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ERROR ESTIMATES FOR THE NUMERICAL APPROXIMATION OF DIRICHLET BOUNDARY CONTROL FOR SEMILINEAR ELLIPTIC EQUATIONS*

EDUARDO CASAS† AND JEAN-PIERRE RAYMOND‡

Abstract. We study the numerical approximation of boundary optimal control problems governed by semilinear elliptic partial differential equations with pointwise constraints on the control. The control is the trace of the state on the boundary of the domain, which is assumed to be a convex, polygonal, open set in $\mathbb{R}^2$. Piecewise linear finite elements are used to approximate the control as well as the state. We prove that the error estimates are of order $O(h^{1-1/p})$ for some $p > 2$, which is consistent with the $W^{1-1/p,p}(\Gamma)$-regularity of the optimal control.

Key words. Dirichlet control, semilinear elliptic equation, numerical approximation, error estimates

AMS subject classifications. 65N30, 65N15, 49M05, 49M25

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1. Introduction.

In this paper we study an optimal control problem governed by a semilinear elliptic equation. The control is the Dirichlet datum on the boundary of the domain. Bound constraints are imposed on the control. The cost functional involves the control in a quadratic way and the state in a general way. The goal is to derive error estimates for the discretization of the control problem.

There are not many papers devoted to the derivation of error estimates for the discretization of control problems governed by partial differential equations; see the pioneering works by Falk [19] and Geveci [21]. However, recently some papers have appeared, providing new methods and ideas. Arada, Casas, and Tröltzsch [1] derived error estimates for the controls in the L^∞ and L^2 norms for distributed control problems. Similar results for an analogous problem, but also including integral state constraints, were obtained by Casas [8]. The case of a Neumann boundary control problem has been studied by Casas, Mateos, and Tröltzsch [11]. The novelty of our paper with respect to the previous ones is twofold. First, here we deal with a Dirichlet problem, the control being the value of the state on the boundary. Second, we consider piecewise linear continuous functions to approximate the optimal control, which is necessary because of the Dirichlet nature of the control, but it introduces some new difficulties. In the previous papers the controls were always approximated by piecewise constant functions. In the present situation we have developed new methods, which can be used in the framework of distributed or Neumann controls to consider piecewise linear approximations. This could lead to better error estimates than those deduced for piecewise controls.

As far as we know, there is another paper dealing with the numerical approximation of a Dirichlet control problem of Navier–Stokes equations, by Gunzburger,

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†Departamento de Matemática Aplicada y Ciencias de la Computación, E.T.S.I. Industriales y de Telecomunicación, Universidad de Cantabria, 39071 Santander, Spain (eduardo.casas@unican.es).
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‡Laboratoire MIP, UMR CNRS 5640, Université Paul Sabatier, 31062 Toulouse Cedex 4, France (raymond@mip.ups-tlse.fr).

Hou, and Svobodny [23]. Their procedure of proof does not work when the controls are subject to bound constraints, as considered in our problem. To deal with this difficulty we assume that sufficient second order optimality conditions are satisfied. We also see that the gap between the necessary and sufficient optimality conditions is very narrow. It is of the same type as in the finite dimensional case.

Let us mention some recent papers which provide some new ideas for deriving optimal error estimates. Hinze [26] suggested discretizing the state equation but not the control space. In some cases, including the case of semilinear equations, it is possible to solve the incompletely discretized problem on a computer. However, we believe this process offers no advantages for our problem because the discretization of the states forces the discretization of the controls. Another idea, due to Meyer and Rösch [33], works for linear-quadratic control problems in the distributed case, but we do not know if it is possible to adapt it to the general case.

In the case of parabolic problems, the theory is far from being complete, but some research has been carried out; see Knowles [27], Lasiecka [28], [29], McKnight and Bosarge [32], Tiba and Tröltzsch [36], and Tröltzsch [38], [39], [40], [41].

In the context of control problems of ordinary differential equations, great work has been done by Hager [24], [25] and Dontchev and Hager [16], [17]; see also the work by Malanowski, Büskens, and Maurer [31]. The reader is also referred to the detailed bibliography in [17].

The plan of the paper is as follows. In section 2 we set the optimal control problem and we establish the results we need for the state equation. In section 3 we write the first and second order optimality conditions. The first order conditions allow us to deduce some regularity results of the optimal control, which are necessary to derive the error estimates of the discretization. The second order conditions are also essential to prove the error estimates. The discrete optimal control problem is formulated in section 4 and the first order optimality conditions are given. To write these conditions we have defined a discrete normal derivative for piecewise linear functions, which are solutions of some discrete equation. Sections 6 and 7 are devoted to the analysis of the convergence of the solutions of the discrete optimal control problems and to the proof of error estimates. The main result is Theorem 7.1, where we establish $\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} = O(h^{1-1/p})$.

The numerical tests we have performed confirm our theoretical estimates. For a detailed report we refer to [12]. A simple example is reported in section 8.

2. The control problem. Throughout this paper, Ω denotes an open convex bounded polygonal set of $\mathbb{R}^2$, and Γ denotes its boundary. In this domain we formulate the following control problem:

$$\text{(P)}\begin{cases} \inf \quad J(u) = \displaystyle\int_\Omega L(x, y_u(x))\,dx \ + \frac{N}{2}\int_\Gamma u^2(x)\,dx \\ \qquad \text{subject to } (y_u, u) \in L^\infty(\Omega) \times L^\infty(\Gamma), \\ u \in U^{ad} = \{u \in L^\infty(\Gamma) \mid \alpha \le u(x) \le \beta \ \text{ a.e. } x \in \Gamma\}, \\ (y_u, u) \ \text{ satisfying the state equation (2.1)}, \end{cases}$$

$$-\Delta y_u(x) = f(x, y_u(x)) \ \text{ in } \Omega, \qquad y_u(x) = u(x) \ \text{ on } \Gamma, \tag{2.1}$$

where $-\infty < \alpha < \beta < +\infty$ and $N > 0$. Here u is the control, while y_u is the associated state. The following hypotheses are assumed about the functions involved in the control problem (P).

(A1) The function $L : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is measurable with respect to the first component and is of class C^2 with respect to the second one, $L(\cdot, 0) \in L^1(\Omega)$, and for all $M > 0$ there exist a function $\psi_{L,M} \in L^{\bar{p}}(\Omega)$ ($\bar{p} > 2$) and a constant $C_{L,M} > 0$ such that

$$\left|\frac{\partial L}{\partial y}(x, y)\right| \le \psi_{L,M}(x), \quad \left|\frac{\partial^2 L}{\partial y^2}(x, y)\right| \le C_{L,M},$$
$$\left|\frac{\partial^2 L}{\partial y^2}(x, y_2) - \frac{\partial^2 L}{\partial y^2}(x, y_1)\right| \le C_{L,M}|y_2 - y_1|,$$

for a.e. $x \in \Omega$ and $|y|, |y_i| \le M$, $i = 1, 2$.

(A2) The function $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is measurable with respect to the first variable and is of class C^2 with respect to the second one,

$$f(\cdot, 0) \in L^{\bar{p}}(\Omega) \ \ (\bar{p} > 2), \quad \frac{\partial f}{\partial y}(x, y) \le 0 \ \text{ a.e. } x \in \Omega \text{ and } y \in \mathbb{R}.$$

For all $M > 0$ there exists a constant $C_{f,M} > 0$ such that

$$\left|\frac{\partial f}{\partial y}(x, y)\right| + \left|\frac{\partial^2 f}{\partial y^2}(x, y)\right| \le C_{f,M} \text{ a.e. } x \in \Omega \text{ and } |y| \le M,$$
$$\left|\frac{\partial^2 f}{\partial y^2}(x, y_2) - \frac{\partial^2 f}{\partial y^2}(x, y_1)\right| < C_{f,M}|y_2 - y_1| \ \text{ a.e. } x \in \Omega \text{ and } |y_1|, |y_2| \le M.$$

Let us finish this section by proving that problem (P) is well defined. We will say that an element $y_u \in L^\infty(\Omega)$ is a solution of (2.1) if

(2.2)
$$\int_\Omega -\Delta w\, y\, dx = \int_\Omega f(x, y(x))w(x)dx - \int_\Gamma u(x)\partial_\nu w(x)dx, \quad \forall w \in H^2(\Omega) \cap H_0^1(\Omega),$$

where ∂_ν denotes the normal derivative on the boundary Γ. This is the classical definition in the transposition sense. To study (2.1), we state an estimate for the linear equation

$$-\Delta z(x) = b(x)z(x) \ \text{ in } \Omega, \qquad z(x) = u(x) \ \text{ on } \Gamma, \tag{2.3}$$

where b is a nonpositive function belonging to $L^\infty(\Omega)$.

LEMMA 2.1. *For every $u \in L^\infty(\Gamma)$, the linear equation* (2.3) *has a unique solution $z \in L^\infty(\Omega)$ (defined in the transposition sense), and it satisfies*

(2.4)
$$\|z\|_{L^2(\Omega)} \le C\|u\|_{H^{-1/2}(\Gamma)}, \ \|z\|_{H^{1/2}(\Omega)} \le C\|u\|_{L^2(\Gamma)} \ \textit{and} \ \|z\|_{L^\infty(\Omega)} \le \|u\|_{L^\infty(\Gamma)}.$$

The proof is standard: the first inequality is obtained by using the transposition method (see Lions and Magenes [30]), the second inequality is deduced by interpolation, and the last one is obtained by applying the maximum principle.

THEOREM 2.2. *For every $u \in L^\infty(\Gamma)$, the state equation* (2.1) *has a unique solution $y_u \in L^\infty(\Omega) \cap H^{1/2}(\Omega)$. Moreover the following Lipschitz properties hold:*

$$\begin{aligned} &\|y_u - y_v\|_{L^\infty(\Omega)} \le \|u - v\|_{L^\infty(\Gamma)}, \\ &\|y_u - y_v\|_{H^{1/2}(\Omega)} \le C\|u - v\|_{L^2(\Gamma)} \quad \forall u, v \in L^\infty(\Gamma). \end{aligned} \tag{2.5}$$

Finally if $u_n \rightharpoonup u$ *weakly** *in* $L^\infty(\Gamma)$, *then* $y_{u_n} \to y_u$ *strongly in* $L^r(\Omega)$ *for all* $r < +\infty$.

Proof. Let us introduce the following problems:

$$-\Delta z = 0 \text{ in } \Omega, \qquad z = u \text{ on } \Gamma, \tag{2.6}$$

and

$$-\Delta \zeta = g(x,\zeta) \text{ in } \Omega, \qquad \zeta = 0 \text{ on } \Gamma, \tag{2.7}$$

where $g : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is given by $g(x,t) = f(x, z(x)+t)$, with z being the solution of (2.6). Lemma 2.1 implies that (2.6) has a unique solution in $L^\infty(\Omega) \cap H^{1/2}(\Omega)$. It is obvious that assumption (A2) is fulfilled by g and that (2.7) is a classically well-set problem having a unique solution in $H^1_0(\Omega) \cap L^\infty(\Omega)$. Moreover, since Ω is convex, we know that $\zeta \in H^2(\Omega)$; see Grisvard [22]. Finally the solution y_u of (2.1) can be written as $y_u = z + \zeta$. Estimates (2.5) follow from Lemma 2.1; see Arada and Raymond [2] for a detailed proof in the parabolic case. The continuous dependence in $L^r(\Omega)$ follows in a standard way by using (2.5) and the compactness of the inclusion $H^{1/2}(\Omega) \subset L^2(\Omega)$ along with the fact that $\{y_{u_n}\}$ is bounded in $L^\infty(\Omega)$, as deduced from the first inequality of (2.5). ∎

Now the following theorem can be proved by standard arguments.

THEOREM 2.3. *Problem* (P) *has at least one solution.*

3. Optimality conditions.

Before writing the optimality conditions for (P) let us state the differentiability properties of J.

