Relevance of a random choice in tests of Bell inequalities with two atomic qubits

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(Received 7 January 2009; revised manuscript received 13 March 2009; published 30 April 2009)

It is pointed out that a loophole may exist in experimental tests of Bell inequalities using atomic qubits, due to possible errors in the angles defining the observables whose correlation is measured. A sufficient condition is derived for closing the loophole.

DOI: 10.1103/PhysRevA.79.044104 PACS number(s): 03.65.Ud, 03.67.Mn, 37.10.Ty, 42.50.Xa

Two experimental tests of Bell inequalities have been performed using atomic qubits. The former, by a group at the National Institute of Standards and Technology (NIST) [1], measured correlations of two entangled Be^+ ions at a distance of 3 μm, the latter, by a group in Maryland [2], used Yb^+ ions distant about 1 m. The reports of both experiments claim that a CHSH [3] (Bell) inequality has been violated, modulo the locality loophole, and that the experiments are relevant because they close the detection loophole. In my opinion it is more than that because they are the first experiments which have actually tested a Bell inequality. In fact previous experiments, in particular those involving optical photon pairs [4], did not test any *genuine* Bell inequality, that is an inequality which is a necessary condition for the existence of local hidden variables (LHV) models. The inequalities tested in those experiments involved additional assumptions so that their violation refutes only restricted families of LHV models, namely, those fulfilling the additional assumption (for details see [5]).

The Maryland experiment is specially important because, although the measurements are not spatially separated in the sense of relativity theory (the locality loophole remains), they are made at a macroscopic distance, which is not the case in the NIST experiment. In contrast the statistics in the NIST experiment is much better, providing a clear violation of the CHSH inequality, while in the Maryland experiment the claimed violation is smaller. Furthermore the authors of the NIST experiment made an analysis of the possible errors in the phases defining the observables measured, something which was not made in the Maryland experiment. The aim of the present Brief Report is to study the possible loophole derived from the errors in the phases and to derive a simple method of locking that loophole.

I will consider experiments, similar to the Maryland one [2], where a pair of atoms (or ions) is prepared in an entangled state. Then Alice performs a rotation of the state of her atom by an angle θ_a and, after a short time, she may detect fluorescence of the atom illuminated by an appropriate laser. Similarly Bob performs a rotation of his atom by an angle θ_b and, after that, he may detect fluorescence too. In the NIST experiment the procedure to prepare the entangled atomic state and the method of measurement are different, but the arguments that follow apply equally. I shall label $p_{++}(\theta_a,\theta_b)$ the probability of coincidence detection and $p_{--}(\theta_a,\theta_b)$ the probability that neither Alice nor Bob detect fluorescence. Similarly $p_{-+}(\theta_a,\theta_b)(p_{+-}(\theta_a,\theta_b))$ will be the probability that only Bob (Alice) detects fluorescence. The correlation function $E(\theta_a,\theta_b)$ is defined by

$$E(\theta_a,\theta_b)=p_{++}(\theta_a,\theta_b)+p_{--}(\theta_a,\theta_b)-p_{+-}(\theta_a,\theta_b)-p_{-+}(\theta_a,\theta_b). \tag{1}$$

The notation used by the authors of the Maryland experiment is, however, somewhat misleading. Instead of Eq. (1) they write

$$E(\theta_a,\theta_b)=p(\theta_a,\theta_b)+p(\theta_a+\pi,\theta_b+\pi)-p(\theta_a,\theta_b+\pi)-p(\theta_a+\pi,\theta_b), \tag{2}$$

where they label $p(\theta_a,\theta_b)$ the quantity which I have labeled $p_{++}(\theta_a,\theta_b)$. Definition Eq. (2), in place of Eq. (1), rests upon assuming the equalities

$$p_{-+}(\theta_a,\theta_b)=p(\theta_a+\pi,\theta_b), p_{+-}(\theta_a,\theta_b)=p(\theta_a+\pi,\theta_b),$$

$$p_{--}(\theta_a,\theta_b)=p(\theta_a+\pi,\theta_b+\pi),$$

which are true according to quantum mechanics, but may not be true in LHV theories. In any case the authors measured $E(\theta_a,\theta_b)$ as defined in Eq. (1) [6]. From the correlations a parameter S is defined by

$$S=|E(\theta_a,\theta_b)+E(\theta_a',\theta_b)|+|E(\theta_a,\theta_b')-E(\theta_a',\theta_b')|, \tag{3}$$

which should fulfill the CHSH [3] inequality $S\leq 2$ if LHV models are possible.

