Mueller matrix differential decomposition for direction reversal: application to samples measured in reflection and backscattering

Noé Ortega-Quijano* and José Luis Arce-Diego

Applied Optical Techniques Group, Electronics Technology, Systems and Automation Engineering Department, University of Cantabria, Avenida de los Castros S/N, 39005 Santander, Cantabria, Spain
**ortegan@unican.es*

Abstract: Mueller matrix differential decomposition is a novel method for analyzing the polarimetric properties of optical samples. It is performed through an eigenanalysis of the Mueller matrix and the subsequent decomposition of the corresponding differential Mueller matrix into the complete set of 16 differential matrices which characterize depolarizing anisotropic media. The method has been proposed so far only for measurements in transmission configuration. In this work the method is extended to the backward direction. The modifications of the differential matrices according to the reference system are discussed. The method is successfully applied to Mueller matrices measured in reflection and backscattering.

OCIS codes: (260.5430) Polarization; (260.2130) Ellipsometry and polarimetry; (120.5410) Polarimetry.

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1. Introduction

The polarimetric characterization of optical media by Mueller matrices has become a widely used technique in many fields, mainly due to its capacity to characterize depolarization and to its suitability for experimental applications. The differential formulation of the Mueller calculus was first presented more than 30 years ago [1]. It constitutes a powerful method for studying the evolution of polarized light propagation in optical media. However, it was only

#146875 - $15.00 USD Received 3 May 2011; revised 31 May 2011; accepted 31 May 2011; published 12 Jul 2011

proposed for non-depolarizing media. Therefore, the ability of Mueller calculus to deal with partially polarized light and depolarization effects has been underutilized so far.

Recently, the complete set of 16 differential Mueller matrices for describing general depolarizing anisotropic media has been presented and discussed [2]. They enable to apply the general differential Mueller calculus to a vast range of theoretical and experimental applications in many fields of interest in Optics. A novel Mueller matrix decomposition method has been proposed based on this theoretical frame [3]. Among its main advantages, it should be emphasized that it is especially adequate for the study of media with simultaneously occurring effects [3,4] and it remarkably constitutes the first unique Mueller matrix decomposition.

Mueller matrix differential decomposition has only been proposed for the forward direction, i.e. for Mueller matrices measured in transmission. However, there is a wide range of applications in which measurements are performed in the backward direction. In this work we discuss the modifications in the basic differential Mueller matrices that are involved in light beam propagation direction reversal, and we apply them to extend the differential decomposition to measurements performed in the backward direction. The decomposition is successfully applied to several media measured in reflection and in backscattering.

2. Mueller matrix differential decomposition for direction reversal

We consider a beam propagating along the z axis in a right-handed Cartesian coordinate system and assume that the beam always travels towards the observer (Fig. 1). According to the differential formulation of Mueller calculus, the Stokes vector satisfies

$$d\vec{\mathbf{S}}/dz = \mathbf{m}\vec{\mathbf{S}}, \tag{1}$$

where $\vec{\mathbf{S}}$ is the Stokes vector that describes the beam, and $\mathbf{m}$ is the 4x4 differential Mueller matrix that characterizes the polarimetric behavior of an infinitesimal slab of the medium [1]. The differential matrix is related to a corresponding macroscopic Mueller matrix by

$$\mathbf{m} = \left(d\mathbf{M}/dz\right)\mathbf{M}^{-1}. \tag{2}$$

$\mathbf{M}$ is the Mueller matrix that describes the medium from z_0 to z. It can be shown that the eigenvalues of the macroscopic and differential Mueller matrices ($\lambda_{\mathbf{M}}$ and $\lambda_{\mathbf{m}}$ respectively) are related in full parallelism with Eq. (2), while remarkably, the eigenvectors of the macroscopic and the differential matrices (grouped by columns in matrices $\mathbf{V_M}$ and $\mathbf{V_m}$) are the same [5]. The eigenvectors matrices will be thus denoted by $\mathbf{V}$. If the initial condition $\mathbf{M}_{\mathbf{z_0=0}} = \mathbf{I}$ is fulfilled ($\mathbf{I}$ being the 4x4 identity matrix), then $\lambda_{\mathbf{M}(z_0=0)} = 1$, and the eigenvalues of $\mathbf{M}$ and $\mathbf{m}$ are related by

$$\lambda_{\mathbf{m}} = \ln\left(\lambda_{\mathbf{M}}\right)/z. \tag{3}$$

Therefore, assuming that the macroscopic Mueller matrix is diagonalizable, the differential Mueller matrix can be obtained from the eigenanalysis of $\mathbf{M}$ by

$$\mathbf{m} = \mathbf{V}\mathbf{m}_{\lambda}\mathbf{V}^{-1}, \tag{4}$$

where the non-zero diagonal elements of $\mathbf{m}_{\lambda}$ are the eigenvalues given in Eq. (3).