THEOREM 3.1. *The mapping* $G : L^\infty(\Gamma) \longrightarrow L^\infty(\Omega) \cap H^{1/2}(\Omega)$ *defined by* $G(u) = y_u$ *is of class* C^2. *Moreover, for all* $u, v \in L^\infty(\Gamma)$, $z_v = G'(u)v$ *is the solution of*

$$-\Delta z_v = \frac{\partial f}{\partial y}(x, y_u) z_v \text{ in } \Omega, \qquad z_v = v \text{ on } \Gamma, \tag{3.1}$$

and for every $v_1, v_2 \in L^\infty(\Omega)$, $z_{v_1 v_2} = G''(u) v_1 v_2$ *is the solution of*

$$\begin{cases} -\Delta z_{v_1 v_2} = \dfrac{\partial f}{\partial y}(x, y_u) z_{v_1 v_2} + \dfrac{\partial^2 f}{\partial y^2}(x, y_u) z_{v_1} z_{v_2} & \text{in } \Omega, \\ z_{v_1 v_2} = 0 & \text{on } \Gamma, \end{cases} \tag{3.2}$$

where $z_{v_i} = G'(u) v_i$, $i = 1, 2$.

Proof. Let us define the space

$$V = \{ y \in H^{1/2}(\Omega) \cap L^\infty(\Omega) : \Delta y \in L^2(\Omega) \}$$

endowed with the natural graph norm. Now we consider the function $F : L^\infty(\Gamma) \times V \longrightarrow L^\infty(\Gamma) \times L^2(\Omega)$ defined by $F(u, y) = (y|_\Gamma - u, \Delta y + f(x, y))$. It is obvious that F is of class C^2 and that for every pair (u, y) satisfying (2.1) we have $F(u, y) = (0, 0)$. Furthermore

$$\frac{\partial F}{\partial y}(u, y) \cdot z = \left(z|_\Gamma, \Delta z + \frac{\partial f}{\partial y}(x, y) z \right).$$

By using Lemma 2.1 we deduce that $(\partial F/\partial y)(u, y) : V \longrightarrow L^\infty(\Gamma) \times L^2(\Omega)$ is an isomorphism. Then the implicit function theorem allows us to conclude that G is of class C^2, and now the rest of the theorem follows easily. ∎

Theorem 3.1, along with the chain rule, leads to the following result.

THEOREM 3.2. *The functional* $J : L^\infty(\Gamma) \to \mathbb{R}$ *is of class* C^2. *Moreover, for every* $u, v, v_1, v_2 \in L^\infty(\Gamma)$,

$$J'(u)v = \int_\Gamma (Nu - \partial_\nu \phi_u)\, v\, dx \tag{3.3}$$

and

$$J''(u)v_1v_2 = \int_\Omega \left[\frac{\partial^2 L}{\partial y^2}(x, y_u) z_{v_1} z_{v_2} + \phi_u \frac{\partial^2 f}{\partial y^2}(x, y_u) z_{v_1} z_{v_2}\right] dx + \int_\Gamma N v_1 v_2\, dx, \tag{3.4}$$

where $z_{v_i} = G'(u)v_i$, $i = 1, 2$, $y_u = G(u)$, *and the adjoint state* $\phi_u \in H^2(\Omega)$ *is the unique solution of the problem*

$$-\Delta\phi = \frac{\partial f}{\partial y}(x, y_u)\phi + \frac{\partial L}{\partial y}(x, y_u) \quad \text{in } \Omega, \qquad \phi = 0 \quad \text{on } \Gamma. \tag{3.5}$$

The first order optimality conditions for problem (P) follow readily from Theorem 3.2.

THEOREM 3.3. *Assume that* $\bar{u}$ *is a local solution of problem* (P) *and let* $\bar{y}$ *be the corresponding state. Then there exists* $\bar{\phi} \in H^2(\Omega)$ *such that*

$$-\Delta\bar{\phi} = \frac{\partial f}{\partial y}(x, \bar{y})\bar{\phi} + \frac{\partial L}{\partial y}(x, \bar{y}) \text{ in } \Omega, \qquad \bar{\phi} = 0 \text{ on } \Gamma, \tag{3.6}$$

and

$$\int_\Gamma \left(N\bar{u} - \partial_\nu\bar{\phi}\right)(u - \bar{u})\, dx \geq 0, \quad \forall u \in U^{ad}, \tag{3.7}$$

which is equivalent to

$$\bar{u}(x) = \mathrm{Proj}_{[\alpha,\beta]}\Big(\frac{1}{N}\partial_\nu\bar{\phi}(x)\Big) = \max\Big\{\alpha, \min\Big\{\beta, \frac{1}{N}\partial_\nu\bar{\phi}(x)\Big\}\Big\}. \tag{3.8}$$

THEOREM 3.4. *Assume that* $\bar{u}$ *is a local solution of problem* (P) *and let* $\bar{y}$ *and* $\bar{\phi}$ *be the corresponding state and adjoint state. Then there exists* $p \in (2, \bar{p}]$ (*with* $\bar{p} > 2$ *as introduced in assumptions* (A1) *and* (A2)) *depending on the measure of the angles of the polygon* Ω *such that* $\bar{y} \in W^{1,p}(\Omega)$, $\bar{\phi} \in W^{2,p}(\Omega)$, *and* $\bar{u} \in W^{1-1/p,p}(\Gamma) \subset C(\Gamma)$.

Proof. From assumption (A1) and using elliptic regularity results, it follows that $\bar{\phi}$ belongs to $W^{2,p}(\Omega)$ for some $p \in (2, \bar{p}]$ depending on the measure of the angles of Γ; see Grisvard [22, Chapter 4]. To prove that $\bar{u}$ belongs to $W^{1-1/p,p}(\Gamma)$ we recall the norm in this space,

$$\|\bar{u}\|_{W^{1-1/p,p}(\Gamma)} = \left\{\int_\Gamma |\bar{u}(x)|^p dx + \int_\Gamma\int_\Gamma \frac{|\bar{u}(x) - \bar{u}(\xi)|^p}{|x - \xi|^p}\, dx\, d\xi\right\}^{1/p},$$

where we have used the fact that $\Omega \subset \mathbb{R}^2$. Due to [22, Theorem 1.5.2.3] and the fact that $\bar{\phi} = 0$ on Γ, it can be shown that $\partial_\nu\bar{\phi}$ belongs to $W^{1-1/p,p}(\Gamma)$. With the relation (3.8) and

$$\left|\mathrm{Proj}_{[\alpha,\beta]}\left(\frac{1}{N}\partial_\nu\bar{\phi}(x)\right) - \mathrm{Proj}_{[\alpha,\beta]}\left(\frac{1}{N}\partial_\nu\bar{\phi}(\xi)\right)\right| \leq \frac{1}{N}|\partial_\nu\bar{\phi}(x) - \partial_\nu\bar{\phi}(\xi)|,$$

one can prove that the integrals in the above norm are finite.

Finally, decomposing (2.1) into two problems as in the proof of Theorem 2.3, we get that $\bar{y} = \bar{z} + \bar{\zeta}$, with $\bar{\zeta} \in H^2(\Omega)$ and $\bar{z} \in W^{1,p}(\Omega)$, which completes the proof. □

In order to establish the second order optimality conditions, we define the cone of critical directions

$$C_{\bar{u}} = \{v \in L^2(\Gamma) \text{ satisfying (3.9) and } v(x) = 0 \text{ if } |\bar{d}(x)| > 0\},$$

$$v(x) = \begin{cases} \geq 0 & \text{where } \bar{u}(x) = \alpha \\ \leq 0 & \text{where } \bar{u}(x) = \beta \end{cases} \quad \text{for a.e. } x \in \Gamma, \tag{3.9}$$

where $\bar{d}$ denotes the derivative $J'(\bar{u})$,

$$\bar{d}(x) = N\bar{u}(x) - \partial_\nu \bar{\phi}(x). \tag{3.10}$$

Now we formulate the second order necessary and sufficient optimality conditions.

THEOREM 3.5. *If $\bar{u}$ is a local solution of* (P), *then $J''(\bar{u})v^2 \geq 0$ holds for all $v \in C_{\bar{u}}$. Conversely, if $\bar{u} \in U^{ad}$ satisfies the first order optimality conditions provided by Theorem* 3.3 *and the coercivity condition*

$$J''(\bar{u})v^2 > 0, \quad \forall v \in C_{\bar{u}} \setminus \{0\}, \tag{3.11}$$

then there exist $\mu > 0$ and $\varepsilon > 0$ such that $J(u) \geq J(\bar{u}) + \mu\|u - \bar{u}\|^2_{L^2(\Gamma)}$ is satisfied for every $u \in U^{ad}$ obeying $\|u - \bar{u}\|_{L^\infty(\Omega)} \leq \varepsilon$.

The necessary condition provided in the theorem is quite easy to get. The sufficient conditions are proved by Casas and Mateos [9, Theorem 4.3] for distributed control problems with integral state constraints. The proof can be translated in a straightforward way into the case of boundary controls; see also Bonnans and Zidani [4].

Remark 3.6. It can be proved (see Casas and Mateos [9, Theorem 4.4]) that the following two conditions are equivalent:

(1) $J''(\bar{u})v^2 > 0$ for every $v \in C_{\bar{u}} \setminus \{0\}$.

(2) There exist $\delta > 0$ and $\tau > 0$ such that $J''(\bar{u})v^2 \geq \delta\|v\|^2_{L^2(\Gamma)}$ for every $v \in C^\tau_{\bar{u}}$, where

$$C^\tau_{\bar{u}} = \{v \in L^2(\Gamma) \text{ satisfying (3.9) and } v(x) = 0 \text{ if } \ |\bar{d}(x)| > \tau\}.$$

It is clear that $C^\tau_{\bar{u}}$ contains strictly $C_{\bar{u}}$, so condition (2) seems to be stronger than (1), but in fact they are equivalent. For the proof of this equivalence, we use the fact that u appears linearly in the state equation and quadratically in the cost functional.

4. Numerical approximation of (P). Let us consider a family of triangulations $\{\mathcal{T}_h\}_{h>0}$ of $\bar{\Omega}$: $\bar{\Omega} = \cup_{T \in \mathcal{T}_h} T$. With each element $T \in \mathcal{T}_h$, we associate two parameters $\rho(T)$ and $\sigma(T)$, where $\rho(T)$ denotes the diameter of the set T, and $\sigma(T)$ is the diameter of the largest ball contained in T. Let us define the size of the mesh by $h = \max_{T \in \mathcal{T}_h} \rho(T)$. For fixed $h > 0$, we denote by $\{T_j\}_{j=1}^{N(h)}$ the family of triangles belonging to $\mathcal{T}_h$ and having a side included in the boundary Γ. If the vertices of $T_j \cap \Gamma$ are x^j_Γ and x^{j+1}_Γ, then $[x^j_\Gamma, x^{j+1}_\Gamma] := T_j \cap \Gamma$, $1 \leq j \leq N(h)$, with $x^{N(h)+1}_\Gamma = x^1_\Gamma$. We will also follow the notation $x^0_\Gamma = x^{N(h)}_\Gamma$. We assume that every vertex of the polygon Ω is one of these boundary points x^j_Γ of the triangulation and that the numbering of the nodes $\{x_\Gamma^j\}_{j=1}^{N(h)}$ is made counterclockwise. The length of the interval $[x_\Gamma^j, x_\Gamma^{j+1}]$ is denoted by $h_j = |x_\Gamma^{j+1} - x_\Gamma^j|$. The following hypotheses on the triangulation are also assumed.

(H1) There exists a constant $\rho > 0$ such that $h/\rho(T) \leq \rho$ for all $T \in \mathcal{T}_h$ and $h > 0$.

(H2) All the angles of all triangles are less than or equal to $\pi/2$.

