Generalized Jones matrices for anisotropic media

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Abstract: The interaction of arbitrary three-dimensional light beams with optical elements is described by the generalized Jones calculus, which has been formally proposed recently [Azzam, J. Opt. Soc. Am. A **28**, 2279 (2011)]. In this work we obtain the parametric expression of the 3×3 differential generalized Jones matrix (dGJM) for arbitrary optical media assuming transverse light waves. The dGJM is intimately connected to the Gell-Mann matrices, and we show that it provides a versatile method for obtaining the macroscopic GJM of media with either sequential or simultaneous anisotropic effects. Explicit parametric expressions of the GJM for some relevant optical elements are provided.

OCIS codes: (260.0260) Physical optics; (260.5430) Polarization; (260.2130) Ellipsometry and polarimetry.

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1. Introduction

In the widely-used Jones calculus, completely polarized light beams are characterized by the Jones vector. Such description assumes transverse plane waves traveling along a fixed propagation direction parallel to the z axis, which is a valid approach in a broad range of physical situations. However, during the last years there is a growing interest in applications that entail propagation direction changes and/or non-transverse electric fields. Some of these applications are light propagation in turbid media, high numerical aperture focusing, near-field optics and light interaction with nanoparticles [1–3]. Completely polarized light beams in such situations can be characterized by the generalized Jones vector (GJV), also called 3D Jones vector [4,5]. A generalized 3×3 Jones calculus has been recently proposed for

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describing the interaction of GJVs with anisotropic optical media and devices [5]. However, no generalized Jones matrices for optical devices or media have been given so far.

In this work we present a method for obtaining the generalized Jones matrices (GJMs) of arbitrary optical media and devices. Our development is based on the differential formulation of the generalized Jones calculus, and assumes transverse light waves. We present the explicit form of the differential generalized Jones matrix (dGJM), which extends the well-known differential Jones matrix to the three-dimensional framework. The GJM of samples with either sequential or simultaneous optical effects can be readily obtained from the dGJM. In order to illustrate the applicability of the proposed method, we provide explicit expressions of the GJM for the major types of optical devices, namely linear retarders and dichroic absorbers, and optically-active rotators.

2. Generalized Jones calculus and the differential generalized Jones matrix

According to the generalized Jones calculus [5], the equation that describes the elastic interaction of an input generalized Jones vector $\vec{\mathbf{E}}^{\mathbf{i}}$ with a sample is:

$$\vec{\mathbf{E}}^{\mathbf{o}} = \mathbf{G}\vec{\mathbf{E}}^{\mathbf{i}}, \tag{1}$$

where $\vec{\mathbf{E}}^{\mathbf{o}}$ is the output GJV and $\mathbf{G}$ is the generalized Jones matrix of the sample, a 3×3 complex matrix that determines the linear relationship between the input and output GJVs. Both GJVs are specified in the same right-handed Cartesian coordinate system xyz . From now on, we assume that both the input and output GJVs correspond to transverse totally polarized light waves, and therefore $\mathbf{G}$ models the linear relationship between input and output transverse waves.

The GJM characterizes an optical element as a whole, basically modeling light-sample interactions as an input-output mechanism. The differential formulation enables to further describe the continuous evolution of polarized light propagation through anisotropic media. In the conventional Jones calculus, the 2×2 differential Jones matrix $\mathbf{j}$ completely characterizes the anisotropic properties of an infinitesimal slab of the medium [6]. The differential Jones matrix is intimately connected with the Pauli matrices [7,8], the generators of the group SU(2), due to the fact that the Jones matrix constitutes a representation of the Lorentz group [9]. Specifically, the differential Jones matrix can be expressed as:

$$\mathbf{j} = \frac{1}{2}\sum_{l=0}^{3} i f_l \boldsymbol{\sigma}_{\mathbf{l}}, \tag{2}$$

where the four coefficients f_l are:

$$f_0 = 2\gamma_i = 2\left(\eta_i - i\,\kappa_i\right), \tag{3}$$

$$f_1 = \gamma_q = \eta_q - i\,\kappa_q, \tag{4}$$

$$f_2 = \gamma_u = \eta_u - i\,\kappa_u, \tag{5}$$

$$f_3 = \gamma_v = \eta_v - i\,\kappa_v, \tag{6}$$

each of them being associated to the corresponding Pauli matrix [7,8]. The well-known general expression for the differential Jones matrix is thus:

$$\mathbf{j} = \frac{1}{2}\begin{bmatrix} 2\kappa_i + \kappa_q + i\left(2\eta_i + \eta_q\right) & \eta_v + \kappa_u + i\left(\eta_u - \kappa_v\right) \\ -\eta_v + \kappa_u + i\left(\eta_u + \kappa_v\right) & 2\kappa_i - \kappa_q + i\left(2\eta_i - \eta_q\right) \end{bmatrix}. \tag{7}$$