In order to show that a loophole might exist in the experiments, in addition to the locality loophole, I begin remembering that, according to Bell [7], a LHV model will contain a set of hidden variables, λ, a positive normalized density function, $\rho(\lambda)$, and two functions $M_a(\lambda,\theta_a)$, $M_b(\lambda,\theta_b)$, θ_a and θ_b being parameters which may be controlled by Alice and Bob, respectively. The latter functions fulfill

$$M_a(\lambda,\theta_a),M_b(\lambda,\theta_b)\in\{0,1\}. \tag{4}$$

In the atomic experiments here studied the parameters θ_a and θ_b are angles defining the observables measured. The probability, $p_{++}(\theta_a,\theta_b)$, that the coincidence measurement of two dichotomic observables, in two separated regions, gives a positive answer for both variables should be obtained in the LHV model by means of the integral,

$$p_{++}(\theta_a,\theta_b)=\int\rho(\lambda)M_a(\lambda,\theta_a)M_b(\lambda,\theta_b)d\lambda. \tag{5}$$

Similarly the probability, $p_{+-}(\theta_a,\theta_b)$, that Alice gets the answer "yes" and Bob the answer "no" is given by

$$p_{+-}(\theta_a,\theta_b) = \int \rho(\lambda) M_a(\lambda,\theta_a)[1 - M_b(\lambda,\theta_b)]d\lambda, \quad (6)$$

and analogous expressions for p_{-+} and p_{--}.

A LHV model for an atomic experiment may be obtained by choosing

$$\rho(\lambda) = \frac{1}{2\pi}, \quad \lambda \in [0,2\pi], \quad M_a(\lambda,\theta_a) = \Theta\left(\frac{\pi}{2} - |\lambda - \theta_a|\right),$$

$$M_b(\lambda,\theta_b) = \Theta\left(\frac{\pi}{2} - |\lambda - \theta_b - \pi|\right), \quad \mathrm{mod}(2\pi), \quad (7)$$

where $\Theta(x)=1$ if $x>0$, $\Theta(x)=0$ if $x<0$. It is easy to see, taking Eqs. (5) and (6) into account, that the model predictions are (assuming $\theta_a,\theta_b \in [0,\pi]$)

$$p_{++}(\theta_a,\theta_b) = p_{--}(\theta_a,\theta_b) = \frac{|\theta_a - \theta_b|}{2\pi},$$

$$p_{+-}(\theta_a,\theta_b) = p_{-+}(\theta_a,\theta_b) = \frac{1}{2} - \frac{|\theta_a - \theta_b|}{2\pi}, \quad (8)$$

whence I get

$$E(\theta_a,\theta_b) = \frac{2}{\pi}|\theta_a - \theta_b| - 1. \quad (9)$$

It is not difficult to show that, for any choice of the angles $\theta_a,\theta_b,\theta'_a,\theta'_b$, the model predicts $S \le 2$ with S given by Eq. (3).

Now let us assume that an experiment is performed so that Alice and Bob start measuring the quantity $E(\theta_a,\theta_b)$ in a sequence of runs of the experiment. After that they measure $E(\theta_a,\theta'_b)$ in another sequence, then they measure $E(\theta'_a,\theta_b)$ and, finally, they measure $E(\theta'_a,\theta'_b)$. Let α be the error in the rotation performed by Bob on his atom in the first sequence of runs, so that the rotation angle is $\theta_b+\alpha$ rather than θ_b in the measurement of $E(\theta_a,\theta_b)$. Similarly I shall assume that the rotation angles are $\theta'_b+\beta$, $\theta_b+\gamma$, and $\theta'_b+\delta$ in the measurements of $E(\theta_a,\theta'_b)$, $E(\theta'_a,\theta_b)$ and $E(\theta'_a,\theta'_b)$, respectively. For simplicity I will neglect the errors made by Alice. The errors are considered small, specifically $|\alpha|,|\beta|,|\gamma|,|\delta| < \pi/4$. I shall prove that, taking into account the errors in the measurement of the angles, the LHV model prediction for the parameter S, Eq. (3) may apparently violate the CHSH [3] inequality $S \le 2$. To do that let us choose, as was made in the Maryland experiment [2],

$$\theta_a = \frac{\pi}{2}, \quad \theta_b = \frac{\pi}{4}, \quad \theta'_a = 0, \quad \theta'_b = \frac{3\pi}{4}. \quad (10)$$