The first assumption is not a restriction in practice and it is the usual one. The second assumption is going to allow us to use the discrete maximum principle and it is actually not too restrictive.

Given two points ξ_1 and ξ_2 of Γ, we denote by $[\xi_1, \xi_2]$ the part of Γ obtained by running the boundary from ξ_1 to ξ_2 counterclockwise. With this convention we have $(\xi_2, \xi_1) = \Gamma \setminus [\xi_1, \xi_2]$. According to this notation,

$$\int_{\xi_1}^{\xi_2} u(x)\,dx \quad \text{and} \quad \int_{\xi_2}^{\xi_1} u(x)\,dx$$

denote the integrals of a function $u \in L^1(\Gamma)$ on the parts of Γ defined by $[\xi_1, \xi_2]$ and $[\xi_2, \xi_1]$, respectively. In particular we have

$$\int_{\xi_1}^{\xi_2} u(x)\,dx = \int_\Gamma u(x)\,dx - \int_{\xi_2}^{\xi_1} u(x)\,dx.$$

Associated with this triangulation, we consider the sets

$$U_h = \Big\{u_h \in C(\Gamma) : u_h|_{[x_\Gamma^j, x_\Gamma^{j+1}]} \in \mathcal{P}_1 \text{ for } \ 1 \leq j \leq N(h)\Big\},$$
$$Y_h = \Big\{y_h \in C(\bar\Omega) : y_h|_T \in \mathcal{P}_1 \ \forall\, T \in \mathcal{T}_h\Big\},$$
$$Y_{h0} = \Big\{y_h \in Y_h : y_h|_\Gamma = 0\Big\},$$

where $\mathcal{P}_1$ is the space of polynomials of degree less than or equal to 1. The space U_h is formed by the restrictions to Γ of the functions of Y_h.

Let us consider the projection operator $\Pi_h : L^2(\Gamma) \longmapsto U_h$,

$$(\Pi_h v, u_h)_{L^2(\Gamma)} = (v, u_h)_{L^2(\Gamma)} \quad \forall u_h \in U_h.$$

The following approximation property of Π_h is well known (see for instance [20, Lemma 3.1]):

$$\|y - \Pi_h y\|_{L^2(\Gamma)} + h^{1/2}\|y - \Pi_h y\|_{H^{1/2}(\Gamma)} \leq Ch^{s-1/2}\|y\|_{H^s(\Omega)}, \quad \forall y \in H^s(\Omega),$$

and for every $1 \leq s \leq 2$. Observing that, for $1/2 < s \leq 3/2$,

$$u \longmapsto \inf_{y|_\Gamma = u} \|y\|_{H^s(\Omega)}$$

is a norm equivalent to the usual one of $H^{s-1/2}(\Gamma)$, we deduce from the above inequality that

$$\|u - \Pi_h u\|_{L^2(\Gamma)} + h^{1/2}\|u - \Pi_h u\|_{H^{1/2}(\Gamma)} \leq Ch^s\|u\|_{H^s(\Gamma)}, \quad \forall u \in H^s(\Gamma), \tag{4.1}$$

and for every $1/2 < s \leq 3/2$.

Let $a : Y_h \times Y_h \longmapsto \mathbb{R}$ be the bilinear form given by

$$a(y_h, z_h) = \int_\Omega \nabla y_h(x) \nabla z_h(x)\, dx.$$

For all $u \in L^\infty(\Gamma)$, we consider the following problem:

$$(4.2)\qquad \begin{cases} \text{Find } y_h(u) \in Y_h, \text{ such that } y_h = \Pi_h u \text{ on } \Gamma, \text{ and} \\ a(y_h(u), w_h) = \displaystyle\int_\Omega f(x, y_h(u)) w_h\, dx \quad \forall w_h \in Y_{h0}. \end{cases}$$

PROPOSITION 4.1. *For every $u \in L^\infty(\Gamma)$, (4.2) admits a unique solution $y_h(u)$.*

Proof. Let z_h be the unique element in Y_h satisfying $z_h = \Pi_h u$ on Γ, and let $z_h(x_i) = 0$ for all vertices x_i of the triangulation $\mathcal{T}_h$ not belonging to Γ. The equation

$$\zeta_h \in Y_{h0}, \quad a(\zeta_h, w_h) = -a(z_h, w_h) + \int_\Omega f(x, z_h + \zeta_h) w_h\, dx, \quad \forall w_h \in Y_{h0},$$

admits a unique solution (it is a consequence of the Minty–Browder theorem; see Brézis [7]). The function $z_h + \zeta_h$ is clearly a solution of (4.2). The uniqueness of the solution to (4.2) also follows from the Minty–Browder theorem. $\square$

Due to Proposition 4.1, we can define a functional J_h in $L^\infty(\Gamma)$ by

$$J_h(u) = \int_\Omega L(x, y_h(u)(x))\, dx + \frac{N}{2} \int_\Gamma u^2(x)\, dx.$$

The finite dimensional control problem approximating (P) is

$$(\mathrm{P}_h) \begin{cases} \min J_h(u_h) = \displaystyle\int_\Omega L(x, y_h(u_h)(x))\, dx + \frac{N}{2} \int_\Gamma u_h^2(x)\, dx \\ \text{subject to } u_h \in U_h^{ad}, \end{cases}$$

where

$$U_h^{ad} = U_h \cap U^{ad} = \{u_h \in U_h \mid \alpha \le u_h(x) \le \beta \ \forall\, x \in \Gamma\}.$$

The existence of a solution of (P_h) follows from the continuity of J_h in U_h and the fact that U_h^{ad} is a nonempty compact subset of U_h. Our next goal is to write the conditions for optimality satisfied by any local solution $\bar{u}_h$. First, we have to obtain an expression for the derivative of $J_h : L^\infty(\Gamma) \to \mathbb{R}$ analogous to the one of J given by formula (3.3). Given $u \in L^\infty(\Gamma)$ we consider the adjoint state $\phi_h(u) \in Y_{h0}$ solution of the equation

$$(4.3)\qquad a(w_h, \phi_h(u)) = \int_\Omega \left[\frac{\partial f}{\partial y}(x, y_h(u))\phi_h(u) + \frac{\partial L}{\partial y}(x, y_h(u))\right] w_h\, dx \quad \forall w_h \in Y_{h0}.$$

To obtain the analogous expression to (3.3) we have to define a discrete normal derivative $\partial_\nu^h \phi_h(u)$.

PROPOSITION 4.2. *Let u belong to $L^\infty(\Gamma)$ and let $\phi_h(u)$ be the solution of* (4.3). *There exists a unique element $\partial_\nu^h \phi_h(u) \in U_h$ verifying*

$$(4.4)\qquad \begin{aligned} &(\partial_\nu^h \phi_h(u), w_h)_{L^2(\Gamma)} = a(w_h, \phi_h(u)) \\ &- \int_\Omega \left[\frac{\partial f}{\partial y}(x, y_h(u))\phi_h(u) + \frac{\partial L}{\partial y}(x, y_h(u))\right] w_h\, dx \quad \forall w_h \in Y_h. \end{aligned}$$

Proof. The trace mapping is a surjective mapping from Y_h on U_h; therefore the linear form

$$L(w_h) = a(w_h, \phi_h(u)) - \int_\Omega \left[\frac{\partial f}{\partial y}(x, y_h(u))\phi_h(u) + \frac{\partial L}{\partial y}(x, y_h(u))\right] w_h\, dx$$

is well defined on U_h, and it is continuous on U_h. Let us remark that if in (4.4) the trace of w_h on Γ is zero, then (4.3) leads to

$$L(w_h) = 0.$$

Hence L can be identified with a unique element of U_h, which proves the above proposition. ∎

Now the function G introduced in Theorem 3.1 is approximated by the function $G_h : L^\infty(\Gamma) \longmapsto Y_h$ defined by $G_h(u) = y_h(u)$. We can easily verify that G_h is of class C^2 and that for $u, v \in L^\infty(\Gamma)$, the derivative $z_h = G_h'(u)v \in Y_h$ is the unique solution of

$$(4.5)\qquad \begin{cases} a(z_h, w_h) = \displaystyle\int_\Omega \frac{\partial f}{\partial y}(x, y_h(u)) z_h w_h\, dx, \quad \forall w_h \in Y_{h0}, \\ z_h = \Pi_h v \quad \text{on } \Gamma. \end{cases}$$

From here we deduce

$$J_h'(u)v = \int_\Omega \frac{\partial L}{\partial y}(x, y_h(u)) z_h\, dx + N \int_\Gamma uv\, dx.$$

Now (4.4) and the definition of Π_h lead to

$$(4.6)\qquad J_h'(u)v = N\int_\Gamma uv\, dx - \int_\Gamma \partial_\nu^h \phi_h(u) \Pi_h v\, dx = \int_\Gamma (Nu - \partial_\nu^h \phi_h(u))v\, dx$$

for all $u, v \in L^\infty(\Gamma)$.

Finally, we can write the first order optimality conditions.

THEOREM 4.3. *Let us assume that $\bar{u}_h$ is a local solution of* (P_h) *and $\bar{y}_h$ the corresponding state; then there exists $\bar{\phi}_h \in Y_{h0}$ such that*

$$(4.7)\qquad a(w_h, \bar{\phi}_h) = \int_\Omega \left[\frac{\partial f}{\partial y}(x, \bar{y}_h)\bar{\phi}_h + \frac{\partial L}{\partial y}(x, \bar{y}_h)\right] w_h\, dx \quad \forall w_h \in Y_{h0}$$

and

$$(4.8)\qquad \int_\Gamma (N\bar{u}_h - \partial_\nu^h \bar{\phi}_h)(u_h - \bar{u}_h)\, dx \geq 0 \quad \forall u_h \in U_h^{ad}.$$

This theorem follows readily from (4.6).

Remark 4.4. The reader could think that a projection property for $\bar{u}_h$ similar to that obtained for $\bar{u}$ in (3.8) can be deduced from (4.8). Unfortunately this property does not hold because $u_h(x)$ cannot be taken arbitrarily in $[\alpha, \beta]$. Functions $u_h \in U_h$ are determined by their values at the nodes $\{x_\Gamma^j\}_{j=1}^{N(h)}$. If we consider the basis of U_h $\{e_j\}_{j=1}^{N(h)}$ defined by $e_j(x_\Gamma^i) = \delta_{ij}$, then we have

$$u_h = \sum_{j=1}^{N(h)} u_{h,j} e_j \quad \text{with } \ u_{h,j} = u_h(x_\Gamma^j), \ \ 1 \le j \le N(h).$$

Now (4.8) can be written

$$\sum_{j=1}^{N(h)} \int_\Gamma (N\bar{u}_h - \partial_\nu^h \bar{\phi}_h) e_j \, dx (u_{h,j} - \bar{u}_{h,j}) \ge 0, \quad \forall \{u_{h,j}\}_{j=1}^{N(h)} \subset [\alpha, \beta], \tag{4.9}$$

where $\bar{u}_{h,j} = \bar{u}_h(x_\Gamma^j)$. Then (4.9) leads to

$$\bar{u}_{h,j} = \begin{cases} \alpha & \text{if } \ \int_\Gamma (N\bar{u}_h - \partial_\nu^h \bar{\phi}_h) e_j \, dx > 0, \\ \beta & \text{if } \ \int_\Gamma (N\bar{u}_h - \partial_\nu^h \bar{\phi}_h) e_j \, dx < 0. \end{cases} \tag{4.10}$$

In order to characterize $\bar{u}_h$ as the projection of $\partial_\nu^h \bar{\phi}_h / N$, let us introduce the operator $\text{Proj}_h : L^2(\Gamma) \longmapsto U_h^{ad}$ as follows. Given $u \in L^2(\Gamma)$, $\text{Proj}_h u$ denotes the unique solution of the problem

$$\inf_{v_h \in U_h^{ad}} \|u - v_h\|_{L^2(\Gamma)},$$

which is characterized by the relation

$$\int_\Gamma (u(x) - \text{Proj}_h u(x))(v_h(x) - \text{Proj}_h u(x)) \, dx \le 0 \ \ \forall v_h \in U_h^{ad}. \tag{4.11}$$

Then (4.8) is equivalent to

$$\bar{u}_h = \text{Proj}_h \left(\frac{1}{N} \partial_\nu^h \bar{\phi}_h \right). \tag{4.12}$$

Let us recall the result in [13, Lemma 3.3], where a characterization of $\text{Proj}_h(u_h)$ is stated. Given $u_h \in U_h$ and $\bar{u}_h = \text{Proj}_h(u_h)$, $\bar{u}_h$ is then characterized by the inequalities

$$\begin{aligned} &h_{j-1}[(u_{h,j-1} - \bar{u}_{h,j-1}) + 2(u_{h,j} - \bar{u}_{h,j})](t - \bar{u}_{h,j}) \\ &\quad + h_j[2(u_{h,j} - \bar{u}_{h,j}) + (u_{h,j+1} - \bar{u}_{h,j+1})](t - \bar{u}_{h,j}) \le 0 \end{aligned}$$

for all $t \in [\alpha, \beta]$ and $1 \le j \le N(h)$.

