Quantum vacuum effects as generalized $f(R)$ gravity: Application to stars

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(Received 31 August 2009; revised manuscript received 15 January 2010; published 24 March 2010)

It is assumed that, for weak space-time curvature, the main gravitational effect of the quantum vacuum stress energy corresponds to adding two terms to the Einstein-Hilbert action, proportional to the square of the curvature scalar and to the contraction of two Ricci tensors, respectively. It is shown that compatibility with terrestrial and Solar System observations implies that the square roots of the coefficients of these terms should be either a few millimeters or a few hundred meters. It is shown that the vacuum contribution increase the stability of massive white dwarfs.

I. INTRODUCTION

In the study of quantum fields in curved space-time it has been established that the quantum vacuum gives rise to a finite, nonzero, energy [1,2]. Furthermore, some effects have been attributed to the gravity of the quantum vacuum, like the observed acceleration in the expansion of the Universe [3].

The gravitational effects of the quantum vacuum may be taken into account introducing a vacuum stress-energy tensor, $T_{\mu\nu}^{\text{vac}}$, in the Einstein equation, which should read

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = k(T_{\mu\nu} + T_{\mu\nu}^{\text{vac}}), \tag{1}$$

where $R_{\mu\nu}$ is the Ricci tensor, R the curvature scalar, and $T_{\mu\nu}$ is the stress-energy tensor of matter either baryonic or dark plus radiation, k is 8π times Newton's' constant, and we shall use units $c = 1$ throughout. We assume that the vacuum tensor, $T_{\mu\nu}^{\text{vac}}$, depends on the space-time curvature, therefore it should be a functional of the Riemann tensor, $R_{\mu\nu\lambda\sigma}$ (and the metric, $g_{\mu\nu}$). It is plausible to derive the functional from a generalized Einstein-Hilbert action

$$S = \frac{1}{2k} \int d^4x \sqrt{-g}(R + F) + S_{\text{mat}}, \tag{2}$$

where F is associated to the vacuum. In general F should be a function of the scalars, which may be obtained by combining the Riemann tensor, $R_{\mu\nu\lambda\sigma}$, and its derivatives, with the metric tensor, $g_{\mu\nu}$. Here, I shall considers scalars more general than the Ricci scalar, whose functions have been extensively explored in recent years under the name of $f(R)$ *gravity* [4,5].

The action Eq. (2) may be interpreted as a modification of general relativity (GR), that is we may put the tensor $T_{\mu\nu}^{\text{vac}}$ on the left side, rather than the right side, of Einstein Eq. (1). Indeed, it is equivalent in practice whether we assume that the quantum vacuum gives rise to some stress energy to be added to the matter one or we assume that GR should be modified by adding to R the term F in the gravitational action. Even if we remain within metric theories of gravity, that is we assume that the stress-energy

tensor of matter produces the curvature of space-time, the arguments leading to the GR choice (the standard Einstein equation) are not compulsory, but small modifications are compatible with observations. That is, we might introduce the sum $R + F$, instead of the Ricci scalar R in the Einstein-Hilbert action. Of course there are strong constraints to the form of F from both observational evidence and the requirements of consistence.

There are a number of proposals for the gravity of the quantum vacuum derived from fundamental arguments involving the quantization of model vacuum fields [6]. They lead to F being a function of scalars like $R_{\mu\nu}R^{\mu\nu}$, $R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}$, and $\Box R$, in addition to R. Most of the proposals have been made for the study of cosmology, in particular, the attempt to explain the observed accelerated expansion of the Universe (the "dark energy"), although it is not necessary to modify GR in order to explain it [7].

The aim of the present paper is to study the influence of modified GR in the structure of Newtonian or weakly relativistic stars. A study of fully relativistic stars has been made recently within $f(R)$ gravity [8], where the authors conclude that neutron stars are not possible (or they require extreme fine-tuning) within $f(R)$ theories if these are compatible with observational constraints. However, the theories studied here are more general than those considered in Ref. [8].

II. VACUUM GRAVITY FOR WEAK CURVATURE

In order to find the most appropriate function F to be put in the action Eq. (2) I shall not attempt to derive it from fundamental arguments, but use a plausible phenomenological approach combining arguments of simplicity with dimensional considerations. The Ricci scalar has dimensions, L^{-2}, of inverse squared length and the theory derived by using it in the action (that is, GR) is known to give very good agreement with observations for a wide range of intensities of the gravitational field (i.e., curvature of space-time.) The purpose of this paper is to study the influence of vacuum gravity on Newtonian or weakly relativistic stars. Therefore, I propose to include in F

only terms with dimensions not departing too much from L^{-2}. Thus, I shall assume that F is a sum of terms with dimensions L^0, L^{-2}, and L^{-4}. For simplicity I will exclude more complicated terms with dimensions intermediate between L^0 and L^{-4}, like $L^{-2} \log L$. Thus, I obtain

$$F = \Lambda + a_0 R + a_1 R^2 + a_2 R_{\mu\nu} R^{\mu\nu} + a_3 \Box R$$
$$+ a_4 \nabla^\mu \nabla^\nu R_{\mu\nu} + a_5 R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma}, \qquad (3)$$

where ∇_μ means the covariant derivative and $\Box \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu$. The constant parameter Λ has dimensions L^{-2}, a_0 is dimensionless, and all remaining coefficients a_j have dimensions L^2. There are no other scalar terms with similar dimensional dependence so that Eq. (3) gives the most general F fulfilling the requirements of dimensionality and simplicity above stated. However, I recognize that the stated criterion of simplicity might be questioned.

In Eq. (3) the term Λ will give rise to a "cosmological constant" in Einstein's equation. It may be relevant in cosmology, but it will have a negligible effect in the structure of stars, and I ignore it in the following. The term $a_0 R$ may be absorbed in the standard GR term, which amounts to a rescaling of Newton constant (the new constant k will be the old one divided by $1 + a_0$). After this rescaling the gravitational action in Eq. (4) contains only the "true vacuum polarization," in the words of Zeldovich [9]. The term with a_3 may be removed because it gives no contribution to the field equations. The same is true for the term with a_4 due to the fact that $\nabla^\nu (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) = 0$, which leads to the equality $\nabla^\mu \nabla^\nu R_{\mu\nu} = \frac{1}{2} \Box R$. Finally, the term with a_5 may be removed taking into account the well-known fact that the Gauss-Bonnet invariant,

$$\mathcal{G} = R^2 - 4 R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma},$$

does not contribute to the field equations, and therefore we may substitute $4 R_{\mu\nu} R^{\mu\nu} - R^2$ for $R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma}$ without changing those equations. Thus, I will study here the action

$$S = \frac{1}{2k} \int d^4 x \sqrt{-g} (R + a R^2 + b R_{\mu\nu} R^{\mu\nu}) + S_{\text{mat}}, \qquad (4)$$

which contains just two free parameters, a and b.

The field equations associated to the action Eq. (4) may be got from Carroll *et al.* [10] leading to

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + 2a[RR_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R^2 - \nabla_\mu \nabla_\nu R + g_{\mu\nu} \Box R]$$
$$+ b[-\frac{1}{2} g_{\mu\nu} R_{\lambda\sigma} R^{\lambda\sigma} + 2 R_\mu^\sigma R_{\sigma\nu} + \Box R_{\mu\nu}$$
$$+ g_{\mu\nu} \nabla_\lambda \nabla_\sigma R^{\lambda\sigma}] - b[\nabla_\sigma \nabla_\nu R_\mu^\sigma + \nabla_\sigma \nabla_\mu R_\nu^\sigma] = k T_{\mu\nu}, \qquad (5)$$

where $T_{\mu\nu}$ is the stress-energy tensor of matter. The last term of the left side may be transformed taking into account the standard rule for the commutation of covariant derivatives of a tensor, that is

$$\nabla_\sigma \nabla_\nu R_\mu^\sigma = g^{\lambda\sigma} \nabla_\sigma \nabla_\nu R_{\mu\lambda}$$
$$= \nabla_\nu \nabla_\sigma R_\mu^\sigma + R_\mu^\lambda R_{\lambda\nu\sigma}^\sigma - R_\lambda^\sigma R_{\mu\nu\sigma}^\lambda$$
$$= R_\mu^\lambda R_{\lambda\nu} - R^{\sigma\lambda} R_{\lambda\mu\nu\sigma} + \nabla_\nu \nabla_\sigma R_\mu^\sigma. \qquad (6)$$

When this is put in Eq. (5) its left-hand side becomes

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + 2a[RR_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R^2 - \nabla_\mu \nabla_\nu R$$
$$+ g_{\mu\nu} \Box R] + b[-\frac{1}{2} g_{\mu\nu} R_{\lambda\sigma} R^{\lambda\sigma} + \Box R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} \Box R$$
$$- \nabla_\mu \nabla_\nu R + 2 R^{\sigma\lambda} R_{\lambda\mu\nu\sigma}],$$

where I have taken into account the equalities

$$R^{\sigma\lambda} R_{\lambda\mu\nu\sigma} = R^{\sigma\lambda} R_{\nu\sigma\lambda\mu} = R^{\lambda\sigma} R_{\lambda\nu\mu\sigma} = R^{\sigma\lambda} R_{\lambda\nu\mu\sigma},$$

and, taking into account that the divergence of Einstein's tensor Eq. (8) is zero,

$$\nabla_\sigma R^{\lambda\sigma} = \frac{1}{2} \nabla^\lambda R, \qquad \nabla_\sigma R_\mu^\sigma = \frac{1}{2} \nabla_\mu R.$$