An infinitesimal slab of a general depolarizing anisotropic medium situated between z and $z + \Delta z$ can be divided into 16 lamellae corresponding to the basic types of optical behavior. This situation is schematically depicted in Fig. 1. Each of them has an associated differential Mueller matrix. The complete set of basic differential Mueller matrices has been previously presented and discussed elsewhere [2]. It is composed of 7 differential matrices corresponding to non-depolarizing effects and 9 additional differential matrices for depolarizing media. Each of them has an associated differential parameter that quantifies the contribution of each

#146875 - $15.00 USD Received 3 May 2011; revised 31 May 2011; accepted 31 May 2011; published 12 Jul 2011

optical effect to the total differential Mueller matrix. Specifically, the first matrix $\mathbf{m_1}$ corresponds to isotropic absorption (associated differential parameter κ_i), matrices $\mathbf{m_{2,3,4}}$ describe linear *x*-*y*, linear ± 45° and circular dichroism, respectively (differential parameters $\kappa_{q,u,v}$) and matrices $\mathbf{m_{5,6,7}}$ account for the three types of birefringence (differential parameters $\eta_{q,u,v}$). Depolarizing differential matrices $\mathbf{m_{8,9,10}}$ are the differential Mueller matrices for diagonal depolarization (characterized by differential parameters $\kappa'_{iq,iu,iv}$), while $\mathbf{m_{11,12,13}}$ and $\mathbf{m_{14,15,16}}$ correspond to the different types of anomalous dichroism and anomalous depolarization (differential parameters $\kappa'_{q,u,v}$ and $\eta'_{q,u,v}$ respectively) [2]. The non-depolarizing differential parameters are directly related to the propagation constant $\tilde{\eta} = \eta + i\kappa$, as already pointed out [2]. The depolarizing differential parameters are given in a generic form. The sign convention used in this work is $w_{q,u,v} = \left(w_{x,+45,rcp} - w_{y,-45,lcp}\right)/2$, where w is either η or κ. The general form of the differential Mueller matrix for depolarizing anisotropic media in the forward direction is the summation of the 16 basic differential matrices

$$\mathbf{m^f} = \sum_{n=1}^{16} \mathbf{m_n^f} = \begin{bmatrix} \kappa_i & \kappa_q + \kappa'_q & \kappa_u + \kappa'_u & \kappa_v + \kappa'_v \\ \kappa_q - \kappa'_q & \kappa_i - \kappa'_{iq} & \eta_v + \eta'_v & \eta_u + \eta'_u \\ \kappa_u - \kappa'_u & -\eta_v + \eta'_v & \kappa_i - \kappa'_{iu} & \eta_q + \eta'_q \\ \kappa_v - \kappa'_v & -\eta_u + \eta'_u & -\eta_q + \eta'_q & \kappa_i - \kappa'_{iv} \end{bmatrix}. \tag{5}$$

Therefore, any differential Mueller matrix can be expanded into a weighted sum of the complete set of basic differential Mueller matrices. The decomposition of $\mathbf{m}$ according to the general expression given in Eq. (5) enables to obtain the 16 parameters associated with each optical effect [3]. It is important to emphasize that the differential matrix is order independent as a consequence of the commutative property of matrix addition and the infinitesimal nature of the differential Mueller matrix [5]. Moreover, it can be easily demonstrated that decomposing the Mueller matrix of a certain homogeneous sample into the 16 basic differential Mueller matrices is equivalent to describing the sample by an infinite number of identical slabs subdivided into 16 lamellae that account for each optical property [5]. This property reinforces the experimental suitability of this decomposition. As a result of the order independence, the differential Mueller matrix decomposition is a unique decomposition, which is particularly important for avoiding decomposition ambiguities in many applications [3]. It should be noted that the calculation of the differential matrix from a Mueller matrix requires to know *z*, i.e. the optical path undergone by the measured photons. The determination of *z* is not readily achievable in many applications. In general, for those situations in which *z* is unknown, the optical path weighted differential matrix $\bar{\mathbf{m}} = \mathbf{m}z$ will be obtained. This matrix contains the information about accumulated effects, instead of effects per unit length. The differential decomposition thus results in accumulated differential parameters, which will be denoted with an upper bar in parallelism with $\bar{\mathbf{m}}$.