The values predicted by the LHV model for the relevant quantities are

$$E(\theta_a,\theta_b+\alpha) = -0.5 - \frac{2\alpha}{\pi}, \quad E(\theta_a,\theta'_b+\beta) = -0.5 + \frac{2\beta}{\pi},$$

$$E(\theta'_a,\theta_b+\gamma) = -0.5 + \frac{2\gamma}{\pi}, \quad E(\theta'_a,\theta'_b+\delta) = 0.5 + \frac{2\delta}{\pi}. \quad (11)$$

The parameter S' actually measured in the experiment would be

$$S' = |E(\theta_a,\theta_b+\alpha) + E(\theta'_a,\theta_b+\gamma)| + |E(\theta_a,\theta'_b+\beta) - E(\theta'_a,\theta'_b+\delta)|. \quad (12)$$

For this parameter the LHV model predicts

$$S' = 2 + \frac{2}{\pi}(\alpha - \beta - \gamma + \delta), \quad (13)$$

which may violate the inequality $S \le 2$ for some values of the parameters α,β,γ and δ. I stress that this does not imply the violation of a Bell inequality by a LHV model because the parameter S' of Eq. (13) is not a CHSH parameter as defined in Eq. (3). This proves the possible existence of a loophole, due to errors in the angles, in Bell tests using atomic qubits. This loophole is probably irrelevant in typical experiments involving optical photon pairs because the errors in the angles, defined by the positions of the polarizers, are small.

In order to study the relevance of the angular errors in the performed experiments I begin pointing out that the predictions of the above LHV model would agree with the results of the Maryland experiment if we choose

$$\frac{2}{\pi}\alpha = 0.018, \quad \frac{2}{\pi}\beta = -0.046,$$

$$\frac{2}{\pi}\gamma = -0.081, \quad \frac{2}{\pi}\delta = 0.073, \quad (14)$$

which correspond to errors between 2° and 7°. With similar errors the results of the NIST experiment might be also reproduced by the LHV model. It is the case that the errors in the Maryland experiment are of that order (see below) while in the NIST experiment are substantially smaller.

In order to test a Bell inequality when there are errors in the rotation angles we should consider Bell inequalities, different from the CHSH one, which include the possibility of errors. A simple inequality is derived as follows. I shall consider an experiment where the quantities E, Eq. (3), are defined for eight (rather than four) angles, so that the parameter S' is

$$S' = |E(\theta_a,\theta_b) + E(\phi'_a,\phi_b)| + |E(\phi_a,\phi'_b) - E(\theta'_a,\theta'_b)|, \quad (15)$$

where the angles ϕ are (slightly) different from the angles θ. Now I introduce the "errors" Δ_j

$$\Delta_1 \equiv E(\phi'_a,\phi_b) - E(\theta'_a,\theta_b), \quad \Delta_2 \equiv E(\phi_a,\phi'_b) - E(\theta_a,\theta'_b), \quad (16)$$

and I get

$$S' = |E(\theta_a,\theta_b) + E(\theta'_a,\theta_b) + \Delta_1| + |E(\theta_a,\theta'_b) - E(\theta'_a,\theta'_b) + \Delta_2|$$
$$\leq |E(\theta_a,\theta_b) + E(\theta'_a,\theta_b)| + |E(\theta_a,\theta'_b) - E(\theta'_a,\theta'_b)| + |\Delta_1| + |\Delta_2|$$
$$\equiv S + |\Delta_1| + |\Delta_2|. \quad (17)$$

A necessary condition for the existence of LHV models of the experiment is that the quantity S fulfils the CHSH inequality [see Eq. (3)], which leads to the Bell inequality for eight different angles

$$S' \leq 2 + |\Delta_1| + |\Delta_2|. \quad (18)$$

If the quantities Δ_j have Gaussian distributions with standard deviation σ, it is straightforward to get the mean and the standard deviations of the sum of their absolute values and we may write

$$|\Delta_1| + |\Delta_2| = \sqrt{\frac{2}{\pi}}\sigma \pm \sqrt{\frac{2\pi - 1}{\pi}}\sigma \simeq 0.80\sigma \pm 1.30\sigma,$$

which leads to the (Bell) inequality

$$S' \leq 2 + 0.80\sigma \pm 1.30\sigma, \quad (19)$$

where S' is given by Eq. (15).