The trace of the field equation is specially simple, namely,

$$(6a + 2b) \Box R - R = k T_\mu^\mu \equiv kT. \qquad (7)$$

It is convenient to rewrite the field equation in terms of Einstein's tensor, $G_{\mu\nu}$, and its trace, G, related to the Ricci tensor, $R_{\mu\nu}$, by

$$R_{\mu\nu} = G_{\mu\nu} - \frac{1}{2} g_{\mu\nu} G, \qquad R = -G. \qquad (8)$$

Also, I shall write the field equation so that it looks like the standard GR Eq. (1), that is,

$$G_{\mu\nu} = k T_{\mu\nu} + k T_{\mu\nu}^{\text{vac}}, \qquad (9)$$

defining

$$T_{\mu\nu}^{\text{vac}} \equiv k^{-1} \{ (2a + b)[g_{\mu\nu} \Box G - \nabla_\mu \nabla_\nu G]$$
$$+ a[GG_{\mu\nu} - \frac{1}{4} g_{\mu\nu} G^2] + b[2 G^{\sigma\lambda} R_{\lambda\mu\nu\sigma}$$
$$- \frac{1}{2} g_{\mu\nu} G_{\lambda\sigma} G^{\lambda\sigma} - \frac{1}{4} g_{\mu\nu} G^2 - \Box G_{\mu\nu}] \}. \qquad (10)$$

I stress that all results of this paper will be independent of whether we assume that $T_{\mu\nu}^{\text{vac}}$ is a quantum vacuum stress energy, or we consider Eqs. (9) and (10) together as a modification of standard GR, maybe with no reference to the vacuum. In any case, I shall use throughout this paper a language appropriate for the former assumption.

III. APPROXIMATE VACUUM FIELD IN A NEWTONIAN STAR

Our task is to solve the field equations for a spherically symmetric body, with mass M and radius R_o, in a static space-time with the condition that the metric is asymptotically flat (Minkowskian). The body may be a metallic sphere in a laboratory experiment, say like that of Eötvös, the Earth, or the Sun, but to be specific I shall speak about a star from now on. I will use standard (curvature) coordinates with metric

$$ds^2 = -\exp(\beta(r))dt^2 + \exp(\alpha(r))dr^2 + r^2 d\Omega^2. \quad (11)$$

In a static problem of spherical symmetry there are only three independent components of Einstein's tensor Eq. (9), namely, $G_t^t, G_r^r, G_\theta^\theta = G_\phi^\phi$, which are well-known functions of the metric parameters α and β [11]. Then we have four independent equations, including a relation between density and pressure, in particular, for a star in equilibrium the latter relation is the equation of state of matter, i.e., $p = p(\rho)$. We also have four unknown functions, namely, $\alpha(r)$, $\beta(r)$, $\rho(r)$, and $p(r)$. The solution of these equations is involved in general and here will be solved only for a few particular cases of Newtonian, or slightly relativistic, stars.

Thus, I shall consider stars where:

(1) The metric coefficient β, Eq. (11), is small compared with unity. In this case, $\beta \simeq 2\Phi$, Φ being the Newton potential which fulfills $|\Phi| \approx kM/R_o \ll 1$.

(2) The matter pressure, p, is small in comparison with the matter density, ρ. Actually, this condition is related to the former because we have $p/\rho \simeq kM/R_o$.

In order to get the three components of $T_{\mu\nu}^{vac}(= k^{-1}G_{\mu\nu} - T_{\mu\nu}^{vac})$, we shall solve Eq. (9). For Newtonian stars the equation may be approximated as follows: First, we may neglect terms quadratic in $G_{\mu\nu}$. In fact, the terms linear in $G_{\mu\nu}$ in the right side of Eq. (10) are of order $ak\rho/R_o^2$, ρ being the typical matter density. In contrast, the terms quadratic in $G_{\mu\nu}$ are of order $ak^2\rho^2$, that is, smaller than the former by $k\rho R_\circ^2 \simeq kM/R_o \ll 1$. Thus, Eq. (9) may be approximated by

$$G_{\mu\nu} - kT_{\mu\nu} \equiv kT_{\mu\nu}^{vac}$$
$$\simeq (2a+b)[g_{\mu\nu}\Box G - \nabla_\mu \nabla_\nu G] - b\Box G_{\mu\nu}. \quad (12)$$

Second, we may neglect the matter pressure in comparison with the density in the interior of the star, that is, we may assume

$$T_t^t = \rho \simeq T, \qquad T_r^r = T_\theta^\theta = T_\phi^\phi = -p \simeq 0. \quad (13)$$

However, I shall retain p in some cases for the sake of clarity. Using the metric Eq. (11), we may write

$$\nabla_\mu \nabla_\nu G = \delta_\mu^1 \delta_\nu^1 \frac{d^2 G}{dr^2} - \Gamma_{\mu\nu}^1 \frac{dG}{dr},$$

where I label 1 the index of the radial coordinate, in order to avoid confusion with the coordinate itself. The affine connections Γ_{tt}^1 and Γ_{11}^1 are of order kM/R_o, whence the terms involving them may be neglected. To the same order we may approximate

$$g^{\theta\theta}\Gamma_{\theta\theta}^1 = g^{\phi\phi}\Gamma_{\phi\phi}^1 \simeq -1/r, \qquad g^{11} \simeq 1,$$
$$\Box G \simeq \nabla^2 G. \quad (14)$$

The term $\Box G_{\mu\nu}$ is more involved, although straightfor-

ward, and I shall not write it in general. For our case, that is, with the metric Eq. (11) and the approximations Eqs. (14), the three independent components of the tensor Eq. (12) become

$$b\nabla^2 G_t^t + G_t^t = (2a+b)\nabla^2 G + k\rho,$$
$$b\nabla^2 G_r^r + \frac{4b}{r^2}(G_\theta^\theta - G_r^r) + G_r^r = (2a+b)\frac{2}{r}\frac{dG}{dr} - kp,$$
$$b\nabla^2 G_\theta^\theta + \frac{2b}{r^2}(G_r^r - G_\theta^\theta) + G_\theta^\theta = (2a+b)\left(\frac{d^2 G}{dr^2} + \frac{1}{r}\frac{dG}{dr}\right)$$
$$- kp. \quad (15)$$

At this moment I point out that, at a difference with general relativity, here local isotropy of the matter stresses does not imply isotropy of the spatial part of the Ricci tensor. That is, although the Einstein Eq. (1) obviously leads to the implication

$$T_r^r = T_\theta^\theta \Rightarrow G_r^r = G_\theta^\theta \Rightarrow R_r^r = R_\theta^\theta,$$

this is no longer true for the more involved field Eq. (10) [or the approximate Eq. (12)]. Indeed, we may have $G_r^r \neq G_\theta^\theta$ (and therefore anisotropy of the vacuum stresses, i.e., $(T^{vac})_r^r \neq (T^{vac})_\theta^\theta$) even if $T_r^r = T_\theta^\theta$. Furthermore, the anisotropy of the gravitational field, i.e., the inequality $G_r^r \neq G_\theta^\theta$, might induce anisotropy of the matter stresses, i.e., $T_r^r \neq T_\theta^\theta$, but I shall not consider that possibility in this paper. Nevertheless, the matter local anisotropy, $T_r^r \neq T_\theta^\theta$, induced by gravitational field anisotropy, $G_r^r \neq G_\theta^\theta$, might be relevant in strong gravitational fields.

IV. THE FIELD OUTSIDE THE STAR

The solution of the field equations outside a spherical star, with a general $f(R)$, has been studied by several authors [12]. The interest in the problem is that it puts constraints on the functions $f(R)$ in order to be compatible with known facts in the Solar System. Here, I shall make a similar calculation for our action Eq. (2) with the purpose of finding the range of values of the parameters a and b compatible with terrestrial and Solar Sytem observations.

Thus, our aim is to get the Einstein tensor, G_μ^ν, outside the star taking into account that $T_\mu^\nu = 0$ is there. Hence, the vacuum stress energy might be easily obtained, that is,

$$\rho^{vac} = k^{-1}G_t^t, \qquad p_r^{vac} = -k^{-1}G_1^1,$$
$$p_\theta^{vac} = -k^{-1}G_\theta^\theta.$$

In order to solve Eqs. (15) I begin obtaining appropriate linear combinations of them. If we add the first equation plus the second one plus 2 times the third, we get the trace equation [compare with Eq. (7)]

$$G - (6a+2b)\nabla^2\rho = kT, \quad (16)$$

where I have approximated $\Box T$ by the flat-space Laplacian of the matter density, $\nabla^2\rho$ [see Eqs. (13) and (14)]. I shall

assume $6a + 2b > 0$, that is, I exclude the case $6a + 2b < 0$ which, leading to an oscillating function $G(r)$, is unphysical. The limiting case $6a + 2b = 0$ will be considered below.