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The 8 differential parameters included in this matrix are directly related to the complex propagation constant of the medium [6], which we denote here as $\gamma = \eta - i\,\kappa$. In particular, η_i and κ_i are the isotropic retardation and absorption, while parameters $\eta_{q,u,v}$ and $\kappa_{q,u,v}$ account for birefringence and dichroism effects. Subscripts indicate the specific type of anisotropy, where the convention $x_{q,u,v} = x_{y,-45°,lcr} - x_{x,45°,rcp}$ has been adopted.

From this theoretical approach, the differential calculus can be extended to the three-dimensional formulation. In order to do that, it is necessary to replace the Pauli matrices by the Gell-Mann matrices [4,8], which form the complete set of infinitesimal generators of the Lie group SU(3):

$$\begin{gathered}
\mathbf{O_0} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{O_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \mathbf{O_2} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
\mathbf{O_3} = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \mathbf{O_4} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad \mathbf{O_5} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \\
\mathbf{O_6} = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix} \quad \mathbf{O_7} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \mathbf{O_8} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}
\end{gathered} \tag{8}$$

Taking these matrices into account, the differential generalized Jones matrix $\mathbf{g}$ can be expressed as the following linear combination:

$$\mathbf{g} = \frac{1}{2} \sum_{m=0}^{8} i p_m \mathbf{O_m}, \tag{9}$$

where the nine coefficients p_m are:

$$\{p_{0\ldots8}\} = \left\{2\gamma_i, \gamma_q^{xy}, \gamma_u^{xy}, \gamma_v^{xy}, 2/\sqrt{3}\,\gamma_q^{z}, \gamma_u^{xz}, \gamma_v^{xz}, \gamma_u^{yz}, \gamma_v^{xz}\right\}. \tag{10}$$

The differential GJM is thus:

$$\mathbf{g} = \frac{1}{2} \begin{bmatrix} \begin{array}{c} 2\kappa_i + \kappa_q^{xy} + 2\kappa_q^{z}/3 \\ +i\left(2\eta_i + \eta_q^{xy} + 2\eta_q^{z}/3\right) \end{array} & \begin{array}{c} \eta_v^{xy} + \kappa_u^{xy} \\ +i\left(\eta_u^{xy} - \kappa_v^{xy}\right) \end{array} & \begin{array}{c} \eta_v^{xz} + \kappa_u^{xz} \\ +i\left(\eta_u^{xz} - \kappa_v^{xz}\right) \end{array} \\ \begin{array}{c} -\eta_v^{xy} + \kappa_u^{xy} \\ +i\left(\eta_u^{xy} + \kappa_v^{xy}\right) \end{array} & \begin{array}{c} 2\kappa_i - \kappa_q^{xy} + 2\kappa_q^{z}/3 \\ +i\left(2\eta_i - \eta_q^{xy} + 2\eta_q^{z}/3\right) \end{array} & \begin{array}{c} \eta_v^{yz} + \kappa_u^{yz} \\ +i\left(\eta_u^{yz} - \kappa_v^{yz}\right) \end{array} \\ \begin{array}{c} -\eta_v^{xz} + \kappa_u^{xz} \\ +i\left(\eta_u^{xz} + \kappa_v^{xz}\right) \end{array} & \begin{array}{c} -\eta_v^{yz} + \kappa_u^{yz} \\ +i\left(\eta_u^{yz} + \kappa_v^{yz}\right) \end{array} & \begin{array}{c} 2\kappa_i - 4\kappa_q^{z}/3 \\ +i\left(2\eta_i - 4\eta_q^{z}/3\right) \end{array} \end{bmatrix}. \tag{11}$$

