \leq C\varepsilon^{1/2} \|\nabla(h u(\tilde{u}_\varepsilon - u + W_\varepsilon h u))\|_{L^2(\Omega)}.$$

In addition, from the smoothness of h , the embedding theorem of $H_0^1(\Omega)$ into $L^6(\Omega)$, the boundedness of W_ε in $H^1(\Omega)$, (10) and (11), it follows that

$$|S_{3a}| \leq C\varepsilon^{1/2} (\|\nabla u\|_{L^4(\Omega)} \|\tilde{u}_\varepsilon - u + W_\varepsilon h u\|_{L^4(\Omega)} + \|u\|_{L^\infty(\Omega)} \|\nabla(\tilde{u}_\varepsilon - u + W_\varepsilon h u)\|_{L^2(\Omega)}) \leq C\varepsilon^{1/2} \|f\|_{L^2(\Omega)}^2.$$

Finally, by the definition of W_ε and h , we can rewrite S_{3b} as

$$S_{3b} = \frac{4}{\varepsilon - 4C_0\varepsilon^2} \int_{S_\varepsilon} h u (u_\varepsilon - u + h u) ds.$$

Thus, computing the area of S_ε and using (11) and (10) we get

$$\begin{aligned} |S_{3b}| &\leq C\varepsilon^{-1} (\|u\|_{L^\infty(\Omega)} |S_\varepsilon|^{1/2} \|u_\varepsilon\|_{L^2(S_\varepsilon)} + \|u\|_{L^\infty(\Omega)}^2 |S_\varepsilon|) \\ &\leq C\varepsilon \|f\|_{L^2(\Omega)}^2. \end{aligned}$$

In a similar way,

$$|S_{3c}| \leq \|\nabla W_\varepsilon\|_{L^2(\Omega)} \|\nabla(hu)\|_{L^4(\Pi_\varepsilon)} \|\tilde{u}_\varepsilon - u + W_\varepsilon h u\|_{L^4(\Pi_\varepsilon)} \\ + |\Pi_\varepsilon|^{1/4} \|\nabla(hu)\|_{L^4(\Pi_\varepsilon)} \|\nabla(\tilde{u}_\varepsilon - u + W_\varepsilon h u)\|_{L^2(\Pi_\varepsilon)},$$

and by the boundedness of W_ε in $H^1(\Omega)$, (11), (10) and (4) we get

$$|S_{3c}| \leq C\varepsilon^{1/8} \|f\|_{L^2(\Omega)}^2.$$

Now, gathering all the above estimates, we conclude that

$$\|\nabla(u_\varepsilon - u + W_\varepsilon h u)\|_{L^2(\Omega_\varepsilon)}^2 + \varepsilon^{-2} \|u_\varepsilon - u + h u\|_{L^2(S_\varepsilon)}^2 \\ \leq C\varepsilon^{1/8} \|f\|_{L^2(\Omega)}^2. \tag{15}$$

To obtain (13) from (15), we consider the Poincaré inequality for the H^1 -extension of $u_\varepsilon - u + W_\varepsilon h u$ to Ω , $\mathcal{P}_\varepsilon(u_\varepsilon - u + W_\varepsilon h u) \in H_0^1(\Omega)$, which satisfies (9) for $w = u_\varepsilon - u + W_\varepsilon h u$.

Finally, from (13), the definition of W_ε , the smoothness of h , (4) and (11), we can write

$$\|u_\varepsilon - u\|_{L^2(\Omega_\varepsilon)}^2 \leq \|u_\varepsilon - u + W_\varepsilon h u\|_{L^2(\Omega_\varepsilon)}^2 + \|W_\varepsilon h u\|_{L^2(\Omega_\varepsilon)}^2 \\ \leq C(\varepsilon^{1/8} \|f\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Pi_\varepsilon)}^2) \\ \leq C(\varepsilon^{1/8} \|f\|_{L^2(\Omega)}^2 + \varepsilon \|\nabla u\|_{L^2(\Omega)}^2) \leq C\varepsilon^{1/8} \|f\|_{L^2(\Omega)}^2.$$