5. Numerical analysis of the state and adjoint equations. Throughout the following, the operator $I_h \in \mathcal{L}(W^{1,p}(\Omega), Y_h)$ denotes the classical interpolation operator [6]. We also need the interpolation operator $I_h^\Gamma \in \mathcal{L}(W^{1-1/p,p}(\Gamma), U_h)$. Since we have

$$I_h^\Gamma(y|_\Gamma) = (I_h y)|_\Gamma, \quad \forall y \in W^{1,p}(\Omega),$$

we shall use the same notation for both interpolation operators. The reader can observe that this abuse of notation does not lead to any confusion.

The goal of this section is to obtain the error estimates of the approximations $y_h(u)$ given by (4.2) to the solution y_u of (2.1). In order to carry out this analysis we decompose (2.1) into two problems, as in the proof of Theorem 2.3. We take $z \in H^{1/2}(\Omega) \cap L^\infty(\Omega)$ and $\zeta \in H^1_0(\Omega) \cap H^2(\Omega)$ as the solutions of (2.6) and (2.7), respectively. Then we have $y_u = z + \zeta$.

Let us consider now the discretizations of (2.6) and (2.7):

$$\begin{cases} \text{Find } z_h \in \ Y_h \text{ such that } \ z_h = \Pi_h u \text{ on } \Gamma \text{ and} \\ a(z_h, w_h) = 0, \quad \forall w_h \in Y_{h0}, \end{cases} \tag{5.1}$$

and

$$\begin{cases} \text{Find } \zeta_h \in Y_{h0} \text{ such that} \\ a(\zeta_h, w_h) = \displaystyle\int_\Omega g_h(x, \zeta_h(x)) w_h(x)\, dx, \quad \forall w_h \in Y_{h0}, \end{cases} \tag{5.2}$$

where $g_h(x,t) = f(x, z_h(x) + t)$. Now the solution $y_h(u)$ of (4.2) is decomposed as follows: $y_h(u) = z_h + \zeta_h$. The following lemma provides the estimates for $z - z_h$.

LEMMA 5.1. *Let $u \in U^{ad}$, and let z and z_h be the solutions of* (2.6) *and* (5.1), *respectively; then*

$$\|z_h\|_{L^\infty(\Omega)} \le \|\Pi_h u\|_{L^\infty(\Gamma)} \le C(\alpha,\beta) \text{ and } \|z_h\|_{W^{1,r}(\Omega)} \le C\|\Pi_h u\|_{W^{1-1/r,r}(\Gamma)}, \tag{5.3}$$

$$\|z_h\|_{L^2(\Omega)} \le C\|\Pi_h u\|_{H^{-1/2}(\Gamma)}, \tag{5.4}$$

where $1 < r \le p$ is arbitrary, with p being as given in Theorem 3.4. *If, in addition, $u \in H^s(\Gamma) \cap U^{ad}$, with $0 \le s \le 1$, then we also have*

$$\|z - z_h\|_{L^2(\Omega)} \le C h^{s+1/2} \|u\|_{H^s(\Gamma)} \quad \forall h > 0 \text{ and } 0 \le s \le 1. \tag{5.5}$$

Proof. The first inequality of (5.3) is proved in Ciarlet and Raviart [14]; we have only to notice that

$$\|\Pi_h u\|_{L^\infty(\Gamma)} \le C\|u\|_{L^\infty(\Gamma)} \le C(\alpha,\beta), \tag{5.6}$$

where C is independent of h and $u \in U^{ad}$; see Douglas, Dupont, and Wahlbin [18].

Inequality (5.5) can be found in French and King [20, Lemma 3.3] by just taking into account that

$$\|z\|_{H^{s+1/2}(\Omega)} \le C\|u\|_{H^s(\Gamma)}.$$

The second inequality of (5.3) is established in Bramble, Pasciak, and Schatz [5, Lemma 3.2] for $r = 2$. Let us prove it for all r in the range $(1, p]$. Let us consider the $z^h \in H^1(\Omega)$ solution of the problem

$$-\Delta z^h = 0 \quad \text{in } \Omega, \qquad z^h = \Pi_h u \quad \text{on } \Gamma.$$

This is a standard Dirichlet problem with the property (see Dauge [15])

$$\|z^h\|_{W^{1,r}(\Omega)} \le C\|\Pi_h u\|_{W^{1-1/r,r}(\Gamma)}.$$

Let us denote by $\hat{I}_h : W^{1,r}(\Omega) \longmapsto Y_h$ the generalized interpolation operator, due to Scott and Zhang [35], that preserves piecewise-affine boundary conditions. More precisely, it has the properties $\hat{I}_h(y_h) = y_h$ for all $y_h \in Y_h$ and $\hat{I}_h(W_0^{1,r}(\Omega)) \subset Y_{h0}$. These properties imply that $\hat{I}_h(z^h) = \Pi_h u$ on Γ. Thus we have

$$-\Delta(z^h - \hat{I}_h(z^h)) = \Delta \hat{I}_h(z^h) \quad \text{in } \Omega, \qquad z^h - \hat{I}_h(z^h) = 0 \quad \text{on } \Gamma,$$

and $z_h - \hat{I}_h(z^h) \in Y_{h0}$ satisfies

$$a(z_h - \hat{I}_h(z^h), w_h) = -a(\hat{I}_h(z^h), w_h) \ \ \forall w_h \in Y_{h0}.$$

Then by using the L^p estimates (see, for instance, Brenner and Scott [6, Theorem 7.5.3]), we get

$$\begin{aligned} &\|z_h - \hat{I}_h(z^h)\|_{W^{1,r}(\Omega)} \le C\|z^h - \hat{I}_h(z^h)\|_{W^{1,r}(\Omega)} \\ &\le C(\|z^h\|_{W^{1,r}(\Omega)} + \|\hat{I}_h(z^h)\|_{W^{1,r}(\Omega)}) \le C\|z^h\|_{W^{1,r}(\Omega)} \le C\|\Pi_h u\|_{W^{1-1/r,r}(\Gamma)}. \end{aligned}$$

Then we conclude the proof as follows:

$$\|z_h\|_{W^{1,r}(\Omega)} \le \|\hat{I}_h(z^h)\|_{W^{1,r}(\Omega)} + \|z_h - \hat{I}_h(z^h)\|_{W^{1,r}(\Omega)} \le C\|\Pi_h u\|_{W^{1-1/r,r}(\Gamma)}.$$

Finally, let us prove (5.4). Using (5.5) with $s = 0$, (2.4), and an inverse inequality, we get

$$\begin{aligned} &\|z_h\|_{L^2(\Omega)} \le \|z^h - z_h\|_{L^2(\Omega)} + \|z^h\|_{L^2(\Omega)} \\ &\le C(h^{1/2}\|\Pi_h u\|_{L^2(\Gamma)} + \|\Pi_h u\|_{H^{-1/2}(\Gamma)}) \le C\|\Pi_h u\|_{H^{-1/2}(\Gamma)}. \qquad \square \end{aligned}$$

Remark 5.2. The inverse estimate used in the proof,

$$\|u\|_{L^2(\Gamma)} \le Ch^{-1/2}\|u\|_{H^{-1/2}(\Gamma)}, \quad \forall u \in U_h,$$

can be derived from the well-known inverse estimate [3],

$$\|u\|_{H^{1/2}(\Gamma)} \le Ch^{-1/2}\|u\|_{L^2(\Gamma)}, \quad \forall u \in U_h,$$

and from the equality

$$\|u\|^2_{L^2(\Gamma)} = \|u\|_{H^{1/2}(\Gamma)}\|u\|_{H^{-1/2}(\Gamma)}.$$

Now we obtain the estimates for $\zeta - \zeta_h$.

LEMMA 5.3. *There exist constants* $C_i = C_i(\alpha, \beta) > 0$ $(i = 1, 2)$ *such that, for all* $u \in U^{ad} \in H^s(\Gamma)$*, the following estimates hold:*

$$\|\zeta_h\|_{L^\infty(\Omega)} \le C_1, \quad \forall h > 0 \text{ and } s = 0, \tag{5.7}$$

$$\|\zeta - \zeta_h\|_{L^2(\Omega)} \le C_2 h^{s+1/2}(1 + \|u\|_{H^s(\Gamma)}), \quad \forall h > 0 \text{ and } 0 \le s \le 1, \tag{5.8}$$

where ζ *and* ζ_h *are the solutions of* (2.7) *and* (5.2)*, respectively.*

Proof. We are going to introduce an intermediate function $\zeta^h \in H^2(\Omega)$ satisfying

$$-\Delta\zeta^h = g_h(x, \zeta^h(x)) \quad \text{in } \Omega, \qquad \zeta^h = 0 \quad \text{on } \Gamma. \tag{5.9}$$

By using classical methods (see for instance Stampacchia [34]), we get the boundedness of ζ and ζ^h in $L^\infty(\Omega)$ for some constants depending on $\|u\|_{L^\infty(\Gamma)}$ and $\|\Pi_h u\|_{L^\infty(\Gamma)}$, which are uniformly estimated by a constant depending only on α and β; see (5.6). On the other hand, from (2.7), (5.9), and assumption (A2), we deduce

$$\begin{aligned}
&C_1\|\zeta-\zeta^h\|^2_{H^1(\Omega)} \le a(\zeta-\zeta^h,\zeta-\zeta^h)\\
&= \int_\Omega [g(x,\zeta(x))-g_h(x,\zeta^h(x))](\zeta(x)-\zeta^h(x))\,dx\\
&= \int_\Omega [g(x,\zeta(x))-g(x,\zeta^h(x))](\zeta(x)-\zeta^h(x))\,dx\\
&\quad+\int_\Omega [g(x,\zeta^h(x))-g_h(x,\zeta^h(x))](\zeta(x)-\zeta^h(x))\,dx\\
&\le \int_\Omega [g(x,\zeta^h(x))-g_h(x,\zeta^h(x))](\zeta(x)-\zeta^h(x))\,dx \le C_2\|z-z_h\|_{L^2(\Omega)}\|\zeta-\zeta^h\|_{L^2(\Omega)}\\
&\le C_3\|z-z_h\|^2_{L^2(\Omega)}+\frac{C_1}{2}\|\zeta-\zeta^h\|^2_{L^2(\Omega)}.
\end{aligned}$$

This inequality, along with (5.5), implies

$$\|\zeta-\zeta^h\|_{H^1(\Omega)} \le Ch^{s+1/2}\|u\|_{H^s(\Gamma)}. \tag{5.10}$$

Thanks to the convexity of Ω, ζ^h belongs to $H^2(\Omega)$ (see Grisvard [22]) and

$$\|\zeta^h\|_{H^2(\Omega)} \le C\|g_h(x,\zeta^h)\|_{L^2(\Omega)} = C(\|u\|_{L^\infty(\Gamma)},\|\Pi_h u\|_{L^\infty(\Gamma)}).$$

Now using the results of Casas and Mateos [10, Lemma 4 and Theorem 1] we deduce that

$$\|\zeta^h-\zeta_h\|_{L^2(\Omega)} \le Ch^2, \tag{5.11}$$

$$\|\zeta^h-\zeta_h\|_{L^\infty(\Omega)} \le Ch. \tag{5.12}$$

Finally (5.8) follows from (5.10) and (5.11), and (5.7) is a consequence of the boundedness of $\{\zeta^h\}_{h>0}$ and (5.12). ∎

THEOREM 5.4. *There exist constants $C_i = C_i(\alpha,\beta) > 0$ $(i = 1, 2)$ such that for every $u \in U^{ad}\cap H^s(\Gamma)$, with $0 \le s \le 1$, the following inequalities hold:*

$$\|y_h(u)\|_{L^\infty(\Omega)} \le C_1, \quad \forall h>0 \text{ and } s=0, \tag{5.13}$$

$$\|y_u-y_h(u)\|_{L^2(\Omega)} \le C_2h^{s+1/2}(1+\|u\|_{H^s(\Gamma)}) \quad \forall h>0 \text{ and } 0\le s\le 1. \tag{5.14}$$

Furthermore if $u_h \rightharpoonup u$ weakly in $L^2(\Gamma)$, $\{u_h\}_{h>0}\subset U^{ad}$, then $y_h(u_h)\to y_u$ strongly in $L^r(\Omega)$ for every $r<+\infty$.