Equation (16) in flat space may be solved by Green's function method as follows: In order to simplify the writing I shall sometimes use a dimensionless position vector, $\mathbf{x}$, and a dimensionless star radius X defined by

$$\mathbf{x} = \frac{\mathbf{r}}{\sqrt{6a + 2b}}, \qquad x = |\mathbf{x}|,$$

$$X = \frac{R_o}{\sqrt{6a + 2b}}, \qquad \gamma = \sqrt{\frac{6a + 2b}{|b|}},$$ (17)

where I have introduced also the parameter γ for latter convenience. Thus, the fundamental solution for the trace Eq. (16) may be written

$$\nabla^2 f(\mathbf{x}) - f(\mathbf{x}) = -4\pi\delta^3(\mathbf{x}) \Rightarrow f = \frac{1}{x}\exp(-x), \quad (18)$$

Hence, the Einstein tensor outside the star may be obtained by integration, giving

$$G_t^t + G_r^r + 2G_\theta^\theta \equiv G(x)$$

$$= \frac{k}{4\pi} \int_{|\mathbf{z}| < X} \frac{\rho(z)d^3\mathbf{z}}{|\mathbf{x} - \mathbf{z}|} \exp(-|\mathbf{x} - \mathbf{z}|)$$

$$= \frac{kM^*}{4\pi(6a + 2b)r} \exp\left(-\frac{r}{\sqrt{6a + 2b}}\right),$$ (19)

where

$$M^* \equiv (6a + 2b)^{3/2} \int_0^X \rho(z)4\pi z \sinh z\, dz. \quad (20)$$

We see that for (small) objects fulfilling $R_o \ll \sqrt{6a + 2b}$, that is, $X \ll 1$, $\sinh z \simeq z$, leading to $M^* \simeq M$. This is not the case for bodies such that $R_o \gtrsim \sqrt{6a + 2b}$.

The physics behind these result may be better understood if we define a new mass parameter

$$M_a \equiv M^* \exp(-X)$$

$$\equiv (6a + 2b)^{3/2} \exp(-X) \int_0^X \rho(z)4\pi z \sinh z\, dz, \quad (21)$$

so that Eq. (19) may be rewritten

$$G(r) = \frac{kM_a}{4\pi(6a + 2b)r} \exp\left(-\frac{r - R_o}{\sqrt{6a + 2b}}\right). \quad (22)$$

We see that the vacuum correction Eq. (22) may be interpreted as if it depends on the distance, $r - R_o$, from the point $\mathbf{r}$, where we measure G to the closest point in the surface of the star and the correction is due to the mass, M_a, contained in some volume of the star most close to the point $\mathbf{r}$. Indeed, the factor $\exp(-X)\sinh z$ in the integral Eq. (21) effectively restricts the range of integration to a region near the surface. In particular, for a body of constant density, like the Earth, or more generally any rocky planet or satellite, Eq. (21) gives

$$\frac{M_a}{M} = \frac{3}{2X^2}[1 + \exp(-2X)] - \frac{3}{2X^3}[1 - \exp(-2X)]$$

$$\simeq \frac{3(6a + 2b)}{2R_0^2} \ll 1,$$ (23)

the latter equality being valid for $6a + 2b \gg R_0^2$. In a star, like the Sun, the ratio M_a/M would be much smaller because the surface density is smaller than on Earth, and the radius much bigger.

In order to proceed with the calculation of the Einstein tensor, I obtain from Eq. (15)

$$b\nabla^2(2G_t^t - G_r^r - 2G_\theta^\theta) + 2G_t^t - G_r^r - 2G_\theta^\theta = 2\rho, \quad (24)$$

again neglecting $p \ll \rho$. The solution outside the star is (compare with Eq. (22))

$$2G_t^t - G_r^r - 2G_\theta^\theta = \frac{kM_b}{2\pi(-b)r} \exp\left(-\frac{r - R_o}{\sqrt{-b}}\right), \quad (25)$$

where

$$M_b \equiv (-b)^{3/2} \exp(-\gamma X) \int_0^{\gamma X} \rho(z)\sinh z\, 4\pi z\, dz. \quad (26)$$

I have assumed $b < 0$ because a positive b would lead to an unphysical oscillating function. For a body with constant density this leads to

$$\frac{M_b}{M} = \frac{3}{2\gamma^2 X^2}[1 + \exp(-2\gamma X)] - \frac{3}{2\gamma^3 X^3}[1 - \exp(-2\gamma X)]$$

$$\simeq \frac{3|b|}{2R_0^2} \ll 1,$$ (27)

the latter equality valid for $|b| \ll R_0^2$.

From Eqs. (22) and (25) we get outside the star

$$\rho^{\mathrm{vac}} = k^{-1}G_t^t$$

$$= \frac{1}{12\pi r^3}[M_a x^2 \exp(X - x) + 2M_b \gamma^2 x^2 \exp(\gamma X - \gamma x)],$$ (28)

$$p_r^{\mathrm{vac}} + 2p_\theta^{\mathrm{vac}} = -k^{-1}(G_r^r + 2G_\theta^\theta)$$

$$= \frac{1}{6\pi r^3}[M_b \gamma^2 x^2 \exp(\gamma X - \gamma x)$$

$$- M_a x^2 \exp(-x)].$$ (29)

In order to obtain separately the two different pressures, p_r^{vac} and p_θ^{vac}, I proceed as follows: The vacuum stress-energy tensor Eq. (10) is divergence-free as may be easily checked. Actually, this property is a consequence of deriving the field equations from an action functional. As the Einstein tensor has also zero divergence, Eq. (9) shows that both the matter and the vacuum stress-energy tensors are

separately divergence-free. The divergence of the vacuum tensor gives the relation

$$\frac{dp_r^{\text{vac}}}{dr} + 2\frac{p_r^{\text{vac}} - p_\theta^{\text{vac}}}{r} = -\beta'(\rho^{\text{vac}} + p_r^{\text{vac}}) \simeq 0, \quad (30)$$

where β' is the radial derivative of the metric coefficient β. The second equality follows from the fact that, in Newtonian stars, $r\beta'$ is of order $kM/R_o \ll 1$, while ρ^{vac}, p_r^{vac}, and p_θ^{vac} have the same order. Hence, we obtain

$$\frac{d}{dr}(r^3 p_r^{\text{vac}}) \simeq r^2(p_r^{\text{vac}} + 2p_\theta^{\text{vac}}),$$

which, taking Eq. (29) into account, gives

$$p_r^{\text{vac}} = \frac{1}{6\pi r^3}[M_a(x+1)\exp(X-x) - M_b(\gamma x + 1) \\ \times \exp(\gamma X - \gamma x)], \quad (31)$$

$$p_\theta^{\text{vac}} = \frac{1}{12\pi r^3}[M_b(\gamma^2 x^2 + \gamma x + 1)\exp(\gamma X - \gamma x) \\ - M_a(x^2 + x + 1)\exp(X-x)], \quad (32)$$

where the labels Eqs. (17) have been used. An integration constant has been fixed so that $r^3 p_r^{\text{vac}} \to 0$ for $r \to \infty$ in order that the absolute value of p_r never surpasses ρ, Eq. (28).

The limiting case

$$3a + b = 0, \qquad -b > 0$$

cannot be studied by the procedure leading to Eqs. (28) to (36) but its trace Eq. (16) is rather simple, namely,

$$G = kT \Rightarrow T^{\text{vac}} = 0,$$

in particular $G = 0$ outside the star.

Also, Eqs. (15) are simple, and we obtain outside the star (here $M_a = M_b$)

$$\rho^{\text{vac}} = k^{-1}G_t^t = \frac{M_a}{18\pi ar}\exp\left(-\frac{r-R_o}{\sqrt{3a}}\right),$$

$$p_r^{\text{vac}} = -k^{-1}G_r^r = -\frac{M_a}{6\pi r^3}\left(\frac{r}{\sqrt{3a}} + 1\right)\exp\left(-\frac{r-R_o}{\sqrt{3a}}\right),$$

$$p_\theta^{\text{vac}} = -k^{-1}G_\theta^\theta \\ = \frac{M_a}{12\pi r^3}\left(\frac{r^2}{3a} + \frac{r}{\sqrt{3a}} + 1\right)\exp\left(-\frac{r-R_o}{\sqrt{3a}}\right).$$

We see that these equations are the limit of Eqs. (28)–(32) when $b \to -3a < 0$. It is interesting that, in this case, the vacuum field looks like a radiation field because its stress-energy tensor is traceless. However, that radiation field is not isotropic, that is, $p_r^{\text{vac}} \neq p_\theta^{\text{vac}}$, p_r^{vac} being negative and p_θ^{vac} positive.

Also, the case $b = 0$ cannot be studied by the procedure leading to Eqs. (28)–(32) but Eqs. (15) may be easily solved taking the trace Eq. (22) into account. We get

$$k\rho^{\text{vac}} \equiv G_t^t = \frac{1}{3}G = \frac{kM_a}{72\pi ar}\exp\left(-\frac{r-R_o}{\sqrt{6a}}\right), \quad (33)$$

$$-kp_r^{\text{vac}} \equiv G_r^r = \frac{1}{3}G - 2a\frac{d^2G}{dr^2} \\ = -\frac{kM_a}{6\pi r^3}\left(1 + \frac{r}{\sqrt{6a}}\right)\exp\left(-\frac{r-R_o}{\sqrt{6a}}\right), \quad (34)$$

$$-kp_\theta^{\text{vac}} \equiv G_\theta^\theta = \frac{1}{3}G - \frac{2a}{r}\frac{dG}{dr} \\ = \frac{kM_a}{12\pi r^3}\left(\frac{r^2}{6a} + \frac{r}{\sqrt{6a}} + 1\right)\exp\left(-\frac{r-R_o}{\sqrt{6a}}\right). \quad (35)$$

It is interesting that, in this case, the mean vacuum pressure is negative fulfilling $p_{\text{mean}}^{\text{vac}} = -\frac{2}{3}\rho^{\text{vac}}$.