The dGJM involves a total of 18 differential parameters that completely characterize the polarimetric properties of the sample. Isotropic retardation and absorption are denoted in the same way (η_i and κ_i). Parameters $\eta_{q,u,v}^{xy}$ and $\kappa_{q,u,v}^{xy}$ are the linear xy, linear $\pm 45^\circ$ and circular birefringence and dichroism in the xy plane respectively. They correspond to those involved in Eq. (7). The rest of parameters complement the three-dimensional anisotropy characterization: $\eta_{u,v}^{xy}$ and $\kappa_{u,v}^{xy}$ are the linear $\pm 45^\circ$ and circular birefringence and dichroism in the xz plane, while $\eta_{u,v}^{yz}$ and $\kappa_{u,v}^{yz}$ are analogously defined for the yz plane. Finally, η_q^{z}

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and κ_q^z quantify the difference in linear retardance and absorption between the z direction and the xy plane.

The dGJM presents some remarkable characteristics. Firstly, comparing Eq. (7) and Eq. (11), it can be observed that the upper-left 2×2 block coincides with $\mathbf{j}$, apart from parameters η_q^z and κ_q^z, which contribute to the expression in the three-dimensional framework. Additionally, the symmetries of the dGJM are similar to those shown by the differential Jones matrix. In the extended matrix $\mathbf{g}$, the positions occupied by the differential parameters for anisotropic effects in the xz plane are restricted to elements $\mathbf{g}(1,3)$ and $\mathbf{g}(3,1)$, while those for the yz plane are contained in $\mathbf{g}(2,3)$ and $\mathbf{g}(3,2)$.

The equation describing the GJV evolution of a transversally polarized light wave along the propagation direction $\hat{\mathbf{n}}$ is:

$$d\vec{\mathbf{E}}/dl = \mathbf{g}\vec{\mathbf{E}}, \tag{12}$$

where l is the distance traveled in such direction. Assuming that the medium is homogeneous, the GJM can be directly obtained from the dGJM by:

$$\mathbf{G} = \exp(\mathbf{g}l). \tag{13}$$

The previous equation is in full parallelism with those involved in both Jones and Mueller calculus [8,10,11], and enables to obtain the GJM of media showing multiple simultaneous optical effects. In the case of sequential optical elements, each of them characterized by $\mathbf{g}_i$, the total GJM is given by:

$$\mathbf{G} = \prod_i \exp(\mathbf{g}_i l), \tag{14}$$

which is applicable to a train of elements with either single or multiple optical effects.

3. Generalized Jones matrices of basic polarization devices

The presented approach is now applied to obtain explicit expressions of the GJMs for the most relevant classes of optical elements. A right-handed laboratory Cartesian coordinate system xyz is assumed (Fig. 1). According to the convention used in [5] and adopted in this work, any other local coordinate system $x'y'z'$ is univocally defined by $z'(\theta,\phi)$ (i.e. by the polar and azimuthal angles of the z' axis), being $x'(\pi/2,\phi-\pi/2)$ and $y'=z'\times x'$. The effect of the coordinate system transformation on the GJV is:

$$\vec{\mathbf{E}}_{x'y'z'} = \mathbf{C}(\theta,\phi)\vec{\mathbf{E}}_{xyz}, \tag{15}$$

where the subscript indicates the coordinate system in which the GJV is specified, and the coordinate system transformation matrix $\mathbf{C}(\theta,\phi)$ is:

$$\mathbf{C}(\theta,\phi) = \begin{bmatrix} \sin\phi & \cos\theta\cos\phi & \sin\theta\cos\phi \\ -\cos\phi & \cos\theta\sin\phi & \sin\theta\sin\phi \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}. \tag{16}$$

We first consider the case of a linear retarder (LR). In general, birefringent media possess three different principal refractive indices, which define the characteristic index ellipsoid of the sample [12]. However, a vast number of optical devices and samples are made of uniaxial materials. Uniaxial media show two equal refractive indices (the ordinary indices) different to

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the third one (called the extraordinary index). In such situations, the index ellipsoid becomes an ellipsoid of revolution, whose orientation can be completely defined by the optic axis $z'(\theta_b, \phi_b)$ (which is aligned with the extraordinary principal axis) as shown in Fig. 1.