The procedure described above is limited to the forward direction. However, there are many applications in which the direction is modified and measurements are performed in the backward direction. The application of the differential decomposition for situations with light propagation reversal requires a detailed analysis.

The modifications of the differential matrices for direction reversal have been previously discussed for differential Jones matrices [6]. In this work we will develop a similar procedure for Mueller differential calculus, and it will be applied to perform the differential Mueller decomposition for measurements performed in the reverse direction.

#146875 - $15.00 USD Received 3 May 2011; revised 31 May 2011; accepted 31 May 2011; published 12 Jul 2011

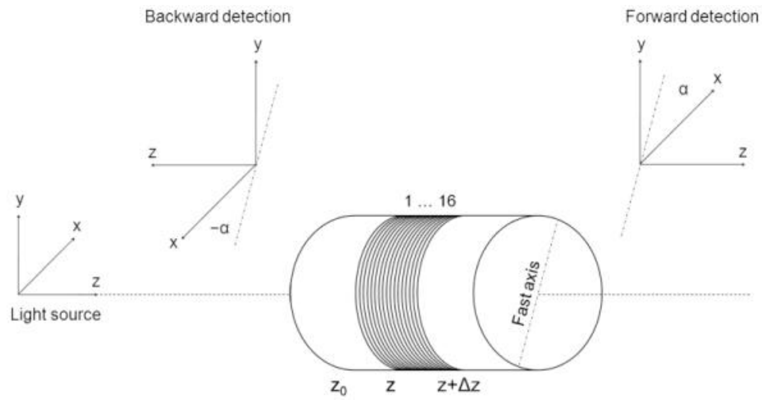

Fig. 1. Definition of the reference system and the conventions used in this work. The 16 lamellae for the infinitesimal slab of the medium between z and $z + \Delta z$ are depicted.

We consider an optical medium with a generic fast axis to illustrate this situation, as shown in Fig. 1. It is assumed that the optical beam always travels towards the observer, in agreement with the reference system and the conventions adopted at the beginning of this work. The azimuth angle α is measured counterclockwise from the x axis. For the specific example considered, the observed azimuth angle is positive when the measurement is performed in transmission, but it remarkably takes the opposite sign when measuring in the backward direction as a result of the reference and sign convention (Fig. 1). Therefore, it is self-evident that the measurement configuration has to be taken into account for the correct analysis of the measured polarimetric properties of the sample. The introduction of these considerations into the basic differential Mueller matrices entails a change of sign of the basic types of optical behavior for the linear ± 45° direction. This result is analogous to the conclusion obtained for differential Jones matrices [6]. In particular, the sign of the differential matrices for ± 45° birefringence, dichroism, anomalous birefringence and anomalous dichroism must be inverted for direction reversal. As a result, the differential Mueller matrix for the backward direction is

$$\mathbf{m^b} = \sum_{n=1}^{16} \mathbf{m_n^b} = \begin{bmatrix} \kappa_i & \kappa_q + \kappa_q^{'} & -\kappa_u - \kappa_u^{'} & \kappa_v + \kappa_v^{'} \\ \kappa_q - \kappa_q^{'} & \kappa_i - \kappa_{iq}^{'} & \eta_v + \eta_v^{'} & -\eta_u - \eta_u^{'} \\ -\kappa_u + \kappa_u^{'} & -\eta_v + \eta_v^{'} & \kappa_i - \kappa_{iu}^{'} & \eta_q + \eta_q^{'} \\ \kappa_v - \kappa_v^{'} & +\eta_u - \eta_u^{'} & -\eta_q + \eta_q^{'} & \kappa_i - \kappa_{iv}^{'} \end{bmatrix}. \tag{6}$$

3. Application to experimental Mueller matrices

The differential decomposition has been previously verified for several types of media in transmission configuration [3]. Now the method is applied to several homogeneous media measured in reflection and backscattering configuration.

The first sample is a nylon target studied in specular reflection [7]. The measured matrix $\mathbf{M_1}$ is included in Table 1. The calculation of the corresponding differential Mueller matrix through the eigenanalysis described above enables one to obtain the accumulated matrix $\bar{\mathbf{m}}_1$ (Table 2). The application of the differential decomposition yields $\bar{\kappa}_{iq}^{'} = 2.0509$, $\bar{\kappa}_{iu}^{'} = 2.0542$ and $\bar{\kappa}_{iv}^{'} = 2.1497$. The other parameters are negligible. The results are included in Table 3. The analysis indicates that it is a depolarizing medium with nearly polarization independent isotropic depolarization.