Finally, in the particular case

$$2a + b = 0,$$

Eqs (28)–(36) hold true, but it is interesting to write them explicitly because they are especially simple, that is,

$$G = G_t^t = k\rho^{\text{vac}} = \frac{kM_a}{8\pi ar}\exp\left(-\frac{r-R_o}{\sqrt{2a}}\right),$$

$$G_r^r = p_r^{\text{vac}} = G_\theta^\theta = p_\theta^{vac} = 0.$$

In summary the vacuum density and stresses outside the star, derived from the action Eqs. (4) are given by Eqs. (28) and (29), or the appropriate limits, for any values of the parameters a and b fulfilling

$$3a \geq -b \geq 0. \quad (36)$$

For values violating these inequalities the solution of the field equation outside a Newtonian spherical body would give an unphysical oscillatory behavior. Of course the values $a = b = 0$ correspond to general relativity (without vacuum field.)

Now I shall calculate the coefficients $\alpha(r)$ and $\beta(r)$ of the metric Eq. (11) taking into account the well-known relations [11]

$$\exp(-\alpha) = 1 - \frac{1}{r}\int_0^r u^2 G_t^t(u)du,$$
$$\frac{d\beta}{dr} \equiv \beta' = \frac{e^\alpha - 1}{r} - re^\alpha G_r^r. \quad (37)$$

From the former it is straightforward to obtain the function $\alpha(r)$ outside the star. We get, taking Eq. (28) into account,

$$\exp(-\alpha) = 1 - \frac{kM}{4\pi r}(1 - \delta\alpha), \quad (38)$$

where

$$\delta\alpha(x) = \frac{1}{3}\frac{M_a}{M}[(x+1)\exp(X-x) - (X+1)]$$
$$+ \frac{2}{3}\frac{M_b}{M}[(\gamma x + 1)\exp(\gamma X - \gamma x) - (\gamma X + 1)].$$
$$(39)$$

We see that the dimensionless function $\delta\alpha$ represents the correction to the GR (Schwarzschild) exterior solution of a spherical star.

In order to get the function $\beta'(r)$, I start expanding the metric coefficient $\exp\alpha$ in powers of the gravitational constant k, a parameter that may be considered small because $kM/R_o \ll 1$. That is,

$$\exp\alpha = 1 + \frac{kM}{4\pi r}(1 - \delta\alpha) + \frac{k^2 M^2}{16\pi^2 r^2}(1 - \delta\alpha)^2 + O(k^3).$$

Inserting this in the second Eq. (37), we obtain β' as an expansion in powers of k, that is,

$$\beta' = \frac{kM}{4\pi r^2}\left(1 - \delta\alpha + \frac{4\pi r^3 p_r^{\text{vac}}}{M}\right)$$
$$+ \frac{k^2 M^2}{16\pi^2 r^3}(1 - \delta\alpha)\left(1 - \delta\alpha + \frac{4\pi r^3 p_r^{\text{vac}}}{M}\right) + O(k^3),$$
$$(40)$$

where $\delta\alpha$ was given in Eq. (39) and

$$1 - \delta\alpha + \frac{4\pi r^3 p_r^{\text{vac}}}{M} = \frac{M_a}{3M}[(x+1)\exp(X-x) + (X+1)]$$
$$+ \frac{M_b}{3M}[-4(\gamma x + 1)\exp(\gamma X - \gamma x)$$
$$+ 2(\gamma X + 1)].$$

This leads to

$$\beta' = \frac{kM}{4\pi r^2} + \frac{kM_a}{12\pi r^2}[(x+1)\exp(X-x) + (X+1)]$$
$$+ \frac{kM_b}{6\pi r^2}[(\gamma X + 1) - 2(\gamma x + 1)\exp(\gamma X - \gamma x)]$$
$$+ O(k^2). \qquad (41)$$

Getting the term proportional to k^2 is straightforward, but the resulting expression is involved, and I will not write it explicitly.

Hence, the parameter β may be easily obtained by means of the integration of Eq. (40) with the condition $\beta(r) \to 0$ when $r \to \infty$. The result may be written

$$\beta = -\frac{kM}{4\pi r}(1 + \delta\beta_1) - \frac{k^2 M^2}{32\pi^2 r^2}(1 + \delta\beta_2) + O(k^3),$$
$$(42)$$

where the function $\delta\beta_1(r)$ is the correction to the GR (Schwarzschild) solution to lowest order in k. It is

$$\delta\beta_1 \equiv \frac{M_a}{3M}[(X+1) + \exp(X-x)]$$
$$+ \frac{M_b}{3M}[2(\gamma X + 1) - 4\exp(\gamma X - \gamma x)]. \qquad (43)$$

The term $\delta\beta_2$ is involved, and I will not write it explicitly. Equation (42) to order $O(k)$ is, therefore,

$$\beta \simeq -\frac{kM}{4\pi r}\left[1 + \frac{M_a}{3M}(X+1) + \frac{2M_b}{3M}(\gamma X + 1)\right]$$
$$- \frac{kM_a}{12\pi r}\exp(X-x) + \frac{M_b}{3\pi r}\exp(\gamma X - \gamma x). \qquad (44)$$

As is well known, the term linear in the gravitational constant, k, in the expansion of the metric parameter β equals twice the Newtonian potential. Therefore, Eq. (44) shows that the theory resting upon the action Eq. (4) predicts: 1) a correction to the mass appearing in the Newtonian potential (the term within square bracket), and 2) two non-Newtonian potentials of Yukawa type, one of them attractive and the other one repulsive.

V. CONSTRAINTS ON THE PARAMETERS a AND b

It is known that in $f(R)$ gravity the coefficient of the term R^2 (noted a in this paper) cannot be greater than a few square millimeters in order not to contradict laboratory experiments [4]. However, this is not the case in the generalized theory studied here. In fact, let us compare the Newtonian field intensity, $g \equiv kM/(8\pi r^2)$ with the field intensity Eq. (41) predicted by our theory, which corresponds to half the term linear in k of Eq. (41), and I shall label it $g + \delta g$. We get

$$\frac{\delta g}{g} = \frac{M_a}{3M}[(x+1)\exp(X-x) + (X+1)]$$
$$+ \frac{2M_b}{3M}[(\gamma X + 1) - 2(\gamma x + 1)\exp(\gamma X - \gamma x)]. \quad (45)$$

In the most interesting case of a body with constant density we may put Eqs. (23) and (27) in Eq. (45), and we get

$$\frac{\delta g}{g} = \frac{X\cosh X - \sinh X}{X^3}[(x+1)\exp(-x) + (X+1)$$
$$\times \exp(-X)] + 2\frac{\gamma X\cosh(\gamma X) - \sinh(\gamma X)}{\gamma^3 X^3}$$
$$\times [(\gamma X + 1)\exp(-\gamma X) - 2(\gamma x + 1)\exp(-\gamma x)].$$
$$(46)$$

This expression is small for both small and large X. Indeed, for $X \approx x \ll 1$ we have, expanding in powers,

$$\frac{\delta g}{g} \simeq 4\gamma^2 x^2 - x^2 - 2\gamma^2 X^2 - X^2$$
$$= \left(\frac{4}{|b|} - \frac{1}{6a + 2b}\right)r^2 - \left(\frac{2}{|b|} + \frac{1}{6a + 2b}\right)R_o^2. \quad (47)$$

For $X \gg 1$ and $x - X \ll X$ (i.e., $r - R_o \ll R_o$), we obtain

$$\frac{\delta g}{g} \simeq \frac{1 + 2\gamma^{-1}}{2X} + \frac{x}{2X^2} \exp(X - x) - \frac{2x}{\gamma X^2} \exp(\gamma X - \gamma x)$$

$$\simeq \frac{\sqrt{6a + 2b} + 2\sqrt{|b|}}{R_o} + \frac{\sqrt{6a + 2b}}{R_o} \exp\left(-\frac{r - R_o}{\sqrt{6a + 2b}}\right)$$

$$- \frac{4\sqrt{|b|}}{R_o} \exp\left(-\frac{r - R_o}{\sqrt{|b|}}\right). \tag{48}$$

It is a fact that precise measurements have been made only for the gravitational field of objects with sizes of meters or smaller (e.g., the Eötvös experiment) or for celestial bodies, in particular, the Earth [13]. Thus, if both $\sqrt{a}$ and $\sqrt{|b|}$ have values between hundred meters and a few kilometers both Eqs. (47) and (48) predict violations of Newtonian gravity of order 10^{-4} or less. The two parameters should be large enough because if $|b|$ is small then the ratio $\delta g/g$ becomes large in Eq. (46), and the theory here developed is refuted by laboratory experiments [13]. It is worth noting here that during the late eighties of the twentieth century there were some claims about the existence of non-Newtonian gravity (the "fifth force"), although a reanalysis of the experiments has lead to a consensus that several uncertainties were not taken into account, and the experiments are actually compatible with Newtonian gravity within errors [14]. In any case, I should mention that some of the experiments apparently showed the existence of two non-Newtonian potentials of Yukawa type, one attractive and the other one repulsive, as in our Eq. (48). For instance, in a tower experiment Eckhardt *et al.* [15] reported a non-Newtonian gravity in the form

$$\frac{\delta g_a}{g} = 1 + \varepsilon_{\mathrm{at}} \exp\left(-\frac{r}{\lambda_{\mathrm{at}}}\right) - \varepsilon_{\mathrm{rep}} \exp\left(-\frac{r}{\lambda_{\mathrm{rep}}}\right),$$

with values

$$\varepsilon_{\mathrm{rep}} - \varepsilon_{\mathrm{at}} \simeq 0.007, \qquad \varepsilon_{\mathrm{at}} \gtrsim 0.03,$$
$$\lambda_{\mathrm{rep}} \sim \lambda_{\mathrm{at}} \sim 100 \text{ m}. \tag{49}$$

These values violate our Eq. (48), which predicts

$$\varepsilon \approx \frac{\lambda}{2R_0} \approx \frac{\lambda}{10^7 \text{ m}}, \qquad \lambda \approx \sqrt{a}.$$

In contrast, this equality is compatible with known bounds provided that $\sqrt{a} < 50$ km and $\varepsilon < 0.005$. (See Fig. 1 of Ref. [14].) I conclude that tests of our theory on the Earth's surface would require experiments with errors several orders smaller than those typical of "fifth force" experiments.