Consequently, (14) holds and the theorem is proved. □

Theorem 5. *Let $\{\lambda_k^\varepsilon\}_{k=1}^\infty$ and $\{\lambda_k\}_{k=1}^\infty$ be the eigenvalues of problem (1) and (3), respectively. Then, for each fixed k there exists a constant C_k independent of ε such that*

$$|\lambda_k^\varepsilon - \lambda_k| \leq C_k \varepsilon^{1/16}, \tag{16}$$

holds for sufficiently small ε . In addition, for any eigenvalue λ_k of (3) with multiplicity s ($\lambda_k = \lambda_{k+1} = \dots = \lambda_{k+s-1}$), and for any u eigenfunction corresponding to λ_k , with $\|u\|_{L^2(\Omega)} = 1$, there exists \tilde{u}^ε , \tilde{u}^ε a linear combination of eigenfunctions $\{u_k^\varepsilon\}_{r=k}^{r=k+s-1}$ of (1) corresponding to $\{\lambda_k^\varepsilon\}_{r=k}^{r=k+s-1}$, such that

$$\|\tilde{u}^\varepsilon - u\|_{L^2(\Omega_\varepsilon)} \leq C_k \varepsilon^{1/16}. \tag{17}$$

Proof. On account of (10) and (14), it is self-evident that the properties (C1)–(C3) of Lemma 2 are satisfied. Let us prove property (C4) in Lemma 2. In order to do this, for the $f^\varepsilon \in L^2(\Omega_\varepsilon)$, as stated in property (C4), we consider $\widehat{f}^\varepsilon \in L^2(\Omega)$ the extension of f^ε by zero inside G_ε . We have that $\|\widehat{f}^\varepsilon\|_{L^2(\Omega)}$ is bounded by a constant independent of ε and consequently, there is a subsequence $\varepsilon' \rightarrow 0$ and a certain $f^0 \in L^2(\Omega)$ such that $\widehat{f}^{\varepsilon'} \rightharpoonup f^0$ in $L^2(\Omega)$. Considering $u_{\varepsilon'} = \mathcal{A}^{\varepsilon'} \mathcal{R}^{\varepsilon'} \widehat{f}^{\varepsilon'}$ and $w^0 \in H_0^1(\Omega)$ solution of (8) for $f = f^0$, we rewrite the proof in Theorem 4 with minor modifications, and we obtain that $\|u_{\varepsilon'} - w^0\|_{L^2(\Omega_{\varepsilon'})} \rightarrow 0$, as $\varepsilon' \rightarrow 0$. Consequently, property (C4) also holds.

Now, applying Lemma 2, we have that for each fixed k ,

$$|(\lambda_k^\varepsilon)^{-1} - (\lambda_k)^{-1}| \leq C_k \sup \|u_{\varepsilon,k} - u_{0,k}\|_{L^2(\Omega_\varepsilon)} \tag{18}$$

where the sup is taken over all the functions f_k in the eigenspace associated with $(\lambda_k)^{-1}$, f_k such that $\|f_k\|_{L^2(\Omega)} = 1$, $u_{\varepsilon,k}$ and $u_{0,k}$ are $u_{\varepsilon,k} = \mathcal{A}^\varepsilon \mathcal{R}^\varepsilon f_k$ and $u_{0,k} = \mathcal{R}^\varepsilon \mathcal{A}^0 f_k$ respectively. But, (14) allows us to assert

$$\|u_{\varepsilon,k} - u_{0,k}\|_{L^2(\Omega_\varepsilon)}^2 \leq C_k \varepsilon^{1/8} \|f_k\|_{L^2(\Omega)}^2 \leq C_k \varepsilon^{1/8} \tag{19}$$

for C_k a certain constant independent of ε . From this last inequality, (18) reads $|(\lambda_k^\varepsilon)^{-1} - (\lambda_k)^{-1}|^2 \leq C_k \varepsilon^{1/8}$ which ensures the boundedness of $(\lambda_k^\varepsilon)^{-1}$ by a constant independent of ε and consequently the estimate for the eigenvalues (16) holds.

Finally, let us note that the estimate for the eigenfunctions (17) also holds applying (6) and (14). □

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