Proof. Remembering that $y_u = z+\zeta$ and $y_h(u) = z_h+\zeta_h$, we see that (5.3), (5.5), (5.7), and (5.8) lead readily to inequalities (5.13) and (5.14). To prove the last part of the theorem, it is enough to use Theorem 2.2 and (5.14) with $s = 0$ as follows:

$$\|y_u-y_h(u_h)\|_{L^2(\Omega)} \le \|y_u-y_{u_h}\|_{L^2(\Omega)}+\|y_{u_h}-y_h(u_h)\|_{L^2(\Omega)} \longrightarrow 0 \quad \text{as } h\longrightarrow 0.$$

The convergence in $L^r(\Omega)$ follows from (5.13). ∎

COROLLARY 5.5. *There exists a constant $C = C(\alpha,\beta) > 0$ such that, for all $u\in U^{ad}$ and $v\in U^{ad}\cap H^s(\Gamma)$, with $0\le s\le 1$, we have*

$$\|y_u-y_h(v)\|_{L^2(\Omega)} \le C\big\{\|u-v\|_{L^2(\Gamma)}+h^{s+1/2}(1+\|v\|_{H^s(\Gamma)})\big\}. \tag{5.15}$$

This corollary is an immediate consequence of the second estimate in (2.5) and of (5.14).

Let us finish this section by establishing some estimates for the adjoint states.

THEOREM 5.6. *Given* $u, v \in U^{ad}$*, let* ϕ_u *and* $\phi_h(v)$ *be the solutions of* (3.5) *and* (4.3) *with* u *replaced by* v *in the last equation. Then there exist some constants* $C_i = C_i(\alpha, \beta) > 0$ $(1 \le i \le 3)$ *such that*

$$\|\phi_h(v)\|_{L^\infty(\Omega)} \le C_1 \quad \forall h > 0, \tag{5.16}$$

$$\|\phi_u - \phi_h(v)\|_{L^2(\Omega)} \le C_2(\|u - v\|_{L^2(\Gamma)} + h^2), \tag{5.17}$$

$$\|\phi_u - \phi_h(v)\|_{L^\infty(\Omega)} + \|\phi_u - \phi_h(v)\|_{H^1(\Omega)} \le C_3(\|u - v\|_{L^2(\Gamma)} + h). \tag{5.18}$$

Proof. All the inequalities follow from the results of Casas and Mateos [10] just by taking into account that

$$\|\phi_u - \phi_h(v)\|_X \le \|\phi_u - \phi_v\|_X + \|\phi_v - \phi_h(v)\|_X \le C(\|y_u - y_v\|_{L^2(\Omega)} + \|\phi_v - \phi_h(v)\|_X),$$

with X equal to $L^\infty(\Omega)$, $L^2(\Omega)$, and $H^1(\Omega)$, respectively. □

Now we provide an error estimate for the discrete normal derivative of the adjoint state defined by Proposition 4.2.

THEOREM 5.7. *There exists a constant* $C = C(\alpha, \beta) > 0$ *such that the following estimate holds:*

$$\|\partial_\nu \phi_u - \partial_\nu^h \phi_h(u)\|_{L^2(\Gamma)} \le \begin{cases} Ch^{1/2} & \forall u \in U^{ad}, \\ C(\|u\|_{H^{1/2}(\Gamma)} + 1)h^{1-1/p} & \forall u \in U^{ad} \cap H^{1/2}(\Gamma). \end{cases} \tag{5.19}$$

Proof. First, let us remember that $\phi_u \in H^2(\Omega)$ and therefore $\partial_\nu \phi_u \in H^{1/2}(\Gamma)$. Observe that the definition of the projection operator Π_h leads to

$$\int_\Gamma \left|\partial_\nu \phi_u - \partial_\nu^h \phi_h(u)\right|^2 = \int_\Gamma \left|\partial_\nu \phi_u - \Pi_h \partial_\nu \phi_u\right|^2 + \int_\Gamma \left|\Pi_h \partial_\nu \phi_u - \partial_\nu^h \phi_h(u)\right|^2 = I_1 + I_2.$$

Since $\partial_\nu^h \phi_h(u)$ belongs to U_h, we can write

$$I_2 = \int_\Gamma (\partial_\nu \phi_u - \partial_\nu^h \phi_h(u))(\Pi_h \partial_\nu \phi_u - \partial_\nu^h \phi_h(u)).$$

Let us introduce $z_h \in Y_h$ as the solution to the variational equation

$$\begin{cases} a(z_h, w_h) = 0 \quad \forall w_h \in Y_{h0}, \\ z_h = \Pi_h \partial_\nu \phi_u - \partial_\nu^h \phi_h(u) \quad \text{on } \Gamma. \end{cases}$$

From (5.3) it follows that

$$\|z_h\|_{H^1(\Omega)} \le C\|\Pi_h \partial_\nu \phi_u - \partial_\nu^h \phi_h(u)\|_{H^{1/2}(\Gamma)}. \tag{5.20}$$

Now using the definition of $\partial_\nu^h \phi_h(u)$ stated in Proposition 4.2 and a Green formula for ϕ_u, we can write

$$\begin{aligned} I_2 = a(z_h, \phi_u - \phi_h(u)) &+ \int_\Omega \left(\frac{\partial f}{\partial y}(x, y_h(u))\phi_h(u) - \frac{\partial f}{\partial y}(x, y_u)\phi_u\right) z_h \\ &+ \int_\Omega \left(\frac{\partial L}{\partial y}(x, y_h(u)) - \frac{\partial L}{\partial y}(x, y_u)\right) z_h. \end{aligned} \tag{5.21}$$

Due to the equation satisfied by z_h,

$$a(z_h, I_h\phi_u) = a(z_h, \phi_h(u)) = 0,$$

we also have

(5.22)
$$I_2 = a(z_h, \phi_u - I_h\phi_u) + \int_\Omega \left(\frac{\partial f}{\partial y}(x, y_h(u)) - \frac{\partial f}{\partial y}(x, y_u)\right)\phi_u z_h + \int_\Omega \frac{\partial f}{\partial y}(x, y_h(u))(\phi_h(u) - \phi_u)z_h + \int_\Omega \left(\frac{\partial L}{\partial y}(x, y_h(u)) - \frac{\partial L}{\partial y}(x, y_u)\right)z_h.$$

From well-known interpolation estimates, the second inequality of (5.3), and an inverse inequality, it follows that

$$\begin{aligned} &a(z_h, \phi_u - I_h\phi_u) \le \|z_h\|_{W^{1,p'}(\Omega)}\|\phi_u - I_h\phi_u\|_{W^{1,p}(\Omega)} \\ &\le Ch\|\phi_u\|_{W^{2,p}(\Omega)}\|z_h|_\Gamma\|_{W^{1-1/p',p'}(\Gamma)} \le Ch\|z_h|_\Gamma\|_{H^{1-1/p'}(\Gamma)} \\ &\le Ch^{1/p'}\|z_h|_\Gamma\|_{L^2(\Gamma)} = Ch^{1/p'}\sqrt{I_2}, \end{aligned} \tag{5.23}$$

where $p' = p/(p-1)$.

From assumptions (A1) and (A2) and inequalities (5.13), (5.14) with $s = 0$, (5.16), and (5.17), we get

$$\left|\int_\Omega \left(\frac{\partial f}{\partial y}(x, y_h(u)) - \frac{\partial f}{\partial y}(x, y_u)\right)\phi_u z_h\right| \le Ch^{1/2}\|z_h\|_{L^2(\Omega)}, \tag{5.24}$$

$$\begin{aligned} &\left|\int_\Omega \frac{\partial f}{\partial y}(x, y_h(u))(\phi_h(u) - \phi_u)z_h\right| \le C\|\phi_h(u) - \phi_u\|_{L^2(\Omega)}\|z_h\|_{L^2(\Omega)} \\ &\le Ch^2\|z_h\|_{L^2(\Omega)}, \end{aligned} \tag{5.25}$$

and

$$\left|\int_\Omega \left(\frac{\partial L}{\partial y}(x, y_h(u)) - \frac{\partial L}{\partial y}(x, y_u)\right)z_h\right| \le Ch^{1/2}\|z_h\|_{L^2(\Omega)}. \tag{5.26}$$

Collecting together the estimates (5.23)–(5.26) and using (5.20) and the fact that $p' < 2$, we obtain

$$\begin{aligned} I_2 &\le Ch^{1/p'}\sqrt{I_2} + Ch^{1/2}\|z_h\|_{L^2(\Omega)} \\ &\le C(h^{1/p'}\sqrt{I_2} + h^{1/2}\|\Pi_h\partial_\nu\phi_u - \partial_\nu^h\phi_h(u)\|_{L^2(\Gamma)}) \le Ch^{1/2}\sqrt{I_2}, \end{aligned} \tag{5.27}$$

which implies that

$$I_2 \le Ch. \tag{5.28}$$

Using again that $\phi_u \in W^{2,p}(\Omega)$, we get that $\partial_\nu\phi_u \in W^{1-1/p,p}(\Gamma) \subset H^{1-1/p}(\Gamma)$. Hence from (4.1) with $s = 1 - 1/p$, we can derive

$$I_1 \le Ch\|\partial_\nu\phi_u\|^2_{H^{1/2}(\Gamma)} \le Ch\|\phi_u\|^2_{H^2(\Omega)} \le Ch^{2(1-1/p)}. \tag{5.29}$$

So the first estimate in (5.19) is proved.