In summary, there are two ranges where the parameters are compatible with all performed experiments on Newtonian gravity, namely, either both $\sqrt{a}$ and $\sqrt{|b|}$ are less than a few millimeters or both have values between hundred meters and a few kilometers. But I shall point out

that, if $\sqrt{a}$ and $\sqrt{|b|}$ are in the latter range, the correction Eq. (46) may be quite important for mass concentrations with sizes of a few kilometers, like mountains. This may lead to experimental tests of the theory, but I shall not discuss them in this paper.

Even if the discrepancies between Newtonian gravity and the predictions of Eq. (40) are too small to be detected, it is interesting to see whether the theory developed here predicts corrections to Newtonian gravity in measurable violation of general relativity. The standard comparison is made using an isotropic metric rather than Eq. (11). However, it is possible to make the comparison also with the latter metric [16]. For a spherical body the coefficients of the metric Eq. (11) may be expanded in powers of the gravitational constant, k, in the form

$$\exp\alpha = 1 + \eta \frac{kM}{4\pi r} + \ldots,$$
$$\exp\beta = 1 - \lambda \frac{kM}{4\pi r} + (\zeta - \lambda\eta) \frac{k^2 M^2}{32\pi^2 r^2} + \ldots \tag{50}$$

General relativity predicts $\lambda = \zeta = \eta = 1$. Furthermore, recent measurements [17] give the bound $\eta/\lambda = 1 + (2.1 \pm 2.3) \times 10^{-5}$.

In order to make the comparison with the GR prediction Eq. (50) we need the expansion

$$\exp\beta = 1 + \beta + \frac{1}{2}\beta^2 + O(k^3)$$

$$\simeq 1 - \frac{kM}{4\pi r}(1 + \delta\beta_1) + \frac{k^2 M^2}{32\pi^2 r^2}(1 + 2\delta\beta_1$$
$$+ (\delta\beta_1)^2 - \delta\beta_2). \tag{51}$$

Now the comparison of Eq. (50) with Eqs. (38) and (51) gives (to lowest order in k)

$$\lambda = 1 - \delta\alpha, \qquad \eta = 1 + \delta\beta_1,$$
$$\zeta = 1 + 3\delta\beta_1 + (\delta\beta_1)^2 - \delta\beta_2 - \delta\alpha - \delta\alpha\delta\beta_1.$$

With arguments similar to those used in the analysis of Eq. (45) we get $\lambda = \eta = \zeta = 1$ with errors smaller than 10^{-3} for the Earth gravity and smaller than 10^{-6} for the gravity of the Sun, provided that the parameters $\sqrt{a}$ and $\sqrt{|b|}$ are less than a few kilometers. The reason for the difference between terrestrial and solar gravity lies in that, according to our predictions, the quantities $\delta\alpha$, $\delta\beta_1$, and $\delta\beta_2$ are proportional to the density near the surface (much smaller in the Sun) and inversely proportional to the radius (100 times greater in the Sun). In particular, I get

$$\left|\frac{\eta}{\lambda} - 1\right| \simeq \left|\frac{2M_b}{3M}(\gamma x - 1)\exp(\gamma X - \gamma x)\right.$$
$$\left. + \frac{M_a}{3M}(x + 2)\exp(X - x)\right| < 2.10^{-5}, \tag{52}$$

in agreement with observations [17].

For the gravitational interactions of bodies with sizes of a few meters or smaller, used in laboratory experiments, the ratio η/λ is of order $R_o^2/a \leq 10^{-6}$ if $\sqrt{a}$ and $\sqrt{|b|}$ are larger than about 1 km.

VI. HYDROSTATIC EQUILIBRIUM

The structure of a spherical star in equilibrium may be obtained from Eqs. (9) and (10) plus the equation of state. The solution of these equations is involved and furthermore the fact that they are fourth order shows that we need four (initial or boundary) conditions, which may give rise to some ambiguity. I will not attempt to solve exactly those equations in this paper, but I shall sketch the solution of the approximate Eqs. (15) inside Newtonian or slightly relativistic stars.

I shall begin solving the trace Eq. (16), via Eq. (19), in the interior of the star, that is, $x < X$. I get

$$G(x) = \frac{k}{2x} \int_0^X \rho(z)[\exp(-|x-z|) - \exp(-z-x)]z\,dz.$$
(53)

For a star where $X \gg 1$ we may make the following three approximations: 1) neglecting the second exponential in comparison with the first one (except near the origin), 2) approximating $\rho(z)$ by

$$\rho(z) \simeq \rho(x) + \left[\frac{d\rho}{dz}\right]_{z=x}(z-x) + \frac{1}{2}\left[\frac{d^2\rho}{dz^2}\right]_{z=x}(z-x)^2,$$

and 3) extending the z integration to the interval $(-\infty, \infty)$. Then the integral is trivial, and we get

$$G(x) = k(T + T^{\mathrm{vac}}) \simeq k\left(\rho(x) + \frac{2}{x}\frac{d\rho}{dx} + \frac{d^2\rho}{dx^2}\right). \quad (54)$$

This same result is obtained near the center, although the approximations involved should be different. It may be realized that the last two terms of Eq. (54) are of order $X^{-2} = (6a + 2b)/R_o^2 \ll 1$, and the terms neglected are of the order of X^{-4}.

The same result may be obtained if we combine Eqs. (16) and (9), the latter with ∇^2 substituted for $\Box$, and work to the same order of approximation. In fact, we get

$$T^{\mathrm{vac}} = (6a + 2b)\nabla^2(T + T^{\mathrm{vac}}) \simeq (6a + 2b)\nabla^2 T$$
$$\simeq (6a + 2b)\nabla^2 \rho. \quad (55)$$

The agreement between both calculations, leading to Eqs. (54) and (55), respectively, reinforces the validity of solving Eqs. (15) by approximating G_μ^ν by kT_μ^ν in all terms that are linear in the small parameters a or b. I point out that this approximation was not valid in the study, made in Sec. IV, of the exterior of the star because $T_\mu^\nu = 0$ there. Thus, Eqs. (15) lead, in the star interior, to

$$\rho^{\mathrm{vac}} \equiv (T^{\mathrm{vac}})_t^t \simeq 2a\left(\frac{2}{r}\frac{d\rho}{dr} + \frac{d^2\rho}{dr^2}\right) \equiv 2a\left(\frac{2}{r}\rho' + \rho''\right),$$

$$p_r^{\mathrm{vac}} \equiv -(T^{\mathrm{vac}})_r^r \simeq -(2a + b)\frac{2}{r}\rho', \quad (56)$$

$$p_\theta^{\mathrm{vac}} \equiv -(T^{\mathrm{vac}})_\theta^\theta \simeq -(2a + b)\left(\frac{1}{r}\rho' + \rho''\right).$$

These results are consistent with Eq. (55) for the trace T^{vac}. It is remarkable that the vacuum density in the interior of the star depends only on the parameter a, and not on b, within our approximations (in particular, neglecting the matter pressure in comparison with the matter density). If $-b < 2a$ the mean vacuum pressure is negative in the interior of the star.

I shall point out that Eqs. (56) include *all vacuum effects* to our order of approximation. In particular, we should not add to Eq. (56) the effect of the Yukawa-type contribution to the Newtonian potential [see Eqs. (40)]. It is remarkable that the density of the vacuum contribution is negative in the central part of the star because both $d\rho/dr$ and $d^2\rho/dr^2$ are negative there. However, the total density is positive everywhere when Eq. (56) is valid because we have assumed that $|\rho^{\mathrm{vac}}| \ll \rho$ in deriving it. The vacuum density is positive near the surface of the star, and the vacuum contribution to the total mass is zero, that is,

$$m^{\mathrm{vac}} = \int_0^r 2a\left(\frac{2}{r}\frac{d\rho}{dr} + \frac{d^2\rho}{dr^2}\right)4\pi r^2\,dr = 8\pi a r^2 \frac{d\rho}{dr} \leq 0,$$

$$M^{\mathrm{vac}} = m^{\mathrm{vac}}(R_o) = 0, \quad (57)$$

provided that $d\rho/dr$ is zero at the star surface. In particular, this is the case in all polytropes with $\gamma < 2$ (see next section). I stress that all these properties are valid only for large spherical static bodies, where Eqs. (56) are a good enough approximation. I shall point out that, strictly speaking, the total mass of the star should include the mass of the vacuum contribution outside the star (with density given by Eq. (28) so that M^{vac} as defined in Eq. (57) is only the mass in the interior of the star. The external mass due to the vacuum may be obtained by integration of the density Eq. (28), giving

$$M_{\mathrm{ext}}^{\mathrm{vac}} \simeq \frac{R_o}{3}\left(\frac{M_a}{\sqrt{6a + 2b}} + 2\frac{M_b}{\sqrt{-b}}\right)$$
$$\simeq \frac{M}{R_o}(\sqrt{6a + 2b} + 2\sqrt{-b}), \quad (58)$$

the latter equality being valid for a celestial body of constant density like the Earth.