In the NIST experiment [1] the quantity σ was derived, from measurements, to be 0.03 for every one of five sets of data. Thus we may estimate $\sigma \simeq 0.013$ for the whole experiment, whence Eq. (19) leads to

$$S' \leq 2.011 \pm 0.017, \quad (20)$$

which, for the result of the experiment, $S' = 2.25 \pm 0.01$, implies a clear violation of the Bell inequality.

In the report of the Maryland experiment no analysis is made of the errors in the rotation angles and making an estimate is hazardous. However, if the typical errors in the rotation angles are as high as 5° [8] in the Bell test of Ref. [2], then the results of that experiment are roughly compatible with LHV models, although the conclusion depends on our (uncertain) estimate. This would imply that no Bell inequality is violated.

In the following I shall prove that the loophole arising from the errors in the rotation angles may be closed by a random choice of the angles to be measured. To begin with, it is easy to see that the LHV model predictions do not violate the inequality $S' \leq 2$ if the error in the measurement, by Bob, of the angle θ_b is the same in all measurements of that angle, and similarly for θ'_b. In fact the inequality is fulfilled if $\alpha = \beta$ and $\gamma = \delta$, as may be seen by looking at Eq. (13). In the following I derive a sufficient condition for the fulfillment of the inequality, $S' \leq 2$, for the actually measurable quantity S', by the predictions of any LHV model.

Let us assume that there is a probability distribution, $f_a(x)$, for the errors when Alice rotates her atom by an angle θ_a and another distribution, $f'_a(y)$, when she rotates her atom by an angle θ'_a. Similarly I shall assume that there are similar distributions $f_b(u)$ and $f'_b(v)$ for the errors in the rotations, by Bob, of the angles θ_b and θ'_b. I shall show that a sufficient condition for the inequality $S' \leq 2$ is that the distributions of errors, in the rotations made by Alice, must be the same independently of what rotation performs Bob on the partner atom. And similarly for the rotations made by Bob. If this is the case the predictions of any LHV model for the quantity S' will be obtained from probabilities defined as follows [compare with Eqs. (5) and (6)]:

$$p_{++}(\theta_a,\theta_b) = \int \rho(\lambda) M_a(\lambda,\theta_a + x) M_b(\lambda,\theta_b + u) d\lambda f_a(x) dx f_b(u) du,$$

$$p_{+-}(\theta_a,\theta_b) = \int \rho(\lambda) M_a(\lambda,\theta_a + x)[1 - M_b(\lambda,\theta_b + u)] d\lambda f_a(x) dx f_b(u) du, \quad (21)$$

and similarly for the other quantities p_{ij} with $i,j = +,-$. Now we may define new quantities,

$$Q_a(\lambda,a) = \int M_a(\lambda,\theta_a + x) f_a(x) dx,$$

$$Q_b(\lambda,b) = \int M_b(\lambda,\theta_b + u) f_b(u) du,$$

$$Q_a(\lambda,a') = \int M_a(\lambda,\theta'_a + y) f'_a(y) dy,$$

$$Q_b(\lambda,b') = \int M_b(\lambda,\theta'_b + v) f'_b(v) dv, \quad (22)$$

which fulfill the conditions [compare with Eq. (4)]

$$0 \leq Q_a(\lambda,a), Q_a(\lambda,a'), Q_b(\lambda,b), Q_b(\lambda,b') \leq 1. \quad (23)$$

The consequence is that we may obtain a new LHV model for the experiment with the quantities Q, Eq. (23), in place of the quantities M, Eq. (4). The existence of that model implies the fulfillment of the inequality $S' \leq 2$.

From our proof it is rather obvious that the essential condition required to block the loophole is that the probability distribution of errors made by Bob are independent of what rotation is performed by Alice in the partner atom, and similarly the errors made by Alice should be independent of the rotation performed by Bob. A simple method to ensure that independence is that, after every preparation of the entangled state of the atom, Alice makes a random choice (with equal probabilities) between the rotation angles θ_a and θ_a' and similarly Bob makes a random choice, *independently of Alice,* between θ_b and θ_b'.

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