To complete the proof let us assume that $u \in H^{1/2}(\Gamma)$; then we can use (5.14) with $s = 1/2$ to estimate $y_u - y_h(u)$ in $L^2(\Omega)$ by Ch. This allows us to change $h^{1/2}$ in (5.24) and (5.26) by h. Therefore (5.27) can be replaced with $I_2 \le Ch^{1/p'} = Ch^{1-1/p}$; thus $I_2 \le Ch^{2(1-1/p)}$. So the second estimate in (5.19) is proved. □

COROLLARY 5.8. *There exists a constant C independent of h such that*

$$\left\{\begin{array}{l} \|\partial_\nu^h \phi_h(u)\|_{H^{1/2}(\Gamma)} \le C \quad \forall u \in U^{ad}, \\ \|\partial_\nu^h \phi_h(u)\|_{W^{1-1/p,p}(\Gamma)} \le C(\|u\|_{H^{1/2}(\Gamma)} + 1) \quad \forall u \in U^{ad} \cap H^{1/2}(\Gamma), \\ \|\partial_\nu \phi_u - \partial_\nu^h \phi_h(v)\|_{L^2(\Gamma)} \le C\left\{\|u - v\|_{L^2(\Gamma)} + h^\kappa\right\} \quad \forall u, v \in U^{ad}, \end{array}\right. \tag{5.30}$$

where $\kappa = 1 - 1/p$ if $v \in H^{1/2}(\Gamma)$ and $\kappa = 1/2$ otherwise.

Proof. Let us make the proof in the case when $u \in U^{ad} \cap H^{1/2}(\Gamma)$. The case when $u \in U^{ad}$ can be treated similarly. We know that

$$\|\partial_\nu \phi_u\|_{W^{1-1/p,p}(\Gamma)} \le C\|\phi_u\|_{W^{2,p}(\Omega)} \le C \quad \forall u \in U^{ad}.$$

On the other hand, the projection operator Π_h is stable in the Sobolev spaces $W^{s,q}(\Gamma)$, for $1 \le q \le \infty$ and $0 \le s \le 1$ (see Casas and Raymond [13]); therefore

$$\|\Pi_h \partial_\nu \phi_u\|_{W^{1-1/p,p}(\Gamma)} \le C\|\partial_\nu \phi_u\|_{W^{1-1/p,p}(\Gamma)}.$$

Finally, with an inverse inequality and the estimate $I_2 \le Ch^{2-2/p}$ obtained in the previous proof, we deduce

$$\begin{aligned} \|\partial_\nu^h \phi_h(u)\|_{W^{1-1/p,p}(\Gamma)} &\le \|\Pi_h \partial_\nu \phi_u - \partial_\nu^h \phi_h(u)\|_{W^{1-1/p,p}(\Gamma)} + \|\Pi_h \partial_\nu \phi_u\|_{W^{1-1/p,p}(\Gamma)} \\ &\le C\|\Pi_h \partial_\nu \phi_u - \partial_\nu^h \phi_h(u)\|_{H^{1-1/p}(\Gamma)} + \|\Pi_h \partial_\nu \phi_u\|_{W^{1-1/p,p}(\Gamma)} \\ &\le Ch^{-1+1/p}\|\Pi_h \partial_\nu \phi_u - \partial_\nu^h \phi_h(u)\|_{L^2(\Gamma)} + \|\partial_\nu \phi_u\|_{W^{1-1/p,p}(\Gamma)} \le C. \end{aligned}$$

The third inequality of (5.30) is an immediate consequence of Theorem 5.7. □

6. Convergence analysis for (P_h). In this section we will prove the strong convergence in $L^2(\Gamma)$ of the solutions $\bar{u}_h$ of discrete problems (P_h) to the solutions of (P). Moreover, we will first prove that $\{\bar{u}_h\}_h$ remains bounded in $H^{1/2}(\Gamma)$, and then that it is also bounded in $W^{1-1/p,p}(\Gamma)$. Finally, we will prove the strong convergence of the solutions $\bar{u}_h$ of discrete problems (P_h) to the solutions of (P) in $C(\Gamma)$.

THEOREM 6.1. *For every $h > 0$ let $\bar{u}_h$ be a global solution of problem (P_h). Then there exist weakly*-converging subsequences of $\{\bar{u}_h\}_{h>0}$ in $L^\infty(\Gamma)$ (still indexed by h). If the subsequence $\{\bar{u}_h\}_{h>0}$ is converging weakly* in $L^\infty(\Gamma)$ to some $\bar{u}$, then $\bar{u}$ is a solution of (P),*

$$\lim_{h\to 0} J_h(\bar{u}_h) = J(\bar{u}) = \inf(P) \quad \textit{and} \quad \lim_{h\to 0} \|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} = 0. \tag{6.1}$$

Proof. Since $U_h^{ad} \subset U^{ad}$ holds for every $h > 0$ and U^{ad} is bounded in $L^\infty(\Gamma)$, $\{\bar{u}_h\}_{h>0}$ is also bounded in $L^\infty(\Gamma)$. Therefore, there exist weakly*-converging subsequences as claimed in the statement of the theorem. Let $\{\bar{u}_h\}$ be one of these subsequences and let $\bar{u}$ be the weak* limit. It is obvious that $\bar{u} \in U^{ad}$. Let us prove that $\bar{u}$ is a solution of (P). Let us take a solution of (P), $\tilde{u} \in U^{ad}$; therefore $\tilde{u} \in W^{1-1/p,p}(\Gamma)$ for some $p > 2$; see Theorem 3.4. Let us take $u_h = I_h\tilde{u}$. Then

$u_h \in U_h^{ad}$ and $\{u_h\}_h$ tends to $\tilde{u}$ in $L^\infty(\Gamma)$; see Brenner and Scott [6]. By taking $u = \tilde{u}$, $v = u_h$, and $s = 0$ in (5.15) we deduce that $y_h(u_h) \to y_{\tilde{u}}$ in $L^2(\Omega)$. Moreover, (5.13) implies that $\{y_h(u_h)\}_{h>0}$ is bounded in $L^\infty(\Omega)$. On the other hand, Theorem 5.4 implies that $\bar{y}_h = y_h(\bar{u}_h) \to \bar{y} = y_{\bar{u}}$ strongly in $L^2(\Omega)$, and $\{\bar{y}_h\}_{h>0}$ is also bounded in $L^\infty(\Omega)$. Then we have

$$J(\bar{u}) \le \liminf_{h\to 0} J_h(\bar{u}_h) \le \limsup_{h\to 0} J_h(\bar{u}_h) \le \limsup_{h\to 0} J_h(I_h\tilde{u}) = J(\tilde{u}) = \inf (P).$$

This proves that $\bar{u}$ is a solution of (P) as well as the convergence of the optimal costs, which leads to $\|\bar{u}_h\|_{L^2(\Gamma)} \longrightarrow \|\bar{u}\|_{L^2(\Gamma)}$; hence we deduce the strong convergence of the controls in $L^2(\Gamma)$. ∎

THEOREM 6.2. *Let $p > 2$ be as in Theorem 3.4, and for every h let $\bar{u}_h$ denote a local solution of (P_h). Then there exists a constant $C > 0$ independent of h such that*

$$\|\bar{u}_h\|_{W^{1-1/p,p}(\Gamma)} \le C \quad \forall h > 0. \tag{6.2}$$

Moreover, the convergence of $\{\bar{u}_h\}_{h>0}$ to $\bar{u}$ stated in Theorem 6.1 holds in $C(\Gamma)$.

Proof. By using the stability in $H^{1/2}(\Gamma)$ of the $L^2(\Gamma)$-projections on the sets U_h^{ad} (see Casas and Raymond [13]) along with (4.12) and the first inequality of (5.30), we get that $\{\bar{u}_h\}_{h>0}$ is uniformly bounded in $H^{1/2}(\Gamma)$. Using now the second inequality of (5.30) and the stability of Π_h in $W^{1-1/p,p}(\Gamma)$, we deduce (6.2). Finally, the convergence is a consequence of the compactness of the imbedding $W^{1-1/p,p}(\Gamma) \subset C(\Gamma)$ for $p > 2$. ∎

7. Error estimates. The goal in this section is to prove the following theorem.

THEOREM 7.1. *Let us assume that $\bar{u}$ is a local solution of* (P) *satisfying the sufficient second order optimality conditions provided in Theorem 3.5, and let $\bar{u}_h$ be a local solution of (P_h) such that $\bar{u}_h \to \bar{u}$ in $L^2(\Gamma)$; see Theorem 6.1. Then the following inequality holds:*

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} \le Ch^{1-1/p}, \tag{7.1}$$

where $p > 2$ is given by Theorem 3.4.

We will prove the theorem arguing by contradiction. The statement of the theorem can be stated as follows. There exists a positive constant C such that for all $0 < h < 1/C$, we have

$$\frac{\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)}}{h^{1-1/p}} \le C.$$

Thus if (7.1) is false, for all $k > 0$, there exists $0 < h_k < 1/k$ such that

$$\frac{\|\bar{u} - \bar{u}_{h_k}\|_{L^2(\Gamma)}}{h_k^{1-1/p}} > k.$$

Therefore there exists a sequence of h such that

$$\lim_{h\to 0} \frac{1}{h^{1-1/p}} \|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} = +\infty. \tag{7.2}$$

We will obtain a contradiction for this sequence. For the proof of this theorem, we need some lemmas.

LEMMA 7.2. *Let us assume that* (7.1) *is false. Let* $\delta > 0$ *as given by Remark* 3.6(2). *Then there exists* $h_0 > 0$ *such that*

$$\frac{1}{2}\min\{\delta, N\}\|\bar{u} - \bar{u}_h\|^2_{L^2(\Gamma)} \leq (J'(\bar{u}_h) - J'(\bar{u}))(\bar{u}_h - \bar{u}) \quad \forall h < h_0. \tag{7.3}$$

Proof. Let $\{\bar{u}_h\}_h$ be a sequence satisfying (7.2). By applying the mean value theorem, we get for some $\hat{u}_h = \bar{u} + \theta_h(\bar{u}_h - \bar{u})$,

$$(J'(\bar{u}_h) - J'(\bar{u}))(\bar{u}_h - \bar{u}) = J''(\hat{u}_h)(\bar{u}_h - \bar{u})^2. \tag{7.4}$$

Let us take

$$v_h = \frac{1}{\|\bar{u}_h - \bar{u}\|_{L^2(\Gamma)}}(\bar{u}_h - \bar{u}).$$

Taking a subsequence if necessary, we can assume that $v_h \rightharpoonup v$ in $L^2(\Gamma)$. Let us prove that v belongs to the critical cone $C_{\bar{u}}$ defined in section 3. First, we remark that every v_h satisfies the sign condition (3.9); hence v also does. Let us prove that $v(x) = 0$ if $\bar{d}(x) \neq 0$, with $\bar{d}$ being defined by (3.10). We will use the interpolation operator $I_h \in \mathcal{L}(W^{1-1/p,p}(\Gamma), U_h)$, with $p > 2$ given in Theorem 3.4. Since $\bar{u} \in U^{ad}$, it is obvious that $I_h\bar{u} \in U_h^{ad}$. For any $y \in W^{1,p}(\Omega)$ such that $y|_\Gamma = \bar{u}$, it is clear that $I_h\bar{u}$ is the trace of $I_h y$ (see the beginning of section 5). Now, by using a result of Grisvard [22, Chapter 1], we get

$$\|\bar{u} - I_h\bar{u}\|^p_{L^p(\Gamma)} \leq C\left(\varepsilon^{1-1/p}\|y - I_h y\|^p_{W^{1,p}(\Omega)} + \varepsilon^{-1/p}\|y - I_h y\|^p_{L^p(\Omega)}\right)$$

for every $\varepsilon > 0$ and for some constant $C > 0$ independent of ε and y. Setting $\varepsilon = h^p$ and using that (see, for instance, Brenner and Scott [6])

$$\|y - I_h y\|_{L^p(\Omega)} \leq C_1 h\|y\|_{W^{1,p}(\Omega)}, \quad \|I_h y\|_{W^{1,p}(\Omega)} \leq C_2\|y\|_{W^{1,p}(\Omega)},$$

and

$$\inf_{y|_\Gamma = \bar{u}} \|y\|_{W^{1,p}(\Omega)} \leq C_3\|\bar{u}\|_{W^{1-1/p}(\Gamma)},$$

we conclude that

$$\|\bar{u} - I_h\bar{u}\|_{L^2(\Gamma)} \leq |\Gamma|^{\frac{p-2}{2p}}\|\bar{u} - I_h\bar{u}\|_{L^p(\Gamma)} \leq Ch^{1-1/p}\|\bar{u}\|_{W^{1-1/p,p}(\Gamma)}. \tag{7.5}$$