The three independent components of the Einstein-type Eq. (9) become

$$G_t^t = k\rho + 2ka\left(\frac{2}{r}\rho' + \rho''\right),$$

$$G_r^r = -kp + (2a + b)k\frac{2}{r}\rho', \qquad (59)$$

$$G_\theta^\theta = -kp + (2a + b)k\left(\frac{1}{r}\rho' + \rho''\right),$$

where $\rho' = d\rho/dr$, $\rho'' = d^2\rho/dr^2$. Equations (59) will be the starting point for all calculations of structure of stars to be made in the following.

The condition that Einstein's tensor G_μ^ν is divergence-free leads to the following hydrostatic equilibrium equation [18]:

$$\frac{dp_r^{\text{eff}}}{dr} + \frac{2}{r}(p_r^{\text{eff}} - p_\theta^{\text{eff}}) = -\frac{k}{2}\frac{m^{\text{eff}} + 4\pi r^3 p_r^{\text{eff}}}{4\pi r^2 - krm^{\text{eff}}}(\rho^{\text{eff}} + p_r^{\text{eff}}), \qquad (60)$$

where I define

$$\rho^{\text{eff}} = \rho + \rho^{\text{vac}}, \qquad p_r^{\text{eff}} = p + p_r^{\text{vac}},$$

$$p_\theta^{\text{eff}} = p + p_\theta^{\text{vac}}, \qquad m^{\text{eff}} = 4\pi\int_0^r r^2\rho^{\text{eff}}dr. \qquad (61)$$

Putting Eqs. (56) into Eq. (60), we get

$$\frac{dp}{dr} = -\frac{k}{2}\frac{m + 4\pi r^3 p - 8\pi(a + b)r^2\frac{d\rho}{dr}}{4\pi r^2 - krm - 8\pi akr^3\frac{d\rho}{dr}}$$
$$\times\left(\rho + p - 2b\frac{1}{r}\frac{d\rho}{dr} + 2a\frac{d^2\rho}{dr^2}\right). \qquad (62)$$

I point out again that this equilibrium equation might be seen as either a modification (to first order in a and b) of the general relativistic equilibrium equation, due to the change from the Einstein-Hilbert action to the action Eq. (2), or as an effect of the quantum vacuum. Both interpretations lead to the same physical consequences.

In going from Eq. (60) to Eq. (62) it is interesting the cancellation

$$\frac{dp_r^{\text{vac}}}{dr} + \frac{2}{r}(p_r^{\text{vac}} - p_\theta^{\text{vac}}) = \frac{dp_r^{\text{vac}}}{dr} + \frac{2}{r}(p_r^{\text{eff}} - p_\theta^{\text{eff}}) = 0, \qquad (63)$$

which actually derives from Eq. (30). A fortunate consequence of this cancellation is that Eq. (62) is an ordinary differential equation of *third order* (in the variable $m(r)$), rather than fourth order, when p is written in terms of ρ, e.g., via the equation of state. Then we may use the initial conditions $m(0) = 0$, $[d\rho/dr]_{r=0} = 0$ and only one more condition is needed, e.g., fixing the central density, $\rho(0)$. Thus, the solutions of Eq. (62) consist of a one-parameter family as in the standard theory of stars in equilibrium.

Assuming that both the relativistic and the vacuum corrections are small, Eq. (62) may be approximated as

$$\frac{dp}{dr} = -\frac{km}{8\pi r^2}\rho(1 + \text{GR} + \text{vacuum}),$$

$$\text{GR} \equiv \frac{4\pi r^3 p}{m} + \frac{km}{4\pi r} + \frac{p}{\rho}, \qquad (64)$$

$$\text{vacuum} \equiv -(a + b)\frac{8\pi r^2}{m}\frac{d\rho}{dr} - 2b\frac{1}{r\rho}\frac{d\rho}{dr} + 2a\frac{1}{\rho}\frac{d^2\rho}{dr^2}.$$

It may be realized that the first factor on the right side represents the Newtonian approximation, the three terms labeled GR are the well-known first order corrections of general relativity and those labeled *vacuum* are corrections due to the vacuum stress energy. The former (latter) are of order kM/R (order a/R^2) with respect to the Newtonian approximation. I have ignored terms that are both vacuum and GR corrections, that is, of order akM/R^3 with respect to the Newtonian approximation. As may be seen, and it is well known, that the three GR corrections are positive, that is, every one contributes to the increase of gravitational effects. In contrast, the second term of *vacuum* is negative (remember that $b < 0$ and $d\rho/dr < 0$), and the third one is negative in the central region of the star because $d^2\rho/dr^2$ is negative there. The first vacuum term would be positive (negative) everywhere if $|b| < a$ ($|b| > a$). Finally, the third term is positive near the surface. Consequently, from Eq. (64) no conclusion seems possible about whether the vacuum corrections increase or decrease the effects of gravity in comparison with the Newtonian approximation.

There is, however, a simple argument that gives the answer. The gravitational interaction energy, in the Newtonian approximation but with the vacuum correction included, is given by

$$V = -\frac{k}{8\pi}\int d^3\mathbf{r}_1\rho^{\text{eff}}(\mathbf{r}_1)\int d^3\mathbf{r}_2\rho^{\text{eff}}(\mathbf{r}_2)|\mathbf{r}_1 - \mathbf{r}_2|^{-1}$$

$$\simeq V_{\text{matter}} - \frac{k}{4\pi}\int d^3\mathbf{r}_1\rho(\mathbf{r}_1)\int d^3\mathbf{r}_2\rho^{\text{vac}}(\mathbf{r}_2)|\mathbf{r}_1 - \mathbf{r}_2|^{-1}$$

$$= V_{\text{matter}} - k\int_0^{R_o}\rho(r)m(r)^{\text{vac}}rdr$$

$$= V_{\text{matter}} + 3ka\int_0^{R_o}\rho^2 4\pi r^2 dr, \qquad (65)$$

where we have taken Eq. (57) into account and neglected the vacuum density in the exterior of the star (but see below). Also, I have performed an integration by parts in the second equality assuming that $\rho = 0$ at the star's surface. The reason why the vacuum correction is positive derives from the fact that $m^{\text{vac}}(r) \leq 0$ [see Eq. (57)] so that the vacuum contribution makes the Newtonian gravitational interaction energy *less negative*. It is as if there were a short range repulsion given by

$$3ka\delta^3(\mathbf{r}_1 - \mathbf{r}_2).$$

Actually, the vacuum density outside the star gives another contribution to the gravitational interaction, which is negative. In fact, taking Eq. (28) into account, we have

$$V_{\text{ext}}^{\text{vac}} \simeq -\frac{R_o}{3}\left(\frac{M_a}{\sqrt{6a+2b}} + 2\frac{M_b}{\sqrt{-b}}\right)\frac{kM}{8\pi R_o}$$

$$\simeq -\frac{M}{R_o}(\sqrt{6a+2b} + 2\sqrt{-b})\frac{kM}{8\pi R_o}, \qquad (66)$$

the latter equality being valid for a celestial body of constant density like the Earth. Considering the two terms of Eq. (65) plus the terms Eqs. (58) and (66), we conclude the following: For the Earth the Newtonian gravitational energy, V, and the vacuum term, $M_{\text{ext}}^{\text{vac}}$, may be of similar size (their ratio is of order $kM/\sqrt{a}$), while the other two terms are much smaller. However, in star, where the surface density is very small, the dominant term may be V, followed by V^{vac}, while the other two terms are much smaller than the latter (for the proof in the case of white dwarfs see next section).

The relative importance of the vacuum and GR corrections may be estimated from the ratio of the first vacuum correction to the first GR correction in Eq. (64). It is

$$\frac{\text{vacuum}}{\text{GR}} \sim \frac{2a+b}{rp}\frac{d\rho}{dr} \sim \frac{a}{R^2} \times \frac{\rho}{p}. \qquad (67)$$

In Newtonian stars the first factor on the right side is small, as was shown in the Sec. II, and the second factor is large. Thus, no conclusion may be reached (in any case both GR and vacuum corrections are small). In compact objects like neutron stars the second factor is of order unity, and the first one might be large, which suggests that the vacuum corrections could be very relevant. However, in these stars many of the approximations leading to Eq. (64) are not valid and a calculation starting from Eqs. (9) and (10) would be necessary.