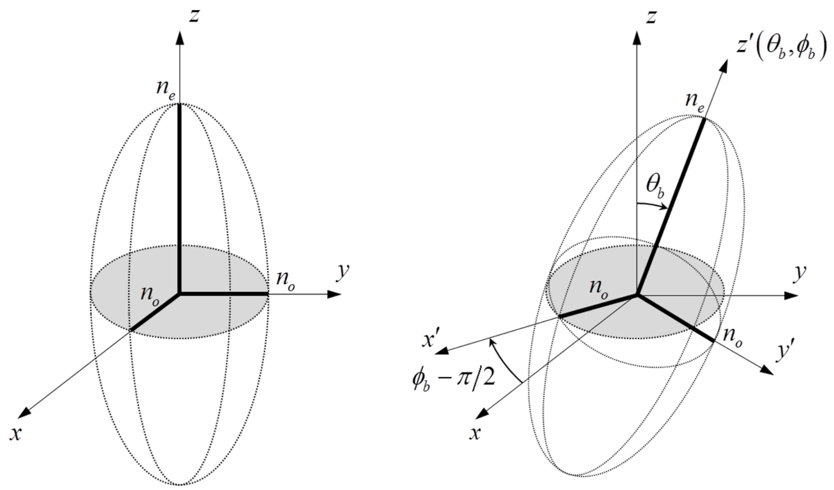

Fig. 1. Index ellipsoid of a uniaxial medium with the optic axis parallel to the laboratory z axis (left) and aligned with the direction $z'(\theta_b, \phi_b)$ of local coordinate system $x'y'z'$ (right).

Considering a linear birefringent medium in which the optic axis is parallel to the z axis (Fig. 1, left), the dGJM in the laboratory system is the following diagonal matrix:

$$\mathbf{g}_{\mathbf{LR}}\Big|_{xyz}^{z} = \frac{i}{\sqrt{3}} \eta_q^z \mathbf{O_4} = \mathrm{diag}\left(i\,\eta_q^z/3, i\,\eta_q^z/3, -i\,2\eta_q^z/3\right), \tag{17}$$

as the only non-null differential parameter is η_q^z, which simply quantifies the difference between the extraordinary and ordinary propagation constants, $\eta_q^z = \Delta\eta = \eta_e - \eta_o$. Substituting Eq. (17) into Eq. (13), the GJM of this device is readily obtained:

$$\mathbf{G}_{\mathbf{LR}}\Big|_{xyz}^{z} = \mathrm{diag}\left(e^{i\,\delta/3}, e^{i\,\delta/3}, e^{-i\,2\delta/3}\right). \tag{18}$$

In this equation, $\delta = l\,\Delta\eta$ is the linear retardance of the sample. At this point, we consider a medium in which the optic axis is arbitrarily oriented along z' (Fig. 1, right). Obviously, the GJM in the local coordinate system verifies $\mathbf{G}_{\mathbf{LR}}\Big|_{x'y'z'}^{z'} = \mathbf{G}_{\mathbf{LR}}\Big|_{xyz}^{z}$. As a consequence, the simplest way to find the GJM of an arbitrarily oriented LR is to perform a coordinate system change, multiply by $\mathbf{G}_{\mathbf{LR}}\Big|_{xyz}^{z}$, and finally apply the inverse coordinate change:

$$\mathbf{G}_{\mathbf{LR}} = \mathbf{C}(\theta_b, \phi_b) \cdot \mathbf{G}_{\mathbf{LR}}\Big|_{xyz}^{z} \cdot \mathbf{C}^{-1}(\theta_b, \phi_b). \tag{19}$$

This expression enables to obtain the GJM of a uniaxial LR with any optic axis orientation, and constitutes the core of a general method for obtaining the GJM of arbitrary elements in the generalized Jones calculus. Note that Eq. (19) parallels the approach used in Jones calculus [8]. As a particular example, the GJM of such device for $z'(\pi/2, \phi_b)$ is:

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$$\mathbf{G_{LR}}\big|_{\theta_b=\pi/2} = \begin{bmatrix} e^{-i2\delta/3}\cos^2\phi_b + e^{i\delta/3}\sin^2\phi_b & \left(e^{-i2\delta/3} - e^{i\delta/3}\right)\cos\phi_b\sin\phi_b & 0 \\ \left(e^{-i2\delta/3} - e^{i\delta/3}\right)\cos\phi_b\sin\phi_b & e^{-i2\delta/3}\sin^2\phi_b + e^{i\delta/3}\cos^2\phi_b & 0 \\ 0 & 0 & e^{i\delta/3} \end{bmatrix}. \quad (20)$$

The same method can be developed for a uniaxial linear dichroic (LD) absorber. In the basic scenario, and according to the same considerations made for the linear birefringent medium:

$$\mathbf{g_{LD}}\big|_{xyz}^{z} = \frac{1}{\sqrt{3}}\kappa_q^z\mathbf{O_4} = \operatorname{diag}\left(\kappa_q^z/3, \kappa_q^z/3, -2\kappa_q^z/3\right), \quad (21)$$

where $\kappa_q^z = \Delta\kappa = \kappa_e - \kappa_o$. The corresponding GJM for the LD is given by:

$$\mathbf{G_{LD}}\big|_{x'y'z'}^{z} = \operatorname{diag}\left(e^{\alpha/3}, e^{\alpha/3}, e^{-2\alpha/3}\right). \quad (22)$$

where $\alpha = l\Delta\kappa$.The GJM of a LD absorber with arbitrary orientation of the extraordinary axis can thus be straightforwardly obtained by introducing Eq. (22) into Eq. (19). The GJM of this optical element for $z'(\pi/2, \phi_d)$ is hereby provided for illustrative purposes:

$$\mathbf{G_{LD}}\big|_{\theta_d=\pi/2} = \begin{bmatrix} e^{-2\alpha/3}\cos^2\phi_d + e^{\alpha/3}\sin^2\phi_d & \left(e^{-2\alpha/3} - e^{\alpha/3}\right)\cos\phi_d\sin\phi_d & 0 \\ \left(e^{-2\alpha/3} - e^{\alpha/3}\right)\cos\phi_d\sin\phi_d & e^{-2\alpha/3}\sin^2\phi_d + e^{\alpha/3}\cos^2\phi_d & 0 \\ 0 & 0 & e^{\alpha/3} \end{bmatrix}. \quad (23)$$

Finally, optical rotators (OR) made of isotropic optically active materials are considered. The non-null differential parameters for this type of medium are $\eta_v^{xy} = \eta_v^{xz} = \eta_v^{yz} = \eta_v$, and its dGJM can be expressed as:

$$\mathbf{g_{OR}} = \frac{1}{2}i\eta_v\left(\mathbf{O_3} + \mathbf{O_6} + \mathbf{O_8}\right) = \frac{1}{2}\begin{bmatrix} 0 & \eta_v & \eta_v \\ -\eta_v & 0 & \eta_v \\ -\eta_v & -\eta_v & 0 \end{bmatrix}. \quad (24)$$

Defining $\psi = \sqrt{3}l\eta_v/2$, the explicit expression of the GJM for the OR is:

$$\mathbf{G_{OR}} = \frac{1}{3}\begin{bmatrix} 2\cos\psi + 1 & \cos\psi + \sqrt{3}\sin\psi - 1 & -\cos\psi + \sqrt{3}\sin\psi + 1 \\ \cos\psi - \sqrt{3}\sin\psi - 1 & 2\cos\psi + 1 & \cos\psi + \sqrt{3}\sin\psi - 1 \\ -\cos\psi - \sqrt{3}\sin\psi + 1 & \cos\psi - \sqrt{3}\sin\psi - 1 & 2\cos\psi + 1 \end{bmatrix}. \quad (25)$$

4. Conclusion

In this work, a method for obtaining the GJM of non-depolarizing anisotropic samples has been presented. Our approach is based on the dGJM, which is obtained by the extension of the differential Jones matrix to the three-dimensional framework. The dGJM provides a versatile method for obtaining the GJM of arbitrary anisotropic optical elements in a simple and elegant way. Explicit parametric expressions of the GJM for some relevant devices have been given. Transverse totally polarized light waves have been assumed. Such approach is useful for applications that entail propagation direction changes (e.g. scattering in the far field), and constitutes a necessary first step toward the development of the 3×3 generalized Jones calculus, which presents a foreseeable potential for a wide range of applications.

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