Let us define

$$\bar{d}_h(x) = N\bar{u}_h(x) - \partial^h_\nu\bar{\phi}_h(x). \tag{7.6}$$

The third inequality of (5.30) implies that $\bar{d}_h \to \bar{d}$ in $L^2(\Gamma)$. Now we have

$$\begin{aligned}\int_\Gamma \bar{d}(x)v(x)\,dx &= \lim_{h\to 0}\int_\Gamma \bar{d}_h(x)v_h(x)\,dx \\ &= \lim_{h\to 0}\frac{1}{\|\bar{u}_h - \bar{u}\|_{L^2(\Gamma)}}\left\{\int_\Gamma \bar{d}_h(I_h\bar{u} - \bar{u})\,dx + \int_\Gamma \bar{d}_h(\bar{u}_h - I_h\bar{u})\,dx\right\}.\end{aligned}$$

From (4.8), (7.2), and (7.5) we deduce

$$\begin{aligned}\int_\Gamma \bar{d}(x)v(x)\,dx &\le \lim_{h\to 0}\frac{1}{\|\bar{u}_h-\bar{u}\|_{L^2(\Gamma)}}\int_\Gamma \bar{d}_h(x)(I_h\bar{u}(x)-\bar{u}(x))\,dx\\ &\le \lim_{h\to 0}\frac{Ch^{1-1/p}}{\|\bar{u}_h-\bar{u}\|_{L^2(\Gamma)}}=0.\end{aligned}$$

Since v satisfies the sign condition (3.9), then $\bar{d}(x)v(x)\ge 0$; hence the above inequality proves that v is zero whenever $\bar{d}$ is not, which allows us to conclude that $v\in C_{\bar{u}}$. Now from the definition of v_h, (3.4), and (3.11) we get

$$\begin{aligned}\lim_{h\to 0}J''(\hat{u}_h)v_h^2 &= \lim_{h\to 0}\left\{\int_\Omega\left[\frac{\partial^2 L}{\partial y^2}(x,y_{\hat{u}_h})+\phi_{\hat{u}_h}\frac{\partial^2 f}{\partial y^2}(x,y_{\hat{u}_h})\right]z_{v_h}^2\,dx+N\right\}\\ &=\int_\Omega\left[\frac{\partial^2 L}{\partial y^2}(x,\bar{y})+\bar{\phi}\frac{\partial^2 f}{\partial y^2}(x,\bar{y})\right]z_v^2\,dx+N\\ &=J''(\bar{u})v^2+N(1-\|v\|^2_{L^2(\Gamma)})\ge N+(\delta-N)\|v\|^2_{L^2(\Gamma)}.\end{aligned}$$

Taking into account that $\|v\|_{L^2(\Gamma)}\le 1$, these inequalities lead to

$$\lim_{h\to 0}J''(\hat{u}_h)v_h^2\ge\min\{\delta,N\}>0,$$

which proves the existence of $h_0>0$ such that

$$J''(\hat{u}_h)v_h^2\ge\frac{1}{2}\min\{\delta,N\}\quad\forall h<h_0.$$

From this inequality, the definition of v_h, and (7.4) we deduce (7.3). □

LEMMA 7.3. *There exists a constant $C>0$ independent of h such that for every $v\in L^\infty(\Gamma)$,*

$$|(J_h'(\bar{u}_h)-J'(\bar{u}_h))v|\le Ch^{1-1/p}\|v\|_{L^2(\Gamma)}. \tag{7.7}$$

Proof. From (3.3), (4.6), (7.6), (6.2), and Theorem 5.7 we get

$$\begin{aligned}&(J_h'(\bar{u}_h)-J'(\bar{u}_h))v=\int_\Gamma(\partial_\nu\phi_{\bar{u}_h}-\partial_\nu^h\bar{\phi}_h)v\,dx\le\|\partial_\nu\phi_{\bar{u}_h}-\partial_\nu^h\bar{\phi}_h\|_{L^2(\Gamma)}\|v\|_{L^2(\Gamma)}\\ &\le C(\|\bar{u}_h\|_{H^{1/2}(\Gamma)}+1)h^{(1-1/p)}\|v\|_{L^2(\Gamma)}\le Ch^{(1-1/p)}\|v\|_{L^2(\Gamma)}.\end{aligned}$$ □

LEMMA 7.4. *There exists a constant $C>0$ independent of h such that for every $v\in L^\infty(\Gamma)$,*

$$|(J_h'(\bar{u}_h)-J'(\bar{u}))v|\le\Big(N\|\bar{u}-\bar{u}_h\|_{L^2(\Gamma)}+Ch^{1-1/p}\Big)\|v\|_{L^2(\Gamma)}. \tag{7.8}$$

Proof. Arguing in a way similar to the previous proof, and using (5.30) and (6.2), we have

$$\begin{aligned}(J_h'(\bar{u}_h)-J'(\bar{u}))v&=\int_\Gamma\Big(N\bar{u}_h-\partial_\nu^h\bar{\phi}_h\Big)\Pi_h v\,dx-\int_\Gamma\Big(N\bar{u}-\partial_\nu\bar{\phi}\Big)v\,dx\\ &=N\int_\Gamma(\bar{u}_h-\bar{u})v\,dx+\int_\Gamma\Big(\partial_\nu\bar{\phi}-\partial_\nu^h\bar{\phi}_h\Big)v\,dx\\ &\le\Big(N\|\bar{u}_h-\bar{u}\|_{L^2(\Gamma)}+Ch^{(1-1/p)}\Big)\|v\|_{L^2(\Gamma)}.\end{aligned}$$ □

One key point in the proof of the error estimates is to get a discrete control $u_h \in U_h^{ad}$ that approximates $\bar{u}$ conveniently and satisfies $J'(\bar{u})\bar{u} = J'(\bar{u})u_h$. Let us find such a control. For every $1 \leq j \leq N(h)$, let us set

$$I_j = \int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} \bar{d}(x)e_j(x)\,dx.$$

Now we define $u_h \in U_h$ with $u_h(x_\Gamma^j) = u_{h,j}$ for every node $x_\Gamma^j \in \Gamma$ by the expression

$$u_{h,j} = \begin{cases} \dfrac{1}{I_j}\displaystyle\int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} \bar{d}(x)\bar{u}(x)e_j(x)\,dx & \text{if } I_j \neq 0, \\[2ex] \dfrac{1}{h_{j-1}+h_j}\displaystyle\int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} \bar{u}(x)\,dx & \text{if } I_j = 0. \end{cases} \tag{7.9}$$

Remember that the measure of $[x_\Gamma^{j-1}, x_\Gamma^{j+1}]$ is $h_{j-1}+h_j = |x_\Gamma^j - x_\Gamma^{j-1}| + |x_\Gamma^{j+1} - x_\Gamma^j|$, which coincides with $|x_\Gamma^{j+1} - x_\Gamma^{j-1}|$ if x_Γ^j is not a vertex of Ω.

In the following lemma, we state that the function u_h defined by (7.9) satisfies our requirements.

LEMMA 7.5. *There exists $h_0 > 0$ such that, for every $0 < h < h_0$, the element $u_h \in U_h$ defined by* (7.9) *obeys the following properties:*

1. $u_h \in U_h^{ad}$.
2. $J'(\bar{u})\bar{u} = J'(\bar{u})u_h$.
3. *The approximation property*

$$\|\bar{u} - u_h\|_{L^2(\Gamma)} \leq Ch^{1-1/p} \tag{7.10}$$

is fulfilled for some constant $C > 0$ independent of h.

Proof. Since $\bar{u}$ is continuous on Γ, there exists $h_0 > 0$ such that

$$|\bar{u}(\xi_2) - \bar{u}(\xi_1)| \leq \frac{\beta-\alpha}{2}, \quad \forall h < h_0, \quad \forall \xi_1, \xi_2 \in [x_\Gamma^{j-1}, x_\Gamma^{j+1}],\ 1 \leq j \leq N(h),$$

which implies that $\bar{u}$ cannot admit both the values α and β on one segment $[x_\Gamma^{j-1}, x_\Gamma^{j+1}]$ for any $h < h_0$. Hence the sign of $\bar{d}$ on $[x_\Gamma^{j-1}, x_\Gamma^{j+1}]$ must be constant due to (3.7). Therefore, $I_j = 0$ if and only if $\bar{d}(x) = 0$ for all $x \in [x_\Gamma^{j-1}, x_\Gamma^{j+1}]$. Moreover if $I_j \neq 0$, then $\bar{d}(x)/I_j \geq 0$ for every $x \in [x_\Gamma^{j-1}, x_\Gamma^{j+1}]$. As a first consequence of this we get that $\alpha \leq u_{h,j} \leq \beta$, which means that $u_h \in U_h^{ad}$. On the other hand,

$$J'(\bar{u})u_h = \sum_{j=1}^{N(h)} \int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} \bar{d}(x)e_j(x)\,dx\,u_{h,j} = \sum_{j=1}^{N(h)} \int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} \bar{d}(x)\bar{u}(x)e_j(x)\,dx = J'(\bar{u})\bar{u}.$$

Finally, let us prove (7.10). Let us remember that $\bar{u} \in W^{1-1/p,p}(\Gamma) \subset H^{1-1/p}(\Gamma)$ and $p > 2$. We note that the norm in $H^s(\Gamma)$, $0 < s < 1$, is given by

$$\|u\|_{H^s(\Gamma)} = \left(\|u\|_{L^2(\Gamma)}^2 + \int_\Gamma\int_\Gamma \frac{|u(x)-u(\xi)|^2}{|x-\xi|^{1+2s}}\,dx\,d\xi\right)^{1/2}. \tag{7.11}$$

Using that $\sum_{j=1}^{N(h)} e_j(x) = 1$ and $0 \le e_j(x) \le 1$ we get

$$\begin{aligned}\|\bar u - u_h\|^2_{L^2(\Gamma)} &= \int_\Gamma \Big| \sum_{j=1}^{N(h)} (\bar u(x) - u_{h,j}) e_j(x) \Big|^2 dx \\ &\le \sum_{j=1}^{N(h)} \int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} |\bar u(x) - u_{h,j}|^2 e_j(x)\, dx \le \sum_{j=1}^{N(h)} \int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} |\bar u(x) - u_{h,j}|^2\, dx.\end{aligned} \tag{7.12}$$

Let us estimate every term of the sum.