As an illustrative example in the following, I study the effect of the vacuum corrections on the internal properties of the Sun. I shall consider, in particular, the change in central pressure due to the correction terms given in Eq. (64). The change in the central pressure will be $\delta p(0)$, where

$$\delta p(r) = -\int_r^{R_\odot} \frac{d\delta p}{dr} dr = \frac{k}{8\pi}\int_r^{R_\odot} \frac{m}{r^2}\rho dr [\text{vacuum}]$$

$$= -(a+b)k\int_r^{R_\odot} \rho\frac{d\rho}{dr} dr - bk\int_r^{R_\odot} \frac{m}{4\pi r^3}\frac{d\rho}{dr} dr$$

$$+ ak\int_r^{R_\odot} \frac{m}{4\pi r^2}\frac{d^2\rho}{dr^2} dr. \qquad (68)$$

An integration by parts of the third term gives an integrated part, a term which combines with the first one and can be integrated easily, and a new term which combines with the second. Thus, we have (assuming that $\rho'(R_o) = 0$)

$$\delta p(0) = \left(a+\frac{b}{2}\right)k[\rho(0)]^2 + (2a-b)k\int_0^{R_\odot} \frac{m}{4\pi r^3}\frac{d\rho}{dr} dr. \qquad (69)$$

The integral in the second term may be performed numeri-

cally using data of the calculations made by Bahcall *et al.* [19], but for our purposes it is enough to find an upper bound. Taking into account that $m/r^3 \leq \frac{4}{3}\pi\rho(0)$ and $d\rho/dr < 0$, the value of the second integral lies in the interval $(-\frac{1}{3}[\rho(0)]^2, 0)$. Thus, we get

$$\left(\frac{1}{3}a + \frac{5}{6}b\right)k[\rho(0)]^2 \leq \delta p(0) \leq \left(a + \frac{b}{2}\right)k[\rho(0)]^2,$$

which shows that

$$|\delta p(0)| \approx (10\text{ km})^2 \times 7.4 \times 10^{-29}\text{ cm/g}$$

$$\times (156\text{ g/cm}^3)^2$$

$$= 1.8 \times 10^{-12}\text{ g/cm}^3,$$

where I have estimated the parameters a and $|b|$ at about 10 km. We see that $\delta p(0)$ may be negative (e.g., for $|b| > 2a$) and positive (e.g., for $|b| < 2a/5$). The central pressure is (in units of mass/volume)

$$p = 1.19\frac{\rho k_B \Theta}{m_H c^2} = 2.6 \times 10^{-4}\text{ g/cm}^3,$$

where k_B is Boltzmann's constant, m_H the mass of the hydrogen atom, and Θ the temperature. The factor 1.19 is the mean number of particles per baryon, which derives from the chemical composition in center of the Sun, that is, 35% hydrogen and 65% helium. Hence, we get

$$\frac{\delta p(0)}{p(0)} = \frac{\delta\Theta(0)}{\Theta(0)} \approx 10^{-8}. \qquad (70)$$

We see that the effect of the vacuum stress energy in the structure of the Sun is negligible. In particular, it gives no change in the prediction of neutrino emission from the center of the Sun.

VII. STABILITY OF $\gamma = 4/3$ POLYTROPES: APPLICATION TO WHITE DWARFS

In the following, I shall apply the theory resting upon the Einstein-type Eqs. (59) to the study of equilibrium and stability of $\gamma = 4/3$ polytropes, that is, Newtonian stars with equation of state of the form $p = K\rho^\gamma$, $\gamma = 4/3$. In order to make more transparent the comparison with the literature, in this section I shall use Newton's constant, G, rather than $k = 8\pi G$. No confusion should arise with the Einstein tensor, which will not be mentioned in this section.

It is well known that polytropes are stable if $\gamma > 4/3$ and unstable if $\gamma < 4/3$. If $\gamma = 4/3$, the stability depends on small corrections, which therefore become relevant. In particular, general relativistic corrections produce unstability, although other corrections may compensate for that. Stars that may be treated as $\gamma = 4/3$ polytropes are white dwarfs and supermassive stars. Both are stable when the internal energy is large enough, but become unstable after radiating a fraction of that energy [20]. In both cases, it is

assumed that a source of unstability are the relativistic corrections, although in some white dwarfs also neutronization of the core may induce unstability. Here, I shall study white dwarfs but not supermassive stars. Indeed, the latter are hypothetical, and there are no observations on them. In addition, the possible corrections due to the quantum vacuum, as they are proposed in this paper, are probably small because typical dimensions of supermassive stars are far larger than the parameters $\sqrt{a}$ and $\sqrt{-b}$.

White dwarfs are formed from ordinary stars after a period of cooling. The theory here considered corresponds to stars sufficiently cold to be treated as if the temperature is zero Kelvin. Also I will consider only massive white dwarfs because small ones are well approximated by $\gamma = 5/3$ polytropes, they are always stable and have no interest here. The theory of white dwarfs in our approach might consists of solving the hydrostatic equilibrium Eq. (62) with the equation of state

$$\rho = m_H n + u(n) \Rightarrow p = n\frac{d\rho}{dn} - \rho = n\frac{du}{dn}, \quad (71)$$

n being the baryon density, ρ the mass density and p the pressure. The mass m_H is close to that of the hydrogen atom (its precise value depends on the chemical composition of the star.) The function $u(n)$ corresponds to a $\gamma = 4/3$ polytriope with small corrections, that is,

$$u(n) = 3K(m_H n)^{4/3} + \delta u(n) \Rightarrow p \simeq K\rho^{4/3},$$
$$u \simeq 3K\rho^{4/3}, \quad (72)$$

where K is a constant. In the Newtonian approximation, Eqs. (62) and (71) become

$$\frac{dp}{dr} = -\frac{Gm}{r^2}\rho,$$
$$p = K\rho^{4/3} \Rightarrow \frac{d\rho}{dr} = -\frac{3}{4K}\frac{Gm(r)}{r^2}\rho(r)^{2/3}. \quad (73)$$

If we introduce in Eq. (73) the appropriate corrections to lowest order we get

$$\frac{d\rho}{dr} = -\frac{3}{4K}\frac{Gm(r)}{r^2}\rho(r)^{2/3}(1 + GR + vacuum + eos), \quad (74)$$

where GR and *vacuum* were given in Eq. (64). The additional term, eos, corresponds to the modification, $\delta u(n)$, of the polytropic equation of state mentioned in Eq. (72), which I will not write explicitly here. Only the central density, $\rho(0) \equiv \rho_c$, is needed in order to fix one solution of Eq. (74) [see comment after Eq. (63)]. Thus, for every value of ρ_c the solution of Eq. (74) provides the functions $\rho(r)$, $p(r)$ and $n(r)$. Hence, we could calculate the baryon number, N, and the mass of the star, M^{eff}, as functions of ρ_c, using the well-known expressions

$$N \equiv \int_0^{R_o} \frac{m_H n(r)4\pi r^2 dr}{\sqrt{1 - 2Gm^{\text{eff}}(r)/r}},$$
$$m^{\text{eff}}(r) \equiv \int_0^r 4\pi x^2 dx \rho^{\text{eff}}(x), \quad (75)$$
$$M^{\text{eff}} = m^{\text{eff}}(\infty),$$

R_o being the radius of the star and ρ^{eff} the sum of the matter and vacuum densities [see Eq. (61)]. The upper limit of the integral giving the total mass, M^{eff}, is ∞, rather than R_o, because we should include the mass associated to the vacuum density both inside and in the neighborhood of the star [see Eq. (58)]. Actually, the quantity of interest is the binding energy defined by

$$E \equiv M^{\text{eff}} - m_H N, \quad (76)$$

which for Newtonian stars (i.e., without the corrections GR, *vacuum*, and *eos*) become

$$E = 3K \int_0^{R_o} \rho^{4/3} 4\pi r^2 dr - G \int_0^{R_o} m\rho 4\pi r dr. \quad (77)$$

In summary, Eqs. (74) and (76) provide a one-parameter family of *solutions of equilibrium*, and the question is which of such solutions correspond to stable equilibrium.

A standard method to study stability is to start from an equilibrium configuration and perform the transformation $r \rightarrow \lambda r$, $\rho \rightarrow \lambda^{-3}\rho$, which leaves the mass unchanged, that is, $M \rightarrow M$. The binding energy, E, becomes a function of λ, and the configuration is stable if the function $E(\lambda)$ has a minimum for $\lambda = 1$. Thus, the equilibrium and stability of white dwarfs may be studied as follows: The solution of the Newtonian Eq. (73) for a central density ρ_c provides the density $\rho(\rho_c; r)$ as a function of the radial coordinate. We may assume that this function is a good approximation for the solution of Eq. (74) with the said central density ρ_c. Thus, we may use it in order to get the baryon number, N, and the binding energy, E, via Eqs. (75) and (76), as functions of the central density. Now we consider star configurations out of equilibrium, but close to the one given by the function $\rho(\rho_c; r)$, by performing the λ transformation above stated. Thus, we get the binding energy, $E(\rho_c, \lambda)$, and the baryon number, $N(\rho_c, \lambda)$, as functions of ρ_c and λ. They correspond to either equilibrium configurations (i.e., fulfilling Eq. (73), when $\lambda = 1$, or nonequilibrium configurations, when $\lambda \neq 1$. Now rather than taking ρ_c and λ as independent variables, we may use N and λ, finding the binding energy as a function $E(N, \lambda)$. It is common to change the variables to $M \equiv m_H N$ and $\rho_c^* \equiv \lambda^{-3}\rho_c$, so that the function should be written $E = E(M, \rho_c^*)$, or simply $E = E(M, \rho_c)$. Then the conditions of equilibrium and stability are

$$\partial E(M, \rho_c)/\partial \rho_c = 0, \qquad \partial^2 E(M, \rho_c)/\partial \rho_c^2 > 0.$$

But we see from these arguments that we must guarantee that M is indeed the number of baryons N times a constant

m_H. The point is important, especially for the calculation of the GR corrections [20].

Both Eqs. (75) and (76) contain three corrections to Newtonian theory. The first one derives from general relativity and for white dwarfs is of order $GM/R_0 \approx 10^{-4}$. The second is the correction for the vacuum contribution developed in this paper, which is of order $a/R_0^2 \approx 10^{-5}$ (remember that $|b| \sim a \lesssim 10$ km and $R_o \approx 4 \times 10^3$ km). The third one is due to the deviation of the equation of state from the polytropic one. The three corrections are small (the latter for massive enough stars), so that we may calculate each of them as if it were alone, and add the corrections at the end. Calculating the GR correction is delicate due to the fact that the proper volume element is not $4\pi r^2 dr$. This problem, however, does not appear in the vacuum correction, which may be treated, in our approximation, as if we solve a purely Newtonian problem with a density and a pressure modified by the vacuum contributions as given in Eq. (56). I will take the corrections GR and *eos* from the literature [20], that is,

$$E = B(M_{\text{Ch}}^{2/3} - M^{2/3})M\rho_c^{1/3} + CM\rho_c^{-1/3} - DM^{7/3}\rho_c^{2/3},$$
(78)

where B, C, D, and the Chandrasekhar mass, $M_{\text{Ch}}^{2/3}$, are positive constants. (I use the notation of Ref. [20].)