#146875 - $15.00 USD Received 3 May 2011; revised 31 May 2011; accepted 31 May 2011; published 12 Jul 2011

Table 1. Experimental Mueller Matrices Considered in this Work

Samples measured in reflection configuration	
$\mathbf{M}_1$	$\mathbf{M}_2$
$\begin{bmatrix} 1.0000 & 0.0063 & 0.0157 & 0.0077 \\ -0.0031 & 0.1286 & -0.0005 & 0.0035 \\ -0.0016 & 0.0008 & -0.1282 & 0.0019 \\ -0.0025 & 0.0010 & -0.0040 & -0.1165 \end{bmatrix}$	$\begin{bmatrix} 1.0000 & 0.1631 & -0.0322 & 0.0802 \\ 0.0083 & 0.4038 & 0.2555 & -0.2158 \\ -0.0026 & 0.4297 & -0.1376 & 0.2016 \\ -0.0116 & 0.0597 & -0.3175 & -0.3690 \end{bmatrix}$
Samples measured in backscattering configuration	
$\mathbf{M}_3$	$\mathbf{M}_4$
$\begin{bmatrix} 1.0000 & -0.0034 & 0.0008 & -0.0058 \\ -0.0049 & 0.2277 & 0.0082 & 0.0016 \\ -0.0017 & 0.0102 & -0.2268 & 0.0045 \\ 0.0007 & 0.0026 & -0.0004 & -0.1001 \end{bmatrix}$	$\begin{bmatrix} 1.000 & -0.115 & -0.066 & 0.023 \\ -0.111 & 0.759 & -0.061 & -0.001 \\ -0.018 & 0.151 & -0.435 & -0.139 \\ -0.046 & 0.006 & 0.128 & -0.334 \end{bmatrix}$

In order to verify these results, we will compare differential decomposition with the well known polar decomposition [8,9]. It quantifies the depolarizing behavior by the depolarization power, which is obtained from the depolarizing component obtained from the decomposition process. In this case it takes a value of 0.8755. In order to verify our results, we obtain our depolarizing matrix from the diagonal depolarization differential matrices weighted by $\kappa_{iq}^{'}$, $\kappa_{iu}^{'}$ and $\kappa_{iv}^{'}$ using the procedure involved in Eq. (3) and Eq. (4). The calculation of the depolarization power of this matrix leads to a value that is identical to the one obtained by the polar decomposition, which corroborates the validity of the results obtained with the differential decomposition. As well as that, it should be noted that a marginal total retardance value of 1.37 degrees is obtained, which is also in excellent agreement with the results reported elsewhere [7]. A similar comparison procedure will be performed for the remaining examples.

Table 2. Differential Mueller Matrices Corresponding to the Experimental Matrices Included in Table 1

Samples measured in reflection configuration	
$\bar{\mathbf{m}}_1$	$\bar{\mathbf{m}}_2$
$\begin{bmatrix} 0.0000 & 0.0149 & 0.0283 & 0.0151 \\ -0.0073 & -2.0508 & -0.0006 & 0.0013 \\ -0.0030 & 0.0001 & -2.0541 & -0.0156 \\ -0.0048 & 0.0003 & 0.0323 & -2.1497 \end{bmatrix}$	$\begin{bmatrix} 0.0015 & 0.1178 & 0.1534 & 0.2332 \\ 0.0019 & -0.4872 & -0.3283 & -0.9559 \\ 0.0146 & 0.2462 & -0.5238 & 2.0819 \\ -0.0246 & 1.2127 & -2.2674 & -1.1306 \end{bmatrix}$
Samples measured in backscattering configuration	
$\bar{\mathbf{m}}_3$	$\bar{\mathbf{m}}_4$
$\begin{bmatrix} 0.0000 & -0.0065 & 0.0009 & -0.0121 \\ -0.0094 & -1.4790 & 0.0000 & 0.0047 \\ -0.0021 & 0.0003 & -1.4828 & -0.0286 \\ 0.0015 & 0.0064 & 0.0030 & -2.3014 \end{bmatrix}$	$\begin{bmatrix} -0.0143 & -0.1200 & -0.1559 & -0.1600 \\ -0.1344 & -0.2789 & -0.1122 & -0.1822 \\ -0.1208 & 0.2666 & -2.0020 & -3.1858 \\ 0.0023 & -0.3654 & 2.8916 & 0.1576 \end{bmatrix}$