Let us start by assuming that $I_j = 0$ so that $u_{h,j}$ is defined by the second relation in (7.9). Then we have

$$\begin{aligned}\int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} |\bar u(x) - u_{h,j}|^2\, dx &= \int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} \Big| \frac{1}{h_{j-1}+h_j} \int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} (\bar u(x) - \bar u(\xi))\, d\xi \Big|^2 dx \\ &\le \int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} \frac{1}{h_{j-1}+h_j} \int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} |\bar u(x) - \bar u(\xi)|^2\, d\xi\, dx \\ &\le (h_{j-1}+h_j)^{2(1-1/p)} \int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} \int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} \frac{|\bar u(x) - \bar u(\xi)|^2}{|x-\xi|^{1+2(1-1/p)}}\, dx\, d\xi \\ &\le (2h)^{2(1-1/p)} \|\bar u\|^2_{H^{1-1/p}(x_\Gamma^{j-1}, x_\Gamma^{j+1})}.\end{aligned} \tag{7.13}$$

Now let us consider the case $I_j \neq 0$:

$$\begin{aligned}\int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} |\bar u(x) - u_{h,j}|^2\, dx &= \int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} \Big| \frac{1}{I_j} \int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} \bar d(\xi) e_j(\xi) (\bar u(x) - \bar u(\xi))\, d\xi \Big|^2 dx \\ &\le \int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} \Big| \int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} \sqrt{\frac{\bar d(\xi) e_j(\xi)}{I_j}} |\bar u(x) - \bar u(\xi)| \sqrt{\frac{\bar d(\xi) e_j(\xi)}{I_j}}\, d\xi \Big|^2 dx \\ &\le \int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} \int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} |\bar u(x) - \bar u(\xi)|^2 \frac{\bar d(\xi) e_j(\xi)}{I_j}\, d\xi\, dx \\ &\le \Big(\int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} \frac{\bar d(\xi) e_j(\xi)}{I_j}\, d\xi \Big) \sup_{\xi \in [x_\Gamma^{j-1}, x_\Gamma^{j+1}]} \int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} |\bar u(x) - \bar u(\xi)|^2\, dx \\ &= \sup_{\xi \in [x_\Gamma^{j-1}, x_\Gamma^{j+1}]} \int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} |\bar u(x) - \bar u(\xi)|^2\, dx.\end{aligned} \tag{7.14}$$

To obtain the estimate for the last term we are going to use Lemma 7.6 stated below, with

$$f(\xi) = \int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} |\bar u(x) - \bar u(\xi)|^2\, dx.$$

Since $H^{1-1/p}(\Gamma) \subset C^{0,\theta}(\Gamma)$ for $\theta = 1/2 - 1/p$ (see, e.g., [37, Theorem 2.8.1]), it is easy to check that

$$|f(\xi_2) - f(\xi_1)| \le \int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} \Big|[\bar u(x) - \bar u(\xi_1)] + [\bar u(x) - \bar u(\xi_2)]\Big| \Big|\bar u(\xi_2) - \bar u(\xi_1)\Big|\, dx$$
$$\le 2(h_{j-1} + h_j)^{1+2\theta} C_{\theta,p} \|\bar u\|^2_{H^{1-1/p}(x_\Gamma^{j-1}, x_\Gamma^{j+1})}.$$

On the other hand, we have

$$\int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} f(\xi)\, d\xi = \int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} \int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} \frac{|\bar u(x) - \bar u(\xi)|^2}{|x-\xi|^{1+2(1-1/p)}} |x-\xi|^{1+2(1-1/p)}\, dx\, d\xi$$
$$\le (h_{j-1} + h_j)^{1+2(1-1/p)} \int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} \int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} \frac{|\bar u(x) - \bar u(\xi)|^2}{|x-\xi|^{1+2(1-1/p)}}\, dx\, d\xi$$
$$\le (h_{j-1} + h_j)^{2+(1-2/p)} \|\bar u\|^2_{H^{1-1/p}(x_\Gamma^{j-1}, x_\Gamma^{j+1})}.$$

Then we can apply Lemma 7.6 to the function f, with

$$M = (h_{j-1} + h_j)^{2\theta} \max\{4C_{\theta,p}, 1\} \|\bar u\|^2_{H^{1-1/p}(x_\Gamma^{j-1}, x_\Gamma^{j+1})} \le Ch^{2\theta} \|\bar u\|^2_{H^{1-1/p}(x_\Gamma^{j-1}, x_\Gamma^{j+1})},$$

to deduce that

$$f(\xi) \le C \|\bar u\|^2_{H^{1-1/p}(x_\Gamma^{j-1}, x_\Gamma^{j+1})} h^{1+2\theta}. \tag{7.15}$$

This inequality, along with (7.14), leads to

$$\int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} |\bar u(x) - u_{h,j}|^2\, dx \le C \|\bar u\|^2_{H^{1-1/p}(x_\Gamma^{j-1}, x_\Gamma^{j+1})} h^{1+2\theta} \tag{7.16}$$

in the case when $I_j \neq 0$.

Since

$$\sum_{j=1}^{N(h)} \|\bar u\|^2_{H^{1-1/p}(x_\Gamma^{j-1}, x_\Gamma^{j+1})} \le 2 \|\bar u\|^2_{H^{1-1/p}(\Gamma)},$$

inequality (7.10) follows from (7.12), (7.13), (7.16), and the fact that $1 + 2\theta = 2(1 - 1/p)$. ∎

LEMMA 7.6. *Given $-\infty < a < b < +\infty$ and $f : [a,b] \longmapsto \mathbb{R}^+$, a function satisfying*

$$|f(x_2) - f(x_1)| \le \frac{M}{2}(b-a) \quad \textit{and} \quad \int_a^b f(x)\, dx \le M(b-a)^2,$$

we have that $f(x) \le 2M(b-a)$ for all $x \in [a,b]$.

Proof. We argue by contradiction and we assume that there exists a point $\xi \in [a,b]$ such that $f(\xi) > 2M(b-a)$; then

$$\int_a^b f(x)\, dx = \int_a^b \{[f(x) - f(\xi)] + f(\xi)\}\, dx > -\frac{M}{2}(b-a)^2 + 2M(b-a)^2 = \frac{3M}{2}(b-a)^2,$$

which contradicts the second assumption on f. ∎

Proof of Theorem 7.1. Setting $u = \bar{u}_h$ in (3.7), we get

$$J'(\bar{u})(\bar{u}_h - \bar{u}) = \int_\Gamma \left(N\bar{u} - \partial_\nu \bar{\phi}\right)(\bar{u}_h - \bar{u})\,dx \geq 0. \tag{7.17}$$

From (4.8) with u_h defined by (7.9), it follows that

$$J_h'(\bar{u}_h)(u_h - \bar{u}_h) = \int_\Gamma \left(N\bar{u}_h - \partial_\nu^h \bar{\phi}_h\right)(u_h - \bar{u}_h)\,dx \geq 0$$

and then that

$$J_h'(\bar{u}_h)(\bar{u} - \bar{u}_h) + J_h'(\bar{u}_h)(u_h - \bar{u}) \geq 0. \tag{7.18}$$

By adding (7.17) and (7.18) and using Lemma 7.5(2), we derive

$$\left(J'(\bar{u}) - J_h'(\bar{u}_h)\right)(\bar{u} - \bar{u}_h) \leq J_h'(\bar{u}_h)(u_h - \bar{u}) = \left(J_h'(\bar{u}_h) - J'(\bar{u})\right)(u_h - \bar{u}).$$

For $h < h_0$, this inequality and (7.3) lead to

$$\begin{aligned} &\frac{1}{2}\min\{N,\delta\}\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)}^2 \leq \left(J'(\bar{u}) - J'(\bar{u}_h)\right)(\bar{u} - \bar{u}_h) \\ &\leq \left(J_h'(\bar{u}_h) - J'(\bar{u}_h)\right)(\bar{u} - \bar{u}_h) + \left(J_h'(\bar{u}_h) - J'(\bar{u})\right)(u_h - \bar{u}). \end{aligned} \tag{7.19}$$

Now from (7.7) and Young's inequality, we obtain

$$\begin{aligned} &|(J_h'(\bar{u}_h) - J'(\bar{u}_h))(\bar{u} - \bar{u}_h)| \leq Ch^{1-1/p}\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} \\ &\leq Ch^{2(1-1/p)} + \frac{1}{8}\min\{N,\delta\}\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)}^2. \end{aligned} \tag{7.20}$$

On the other hand, using again Young's inequality, (7.8), and (7.10), we deduce

$$\begin{aligned} |(J_h'(\bar{u}_h) - J'(\bar{u}))(u_h - \bar{u})| &\leq \Big(N\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} + Ch^{1-1/p}\Big)\|\bar{u} - u_h\|_{L^2(\Gamma)} \\ &\leq \Big(N\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} + Ch^{1-1/p}\Big)h^{1-1/p} \\ &\leq \frac{1}{8}\min\{N,\delta\}\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)}^2 + Ch^{2(1-1/p)}. \end{aligned} \tag{7.21}$$

From (7.19)–(7.21) we get

$$\frac{1}{4}\min\{N,\delta\}\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)}^2 \leq Ch^{2(1-1/p)},$$

which contradicts (7.2). ∎

8. Numerical tests. In this section we present some numerical tests which illustrate our theoretical results. Let Ω be the unit square $(0,1)^2$. Consider

$$y_d(x_1, x_2) = \frac{1}{(x_1^2 + x_2^2)^{1/3}}.$$

We are going to solve the following problem:

$$\text{(P)} \begin{cases} \text{Min } J(u) = \dfrac{1}{2}\displaystyle\int_\Omega (y_u(x) - y_d(x))^2 dx + \dfrac{1}{2}\displaystyle\int_\Gamma u(x)^2 dx, \\ u \in U_{ad} = \{u \in L^2(\Gamma) :\ -1 \leq u(x) \leq 2 \text{ a.e. } x \in \Gamma\}, \\ -\Delta y_u = 0 \text{ in } \Omega,\ y_u = u \text{ on } \Gamma. \end{cases}$$

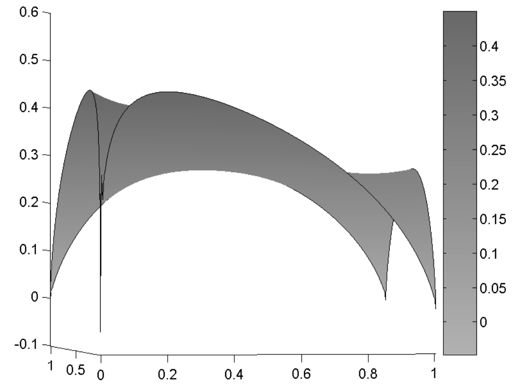

FIG. 8.1.

We remark that $y_d \in L^p(\Omega)$ for all $p < 3$, but $y_d \notin L^3(\Omega)$; therefore the optimal adjoint state $\bar{\varphi}$ is actually in $W^{2,p}(\Omega)$ for $p < 3$. Consequently we can deduce that the optimal control belongs to $W^{1-1/p,p}(\Gamma)$, but $W^{1-1/p,p}(\Gamma)$ is not included in $H^1(\Gamma)$. There is no reason for the normal derivative $\partial_\nu \bar{\varphi}$ to be more regular than $W^{1-1/p,p}(\Gamma)$. For our problem, the plot in Figure 8.1 shows that the optimal control has a singularity in the corner at the origin, and it seems that $\bar{u} \notin H^1(\Gamma)$. So we cannot hope to have a convergence order of $O(h)$. Instead of that, we have a convergence of order $O(h^{1-1/p})$ for some $p > 2$, as predicted by the theory.

Since we do not have an exact solution for (P), we have solved it numerically for $h = 2^{-9}\sqrt{2}$, and we have used this solution for comparison with other solutions for bigger values of h. We have solved it using an active set strategy, as is explained in [11]. Figure 8.1 shows a plot of the optimal solution. The control constraints are not active at the optimal control. In Table 8.1 we show the norm in $L^2(\Gamma)$ of the error of the control and the order of convergence step by step. The order of convergence is measured as

$$o_i = \frac{\log(\|\bar{u}_{h_i} - \bar{u}\|_{L^2(\Gamma)}) - \log(\|\bar{u}_{h_{i-1}} - \bar{u}\|_{L^2(\Gamma)})}{\log(h_i) - \log(h_{i-1})}.$$

Let us remark that $1 - 1/p < 2/3$ for $p < 3$. The values o_i are approximately $2/3$. We believe that the order of convergence could be closer to $2/3$ if we could compare the computed controls with the true optimal control instead of with its numerical approximation. We refer to [12] for more details and numerical tests.

TABLE 8.1

$h_i/\sqrt{2}$	$\|\bar{u}_{h_i} - \bar{u}\|_{L^2(\Gamma)}$	o_i
2^{-3}	0.1055	—
2^{-4}	0.0652	0.6944
2^{-5}	0.0393	0.7302
2^{-6}	0.0237	0.7314
2^{-7}	0.0146	0.7008
2^{-8}	0.0093	0.6493

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