In order to calculate the vacuum correction, I begin writing the binding energy, Eq. (76), taking M^{eff} and N from Eq. (75). The result should be written to first order in G and K (which excludes general relativistic corrections, which are of order G^2 or GK). The quantities ρ^{eff} and m^{eff} are taken from Eqs. (56) and (57), respectively, which includes the vacuum correction to first order in the parameters a and b. This gives

$$E = \int_0^{R_o} 4\pi r^2 dr[(\rho - m_H n) - Gmr^{-1}m_H n]$$
$$+ \int_0^{R_o} 4\pi r^2 dr\left[2a\left(\frac{2}{r}\frac{d\rho}{dr} + \frac{d^2\rho}{dr^2}\right)\right.$$
$$\left. - 8\pi Gar\frac{d\rho}{dr}m_H n\right] + \int_{R_o}^{\infty} \rho_{\text{ext}}4\pi r^2 dr,$$
(79)

the first term of each one of the first two integrals being the internal energy and the second one the gravitational energy. The vacuum correction is given by the last two integrals in Eq. (79). I shall start estimating the third integral. The density outside the star is given by Eq. (28) with M_a and M_b as in Eqs. (21) and (26), respectively. It is known from the Lane-Endem solution of the Newtonian Eq. (73) that the density near the star surface is of the form

$$\rho(r) \simeq C\rho_c^2\left(\frac{G}{K}\right)^{3/2}(R_o - r)^3,$$

where C is a numerical constant. If this is put in the expression of M_a, Eq. (21), we get

$$M_a \simeq 24\pi C\rho_c^2\left(\frac{G}{K}\right)^{3/2}(6a + 2b)^{5/2}R_o.$$

Similarly, we obtain

$$M_b \simeq 24\pi C\rho_c^2\left(\frac{G}{K}\right)^{3/2}(-b)^{5/2}R_o.$$

If we put these expressions in the vacuum density outside the star, Eq. (28), we get

$$\int_{R_o}^{\infty} \rho_{\text{ext}}4\pi r^2 dr = 8\pi C\rho_c^2\left(\frac{G}{K}\right)^{3/2}[(6a + 2b)^2 + 2(-b)^2]R_o^2.$$

The relevant result is that the integral is of order a^2, therefore negligible in comparison with the second integral of Eq. (79), which gives therefore the main contribution to the vacuum correction E^{vac} (the first integral in Eq. (79) is the Newtonian binding energy).

The function $\rho(r)$ is the density of a $\gamma = 4/3$ polytrope and, for the calculation of the integral, we may approximate $m_H n(r) \simeq \rho(r)$. We get

$$E^{\text{vac}} \simeq 2a \int_0^{R_o} 4\pi r dr\left[\frac{d^2(r\rho)}{dr^2} - G\rho 4\pi r^2\frac{d\rho}{dr}\right].$$
(80)

The first term is zero [it equals M^{vac}, see Eq. (57)] and the second one leads, after an integration by parts, to [see Eq. (65)]

$$E^{\text{vac}} \simeq 12\pi aG \int_0^{R_o} \rho^2 4\pi r^2 dr.$$
(81)

It is interesting that the vacuum correction depends only on the parameter a, but not on b.

The integral Eq. (81) may be performed numerically in terms of the central density, ρ_c, and the Lane-Emden variables ξ and θ. We get

$$E^{\text{vac}} = 12\pi aGM\rho_c\xi_1^{-2}|\theta'(\xi_1)|^{-1} \int_0^{\xi_1} \theta^6\xi^2 d\xi$$
$$\simeq 23.4aGM\rho_c.$$
(82)

Adding the vacuum correction, Eq. (82), to the standard expression for the energy we obtain from Eq. (79),

$$E = B(M_{\text{Ch}}^{2/3} - M^{2/3})M\rho_c^{1/3} + CM\rho_c^{-1/3} - DM^{7/3}\rho_c^{2/3} + FM\rho_c,$$
(83)

with

$$F \equiv 23.4aG.$$

The first term in Eq. (83) is the Newtonian energy, the second is due to the departure of the equation of state from a $\gamma = 4/3$ polytrope, the third one is the correction of general relativity, and the last term is the vacuum correction. For a given mass, the central density of equilibrium is obtained when $dE/d\rho_c = 0$, which leads to

$$B(M_{\rm Ch}^{2/3} - M^{2/3}) - C\rho_c^{-2/3} - 2DM^{4/3}\rho_c^{1/3} + 3F\rho_c^{2/3} = 0. \tag{84}$$

Taking into account that the parameters B, C, D are positive, it is easy to see that if $F = 0$ no value (positive) of ρ_c fulfills Eq. (84) whenever $M > M_{\rm Ch}$, that is, equilibrium is not possible. In sharp contrast a value of ρ_c fulfilling Eq. (84) exists for any mass if $F > 0$. In fact, the quantity in the left side of Eq. (84) approaches $+\infty$ when $\rho_c \to \infty$ and $-\infty$ when $\rho_c \to 0$, so that it is zero by continuity for some positive value of ρ_c.

For a mass, M, well below the Chandrasekhar limit the terms with D and F may be considered small in comparison with those with B and C, whence the central density of equilibrium is

$$\rho_c \simeq \rho_0 + \frac{3DM^{4/3}}{C}\rho_0^2 - \frac{9F}{2C}\rho_0^{7/3}, \tag{85}$$

$$\rho_0 \equiv \left(\frac{C}{B}\right)^{3/2}(M_{\rm Ch}^{2/3} - M^{2/3})^{-3/2}.$$

When M approaches the Chandrasekhar limit, $M_{\rm Ch}$, Eq. (85) diverges and the approximations leading to it are not valid. I shall not attempt to solve Eq. (84) for M close to $M_{\rm Ch}$, which would be cumbersome and not very interesting. When $M \gg M_{\rm Ch}$ an approximate solution of Eq. (84) is again possible, leading to the simple expression

$$\rho_c \simeq \left[\frac{B(M^{2/3} - M_{\rm Ch}^{2/3})}{3F}\right]^{3/2}$$

$$\simeq 8.7 \times 10^{-4} a^{-3/2}(M^{2/3} - M_{\rm Ch}^{2/3})^{3/2}. \tag{86}$$

However, the central density, ρ_c, obtained this way is so big that the approximations leading to it do not apply (e.g., for $M/M_{\rm Ch} = 1.01$ we get $\rho_c \approx 10^{19}$ kg/m^3, greater than neutron star densities.)

Stability requires that $d^2E/d\rho_c^2 > 0$. When $dE/d\rho_c = 0$ the stability condition leads to

$$C\rho_c^{-1} - DM^{4/3} + 3F\rho_c^{1/3} \geq 0. \tag{87}$$

If $F = 0$, equilibrium is stable for any mass below a limit smaller than but close to the Chandrasekhar mass. If $F > 0$ stars may be in equilibrium for any mass, as said above. That hydrostatic equilibrium is always stable, i.e., for any mass, M, and for any value of $F > 0$. In fact, Eq. (84) may be rewritten

$$3FM\rho_c^{2/3} = B(M^{2/3} - M_{\rm Ch}^{2/3})M + CM\rho_c^{-2/3}$$

$$+ 2DM^{7/3}\rho_c^{1/3} > DM^{4/3} - C\rho_c^{-1},$$

so that Eq. (87) holds true (taking into account that B, C, $D > 0$, $M > M_{\rm Ch}$). Actually, the derived results for stars above the Chandrasekhar limit are rather academic because for the probable value of F, that is, when the parameter $\sqrt{a}$ lies below the limit derived in Sec. IV, the central density of equilibrium of stars with M close to or larger than $M_{\rm Ch}$ would be so large that the star becomes unstable against neutronization before reaching hydrostatic equilibrium.

In summary, vacuum corrections following from the action Eq. (4) might give rise to dramatic changes in the equilibrium and stability of white dwarfs, but these changes would produce very small observable effects because they are hidden by the existence of instabilities due to neutronization.

VIII. CONCLUSIONS

If we believe that the quantum vacuum in curved space-time gives rise to some stress energy, it is plausible that the gravitational effect of this contribution is equivalent to adding the term $aR^2 + bR_{\mu\nu}R^{\mu\nu}$ to the standard term, R, of the Einstein-Hilbert action, at least if the curvature scalar R is not large. The presence of this term might give rise to effects observable, in principle, in terrestrial and the Solar System observations. Then the present knowledge puts bounds on the possible values of the parameters a and b, namely, $\sqrt{a}$, $\sqrt{-b} < 50$ km. Also, the vacuum stress energy will produce some effects in the structure of Newtonian or slightly relativistic stars. In particular, in white dwarfs it might produce stable hydrostatic equilibrium in stars above the Chandrasekhar limit, but the effect could not be observed because such stars would be unstable against neutronization. The effect of the vacuum stress energy on the internal structure of the Sun would be too small to be detected by observation. Our calculations suggest that the vacuum effects should be important in compact bodies like neutron stars.

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