The second sample is a steel specimen with a film of MgF_2. The measurement was also performed in reflection [10]. The Mueller matrix of this sample is $\mathbf{M}_2$ (Table 1). The decomposition of this matrix presents two dominant types of behavior, namely birefringence

#146875 - $15.00 USD Received 3 May 2011; revised 31 May 2011; accepted 31 May 2011; published 12 Jul 2011

and polarization dependent isotropic depolarization (diagonal depolarization). The differential parameters obtained for this medium are shown in Table 3. It can be observed the medium shows a significant total linear birefringence (2.43 radians). There is also some optical activity. Regarding depolarization it can be seen that it depolarizes circularly polarized light significantly over linearly polarized light. In this case, the coincidence with the values obtained by polar decomposition is also very good (we obtain 2.429 radians of total linear retardance and 0.491 depolarization power from the calculated depolarizing matrix, while the values obtained with polar decomposition are 2.377 radians and 0.492, respectively).

The method can also be applied to measurements in backscattering configuration. We first consider the Mueller matrix $\mathbf{M_3}$ (Table 1) of a dielectric sample [7]. Again, the accumulated differential Mueller matrix $\bar{\mathbf{m}}_3$ is included in Table 2. This matrix only presents three non-zero accumulated differential parameters, which correspond to diagonal depolarization (Table 3). In this case, the calculation of the depolarizing power by the two methods described above are identical, with a value of 0.8150 for both cases.

Table 3. Accumulated Differential Parameters for the Experimental Mueller Matrices $\mathbf{M_1}$, $\mathbf{M_2}$, $\mathbf{M_3}$, and $\mathbf{M_4}$

	$\bar{\kappa}_i$	$\bar{\kappa}_q$	$\bar{\kappa}_u$	$\bar{\kappa}_v$	$\bar{\eta}_q$	$\bar{\eta}_u$	$\bar{\eta}_v$	$\bar{\kappa}'_{iq}$	$\bar{\kappa}'_{iu}$	$\bar{\kappa}'_{iv}$	$\bar{\kappa}'_q$	$\bar{\kappa}'_u$	$\bar{\kappa}'_v$	$\bar{\eta}'_q$	$\bar{\eta}'_u$	$\bar{\eta}'_v$
$\mathbf{M_1}$	0.00	0.00	−0.01	0.01	−0.02	−0.00	−0.00	2.05	2.05	2.15	0.01	−0.02	0.01	0.01	−0.00	−0.00
$\mathbf{M_2}$	0.00	0.06	−0.08	0.10	2.17	1.08	−0.29	0.49	0.53	1.13	0.06	−0.07	1.13	−0.09	−0.13	−0.04
$\mathbf{M_3}$	0.00	−0.01	0.00	−0.01	0.01	0.00	−0.00	1.47	1.48	2.30	0.00	−0.00	−0.01	−0.01	−0.00	0.00
$\mathbf{M_4}$	−0.02	−0.13	0.14	−0.08	−3.04	−0.09	−0.19	0.26	1.99	−0.17	0.01	0.02	−0.08	−0.15	0.27	0.08

Finally, we consider a birefringent chiral turbid sample measured in backscattering ($\mathbf{M_4}$, Table 1). The sample is an aqueous suspension of polystyrene microspheres with a 5M concentration of glucose [11]. This medium is intended for its use as a biological tissue phantom. The results of applying the differential decomposition to this matrix are included in Table 3. They reveal that the sample presents linear retardance and diagonal depolarization. The optical activity of the sample nearly vanishes due to the reasons pointed out in [11]. The calculated total linear retardance is 3.04 radians, while polar decomposition gives a value of 2.81 radians. The depolarizing power of polar decomposition is 0.4725. If that value is obtained by the method described above, we get a value of 0.3026. Although the differential decomposition appropriately characterizes the optical behavior of the sample, small discrepancies are observed in this case with polar decomposition. Further studies may be required to study the differences between both decompositions for turbid media with simultaneously occurring optical effects.

4. Conclusion

The extension of the Mueller matrix differential decomposition to systems with direction reversal has been presented. The validation of this method has been demonstrated for several types of media measured in reflection and backscattering configuration. The results of this work can significantly broaden the potential of the differential decomposition to applications where measurements have to be made in a non-invasive way, such as optical monitorization of fragile materials or in vivo polarimetric imaging of biological tissues.

Acknowledgments

This work has been partially funded by the San Cándido Foundation, Santander (Spain).

#146875 - $15.00 USD Received 3 May 2011; revised 31 May 2011; accepted 31 May 2011; published 12 Jul 2011